Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 101

# Characteristic Function of a Pure Contractive Tuple

T. Bhattacharyya, J. Eschmeier and J. Sarkar

Saarbrücken 2003

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 101
submitted: December 15, 2003

# Characteristic Function of a Pure Contractive Tuple

**T. Bhattacharyya**
Department of Mathematics
Indian Institute of Science
Bangalore 560012
India
tirtha@math.iisc.ernet.in

**J. Eschmeier**
Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
eschmei@math.uni-sb.de

**J. Sarkar**
Department of Mathematics
Indian Institute of Science
Bangalore 560012
India
jaydeb@math.iisc.ernet.in

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

# CHARACTERISTIC FUNCTION OF A PURE CONTRACTIVE TUPLE

T. BHATTACHARYYA, J. ESCHMEIER, AND J. SARKAR

Abstract. A theorem of Sz.-Nagy and Foias [9] shows that the characteristic function $\theta_T(z) = -T + zD_{T^*}(1_{\mathcal{H}} - zT^*)^{-1}D_T$ of a completely non-unitary contraction $T$ is a complete unitary invariant for $T$. In this note we extend this theorem to the case of a pure commuting contractive tuple using a natural generalization of the characteristic function to an operator-valued analytic function defined on the open unit ball of $\mathbb{C}^n$. This function is related to the curvature invariant introduced by Arveson [3].

## 1. Introduction

A contraction $T$ acting on a Hilbert space $\mathcal{H}$ is said to be completely non-unitary (c.n.u.) if there is no non-zero reducing subspace $\mathcal{M}$ of $\mathcal{H}$ such that $T|\mathcal{M}$ is a unitary operator. The class of completely non-unitary operators plays an important role in understanding general contractions because, given any contraction $T$ on a Hilbert space $\mathcal{H}$, there is a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of $\mathcal{H}$ into orthogonal subspaces each of which is a reducing subspace for $T$ such that $T_0 = T|\mathcal{H}_0$ is unitary while $T_1 = T|\mathcal{H}_1$ is a c.n.u. contraction. A key ingredient for studying contraction operators on Hilbert spaces is the following analytic operator-valued function, called the characteristic function of $T$ and introduced by Sz.-Nagy and Foias in [9]:

$$\theta_T(z) = -T + zD_{T^*}(1_{\mathcal{H}} - zT^*)^{-1}D_T, \quad z \in \mathbb{D}. \tag{1.1}$$

Here $\mathbb{D}$ is the open unit disk in the complex plane. The operators $D_T$ and $D_{T^*}$ are the so-called defect operators $(1_{\mathcal{H}} - T^*T)^{1/2}$ and $(1_{\mathcal{H}} - TT^*)^{1/2}$ of $T$ and $T^*$, respectively. By virtue of the relation $TD_T = D_{T^*}T$ (see Section I.3 in [9]), the values $\theta_T(z)$ of the characteristic function can be regarded as bounded operators from $\mathcal{D}_T = \overline{\mathrm{Ran}} D_T$ into $\mathcal{D}_{T^*} = \overline{\mathrm{Ran}} D_{T^*}$.

It is shown in [9] that $\theta_T(z)$ is contraction valued and that $\|\theta_T(0)\xi\| < \|\xi\|$ for all $\xi \in \mathcal{D}_T$. The characteristic functions $\theta_T$ and $\theta_R$ of two contractions $T$ and $R$ are said to coincide if there are unitary operators $\sigma_1 : \mathcal{D}_T \to \mathcal{D}_R$ and $\sigma_2 : \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ such that

$$\theta_T(z) = \sigma_2^{-1}\theta_R(z)\sigma_1 \quad \text{for all} \quad z \in \mathbb{D}. \tag{1.2}$$

It is easy to see that if $T$ and $R$ are two unitarily equivalent contractions, i.e., if there is a unitary operator $U$ such that $T = URU^*$, then the characteristic functions $\theta_T$ and $\theta_R$ coincide. One can easily construct examples to show that the converse of this is not true in this generality (see page 240 in [9]). However, the converse is true if both $T$ and $R$ are c.n.u. contractions.

THEOREM 1.1. ***(Sz.-Nagy and Foias)*** *Two completely non-unitary contractions are unitarily equivalent if and only if their characteristic functions coincide.*

This theorem shows that the characteristic function is a complete unitary invariant for c.n.u. contractions. The route to prove the theorem is via constructing a functional model for c.n.u. contractions which is also of independent interest. We briefly recall some essential features of this model theory relevant to us here. Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{C}^n$. If $\mathcal{E}$ is a complex Hilbert space, we follow the notation of [4] and define $\mathcal{O}(\mathbb{B}^n, \mathcal{E})$ to be the class of all $\mathcal{E}$-valued analytic functions on $\mathbb{B}^n$. For any multi-index $k = (k_1, \dots, k_n) \in \mathbb{N}^n$, we write $|k| = k_1 + \cdots + k_n$. Then consider the Hilbert space

$$H(\mathcal{E}) = \{f \in \mathcal{O}(\mathbb{B}, \mathcal{E}) : \ f = \sum_{k \in \mathbb{N}^n} a_k z^k \text{ with } a_k \in \mathcal{E} \text{ and } \|f\|^2 = \sum_{k \in \mathbb{N}^n} \frac{\|a_k\|^2}{\gamma_k} < \infty\}, \tag{1.1}$$

where $\gamma_k = |k|!/k!$. One can show that $H(\mathcal{E})$ is the $\mathcal{E}$-valued functional Hilbert space given by the reproducing kernel $(1 - \langle z, w\rangle)^{-1} 1_{\mathcal{E}}$. Of course, when $n = 1$ and $\mathcal{E} = \mathbb{C}$, this is the usual Hardy space on the disk. Given complex Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, the multiplier space $M(\mathcal{E}, \mathcal{E}_*)$ consists of all $\varphi \in \mathcal{O}(\mathbb{B}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ such that $\varphi H(\mathcal{E}) \subset H(\mathcal{E}_*)$. By the closed graph theorem, for each function $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$,the induced multiplication operator $M_\varphi : H(\mathcal{E}) \to H(\mathcal{E}_*)$, $f \mapsto \varphi f$ is continuous.
The Sz.-Nagy and Foias model theory works for c.n.u. contractions $T$. Here we shall confine ourselves to a more restricted class. The characteristic function of a single contraction $T$ is a multiplier from the Hardy space $H(\mathcal{D}_T)$ to the Hardy space $H(\mathcal{D}_{T^*})$. A contraction $T$ is said to be of class $C_{\cdot 0}$ if $T^m$ converges strongly to 0 as $m \to \infty$. It is easy to see that each $C_{\cdot 0}$ contraction is completely non-unitary. If $T$ is a $C_{\cdot 0}$ contraction acting on a Hilbert space $\mathcal{H}$, then there is a unitary operator $U$ from $\mathcal{H}$ onto $\mathbb{H} = H(\mathcal{D}_{T^*}) \ominus M_{\theta_T} H(\mathcal{D}_T)$ such that $UTU^* = P_{\mathbb{H}} M_z|\mathbb{H}$ where $M_z$ is the multiplication operator with the independent variable $z$ on $H(\mathcal{D}_{T^*})$. Thus any $C_{\cdot 0}$ contraction can be realized as $P_{\mathbb{H}} M_z|\mathbb{H}$ where the model space $\mathbb{H}$ is the orthocomplement of the range of $M_{\theta_T}$.
In this note, we generalize Theorem 1.1 to the case of pure commuting contractive tuples. So we construct an operator-valued holomorphic function on the open unit ball in $\mathbb{C}^n$ and show that it is a complete unitary invariant for a pure commuting contractive tuple. En route we also construct a functional model for such a tuple.
Previously, Frazho [5] and Popescu [8] have considered characteristic functions for tuples of non-commuting operators. Since they are dealing with non-commuting families of operators, the characteristic function is actually an operator. The characteristic function in that case is a complete unitary invariant for a completely non-coisometric contractive family [8]. It is not clear how the characteristic function of a not necessarily commuting tuple is related to the one defined below in case the tuple consists of commuting operators.

## 2. Definition of the Characteristic Function

A commuting tuple of bounded operators $T = (T_1, \dots, T_n)$ acting on a Hilbert space $\mathcal{H}$ is called contractive if $\|T_1 h_1 + \cdots + T_n h_n\|^2 \le \|h_1\|^2 \cdots + \|h_n\|^2$ for all $h_1, \dots h_n$ in $\mathcal{H}$. This is equivalent to demanding that $\sum_{i=1}^n T_i T_i^* \le 1_{\mathcal{H}}$. The positive operator $(1_{\mathcal{H}} - \sum_{i=1}^n T_i T_i^*)^{1/2}$ and the closure of its range will be called the *defect operator* $D_{T^*}$ and the *defect space* $\mathcal{D}_{T^*}$ of $T^*$.
We shall also denote by $T$ the bounded operator from $\mathcal{H}^n$ to $\mathcal{H}$ which maps $(h_1, h_2, \dots, h_n)$ to $T_1 h_1 + T_2 h_2 + \cdots + T_n h_n$. The adjoint $T^* : \mathcal{H} \to \mathcal{H}^n$ maps $h$ to the column vector $(T_1^* h, T_2^* h, \dots, T_n^* h)$ and, in fact, $T$ is a contractive tuple if and only if the operator $T$ is a contraction. Thus for a contractive tuple $T$ one can also consider the defect operator $D_T = (1_{\mathcal{H}^n} - T^* T)^{1/2} = ((\delta_{ij} 1_{\mathcal{H}} - T_i^* T_j))^{1/2}$ in $\mathcal{B}(\mathcal{H}^n)$ and the associated defect space $\mathcal{D}_T = \overline{\mathrm{Ran}} D_T \subset \mathcal{H}^n$.

Lemma 2.1. *For any commuting contractive tuple $T$, we obtain the identity*

$$TD_T = D_{T^*} T.$$

Proof. Using the definitions of the defect operators we obtain that

$$\begin{aligned} TD_T^2 &= (T_1, T_2, \dots, T_n)((\delta_{ij} 1_{\mathcal{H}} - T_i^* T_j)) \\ &= (T_1 - \sum T_i T_i^* T_1, T_2 - \sum T_i T_i^* T_2, \dots, T_n - \sum T_i T_i^* T_n) \\ &= ((1_{\mathcal{H}} - \sum T_i T_i^*) T_1, (1_{\mathcal{H}} - \sum T_i T_i^*) T_2, \dots, (1_{\mathcal{H}} - \sum T_i T_i^*) T_n) \\ &= D_{T^*}^2 T. \end{aligned}$$

Thus, for any polynomial $p$, we have

$$T\, p(D_T^2) = p(D_{T^*}^2) T.$$

Now the result follows by taking a sequence of polynomials $p_m$ approximating $\sqrt{x}$ uniformly on the interval $0 \le x \le 1$. ∎

Note that, for $z = (z_1, \dots, z_n) \in \mathbb{B}^n$, the operator $Z$ from $\mathcal{H}^n$ to $\mathcal{H}$ which maps $(h_1, \dots, h_n)$ to $z_1 h_1 + \cdots + z_n h_n$ is a contraction because $ZZ^* = \sum |z_i|^2 1_{\mathcal{H}}$. Thus $Z = (z_1 1_{\mathcal{H}}, \dots, z_n 1_{\mathcal{H}})$ is a commuting contractive tuple on $\mathcal{H}$ with $\|Z\| = (\sum |z_i|^2)^{1/2}$. Hence, given a commuting contractive tuple $T$, the operator $ZT^*$ is a strict contraction for $z \in \mathbb{B}^n$ and $1_{\mathcal{H}} - ZT^*$ is invertible. We define the characteristic function of $T$ to be the analytic operator-valued function $\theta_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ with

$$\theta_T(z) = -T + D_{T^*}(1_{\mathcal{H}} - ZT^*)^{-1} Z D_T. \tag{2.1}$$

LEMMA 2.2. *Given a commuting contractive tuple $T$, its characteristic function $\theta_T$ is a multiplier, that is $\theta_T \in M(\mathcal{D}_T, \mathcal{D}_{T*})$, with $\|M_{\theta_T}\| \leq 1$. For $z, w \in \mathbb{B}^n$, the identity*

$$1 - \theta_T(w)\theta_T(z)^* = (1 - WZ^*)D_{T^*}(1 - WT^*)^{-1}(1 - TZ^*)^{-1}D_{T^*} \tag{2.2}$$

*holds.*

Proof. It is an elementary exercise to check that

$$U = \begin{pmatrix} T^* & D_T \\ D_{T^*} & -T \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{D}_T, \mathcal{H}^n \oplus \mathcal{D}_{T^*})$$

defines a unitary matrix operator. By Proposition 1.2 in [4] the transfer function of $U$, that is, the analytic operator-valued function $\theta_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{H} \otimes \mathcal{D}_T, \mathcal{H}^n \otimes \mathcal{D}_{T^*})$,

$$\theta_T(z) = -T + D_{T^*}(1_{\mathcal{H}} - ZT^*)^{-1}ZD_T$$

defines a multiplier $\theta_T \in M(\mathcal{D}_T, \mathcal{D}_{T*})$ with $\|M_{\theta_T}\| \leq 1$ such that formula (2.2) holds. ■

For $z = w$, the right-hand side of formula (2.2) defines a positive operator. Thus we obtain the following corollary.

COROLLARY 2.3. *Given a commuting contractive tuple $T$, its characteristic function $\theta_T$ is a bounded analytic function on $\mathbb{B}^n$ with* $\sup_{z \in \mathbb{B}^n} \|\theta_T(z)\| \leq 1$.

## 3. FUNCTIONAL MODEL OF A PURE COMMUTING CONTRACTIVE TUPLE

The purpose of this section is to produce functional models for pure commuting contractive tuples. This functional model generalizes the corresponding model for $C_{\cdot 0}$ contractions (Theorem VI. 2.3 in [9]) to the multivariable case and reflects very clearly the important role that the characteristic function plays.
A prototype of a commuting contractive tuple is the so-called $n$-shift which we simply call the *shift* as long as the dimension $n$ is fixed. By definition this is the commuting tuple $M_z = (M_{z_1}, \dots, M_{z_n})$ on the scalar-valued functional Hilbert space $H(\mathbb{C})$ consisting of the multiplication operators $M_{z_i}$ with the coordinate functions $z_i$. It is not difficult to see that $\sum_{i=1}^n M_{z_i}M_{z_i}^* = 1 - E_0$ where 1 is the identity operator on $H(\mathbb{C})$ and $E_0$ is the projection onto the one-dimensional subspace consisting of all constant functions (see [2]). Hence the shift is a commuting contractive tuple. It is not hard to show that

$$\text{SOT} - \lim_{k \to \infty} \sum_{1 \leq i_1, i_2, \dots i_k \leq n} M_{z_{i_1}} M_{z_{i_2}} \dots M_{z_{i_k}} M_{z_{i_k}}^* \dots M_{z_{i_2}}^* M_{z_{i_1}}^* = 0.$$

Thus the shift is an example of a *pure* commuting contractive tuple in the sense of the following definition.

DEFINITION 3.1. *For a commuting contractive tuple $T$ on a Hilbert space $\mathcal{H}$, define a completely positive map $P_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $P_T(X) = \sum_{i=1}^n T_i X T_i^*$. We denote by $A_\infty \in \mathcal{B}(\mathcal{H})$ the strong limit of the decreasing sequence of positive operators $I \geq P_T(I) \geq P_T^2(I) \geq ... \geq 0$. The commuting contractive tuple $T$ is called pure if $A_\infty = 0$.*

It is interesting to observe that the norm of $A_\infty$ is either 0 or 1. For the proof, first define for any integer $m \geq 1$, the operator $T^m \in \mathcal{B}(\mathcal{H}^{n^m}, \mathcal{H})$ which sends an element $\underline{h}$ of $\mathcal{H}^{n^m}$ to the sum $\sum_{1 \leq i_1, ..., 1_m \leq n} T_{i_1} ... T_{i_m} h_{i_1 ... i_m}$. Its adjoint $T^{m*} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^{n^m})$ maps a vector $h$ to the $n^m$ column vector $(T_{i_1}^* ... T_{i_m}^* h)_{1 \leq i_1, ..., i_n}$ in $\mathcal{H}^{n^m}$. By the above definition, $T^m T^{m*} = P_T^m(1)$. Thus we find that

$$\|A_\infty^{1/2} h\|^2 = \langle A_\infty h, h \rangle = \lim_{m \to \infty} \langle P_T^m(1) h, h \rangle = \lim_{m \to \infty} \langle T^m T^{m*} h, h \rangle = \lim_{m \to \infty} \|T^{m*} h\|^2.$$

Let $\underline{A}$ denote the operator $A_\infty \oplus A_\infty \oplus \cdots \oplus A_\infty : \mathcal{H}^{n^m} \to \mathcal{H}^{n^m}$. Then $T^m \underline{A} T^{m*} = P_T^m(A_\infty) = A_\infty$. It follows that

$$\|A_\infty{}^{\frac{1}{2}} h\|^2 = \langle A_\infty h, h \rangle = \langle T^m \underline{A} T^{m*} h, h \rangle = \|\underline{A}^{\frac{1}{2}} T^{m*} h\|^2$$

$$\leq \|\underline{A}^{\frac{1}{2}}\|^2 \, \|T^{m*} h\|^2 = \|A_\infty\| \, \|T^{m*} h\|^2 \xrightarrow{m} \|A_\infty\| \, \|A_\infty{}^{\frac{1}{2}} h\|^2.$$

Hence either $A_\infty{}^{1/2} = 0$ or $\|A_\infty\| \geq 1$. But $A_\infty$ being a contraction, this means that $\|A_\infty\| = 1$.

REMARK 3.2. *In the case $n = 1$ a contraction $T \in \mathcal{B}(\mathcal{H})$ is pure in the above sense if and only if it is of class $C_{\cdot 0}$.*

Arveson proved the following theorem for commuting contractive tuples in [2] (Theorem 4.5). In a way, the operator $L$ below is a precursor of the functional model that we are going to construct.

THEOREM 3.3. *Let $T$ be a commuting contractive tuple of operators on some Hilbert space $\mathcal{H}$. Then there exists a unique bounded linear operator $L : H(\mathbb{C}) \otimes \mathcal{D}_{T^*} \to \mathcal{H}$ satisfying*

$$L(f \otimes \xi) = f(T) D_{T^*} \xi$$

*for all $f$ in $\mathbb{C}[z_1, ..., z_n]$, and $\xi$ in $\mathcal{D}_{T^*}$. Furthermore, we have $LL^* = 1_{\mathcal{H}} - A_\infty$ and the identity $L(f(M_z) \otimes 1_{\mathcal{D}_{T^*}}) = f(T) L$ holds for all $f$ in $\mathbb{C}[z_1, \ldots, z_n]$.*

REMARK 3.4. *The tuple $T$ is pure if and only if $L$ is a co-isometry.*

Given a Hilbert space $\mathcal{E}$, we denote by $M_z^{\mathcal{E}} = (M_{z_1}^{\mathcal{E}}, \ldots, M_{z_n}^{\mathcal{E}}) \in \mathcal{B}(H(\mathcal{E}))^n$ the tuple of multiplication operators induced by the coordinate functions $z_i$. There is a canonical unitary operator $U_{\mathcal{E}} : H(\mathbb{C}) \otimes \mathcal{E} \to H(\mathcal{E})$ with $U_{\mathcal{E}}(f \otimes x) = fx$ for $f \in H(\mathbb{C})$ and $x \in \mathcal{E}$. In the following we shall identify the spaces $H(\mathbb{C}) \otimes \mathcal{E}$ and $H(\mathcal{E})$ via this unitary operator $U_{\mathcal{E}}$. In this way each multiplier $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ induces a bounded operator $M_\varphi : H(\mathbb{C}) \otimes \mathcal{E} \to H(\mathbb{C}) \otimes \mathcal{E}_*$.

As observed by Arveson in [2] (Proposition 1.12), the space $H(\mathbb{C})$ is a functional Hilbert space with reproducing kernel

$$K : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}, \quad K(z, w) = (1 - \langle z, w\rangle)^{-1}.$$

In particular, the space $H(\mathbb{C})$ is the closed linear span of the functions $k_w = K(\cdot, w)$ ($w \in \mathbb{B}^n$).

LEMMA 3.5. *Let $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ be a multiplier. Then the identity*

$$M_{\varphi^*}(k_z \otimes x) = k_z \otimes \varphi(z)^* x$$

*holds for all $z \in \mathbb{B}^n$ and $x \in \mathcal{E}_*$.*

Proof. Fix $z \in \mathbb{B}^n$ and $x \in \mathcal{E}_*$. Note first that

$$\langle f \otimes y, k_z \otimes x\rangle = f(z)\langle y, x\rangle = \langle (fy)(z), x\rangle$$

holds for all $f \in H(\mathbb{C})$ and $y \in \mathcal{E}_*$. Hence it follows that $\langle f, k_z \otimes x\rangle = \langle f(z), x\rangle$ for each function $f \in H(\mathcal{E}_*)$. Using this identity twice (for $\mathcal{E}$- and $\mathcal{E}_*$-valued functions), we obtain that

$$\langle f, M_\varphi^*(k_z \otimes x)\rangle = \langle \varphi(z) f(z), x\rangle = \langle f, k_z \otimes \varphi(z)^* x\rangle$$

for each function $f \in h(\mathcal{E})$. ∎

Next we relate the operator $L$ described in Theorem 3.3 with the characteristic function.

LEMMA 3.6. *Given a commuting contractive tuple $T$, we obtain the identity*

$$L^* L + M_{\theta_T} M_{\theta_T}^* = 1_{H(\mathbb{C}) \otimes \mathcal{D}_{T^*}}.$$

Proof. As observed by Arveson in the proof of Theorem 1.2 in [3], the operator $L$ satisfies the identity

$$L(k_z \otimes \xi) = (1 - TZ^*)^{-1} D_{T^*} \xi \quad (z \in \mathbb{B}^n, \xi \in \mathcal{D}_{T^*}).$$

Therefore, for $z, w$ in $\mathbb{B}^n$ and $\xi, \eta$ in $\mathcal{D}_{T^*}$, we obtain that

$$\begin{aligned}
& \langle (L^* L + M_{\theta_T} M_{\theta_T}^*) k_z \otimes \xi, k_w \otimes \eta\rangle \\
= \ & \langle L(k_z \otimes \xi), L(k_w \otimes \eta)\rangle + \langle M_{\theta_T}^*(k_z \otimes \xi), M_{\theta_T}^*(k_w \otimes \eta)\rangle \\
= \ & \langle (1 - TZ^*)^{-1} D_{T^*} \xi, (1 - TW^*)^{-1} D_{T^*} \eta\rangle + \langle k_z \otimes \theta_T(z)^* \xi, k_w \otimes \theta_T(w)^* \eta\rangle \\
= \ & \langle D_{T^*}(1 - WT^*)^{-1}(1 - TZ^*)^{-1} D_T^* \xi, \eta\rangle + \langle k_z, k_w\rangle \langle \theta_T(w) \theta_T(z)^* \xi, \eta\rangle \\
= \ & \langle k_z \otimes \xi, k_w \otimes \eta\rangle.
\end{aligned}$$

To verify the last equality, the reader should use the formula obtained in Lemma 2.2. Using the fact that the vectors $k_z$ form a total set in $H(\mathbb{C})$, the assertion follows. ∎

In [3] Arveson used abstract factorization results to prove the existence of a multiplier $\varphi \in M(\mathcal{D}, \mathcal{D}_{T^*})$ such that

$$1_{H(\mathcal{C})\otimes\mathcal{D}_{T^*}} - L^*L = M_\varphi M_\varphi^*.$$

The above Lemma 3.6 shows that $\varphi$ can be choses as the characteristic function of $T$. As usual we call two commuting tuples $T = (T_1, \ldots, T_n)$ and $R = (R_1, \ldots, R_n)$ of bounded operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ unitarily equivalent if there exists a unitary operator $U$ from $\mathcal{H}$ to $\mathcal{K}$ such that $R_i = UT_iU^*$ holds for all $i = 1, ..., n$.
Now we are ready to prove the main theorem of this section.

THEOREM 3.7. *Every pure commuting contractive tuple $T$ on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the commuting tuple $\mathbb{T} = (\mathbb{T}_1, \ldots, \mathbb{T}_n)$ on the functional space $\mathbb{H}_T = (H(\mathbb{C})\otimes \mathcal{D}_{T^*}) \ominus M_{\theta_T}(H(\mathbb{C}) \otimes \mathcal{D}_T)$ defined by $\mathbb{T}_i = P_{\mathbb{H}_T}(M_{z_i} \otimes 1_{\mathcal{D}_{T^*}})|\mathbb{H}_T$ for $1 \le i \le n$.*

Proof Since $T$ is pure, the map

$$L^* : \mathcal{H} \to H(\mathbb{C}) \otimes \mathcal{D}_{T^*}$$

is an isometry. Thus $\mathcal{H}$ is isometrically embedded into $H(\mathbb{C}) \otimes \mathcal{D}_{T^*}$ via the identification of $\mathcal{H}$ with the closed subspace $L^*\mathcal{H}$. Now $L^*L$ is the projection of $H(\mathbb{C}) \otimes \mathcal{D}_{T^*}$ onto the closed subspace $L^*\mathcal{H}$. But then by Lemma 3.6, the operators $L^*L$ and $M_{\theta_T}M_{\theta_T}^*$ are mutually orthogonal projections which add up to identity. Therefore the subspace $L^*\mathcal{H}$ is the orthocomplement of the range of $M_{\theta_T}$:

$$L^*\mathcal{H} = (H(\mathbb{C}) \otimes \mathcal{D}_{T^*}) \ominus M_{\theta_T}(H(\mathbb{C}) \otimes \mathcal{D}_T).$$

Now by Theorem 3.3, $L^*T_i^* = (M_{z_i} \otimes 1_{\mathcal{D}_{T^*}})^*L^*$. Thus the subspace $L^*\mathcal{H}$ is co-invariant for the shift and, via the identification of $\mathcal{H}$ with $L^*\mathcal{H}$, the operators $T_i$ in $\mathcal{B}(\mathcal{H})$ coincide with the compressions of the operators $M_{z_i} \otimes 1_{\mathcal{D}_{T^*}}$ to the space $\mathbb{H}_T$. ■

So every pure commuting contractive tuple $T$ on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the commuting tuple $P_{\mathbb{H}_T}(M_z \otimes 1_{\mathcal{D}_{T^*}})|\mathbb{H}_T$, where $\mathbb{H}_T$ is the $M_z^*$-invariant subspace $(H(\mathbb{C}) \otimes \mathcal{D}_{T^*}) \ominus M_{\theta_T}(H(\mathbb{C}) \otimes \mathcal{D}_T)$ of $H(\mathbb{C}) \otimes \mathcal{D}_{T^*}$.

## 4. THE CHARACTERISTIC FUNCTION AS A COMPLETE UNITARY INVARIANT

DEFINITION 4.1. *Given two commuting contractive tuples $T$ and $R$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the characteristic functions of $T$ and $R$ are said to coincide if there exist unitary operators $\tau : \mathcal{D}_T \to \mathcal{D}_R$ and $\tau_* : \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ such that the following diagram commutes for all $z$ in $\mathbb{B}^n$*

$$\begin{tikzcd} \mathcal{D}_T \arrow[r, "\theta_T(z)"] \arrow[d, "\tau"'] & \mathcal{D}_{T^*} \arrow[d, "\tau_*"] \\ \mathcal{D}_R \arrow[r, "\theta_R(z)"'] & \mathcal{D}_{R^*} \end{tikzcd}$$

In this section, we prove that the characteristic function of a pure commuting contractive tuple is a complete unitary invariant.

PROPOSITION 4.2. *The characteristic functions of two unitarly equivalent commuting tuples coincide.*

*Proof.* Let $T$ and $R$ be two commuting contractive tuples on $\mathcal{H}$ and $\mathcal{K}$, respectively, such that there is a unitary operator $\sigma : \mathcal{H} \to \mathcal{K}$ satisfying $\sigma T_i \sigma^* = R_i$ for all $i$. Denote by $\underline{\sigma}$ and $\underline{\sigma}^*$ the operators

$$\oplus_{i=1}^n \sigma : \mathcal{H}^n \to \mathcal{K}^n \text{ and } i \oplus_{i=1}^n \sigma^* : \mathcal{K}^n \to \mathcal{H}^n.$$

Then it is easy to see that $\underline{\sigma} D_T^2 \underline{\sigma}^* = D_R^2$ and $\sigma D_{T^*}^2 \sigma^* = D_{R^*}^2$. Thus $\underline{\sigma} D_T \underline{\sigma}^* = D_R$ and $\sigma D_{T^*} \sigma^* = D_{R^*}$. Hence $\underline{\tau} : \mathcal{D}_T \to \mathcal{D}_R$ defined by $\underline{\tau} = \underline{\sigma} \mid_{\mathcal{D}_T}$ is a unitary operator between $\mathcal{D}_T$ and $\mathcal{D}_R$. Similarly, the restriction $\tau_* = \sigma \mid_{\mathcal{D}_{T^*}}$ defines a unitary operator from $\mathcal{D}_{T^*}$ to $\mathcal{D}_{R^*}$. Finally, note that

$$\begin{aligned} \theta_R(z)\underline{\tau} &= (-R + D_{R^*}(1 - ZR^*)^{-1} Z D_R)\underline{\sigma} \mid_{\mathcal{D}_T} . \\ &= -\sigma T + D_{R^*}(1 - ZR^*)^{-1} \underline{Z}\, \underline{\sigma} D_T. \\ &= -\sigma T + D_{R^*}(1 - ZR^*)^{-1} \sigma Z D_T \\ &= -\sigma T + \sigma D_{T^*}(1 - ZT^*) Z D_T \\ &= \tau_* \theta_T(z), \end{aligned}$$

for all $z \in \mathbb{B}^n$. Hence the two characteristic functions $\theta_T$ and $\theta_R$ coincide. ∎

Next we prove the converse of the above proposition for the case of pure tuples.

PROPOSITION 4.3. *Let $T$ and $R$ be two pure commuting contractive tuples on $\mathcal{H}$ and $\mathcal{K}$, respectively. If their characteristic functions $\theta_T$ and $\theta_R$ coincide, then the tuples $T$ and $R$ are unitarily equivalent.*

*Proof.* Let $\tau' : \mathcal{D}_T \to \mathcal{D}_R$ and $\tau_*^{'} : \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ be two unitary operators such that the diagram

$$\begin{tikzcd} \mathcal{D}_T \arrow[r, "\theta_T(z)"] \arrow[d, "\tau'"'] & \mathcal{D}_{T^*} \arrow[d, "\tau'_*"] \\ \mathcal{D}_R \arrow[r, "\theta_R(z)"'] & \mathcal{D}_{R^*} \end{tikzcd}$$

commutes for all $z$ in $\mathbb{B}^n$. The operators $\tau'$ and $\tau'_*$ give rise to unitary operators $\tau = 1 \otimes \tau' : H(\mathbb{C}) \otimes \mathcal{D}_T \to H(\mathbb{C}) \otimes \mathcal{D}_R$ and $\tau_* = 1 \otimes \tau'_* : H(\mathbb{C}) \otimes \mathcal{D}_{T^*} \to H(\mathbb{C}) \otimes \mathcal{D}_{R^*}$ which satisfy the intertwining relation

$$M_{\theta_R}\tau = \tau_* M_{\theta_T}.$$

We conclude that

$$\tau_*(\mathbb{H}_T) = \tau_*((\mathrm{Ran} M_{\theta_T})^\perp) = \tau_*(\mathrm{Ran} M_{\theta_T})^\perp = (\mathrm{Ran} M_{\theta_R})^\perp = \mathbb{H}_R,$$

where $\mathbb{H}_T$ and $\mathbb{H}_R$ are the model spaces for $T$ and $R$ as in Theorem 3.7. Since the operator $\tau_*$ interwines the tuples $(M_z \otimes 1_{\mathcal{D}_{T^*}})^*$ and $(M_z \otimes 1_{\mathcal{D}_{R^*}})^*$ componentwise, the induced unitary operators $\tau_* : \mathbb{H}_T \to \mathbb{H}_R$ intertwines the adjoints of the restrictions of these tuples, which are precisely the model tuples $P_{\mathbb{H}_T}(M_z \otimes 1_{\mathcal{D}_{T^*}})|\mathbb{H}_T$ and $P_{\mathbb{H}_R}(M_z \otimes 1_{\mathcal{D}_{R^*}})|\mathbb{H}_R$. But then Theorem 3.7 shows that $T$ and $R$ are unitarily equivalent. ∎

Summarizing the last two propositions we obtain the main result of this paper.

THEOREM 4.4. *Two pure commuting contractive tuples $T$ and $R$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are unitarily equivalent if and only if their characteristic functions coincide.*

Let $T \in \mathcal{B}(\mathcal{H})^n$ be a pure commuting contractive tuple on a separable Hilbert space $\mathcal{H}$. Arveson used in [3] the abstract solution of the factorization problem

$$1_{H(\mathbb{C})\otimes\mathcal{D}_{T^*}} - L^*L = M_\varphi M_\varphi^*$$

to construct an invariant for pure commuting contractive tuples $T \in \mathcal{B}(\mathcal{H})^n$ with finite defect, that is, with $\dim(\mathcal{D}_{T^*}) < \infty$, called the *curvature invariant.* Since we know that the characteristic function $\theta_T$ of $T$ can be used for $\varphi$, we see that the curvature invariant is completely determined by the characteristic function of $T$. We end this paper by briefly indicating this connection between the characteristic function and the curvature invariant. By Corollary 2.3 the characteristic function $\theta_T$ is a bounded analytic function with values in $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ and supremum norm bounded by one. Suppose that the number $d = \dim(\mathcal{D}_{T^*})$ is finite. Then $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ is topologically isomorphic to a separable Hilbert space, and therefore $\theta_T$ has a pointwise radial limit almost everywhere defining a function $\tilde{\theta}_T : \partial\mathbb{B}^n \to \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ belonging to the unit ball of $\mathrm{L}^\infty(\partial\mathbb{B}^n, \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*}))$. Define $k_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ by

$$k_T(z) = (1 - TZ^*)^{-1} D_{T^*}.$$

It follows from Lemma 2.2 that

$$1-\theta_T(z)\theta_T(z)^* = (1-\|z\|^2)k_T(z)^*k_T(z) \quad (z \in \mathbb{B}^n).$$

Using the definition given by Arveson in [3] we obtain the following representation of the curvature invariant of $T$ in terms of the characteristic function

$$\begin{aligned} K(T) &= \lim_{r\uparrow 1}(1-r^2)\int_S \operatorname{trace} k_T(rz)^*k_T(rz)\mathrm{d}\sigma(z) \\ &= \int_S \operatorname{trace}\,(1_{\mathcal{D}_{T^*}} - \tilde{\theta}_T(z)\tilde{\theta}_T(z)^*)\mathrm{d}\sigma(z). \end{aligned}$$

Here $S=\partial\mathbb{B}^n$ is the unit sphere and $\sigma$ denotes the normalized surface measure on $S$.

Acknowledgement: Work of the first named author is supported by DST grant no. SR/ FTP/ MS-16/ 2001. The third named authors research work is supported by a UGC fellowship.

## References

[1] C. Ambrozie and J. Eschmeier, *A commutant lifting theorem on analytic polyhedra*, preprint.

[2] W. B. Arveson, *Subalgebras of $C^*$-algebras III, Multivariable operator theory*, Acta Math., (2) 181 (1998),159-228.

[3] W. Arveson, *The curvature invariant of a Hilbert module over $\mathbb{C}[z_1,\dots,z_d]$*, J. Reine Angew. Math. 522 (2000), 173–236.

[4] J. Eschmeier and M. Putinar, *Spherical contractions and interpolation problems on the unit ball*, J. Reine angew. Math. 542 (2002), 219-236.

[5] A. E. Frazho, *Models for noncommuting operators*, J. Funct. Anal. 48 (1982), 1–11.

[6] D. Greene, S. Richter and C. Sundberg, *The Structure of Inner Multipliers on Spaces with Complete Nevanlinna Pick Kernels*, J. Funct. Anal. 194 (2002), 311-331.

[7] S. McCullough and T. T. Trent, *Invariant subspaces and Nevanlinna-Pick kernels*, J. Funct. Anal. 178 (2000), 226-249.

[8] G. Popescu, *Characteristic functions for infinite sequences of noncommuting operators*, J. Operator Th., 22 (1989), 51 - 71.

[9] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, 1970.

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
*E-mail address*: `tirtha@math.iisc.ernet.in`

Fachbereich Mathematik, Universität des Saarlandes, 66123 Saarbrücken, Germany
*E-mail address*: `eschmei@math.uni-sb.de`

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
*E-mail address*: `jaydeb@math.iisc.ernet.in`