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Abstract

We establish several smoothness results for local minimizers of non-autonomous variational integrals with anisotropic growth conditions.

1 Introduction

In a recent paper ([ELM]) Esposito, Leonetti and Mingione discuss higher integrability theorems for minimizers of functionals of the form $J[u] = \int_\Omega f(\cdot, \nabla u)\, dx$, where the integrand f is of anisotropic (p, q)-growth with respect to the second argument. Let us summarize some of their results: suppose that the function $D_P f(x, P)$ is α-Hölder continuous with respect to the variable x and that certain natural growth and ellipticity assumptions are satisfied. Then one is interested in the following question: do (local) minimizers u actually belong to the space $W^1_{q,loc}(\Omega; \mathbb{R}^N)$? As shown in Section 3 of [ELM] one can only hope for a positive answer if ($\Omega \subset \mathbb{R}^n$)

$$\frac{q}{p} < \frac{1}{n}(n + \alpha) \tag{1.1}$$

is satisfied. Assuming (1.1) they then exhibit in Section 4 of their paper a sufficient condition for higher integrability: if the Lavrentiev gap functional $\mathcal{L}$ relative to the energy J (see [ELM], Section 2.1) vanishes for all balls $B_R \Subset \Omega$, then Theorem 4 of [ELM] gives local integrability of ∇u for exponents even bigger than q. However, it seems to be a very delicate problem to decide in a general way if the Lavrentiev gap functional vanishes or not. To overcome this difficulty, Esposito, Leonetti and Mingione present a list of explicit examples and prove $\mathcal{L} \equiv 0$ in these concrete cases. Here a possible occurence of a local Lavrentiev phenomenon is excluded via a subtile study of the behavior of f w.r.t. the x-dependence in comparison to the (p, q)-growth in ∇u.

In [CGM] the authors follow a different approach and consider energy densities depending on the modulus of the second argument. With this additional assumption it is possible to introduce some kind of regularization from below in order to prove local Lipschitz continuity of local minimizers.

The main purpose of our paper is to give a rather complete $C^{1,\alpha}$-regularity theory provided that we have some starting $W^1_{\overline{q}}$ integrability of the minimizer. This is discussed in Section 2. In Section 3 we then adapt the two approaches given in [ELM] and [CGM] to the situation at hand and obtain the right starting integrability under some particular assumptions. It remains an open problem to remove these hypotheses.

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Let us give a detailed formulation:

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, denote a bounded domain and consider an energy density $f = f(x,P) \geq 0$, $x \in \overline{\Omega}$, $P \in \mathbb{R}^{nN}$, which satisfies with exponents $1 < p \leq \overline{q} < \infty$

ASSUMPTION 1.1 *There are positive constants λ, Λ, c_1 such that for any $x \in \overline{\Omega}$ and all U, $P \in \mathbb{R}^{nN}$:*

$$\lambda(1+|P|^2)^{\frac{p-2}{2}}|U|^2 \leq D_P^2 f(x,P)(U,U) \leq \Lambda(1+|P|^2)^{\frac{\overline{q}-2}{2}}|U|^2\,, \tag{1.2}$$

$$|D_x D_P f(x,P)| \leq c_1(1+|P|^2)^{\frac{\overline{q}-1}{2}}\,. \tag{1.3}$$

Here f is assumed to be sufficiently smooth which means that we require the partial derivatives $D_P^2 f$ and $D_x D_P f$ to be at least continuous. Note that (1.2) implies the anisotropic growth condition (a, $A > 0$, $b \in \mathbb{R}$)

$$a|P|^p - b \leq f(x,P) \leq A(|P|^{\overline{q}} + 1)\,.$$

For open subsets Ω' of Ω let us define the energy of a function u: $\Omega' \to \mathbb{R}^N$ via

$$J[u,\Omega'] := \int_{\Omega'} f(\cdot, \nabla u)\,\mathrm{d}x\,.$$

The following definition is natural in our setting.

DEFINITION 1.1 *A function $u \in W^1_{1,loc}(\Omega;\mathbb{R}^N)$ is termed a local J-minimizer iff*

i) $J[u,\Omega'] < \infty$ *for any domain* $\Omega' \Subset \Omega$ *and*

ii) $J[u,\Omega'] \leq J[v,\Omega']$ *for any* $\Omega' \Subset \Omega$ *and all* $v \in W^1_{1,loc}(\Omega;\mathbb{R}^N)$ *with* $\operatorname{spt}(u-v) \subset \Omega'$.

Now let us suppose that local J-minimizers are of class $W^1_{\overline{q},loc}$. We then have the following theorem on higher regularity.

THEOREM 1.1 *Let Assumption 1.1 hold together with*

$$\overline{q} < p\frac{n+1}{n}\,. \tag{1.4}$$

Suppose further that u is a local J-minimizer which is of class $W^1_{\overline{q},loc}(\Omega;\mathbb{R}^N)$. Then we have

i) *There exists an open subset $\Omega_0 \subset \Omega$ such that $|\Omega - \Omega_0| = 0$ and $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^N)$ for any $\alpha \in (0,1)$.*

ii) *If $n = 2$, then $\Omega_0 = \Omega$.*

iii) *If $N = 1$ or if f is of special structure, i.e. $f(x,P) = g(x,|P|^2)$, and if in addition for $N > 1$*

$$|D_P^2 f(x,P) - D_P^2 f(x,Q)| \leq c(1+|P|^2+|Q|^2)^{\frac{\overline{q}-2-\gamma}{2}}|P-Q|^\gamma \tag{1.5}$$

holds with some $0 < \gamma < 1$ and for all $x \in \overline{\Omega}$, P, $Q \in \mathbb{R}^{nN}$, then u is of class $C^{1,\alpha}$ in the interior of Ω.

Our second theorem deals with locally bounded minimizers. As a consequence, condition (1.4) can be weakend if $p < n$. Note that on account of Sobolev's embedding theorem, we cannot expect to improve (1.4) in the case $p > n$ since then the boundedness of minimizers is no additional assumption at all (compare Remark 5.5 of [Bi]).

THEOREM 1.2 *Let u denote a local J-minimizer of class $W^1_{\overline{q},loc}(\Omega;\mathbb{R}^N)$ and let Assumption 1.1 hold. If $N = 1$ or if f is of special structure, i.e. $f(x,P) = g(x,|P|^2)$ and if in addition in the case $N > 1$ we have (1.5), then u has Hölder continuous first derivatives in the interior of Ω, provided we assume*

$$u \in L^\infty_{loc}(\Omega;\mathbb{R}^n) \tag{1.6}$$

together with

$$\overline{q} < p+1\,. \tag{1.7}$$

REMARK 1.1 i) *The counterexamples of [ELM] and [FMM] satisfy $\overline{q} > p+1$. Since the solutions constructed there are locally bounded, we see that (1.7) is a rather natural condition for regularity.*

ii) *Due to the counterexamples of [ELM], [FMM], [Mi] we cannot expect to weaken the conditions (1.4) and (1.7), respectively. On the other hand, in the autonomous case the counterpart of (1.4) is $\overline{q} < p(n+2)/n$, whereas (1.7) reads as $\overline{q} < p+2$ in the autonomous case. A first Ansatz to close this gap with some suitable additional assumption on the energy density can be made analogous to Section 4.2.2.2 of [Bi] which, in fact, leads to higher integrability results. We omit further details since it is not clear, whether for instance DeGeorgi-type arguments can be improved with this Ansatz, i.e. the gap to the autonomous case is not understood up to now.*

Theorem 1.1 and Theorem 1.2 are established in Section 2. In Section 3 we will remove the assumption $u \in W^1_{\overline{q},loc}(\Omega;\mathbb{R}^N)$ for some special cases. The results are summarized in Lemma 3.1 and Lemma 3.2.

Throughout this paper summation w.r.t. repeated indices always is assumed. Moreover, positive constants are usually just denoted by c, not necessarily being the same in different occurrencies.

2 Smoothness properties of $W^1_{\overline{q},loc}$-minimizers

In this section we are going to prove Theorem 1.1 and Theorem 1.2. Of course we mainly follow the ideas used in the autonomous case (compare, for instance, [Se], [Ma], [MS], [BF1], [Bi] and the references quoted therein), thus we just give a short summary of the known steps and emphasize the modifications which are needed to handle the non-autonomous case.

2.1 Proof of Theorem 1.1

Step 1. Approximation.
We fix a ball $B_{2R} = B_{2R}(x_0) \Subset \Omega$ and define for $0 < \delta < 1$

$$f_\delta(x,P) = \delta(1+|P|^2)^{\frac{q}{2}} + f(x,P)\,, \quad x \in \overline{\Omega}\,,\ \ P \in \mathbb{R}^{nN}\,,$$

where the exponent q is chosen according to

$$\overline{q} < q < p\Big(1 + \frac{2}{n}\Big) \, . \tag{2.1}$$

Note that f_δ still satisfies (1.3), whereas (1.2) holds with exponent $\overline{q}$ replaced by q. Define u_ε as the mollification of u with parameter $\varepsilon > 0$ and let $v_{\varepsilon,\delta}$ denote the unique solution of the minimization problem

$$J_\delta[w, B_{2R}] := \int_{B_{2R}} f_\delta(\cdot, \nabla w)\, \mathrm{d}x \to \min \qquad \text{in } \; u_\varepsilon + \overset{\circ}{W}{}^1_q(B_{2R}; \mathbb{R}^N)\, .$$

We have the following convergence results:

LEMMA 2.1 *Suppose that the hypotheses of Theorem 1.1 hold. If ε and δ are related via*

$$\delta \;=\; \delta(\varepsilon) \;:=\; \frac{1}{1 + \varepsilon^{-1} + \|\nabla u_\varepsilon\|^{2q}_{L^q(B_{2R})}}$$

and if we abbreviate $v_\varepsilon = v_{\varepsilon,\delta(\varepsilon)}$, $f_\varepsilon = f_{\delta(\varepsilon)}$, then we have as $\varepsilon \to 0$:

$$\begin{array}{lrcl}
\text{i)} & v_\varepsilon & \rightharpoondown & u \quad \textit{in } W^1_1\big(B_{2R}, \mathbb{R}^N\big)\, , \\
\text{ii)} & \delta(\varepsilon) \displaystyle\int_{B_{2R}} (1 + |\nabla v_\varepsilon|^2)^{\frac{q}{2}}\, \mathrm{d}x & \to & 0\, , \\
\text{iii)} & \displaystyle\int_{B_{2R}} f(\cdot, \nabla v_\varepsilon)\, \mathrm{d}x & \to & \displaystyle\int_{B_{2R}} f(\cdot, \nabla u)\, \mathrm{d}x\, , \\
\text{iv)} & \displaystyle\int_{B_{2R}} f_\varepsilon(\cdot, \nabla v_\varepsilon)\, \mathrm{d}x & \to & \displaystyle\int_{B_{2R}} f(\cdot, \nabla u)\, \mathrm{d}x\, .
\end{array}$$

Proof. We have by the minimality of v_ε

$$\begin{aligned}
\int_{B_{2R}} f(\cdot, \nabla v_\varepsilon)\, \mathrm{d}x \;&\le\; \int_{B_{2R}} f_\varepsilon(\cdot, \nabla v_\varepsilon)\, \mathrm{d}x \;\le\; \int_{B_{2R}} f_\varepsilon(\cdot, \nabla u_\varepsilon)\, \mathrm{d}x \\
&=\; \delta(\varepsilon) \int_{B_{2R}} (1 + |\nabla u_\varepsilon|^2)^{\frac{q}{2}}\, \mathrm{d}x + \int_{B_{2R}} f(\cdot, \nabla u_\varepsilon)\, \mathrm{d}x\, .
\end{aligned} \tag{2.2}$$

Here the choice of $\delta(\varepsilon)$ implies that the first term on the r.h.s. converges to 0 as $\varepsilon \to 0$. Next we recall that f is at most of growth order $\overline{q}$, moreover we have that ∇u is of class $L^{\overline{q}}_{loc}$, hence

$$\nabla u_\varepsilon \overset{\varepsilon \to 0}{\to} \nabla u \quad \text{in } \; L^{\overline{q}}(B_{2R}; \mathbb{R}^{nN})\, .$$

This in turn gives

$$\int_{B_{2R}} f(\cdot, \nabla u_\varepsilon)\, \mathrm{d}x \overset{\varepsilon \to 0}{\to} \int_{B_{2R}} f(\cdot, \nabla u)\, \mathrm{d}x\, . \tag{2.3}$$

In fact, to verify (2.3), we may consider the convex function

$$H : \, W^1_{\overline{q}}(B_{2R}, \mathbb{R}^N) \ni v \mapsto \int_{B_{2R}} f(\cdot, \nabla v)\, \mathrm{d}x$$

which is locally bounded from above, hence locally Lipschitz (compare, for instance, [Da], Theorem 2.3, p. 29). This gives (2.3). Then we conclude from (2.2) that $\int_{B_{2R}} f(\cdot, \nabla v_\varepsilon)\,\mathrm{d}x \le$ *const*, hence

$$v_\varepsilon \overset{\varepsilon\to 0}{\rightharpoondown :} v \quad \text{in } W^1_p(B_{2R};\mathbb{R}^N)\,, \qquad v = u \quad \text{on } \partial B_{2R}\,.$$

The lower semicontinuity of J and the uniqueness of minimizers finally prove $v = u$ on B_{2R}, i.e. the lemma is established. $\square$

Step 2. Caccioppoli-type inequalities and higher integrability.
In the following we use the notation from above and observe that v_ε solves the Euler equation

$$\int_{B_{2R}} D_P f_\varepsilon(\cdot, \nabla v_\varepsilon) : \nabla\varphi\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \overset{\circ}{W}{}^1_q(B_{2R};\mathbb{R}^N)\,. \tag{2.4}$$

Here and in the following ":" denotes the standard scalar product in $\mathbb{R}^{nN}$. We have

LEMMA 2.2 *There is a real number $c > 0$ such that for all $\eta \in C^1_0(B_{2R})$, $0 \le \eta \le 1$, and for all $Q \in \mathbb{R}^{nN}$*

$$\begin{aligned}\int_{B_{2R}} \eta^2 \Gamma_\varepsilon^{\frac{p-2}{2}} |\nabla^2 v_\varepsilon|^2\,\mathrm{d}x \quad &\le \quad c\Bigg[\|\nabla\eta\|^2_\infty \int_{\operatorname{spt}\nabla\eta} \Gamma_\varepsilon^{\frac{q-2}{2}} |\nabla v_\varepsilon - Q|^2\,\mathrm{d}x \\ &\qquad + \int_{\operatorname{spt}\eta} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}}\,\mathrm{d}x\Bigg]\,, \end{aligned} \tag{2.5}$$

where $\Gamma_\varepsilon := 1 + |v_\varepsilon|^2$.

Proof. Using the method of difference quotients in equation (2.4) (see e.g. [AF], Proposition 2.4 and Lemma 2.5, [GM], [Ca] or [To] for further details in a related setting; note that Lemma 4.1 of [To] works under our hypotheses) we obtain weak differentiability of ∇v_ε together with

$$\Gamma_\varepsilon^{\frac{p-2}{4}} |\partial_\gamma \nabla v_\varepsilon| \in L^2_{loc}(B_{2R})\,.$$

Then, as outlined in the proof of Lemma 3.1 in [BF1], we deduce from the above integrability property (again using the method of difference quotients and passing to the limit) the inequality

$$\begin{aligned} &\int_{B_{2R}} D^2_P f_\varepsilon(\cdot, \nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma \nabla v_\varepsilon)\eta^2\,\mathrm{d}x \\ &\le \quad -2\int_{B_{2R}} D^2_P f_\varepsilon(\cdot, \nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma(v_\varepsilon - Qx)\otimes\nabla\eta)\eta\,\mathrm{d}x \\ &\qquad -2\int_{B_{2R}} (\partial_\gamma D_P f_\varepsilon)(\cdot, \nabla v_\varepsilon) : \partial_\gamma(v_\varepsilon - Qx)\otimes\nabla\eta\eta\,\mathrm{d}x \\ &\qquad -\int_{B_{2R}} (\partial_\gamma D_p f_\varepsilon)(\cdot, \nabla v_\varepsilon) : \partial_\gamma \nabla v_\varepsilon \eta^2\,\mathrm{d}x\,, \end{aligned} \tag{2.6}$$

being valid for any matrix $Q \in \mathbb{R}^{nN}$. With the help of Young's inequality we get (2.5) by absorbing terms after suitable application of (1.2) and (1.3). Note that (2.5) just follows from our assumptions (1.2) and (1.3), the hypotheses (1.4) and (2.1) do not enter. $\square$

REMARK 2.1 *We can arrange that*

$$\overline{q} - \frac{p}{2} < \frac{q}{2}\,. \tag{2.7}$$

In fact, up to now q was chosen according to $q > \overline{q}$ and $q < p(1+2/n)$. Here we observe that (1.4) gives

$$2\Big(\overline{q} - \frac{p}{2}\Big) < 2\Big(\frac{n+1}{n}p - \frac{p}{2}\Big) = p\frac{n+2}{n}$$

which means that it is possible to choose q in $(2(\overline{q}-p/2), p(n+2)/n)$ by the way satisfying (2.7) which will be assumed from now on.

As already remarked local higher integrability of $\nabla\bar{u}$ up to a certain exponent is established in Theorem 4 of [ELM]. We give a slight improvement which in particular is needed to discuss the case $n = 2$.

LEMMA 2.3 *(compare [BF1], Lemma 3.4) Let $\chi := n/(n-2)$, if $n > 2$, for $n = 2$ let $\chi > 2p/(2p-q)$. Then we have*

$$\nabla v_\varepsilon \in L^{p\chi}_{loc}(B_{2R};\mathbb{R}^{nN})$$

uniformly w.r.t. ε, in particular we find

$$\nabla u \in \begin{cases} L^{pn/(n-2)}_{loc}(\Omega;\mathbb{R}^{nN})\,, & \textit{if } n \geq 3\,,\\ \textit{any } L^{s}_{loc}(\Omega;\mathbb{R}^{nN})\,,\quad s < \infty\,, & \textit{if } n = 2\,. \end{cases}$$

Proof of Lemma 2.3. We consider the case $n \geq 3$, the calculations for $n = 2$ have to be adjusted according to [BF1] or [Bi]. Let

$$\alpha := \frac{p}{2}\frac{n}{n-2} = \frac{p}{2}\chi$$

and observe that by (1.4) we have

$$\overline{q} - \frac{p}{2} < \alpha\,. \tag{2.8}$$

Let us fix radii r and ρ such that $R < r < \frac{3}{2}R$ and $0 < \rho < \frac{R}{2}$. Moreover, let $\eta \in C^1_0(B_{r+\rho/2})$, $\eta = 1$ on B_r, $|\nabla\eta| \leq c/\rho$. Using (2.5), the calculations from the proof of [BF1], Lemma 3.4, lead to the inequality (compare [Bi], second inequality on p. 60)

$$\int_{B_r}\Gamma_\varepsilon^\alpha\,dx \leq c\rho^{-\beta}\Bigg[\int_{B_{2R}}\Gamma_\varepsilon^{\frac{p}{2}}\Bigg]^{\bar{\beta}} + c\Bigg[\int_{B_{r+\rho}}\Gamma_\varepsilon^{\overline{q}-\frac{p}{2}}\,dx\Bigg]^{\chi} + \vartheta\int_{B_{r+\rho}}\Gamma_\varepsilon^\alpha\,dx \tag{2.9}$$

with positive constants β, $\bar{\beta}$, a positive constant c and another constant $\vartheta < 1$ being all independent of ε. The second term on the r.h.s. of (2.9) is new but can be handled via interpolation: note that (2.8) implies that $2\overline{q} - p < 2\alpha = p\chi$, and since $2\overline{q} - p > p$ we have with $\mu \in (0,1)$

$$\frac{1}{2\overline{q}-p} = \frac{\mu}{p} + \frac{1-\mu}{p\chi}\,,$$

hence

$$\|\nabla v_\varepsilon\|_{L^{2\overline{q}-p}} \leq \|\nabla v_\varepsilon\|^\mu_{L^p}\|\nabla v_\varepsilon\|^{1-\mu}_{L^{p\chi}}\,,$$

where the norms are taken w.r.t. $B_{r+\rho}$. Recalling the boundedness of ∇v_ε in $L^p(B_{2R})$, we get

$$\left[\int_{B_{r+\rho}} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}}\,\mathrm{d}x\right]^\chi \leq c\left[\int_{B_{r+\rho}} \Gamma_\varepsilon^\alpha\,\mathrm{d}x\right]^{(1-\mu)\frac{1}{p}(2\overline{q}-p)}.$$

The definition of μ together with (1.4) implies

$$(1-\mu)\frac{1}{p}(2\overline{q}-p) < 1\,,$$

thus Young's inequality gives

$$\left[\int_{B_{r+\rho}} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}}\,\mathrm{d}x\right]^\chi \leq \tau\int_{B_{r+\rho}} \Gamma_\varepsilon^\alpha\,\mathrm{d}x + c(\tau)\,.$$

Inserting this into (2.9) and choosing τ small enough we find

$$\int_{B_r} \Gamma_\varepsilon^\alpha\,\mathrm{d}x \leq c\rho^{-\beta}\left[\int_{B_{2R}} \Gamma_\varepsilon^{\frac{p}{2}}\,\mathrm{d}x\right]^{\tilde{\beta}} + \tilde{\vartheta}\int_{B_{r+\rho}} \Gamma_\varepsilon^\alpha\,\mathrm{d}x$$

with $\tilde{\vartheta} \in (0,1)$. Now the proof of Lemma 2.3 can be completed along well known lines using Lemma 5.1, p. 81, from [Gi]. The last claim of Lemma 2.3 follows from Lemma 2.1 and a covering argument combined with the first part of Lemma 2.3. □

The next result can be established as in [BF1], Proposition 3.5, or as in [Bi], Proposition 3.29.

LEMMA 2.4 *Let $h_{(\varepsilon)} := \Gamma_{(\varepsilon)}^{\frac{p}{4}}$, where $\Gamma := 1+|\nabla u|^2$. Then we have*

i) *$h \in W^1_{2,loc}(B_{2R})$;*

ii) *$h_\varepsilon \rightharpoondown h$ in $W^1_{2,loc}(B_{2R})$;*

iii) *$\nabla v_\varepsilon \to \nabla u$ a.e. on B_{2R} as $\varepsilon \to 0$.*

Together with the higher integrability result from Lemma 2.3, part iii) of Lemma 2.4 is essential for proving a limit version of the ε-Caccioppoli inequality stated in Lemma 2.2.

LEMMA 2.5 *There exists a constant (depending on R) such that for all balls $B_{2r}(\bar{x}) \subset B_R$ we have*

$$\int_{B_r(\bar{x})} |\nabla h|^2\,\mathrm{d}x \leq c\left[r^{-2}\int_{B_{2r}(\bar{x})-B_r(\bar{x})} \Gamma^{\frac{q-2}{2}}|\nabla u - Q|^2\,\mathrm{d}x + \int_{B_{2r}(\bar{x})} \Gamma^{\overline{q}-\frac{p}{2}}\right],$$

where $Q \in \mathbb{R}^{nN}$ is arbitrary.

REMARK 2.2 *On the l.h.s. $|\nabla h|^2$ may be replaced by $\Gamma^{\frac{p-2}{2}}|\nabla^2 u|^2$.*

Proof of Lemma 2.5. In (2.5) we choose $\eta \in C_0^1(B_{2r}(\bar{x}))$ such that $\eta \equiv 1$ on $B_r(\bar{x})$, $0 \le \eta \le 1$, and $|\nabla \eta| \le 2/r$. Then, on the l.h.s. we use lower semicontinuity, the first term on the r.h.s. is handled as in the proof of Lemma 3.6 in [BF1]. By (2.7), the second term from the r.h.s. of (2.5) is dominated by $\int_{B_{2r}(\bar{x})} \Gamma_\varepsilon^{q/2}\, dx$ and on account of $\Gamma_\varepsilon \to \Gamma$ a.e. together with the higher integrability of Γ_ε we may pass to the limit as well. □

Step 3. Blow up and proof of Theorem 1.1 i).
Once having established Lemma 2.5, we can follow the arguments of [BF1], Section 4, (compare also [Bi]) by introducing the excess function for balls $B_r(x) \subset B_R$. With

$$E(x,r) := \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2\, dy + \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^q\, dy\,, \quad \text{if } q \ge 2\,,$$

$$E(x,r) := \fint_{B_r(x)} |V(\nabla u) - V((\nabla u)_{x,r})|^2\, dy\,, \quad V(\xi) := (1+|\xi|^2)^{\frac{q-2}{4}}\xi\,, \quad \text{if } q < 2\,,$$

we have to formulate the blow-up Lemma 4.1 from [BF1] in the following way:

LEMMA 2.6 *Fix $L > 0$. Then there exists a constant $C_*(L)$ such that for every $0 < \tau < 1/4$ there is an $\varepsilon = \varepsilon(L,\tau)$ satisfying: if $B_r(x) \Subset B_R$ and if we have*

$$|(\nabla u)_{x,r}| \le L\,, \quad E(x,r) + r^{\gamma^*} < \varepsilon(L,\tau)\,,$$

then

$$E(x,\tau r) \le C_*(L)\tau^2 \big[E(x,r) + r^{\gamma^*}\big]\,.$$

Here γ^* denotes some arbitrary number in $(0,2)$.

Let us give a short comment: if we follow the arguments from [BF1], Section 4, and introduce the function ψ_m as done there, then we have to bound the quantity $\int_{B_\rho} |\nabla \psi_m|^2\, dz$ for $\rho < 1$ which can be done with the scaled version of Lemma 2.5 leading to the inequality (recall (2.7))

$$\begin{aligned}\int_{B_\rho} |\nabla \psi_m|^2\, dz &\le c(\rho)\left[1 + \lambda_m^{-2} r_m^2 \fint_{B_{r_m}(x_m)} \Gamma^{\overline{q}-\frac{p}{2}}\, dx\right] \\ &\le c(\rho)\left[1 + \lambda_m^{-2} r_m^2 \fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}}\, dx\right]. \qquad (2.10)\end{aligned}$$

We now let for any $1 < t < \infty$

$$V_t(\xi) := (1+|\xi|^2)^{\frac{t-2}{4}}\xi\,, \quad H_t(\xi) := (1+|\xi|^2)^{\frac{t}{2}}\,.$$

By Lemma 2.3 of [Ha] we then have

$$\Big|\sqrt{H_t(\xi)} - \sqrt{H_t(\bar{\xi})}\Big| \le c\big|V_t(\xi) - V_t(\bar{\xi})\big|\,. \qquad (2.11)$$

By assumption, $|(\nabla u)_{x_m,r_m}| \le L$, hence we obtain from (2.11)

$$\begin{aligned}
\fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}}\,\mathrm{d}x &= \fint_{B_{r_m}(x_m)} \Big[\Gamma^{\frac{q}{4}}\Big]^2\,\mathrm{d}x \\
&\le c \fint_{B_{r_m}(x_m)} \Big[\Big|\sqrt{H_q(\nabla u)} - \sqrt{H_q((\nabla u)_{x_m,r_m})}\Big| + \sqrt{H_q((\nabla u)_{x_m,r_m})}\Big]^2\,\mathrm{d}x \\
&\le c \fint_{B_{r_m}(x_m)} \Big|\sqrt{H_q(\nabla u)} - \sqrt{H_q((\nabla u)_{x_m,r_m})}\Big|^2\,\mathrm{d}x + c(L) \\
&\le c \fint_{B_{r_m}(x_m)} |V_q(\nabla u) - V_q((\nabla u)_{x_m,r_m})|^2\,\mathrm{d}x + c(L) = cE(x_m,r_m) + c(L)\,,
\end{aligned}$$

where the last identity follows from the definition of E in the case $q \le 2$. If $q > 2$, then we simply estimate

$$\begin{aligned}
\fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}}\,\mathrm{d}x &\le c\Bigg[1 + \fint_{B_{r_m}(x_m)} |\nabla u|^q\,\mathrm{d}x\Bigg] \\
&\le c\Bigg[1 + \fint_{B_{r_m}(x_m)} |\nabla u - (\nabla u)_{x_m,r_m}|^q\,\mathrm{d}x + c(L)\Bigg] \\
&\le cE(x_m,r_m) + c(L)\,,
\end{aligned}$$

thus (2.10) gives in both cases

$$\int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z \le c(\rho)\Big[1 + r_m^2 + \lambda_m^{-2} r_m^2 c(L)\Big]\,.$$

Recalling the choice of γ^* we observe that as $m \to \infty$

$$\lambda_m^{-2} r_m^2 \to 0\,,$$

hence the boundedness of $\int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z$ follows, and the proof can be completed as in [BF1].

Step 4. Proof of Theorem 1.1 ii).
If $n = 2$, then we know by Lemma 2.3 that $\nabla v_\varepsilon \in L^t_{loc}(B_{2R};\mathbb{R}^{nN})$ for any $t < \infty$ uniform w.r.t. ε. Now we quote [BF2], proof of Theorem 1: on the r.h.s. of (9) from [BF2] we have to add

$$-\int D_{x_s} D_P f_\varepsilon(\cdot,\nabla v_\varepsilon) : \nabla(\eta^2 \partial_s[v_\varepsilon - Qx])\,\mathrm{d}x$$

and by using the growth properties of $D_x D_P f$ together with Young's inequality and the higher integrability of ∇v_ε it is easy to see that we have (14) of [BF2] with an extra additive term of the form *const* r^β, $0 < \beta < 1$, on the r.h.s. But as outlined in [BF3] or [ABF] this term does not affect the application of the Frehse-Seregin Lemma (see [FS]) and the claim follows as before with the help of Frehse's variant of the Dirichlet-Growth

Theorem (see [Fr]).

Step 5. Proof of Theorem 1.1 iii).
We are first going to prove the following auxiliary lemma which gives good initial regularity for our regularizing sequence in the vector case $N > 1$ together with the special structure $f = g(x, |P|^2)$.

LEMMA 2.7 *Assume that $F(x, P)$ satisfies with some given $1 < t < \infty$ for all $x \in \overline{\Omega}$, P, $U \in \mathbb{R}^{nN}$ and with positive constants λ, Λ, c*

$$\begin{aligned}
\lambda(1+|P|^2)^{\frac{t-2}{2}}|U|^2 &\le D_P^2 F(x,P)(U,U) \le \Lambda(1+|P|^2)^{\frac{t-2}{2}}|U|^2\,; && (2.12)\\
|D_x D_P F(x,P)| &\le c(1+|P|^2)^{\frac{t-1}{2}}\,; && (2.13)\\
F(x,P) &= G(x,|P|^2)\,. && (2.14)
\end{aligned}$$

Here G: $\overline{\Omega} \times \mathbb{R} \to [0, \infty)$ is a function of class C^2. Moreover we assume that for some $\gamma > 0$

$$|D_P^2 F(x,P) - D_P^2 F(x,Q)| \le c(1+|P|^2+|Q|^2)^{\frac{t-2-\gamma}{2}}|P-Q|^\gamma\,.$$

Then, if $u \in W^1_{t,loc}(\Omega; \mathbb{R}^N)$ is a local minimizer of $\int_\Omega F(x, \nabla u)\,\mathrm{d}x$, u is of class $C^{1,\kappa}(\Omega; \mathbb{R}^N)$ for any $0 < \kappa < 1$.

REMARK 2.3 *If $N = 1$, then the statement of course holds without (2.14), see [LU]. Once having established the $C^{1,\kappa}$-regularity of the solution u studied in Lemma 2.7, we immediately obtain $u \in W^2_{2,loc}(\Omega; \mathbb{R}^N)$. Combining both facts and using potential theory for linear elliptic systems with continuous coefficients we arrive at $u \in W^2_{\tilde{t},loc}(\Omega; \mathbb{R}^N)$ for any finite $\tilde{t}$.*

Proof of Lemma 2.7. We concentrate on the case $t \ge 2$. In the case $1 < t < 2$ the following arguments have to be modified using Proposition 2.11 in [AF]. Note that for both cases the above Hölder condition for $D_p^2 F(x, \cdot)$ implies the corresponding ones in [AF] and [GM], respectively, if x is considered as fixed. Let $B_R(x_0) \Subset \Omega$, $R \le R_0$, where R_0 is fixed later on. We denote by v the unique solution of the variational problem

$$\int_{B_R(x_0)} F_0(\nabla w)\,\mathrm{d}x \to \min \quad \text{in } u_{|B_R(x_0)} + \overset{\circ}{W}{}^1_t(B_R(x_0); \mathbb{R}^N)\,,$$

where $F_0 := F(x_0, \cdot)$. Then inequality (3.1) of Theorem 3.1 in [GM] gives together with the minimality of v and the growth of F_0:

$$\|\nabla v\|^t_{L^\infty(B_{r/2})} \le c \fint_{B_R} (1+|\nabla v|^2)^{\frac{t}{2}}\,\mathrm{d}x \le c \fint_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x\,. \qquad (2.15)$$

We define $V(\xi) = V_t(\xi)$ as in the third step and recall Lemma 2.3 of [Ha] to obtain for $\rho \le R/2$

$$\begin{aligned}
\int_{B_\rho} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x &\le c\left[\int_{B_\rho} (1+|\nabla v|^2)^{\frac{t}{2}}\,\mathrm{d}x + \int_{B_\rho} \left|(1+|\nabla u|^2)^{\frac{t}{4}} - (1+|\nabla v|^2)^{\frac{t}{4}}\right|^2 \mathrm{d}x\right.\\
&\le c\int_{B_\rho} (1+|\nabla v|^2)^{\frac{t}{2}}\,\mathrm{d}x + c\int_{B_\rho} |V(\nabla u) - V(\nabla v)|^2\,\mathrm{d}x\,.
\end{aligned}$$

Hence, (2.15) implies

$$\int_{B_\rho} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x \le c\Big(\frac{\rho}{R}\Big)^n \int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x + c\int_{B_\rho} |V(\nabla u)-V(\nabla v)|^2\,\mathrm{d}x\,. \quad (2.16)$$

Then (2.3) of [Ha] and (2.1) of [GM] yield

$$\begin{aligned}\int_{B_\rho} |V(\nabla u)-V(\nabla v)|^2\,\mathrm{d}x &\le c\int_{B_R} (1+|\nabla u|^2+|\nabla v|^2)^{\frac{t-2}{2}}|\nabla u-\nabla v|^2\,\mathrm{d}x\\ &\le c\int_{B_R} \underbrace{\int_0^1 (1+|\nabla v+t(\nabla u-\nabla v)|^2)^{\frac{t-2}{2}}|\nabla u-\nabla v|^2\,\mathrm{d}t\,\mathrm{d}x}_{=:(*)}\,.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\big(DF_0(\nabla u)-DF_0(\nabla v)\big):(\nabla u-\nabla v)\\ &= \int_0^1 D^2F_0(\nabla v+t(\nabla u-\nabla v))(\nabla u-\nabla v,\nabla u-\nabla v)\,\mathrm{d}t \ge \lambda(*)\,.\end{aligned}$$

Putting together these two inequalities, using the equations for u, v and recalling the growth condition (2.13) one has (again see [Gi], p. 151)

$$\begin{aligned}\int_{B_R} |V(\nabla u)-V(\nabla v)|^2\,\mathrm{d}x &\le c\int_{B_R} \big(DF_0(\nabla u)-DF_0(\nabla v)\big):(\nabla u-\nabla v)\,\mathrm{d}x\\ &= c\int_{B_R} \big(DF_0(\nabla u)-D_PF(x,\nabla u)\big):(\nabla u-\nabla v)\,\mathrm{d}x\\ &\le cR\int_{B_R} (1+|\nabla u|^2)^{\frac{t-1}{2}}|\nabla u-\nabla v|\,\mathrm{d}x\\ &\le \varepsilon\int_{B_R} (1+|\nabla u|^2)^{\frac{t-2}{2}}|\nabla u-\nabla v|^2\,\mathrm{d}x\\ &\quad +c(\varepsilon)R^2\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x\\ &\le c\varepsilon\int_{B_R} |V(\nabla u)-V(\nabla v)|^2\,\mathrm{d}x\\ &\quad +c(\varepsilon)R^2\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x\,.\end{aligned}$$

Now, if $\varepsilon>0$ is sufficiently small, then it is shown that

$$\int_{B_R} |V(\nabla u)-V(\nabla v)|^2\,\mathrm{d}x \le cR^2\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x\,. \quad (2.17)$$

Inserting this in (2.16) we arrive at

$$\int_{B_\rho} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x \le c\Big[\Big(\frac{\rho}{R}\Big)^n + R^2\Big]\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x\,. \quad (2.18)$$

Note that (2.18) was just shown in case $\rho \le R/2$, for $R/2<\rho<R$ the estimate is trivial.

Next we choose $\beta < n$ which may be arbitrarily close to n. With a suitable choice of R_0 we may apply Lemma 2.1 from [Gi] to (2.18). As a consequence, for all radii $\rho^* \le R^* \le R_0$ which are sufficiently small we have

$$\int_{B_{\rho^*}} (1+|\nabla u|^2)^{\frac{t}{2}} \,\mathrm{d}x \le c\Big(\frac{\rho^*}{R^*}\Big)^\beta \int_{B_{R^*}} (1+|\nabla u|^2)^{\frac{t}{2}} \,\mathrm{d}x\,.$$

Choosing $\rho^* = R$ and $R^* = R_0$ it is shown in particular that

$$\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}} \,\mathrm{d}x \le c\Big(\frac{R}{R_0}\Big)^\beta \int_{B_{R_0}} (1+|\nabla u|^2)^{\frac{t}{2}} \,\mathrm{d}x\,. \tag{2.19}$$

Finally we make use of [GM], formula (3.2), i.e. for some exponent $\sigma > 0$ it holds

$$\fint_{B_\rho} |V(\nabla v) - (V(\nabla v))_{x_0,\rho}|^2 \,\mathrm{d}x \le c\Big(\frac{\rho}{R}\Big)^\sigma \fint_{B_R} |V(\nabla v) - (V(\nabla v))_{x_0,R}|^2 \,\mathrm{d}x\,. \tag{2.20}$$

Note that (2.20) implies as in [GM], (5.6),

$$\begin{aligned}\fint_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2 \,\mathrm{d}x &\le c\Big(\frac{\rho}{R}\Big)^\sigma \fint_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2 \,\mathrm{d}x \\ &\quad + \Big(\frac{R}{\rho}\Big)^n \fint_{B_R} |V(\nabla u) - V(\nabla v)|^2 \,\mathrm{d}x\,,\end{aligned}$$

hence (2.17) and (2.19) imply

$$\begin{aligned}\int_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2 \,\mathrm{d}x &\le c\Bigg[\Big(\frac{\rho}{R}\Big)^{n+\sigma} \int_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2 \,\mathrm{d}x \\ &\quad + R^2 \int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}} \,\mathrm{d}x\Bigg] \\ &\le c\Bigg[\Big(\frac{\rho}{R}\Big)^{n+\sigma} \int_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2 \,\mathrm{d}x + R^{2+\beta}\Bigg]\,.\end{aligned}$$

Now

$$\Psi:\quad \rho \mapsto \Psi(\rho) := \int_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2 \,\mathrm{d}x$$

clearly is an increasing function. From [Gi], p. 86, we infer (choosing $n < 2+\beta < n+\sigma$) that Ψ growth like $\rho^{2+\beta}$. Since $2+\beta > n$, this gives Hölder continuity of $V(\nabla u)$, in particular ∇u is of class C^0. We then let $w = \partial_s u$ and observe that w solves an elliptic system with continuous coefficients. Theorem 3.1 of [Gi], p. 87, then proves our claim. □

For the proof of Theorem 1.1 iii) we will now use DeGiorgi type arguments as done in the proof of Theorem 3.16 in [Bi] which has to be adjusted to the situation at hand. W.l.o.g. we may assume that $n \ge 3$, since by the second part of the theorem regularity in the two-dimensional case holds without structure condition. We still work on the ball

B_{2R} and choose $B_r(\bar{x}) \subset B_R$ and $\eta \in C_0^1(B_r(\bar{x}), [0,1])$. We further let $\omega_\varepsilon = \ln(\Gamma_\varepsilon)$, $\Gamma_\varepsilon = 1 + |\nabla v_\varepsilon|^2$, and consider the sets

$$A(h,r) := \{x \in B_r(\bar{x}) : \omega_\varepsilon > h\}.$$

From Lemma 2.7 we deduce $v_\varepsilon \in W^1_{\infty,loc}(B_{2R};\mathbb{R}^N)$ (and therefore $\nabla v_\varepsilon \in W^1_{2,loc}(B_{2R};\mathbb{R}^{nN})$) which enables us to use the same test functions as in [Bi]. Thus we have (30), p. 62, of [Bi], where on the r.h.s. we have to add the quantity

$$I := \int_{A(k,r)} |D_x D_P f_\varepsilon(\cdot, \nabla v_\varepsilon)|\, |\nabla(\eta^2 \nabla v_\varepsilon(\omega_\varepsilon - k))|\, \mathrm{d}x\,.$$

I itself splits into a sum of three integrals, one of them being

$$\begin{aligned}\int_{A(k,r)} |D_x D_P f_\varepsilon(\cdot, \nabla v_\varepsilon)| \eta^2 (\omega_\varepsilon - k) |\nabla^2 v_\varepsilon|\, \mathrm{d}x \;&\leq\; \gamma \int_{A(k,r)} \Gamma_\varepsilon^{\frac{p-2}{2}} \eta^2 |\nabla^2 v_\varepsilon|^2 (\omega_\varepsilon - k)\, \mathrm{d}x \\ &\quad + c(\gamma) \int_{A(k,r)} \Gamma_\varepsilon^{\frac{2-p}{2}+\bar{q}-1} (\omega_\varepsilon - k)\, \mathrm{d}x\,,\end{aligned}$$

where we used condition (1.3) and Young's inequality. If γ is small enough, then the first integral on the r.h.s. can be absorbed in the first integral on the l.h.s of (30), p. 62, in [Bi]. Then (34), p. 63, of [Bi] reads:

$$\begin{aligned}\int_{A(k,r)} \Gamma_\varepsilon^{\frac{p}{2}} \eta^2 |\nabla \omega_\varepsilon|^2\, \mathrm{d}x \;&\leq\; c\left[\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} (\omega_\varepsilon - k)^2 |\nabla \eta|^2\, \mathrm{d}x + \xi\right], \\ \xi \;&:=\; \int_{A(k,r)} \Gamma_\varepsilon^{\bar{q}-\frac{p}{2}}\, \mathrm{d}x + \int_{A(k,r)} \Gamma_\varepsilon^{\frac{\bar{q}}{2}}\, \mathrm{d}x + \int_{A(k,r)} \Gamma_\varepsilon^{\bar{q}-\frac{p}{2}} (\omega_\varepsilon - k)\, \mathrm{d}x\,.\end{aligned} \tag{2.21}$$

In the same way we use (35), p. 63, of [Bi] with the extra term

$$\int_{A(k,r)} |D_x D_P f_\varepsilon(\cdot, \nabla v_\varepsilon)|\, |\nabla(\eta^2 \nabla v_\varepsilon(\omega_\varepsilon - k)^2|\, \mathrm{d}x$$

on the right-hand side, this time we get

$$\begin{aligned}\int_{A(k,r)} \Gamma_\varepsilon^{\frac{p-2}{2}} (\omega_\varepsilon - k)^2 |\nabla^2 v_\varepsilon|^2 \eta^2\, \mathrm{d}x \;&\leq\; c\left[\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} (\omega_\varepsilon - k)^2 |\nabla \eta|^2\, \mathrm{d}x + \bar{\xi}\right], \\ \bar{\xi} \;&:=\; \int_{A(k,r)} \Gamma_\varepsilon^{\bar{q}-\frac{p}{2}}\, \mathrm{d}x + \int_{A(k,r)} \Gamma_\varepsilon^{\frac{\bar{q}}{2}} (\omega_\varepsilon - k)^2\, \mathrm{d}x \\ &\quad + \int_{A(k,r)} \Gamma_\varepsilon^{\bar{q}-\frac{p}{2}} (\omega_\varepsilon - k)^2\, \mathrm{d}x\,.\end{aligned} \tag{2.22}$$

By combining (2.21) and (2.22) we obtain the following version of (27), p. 61, in [Bi]:

$$\begin{aligned}&\int_{A(k,r)} \Gamma_\varepsilon^{\frac{p}{2}} \eta^2 |\nabla \omega_\varepsilon|^2\, \mathrm{d}x + \int_{A(k,r)} \Gamma_\varepsilon^{\frac{p-2}{2}} (\omega_\varepsilon - k)^2 \eta^2 |\nabla v_\varepsilon|^2\, \mathrm{d}x \\ &\quad \leq\; c\left[\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} |\nabla \eta|^2 (\omega_\varepsilon - k)^2\, \mathrm{d}x + \xi + \bar{\xi}\right].\end{aligned} \tag{2.23}$$

For handling $\xi + \bar{\xi}$ we use (2.7). If we let

$$a(k,r) := \int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} \,\mathrm{d}x\,,$$

then we have

$$\xi + \bar{\xi} \leq c a(k,r)\,. \tag{2.24}$$

Let us further set

$$\tau(k,r) := \int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} (\omega_\varepsilon - k)^2 \,\mathrm{d}x\,.$$

Next we fix numbers $h > k > 0$ and radii $r < \bar{r}$ such that $B_{\bar{r}}(\bar{x}) \subset B_R$. Then, as in [Bi], we deduce from (2.21) – (2.24):

$$\tau(h,r) \leq c\Big[(h-k)^{-2\frac{\chi-1}{\chi}-2} + (h-k)^{-2\frac{\chi-1}{\chi}}\Big](r'-r)^{-2}\tau(k,r')^{1+\frac{\chi-1}{\chi}}$$

provided we assume w.l.o.g. that $R \leq 1$. For the application of the Stamppachia Lemma it is sufficient to study the case $h - k \leq 1$, thus we can replace the quantity $[\dots]$ by $(h-k)^{-2-2(\chi-1)/\chi}$ and argue as in [Bi] with the result that the functions v_ε are locally Lipschitz on B_R uniform w.r.t. ε. As a consequence we get $u \in W^1_{\infty,loc}(\Omega;\mathbb{R}^N)$. Let us fix $\Omega' \Subset \Omega$ and a constant $M > 0$ s.t. $|\nabla u(x)| \leq M$ for a.a. $x \in \Omega'$. Then, as outlined in [MS], we can construct an integrand F on $\overline{\Omega'} \times \mathbb{R}^{nN}$ satisfying (2.12)–(2.14) for a suitable t and s.t.

$$F(x,P) = f(x,P)$$

for $x \in \overline{\Omega'}$ and $P \in \mathbb{R}^{nN}$, $|P| \leq 2M$. But then u is a local minimizer of $\int_{\Omega'} F(\cdot,\nabla v)\,\mathrm{d}x$ on Ω', hence of class $C^{1,\alpha}$ by Lemma 2.7. The reader should note that the Hölder condition for $D_P^2 F(x,\cdot)$ required for the application of Lemma 2.7 is a consequence of the corresponding condition for $D_P^2 f(x,\cdot)$ as stated in the hypotheses of Theorem 1.1 iii) if the vector case is considered. $\square$

2.2 Proof of Theorem 1.2

We use the same regularization as in Step 1 of Section 2.1 where the exponent q is now chosen in $(\overline{q}, p+2)$ sufficiently close to $p+2$ s.t.

$$\overline{q} \leq \frac{1}{2}(p+q)\,. \tag{2.25}$$

Note that such a choice is possible on account of (1.7). Note also that Lemma 2.1 continues to hold since again we assume $\nabla u \in L^{\overline{q}}_{loc}(\Omega;\mathbb{R}^{nN})$. From (1.6) together with the maximum principle it follows that

$$\sup_{0<\varepsilon<1} \|v_\varepsilon\|_{L^\infty(B_{2R})} \leq \sup_{B_{2R}} |u| < \infty\,. \tag{2.26}$$

Step 1. Higher integrability.
We follow [Bi], proof of Theorem 5.21, and show

LEMMA 2.8 *There is a constant c independent of ε such that*

$$\int_{B_r(\bar{x})} |\nabla v_\varepsilon|^s \le c$$

for any ball $B_r(\bar{x}) \Subset B_{2R}$ and any $s \in (1,\infty)$. The constant c depends on the location of the ball, the constants appearing in (1.2) and (1.3), on s and on $\sup_{B_{2R}} |u|$.

Proof. Let $\alpha \ge 0$ denote a fixed real number and define the quantities $\beta := 2 + p - q$,

$$0 < \sigma := \frac{\alpha}{2} + \frac{q}{2} < 1 + \frac{\alpha}{2} + \frac{p}{2} =: \sigma' .$$

For $k \in \mathbb{N}$ large enough we have

$$2k\frac{\sigma}{\sigma'} < 2k - 2 .$$

Finally, we consider $\eta \in C_0^\infty(B_{2R})$, $0 \le \eta \le 1$, and obtain with exactly the same arguments as in [Bi], inequality (19) on p. 155 (by letting $h \equiv 1$ during this calculation and by using (2.26))

$$\begin{aligned} \int_{B_{2R}} \eta^{2k} \Gamma_\varepsilon^{\frac{\alpha+p}{2}+1}\, \mathrm{d}x &\le c\left[1 + \int_{B_{2R}} |\nabla^2 v_\varepsilon|^2 \Gamma_\varepsilon^{\frac{\alpha+p}{2}} \eta^{2k}\, \mathrm{d}x + \int_{B_{2R}} \eta^{2k-1} |\nabla \eta| \Gamma_\varepsilon^{\frac{1+\alpha+p}{2}}\, \mathrm{d}x\right] \\ &=: c[1 + I + II] . \end{aligned} \tag{2.27}$$

If $\operatorname{spt}\eta \subset B_{\rho'} = B_{\rho'}(x_0)$, $\eta = 1$ on $B_\rho = B_\rho(x_0)$ and $|\nabla\eta| \le c/(\rho' - \rho)$, then we can use (20), p. 155 in [Bi] to handle II, i.e. we have

$$II \le \tau \int_{B_{2R}} \eta^{2k} \Gamma_\varepsilon^{1+\frac{\alpha+p}{2}}\, \mathrm{d}x + c(\rho' - \rho)^{-2}\tau^{-1} \int_{B_{2R}} \Gamma_\varepsilon^{\frac{\alpha+p}{2}} \eta^{2k-2}\, \mathrm{d}x \tag{2.28}$$

valid for any $\tau \in (0,1)$, where for τ small enough the first term on the r.h.s. of (2.28) can be absorbed on the l.h.s. of (2.27). For I we observe

$$\begin{aligned} I &\le \tau \int_{B_{2R}} \eta^{2k+2} \Gamma_\varepsilon^{\frac{p-2}{2}+\frac{\alpha+\beta}{2}} |\nabla^2 v_\varepsilon|^2\, \mathrm{d}x + \tau^{-1} \int_{B_{2R}} \eta^{2k-2} \Gamma_\varepsilon^{\frac{\alpha+q}{2}}\, \mathrm{d}x \\ &=: \tau I_1 + \tau^{-1} I_2 . \end{aligned} \tag{2.29}$$

As we shall prove below the quantity I_1 can be bounded in the following form:

$$I_1 \le c(\rho' - \rho)^{-2} \int_{B_{2R}} \eta^{2k} \Gamma_\varepsilon^{\frac{\alpha+\beta}{2}} \Gamma_\varepsilon^{\frac{q}{2}}\, \mathrm{d}x , \tag{2.30}$$

where c also depends on α. We insert (2.30) into (2.29) and replace τ in (2.29) by $\tau'(\rho'-\rho)^2$ for some $\tau' > 0$. Since

$$\frac{\alpha+\beta}{2} + \frac{q}{2} = \frac{\alpha+p}{2} + 1 ,$$

we see that for $\tau' \ll 1$ the term corresponding to τ' can be absorbed on the l.h.s. of (2.27). Moreover, we have with Young's inequality

$$\begin{aligned} (\tau')^{-1}(\rho' - \rho)^{-2} I_2 &= (\tau')^{-1}(\rho' - \rho)^{-2} \int_{B_{2R}} \eta^{2k-2} \Gamma_\varepsilon^{\frac{\alpha+q}{2}}\, \mathrm{d}x \\ &\le (\tau')^{-1}(\rho' - \rho)^{-2} \left[\tau'' \int_{B_{2R}} \left[\eta^{2k-2} \Gamma_\varepsilon^{\frac{\alpha+q}{2}}\right]^{\frac{\sigma'}{\sigma}} \mathrm{d}x + (\tau'')^{-\frac{\sigma}{\sigma'-\sigma}} |B_{2R}|\right] \\ &\le (\tau')^{-1}(\rho' - \rho)^{-2} \left[\tau'' \int_{B_{2R}} \eta^{2k} \Gamma_\varepsilon^{\frac{\alpha+p+2}{2}} + (\tau'')^{-\frac{\sigma}{\sigma'-\sigma}} |B_{2R}|\right] . \end{aligned}$$

If we let $\tau'' = \tau'(\rho' - \rho)^2\tau^*$ and if τ^* is small enough, the first term on the r.h.s. of the above inequality can be absorbed on the l.h.s. of (2.27). Putting together our results we have inequality (23), p. 156, of [Bi], i.e.

$$\int_{B_{2R}} \eta^{2k}\Gamma_\varepsilon^{\frac{\alpha+p+2}{2}}\,\mathrm{d}x \le c\left[1 + \int_{B_{2R}} \eta^{2k-2}\Gamma_\varepsilon^{\frac{\alpha+p}{2}}\,\mathrm{d}x\right]$$

with c also depending on α, ρ and ρ' but independent of ε. Now the same iteration as in [Bi] gives

$$\int_{B_r(x_0)} |\nabla v_\varepsilon|^s\,\mathrm{d}x \le const$$

for any $s < \infty$ and $r < 2R$. It remains to prove the inequality (2.30). But this follows from an appropriate version of Lemma 5.20 i) of [Bi]. Note that (2.30) is the only place where we use the fact that v_ε solves a variational problem. To be more precise, we take

$$\varphi = \eta^{2k+2}\partial_\gamma v_\varepsilon \Gamma_\varepsilon^s$$

as test function in

$$\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\nabla\varphi)\,\mathrm{d}x = -\int_{B_{2R}} D_{x_\gamma}D_P f_\varepsilon(\cdot,\nabla v_\varepsilon) : \nabla\varphi\,\mathrm{d}x\,,$$

where s is some exponent ≥ 0 and k denotes some integer ≥ 1. The admissibility of φ follows from Lemma 2.7 and Remark 2.3. We get

$$\begin{aligned}
&\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma\nabla v_\varepsilon)\eta^{2k+2}\Gamma_\varepsilon^s\,\mathrm{d}x\\
&\quad + \int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma v_\varepsilon\otimes\nabla\Gamma_\varepsilon^s)\eta^{2k+2}\,\mathrm{d}x\\
&\quad = -(2k+2)\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\nabla\eta\otimes\partial_\gamma v_\varepsilon)\eta^{2k+1}\Gamma_\varepsilon^s\,\mathrm{d}x\\
&\qquad -\int_{B_{2R}} D_{x_\gamma}D_P f_\varepsilon(\cdot,\nabla v_\varepsilon) : \nabla(\eta^{2k+2}\partial_\gamma v_\varepsilon\Gamma_\varepsilon^s)\,\mathrm{d}x\,.
\end{aligned}\tag{2.31}$$

To the first integral on the r.h.s. we apply the Cauchy-Schwarz inequality (for the bilinear form $D_P^2(x,\nabla v_\varepsilon(x))$) and then use Young's inequality to get the bound

$$\begin{aligned}
&\tau\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma\nabla v_\varepsilon)\eta^{2k+2}\Gamma_\varepsilon^s\,\mathrm{d}x\\
&\quad +c(\tau)\int_{B_{2R}} |\nabla\eta|^2\eta^{2k}|D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)|\Gamma_\varepsilon^{1+s}\,\mathrm{d}x\,,
\end{aligned}\tag{2.32}$$

and for τ small the first term can be absorbed on the l.h.s. of (2.31). For the second integral on the r.h.s. of (2.31) we use (1.3), thus

$$\begin{aligned}
\text{l.h.s. of (2.31)} \;\le\; & c\Bigg[\int_{B_{2R}} \Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\Gamma_\varepsilon^s|\nabla\eta|\eta^{2k+1}|\nabla v_\varepsilon|\,\mathrm{d}x + \int_{B_{2R}} \Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\Gamma_\varepsilon^s\eta^{2k+2}|\nabla^2 v_\varepsilon|\,\mathrm{d}x\\
&+\int_{B_{2R}} \Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\Gamma_\varepsilon^{s-1}\eta^{2k+2}|\nabla v_\varepsilon||\nabla\Gamma_\varepsilon|\,\mathrm{d}x\Bigg]\\
=: \;& c[J_1 + J_2 + J_3]\,.
\end{aligned}$$

We have (since $0 \le \eta \le 1$, $|\nabla\eta| \le (\rho'-\rho)^{-1}$)

$$\begin{aligned} J_1 &\le c(\rho'-\rho)^{-2}\int_{B_{2R}} \eta^{2k}\Gamma_\varepsilon^{\frac{\overline{q}}{2}+s}\,\mathrm{d}x \\ &= c(\rho'-\rho)^{-2}\int_{B_{2R}} \eta^{2k}\Gamma_\varepsilon^{\frac{\overline{q}-2}{2}+1+s}\,\mathrm{d}x \end{aligned}$$

which means that we obtain the same bound as for the second term in (2.32). With $\kappa > 0$ arbitrary we have

$$J_2 \le \kappa \int_{B_{2R}} \Gamma_\varepsilon^{\frac{p-2}{2}}\Gamma_\varepsilon^s \eta^{2k+2}|\nabla^2 v_\varepsilon|^2\,\mathrm{d}x + c(\kappa)\int_{B_{2R}} \Gamma_\varepsilon^{s+1}\Gamma_\varepsilon^{-\frac{p}{2}+\overline{q}-1}\eta^{2k+2}\,\mathrm{d}x\,.$$

By (1.2) and by choosing κ small enough the first term can be absorbed in the first integral on the l.h.s. of (2.31). For the second term we use $\eta^{2k+2} \le \eta^{2k}$ and observe $-\frac{p}{2}+\overline{q}-1 \le \frac{q-2}{2}$ which is a consequence of (2.25). In order to handle J_3 we observe that the second integral on the l.h.s. of (2.31) can be written as

$$\frac{1}{2}\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(e_\gamma\otimes\nabla\Gamma_\varepsilon^s, e_\gamma\otimes\nabla\Gamma_\varepsilon^s)\eta^{2k+2}\,\mathrm{d}x$$

which is obvious if $N = 1$, whereas in the vector-case we use the special structure. By ellipticity we therefore obtain the lower bound

$$J_4 := c\int_{B_{2R}} \Gamma_\varepsilon^{s-1}\eta^{2k+2}\Gamma_\varepsilon^{\frac{p-2}{2}}\left|\nabla|\nabla v_\varepsilon|^2\right|^2\mathrm{d}x$$

for this term. On the other hand

$$\begin{aligned} J_3 &\le c\int_{B_{2R}} \eta^{2k+2}\Gamma_\varepsilon^{\frac{\overline{q}}{2}}\Gamma_\varepsilon^{s-1}\left|\nabla|\nabla v_\varepsilon|^2\right|^2\mathrm{d}x \\ &\le \kappa\int_{B_{2R}} \eta^{2k+2}\Gamma_\varepsilon^{s-1}\Gamma_\varepsilon^{\frac{p-2}{2}}\left|\nabla|\nabla v_\varepsilon|^2\right|^2\mathrm{d}x \\ &\quad + c(\kappa)\int_{B_{2R}} \eta^{2k+2}\Gamma_\varepsilon^{s-1+\overline{q}+\frac{2-p}{2}}\,\mathrm{d}x\,, \end{aligned}$$

and for all κ small enough the first term is absorbed in J_4. For the second one we use $\eta^{2k+2} \le \eta^{2k}$ and observe that by (2.25)

$$s-1+\overline{q}+\frac{2-p}{2} = s+1+\overline{q}-\frac{p}{2}-1 \le s+1+\frac{q-2}{2}\,.$$

Altogether we have shown that

$$\begin{aligned} \int_{B_{2R}} \eta^{2k+2}|\nabla^2 v_\varepsilon|^2\Gamma_\varepsilon^{s+\frac{p-2}{2}}\,\mathrm{d}x &\le c(\rho'-\rho)^{-2}\int_{B_{2R}} \Gamma_\varepsilon^{\frac{q-2}{2}}\Gamma_\varepsilon^{1+s}\eta^{2k}\,\mathrm{d}x \\ &= c(\rho'-\rho)^{-2}\int_{B_{2R}} \Gamma_\varepsilon^{\frac{q}{2}}\Gamma_\varepsilon^s\eta^{2k}\,\mathrm{d}x\,, \end{aligned}$$

and (2.30) is established by choosing $s = \frac{1}{2}(\alpha+\beta)$. □

Step 2. Uniform local gradient bounds

LEMMA 2.9 *There is a finite local constant independent of ε s.t.*

$$|\nabla v_\varepsilon| \le c \quad \text{on } B_r \Subset B_{2R}\,.$$

Proof. We modify the proof of Theorem 5.22 in [Bi]. To this purpose let us fix radii $0 < r < \hat r < 2R$ and consider $\eta \in C_0^\infty(B_{\hat r})$ with the usual properties where all balls are centered at x_0. Moreover, for $k > 0$ we let

$$A(k,r) = \{x \in B_r : \Gamma_\varepsilon \ge k\}\,.$$

By elementary calculations (see [Bi], p. 157) we obtain

$$\int_{A(k,r)} (\Gamma_\varepsilon - k)^{\frac{n}{n-1}}\,\mathrm{d}x \le c[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}]\,, \tag{2.33}$$

where

$$\begin{aligned} I_1^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat r)} |\nabla\eta|(\Gamma_\varepsilon - k)\,\mathrm{d}x\right]^{\frac{n}{n-1}} \\ &\le c(\hat r - r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon - k)^2\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}} \\ &\quad \cdot\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{2-p}{p}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}, \end{aligned} \tag{2.34}$$

$$\begin{aligned} I_2^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat r)} \eta|\nabla\Gamma_\varepsilon|\,\mathrm{d}x\right]^{\frac{n}{n-1}} \\ &\le c\left[\int_{A(k,\hat r)} \eta^2|\nabla\Gamma_\varepsilon|^2\Gamma_\varepsilon^{\frac{p-2}{2}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{2-p}{2}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}. \end{aligned} \tag{2.35}$$

We claim the validity of

$$\begin{aligned} \int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla\Gamma_\varepsilon|^2\eta^2\,\mathrm{d}x &\le c\left[\int_{A(k,\hat r)} |\nabla\eta|^2\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon - k)^2\,\mathrm{d}x\right. \\ &\quad \left. + \int_{A(k,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}\eta^2\,\mathrm{d}x\right]. \end{aligned} \tag{2.36}$$

Accepting (2.36) for the moment, we get by combining (2.33)–(2.36)

$$\begin{aligned} \int_{A(k,r)} (\Gamma_\varepsilon - k)^{\frac{n}{n-1}}\,\mathrm{d}x &\le c(\hat r - r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon - k)^2\,\mathrm{d}x + \int_{A(k,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}} \\ &\quad \cdot\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{2-p}{2}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}, \end{aligned} \tag{2.37}$$

which corresponds to the inequality (24) on p. 157 of [Bi]. Let s and t denote real numbers > 1. With Hölder's inequality we deduce from Lemma 2.8

$$\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}} (\Gamma_\varepsilon - k)^2 \,\mathrm{d}x \le c \Bigg[\int_{A(k,r)} (\Gamma_\varepsilon - k)^{\frac{n}{n-1}} \,\mathrm{d}x \Bigg]^{\frac{1}{s}}$$

and

$$\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{2-p}{2}} \,\mathrm{d}x \le c \Bigg[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}} \,\mathrm{d}x \Bigg]^{\frac{1}{t}} ,$$

where c now is a local constant and we assume $\hat r \le R_0$ for some $R_0 < R$. Inserting the above inequalities into (2.37) we end up with

$$\begin{aligned}
&\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}} (\Gamma_\varepsilon - k)^2 \,\mathrm{d}x \\
&\le \; c(\hat r - r)^{-\frac{1}{s}\frac{n}{n-1}} \Bigg[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}} (\Gamma_\varepsilon - k)^2 \,\mathrm{d}x + \int_{A(k,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1} \,\mathrm{d}x \Bigg]^{\frac{1}{2s}\frac{n}{n-1}} \\
&\qquad \cdot \Bigg[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}} \,\mathrm{d}x \Bigg]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}} .
\end{aligned} \tag{2.38}$$

Let $h > k$ and define

$$\begin{aligned}
\tau(k,r) &:= \int_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}} (\Gamma_\varepsilon - k)^2 \,\mathrm{d}x \,, \\
a(k,r) &:= \int_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}} \,\mathrm{d}x \,.
\end{aligned}$$

Clearly $a(h,r) \le (h-k)^{-2}\tau(k,r)$ and from (2.38) (with k replaced by h) it follows

$$\begin{aligned}
\tau(h,r) &\le c(\hat r - r)^{-\gamma} \Bigg[\tau(h,\hat r) + \int_{A(h,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1} \,\mathrm{d}x \Bigg]^{\frac{1}{2s}\frac{n}{n-1}} a(h,\hat r)^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}} \\
&\le c(\hat r - r)^{-\gamma} (h-k)^{-\alpha} \tau(k,\hat r)^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}} \\
&\qquad \Bigg[\tau(h,\hat r) + \int_{A(h,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1} \,\mathrm{d}x \Bigg]^{\frac{1}{2s}\frac{n}{n-1}}
\end{aligned} \tag{2.39}$$

with positive exponents γ and α. By (2.25) we have $\overline{q} \le \frac{1}{2}(p+q)$, i.e. $\overline{q} - \frac{p}{2} + 1 \le \frac{1}{2}(q+2)$. If we choose $m > 1$, quote Lemma 2.8 and use Hölder's inequality we therefore get

$$\begin{aligned}
\int_{A(h,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1} \,\mathrm{d}x &\le \int_{A(h,\hat r)} \Gamma_\varepsilon^{\frac{q+2}{2}} \,\mathrm{d}x \\
&= \int_{A(h,\hat r)} \Gamma_\varepsilon^{\frac{1}{m}\frac{q-2}{2}} \Gamma_\varepsilon^{\frac{q+2}{2}-\frac{1}{m}\frac{q-2}{2}} \,\mathrm{d}x \\
&\le c \Bigg[\int_{A(h,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}} \,\mathrm{d}x \Bigg]^{\frac{1}{m}} = c a(h,\hat r)^{\frac{1}{m}} \\
&\le c(h-k)^{-\frac{2}{m}} \tau(k,\hat r)^{\frac{1}{m}} .
\end{aligned}$$

W.l.o.g. we may assume $h-k\leq 1$. Then, with some suitable new positive exponent α (depending on the parameters!) we obtain from (2.39)

$$\begin{aligned}\tau(h,r) &\leq c(\hat{r}-r)^{-\gamma}(h-k)^{-\alpha}\tau(k,\hat{r})^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}}\\ &\quad\cdot\left[\tau(k,\hat{r})+\tau(k,\hat{r})^{\frac{1}{m}}\right]^{\frac{1}{2s}\frac{n}{n-1}}.\end{aligned}$$

Let us finally assume that R_0 is chosen in such a way that

$$\int_{B_{R_0}}\Gamma_\varepsilon^{\frac{q}{2}+1}\,\mathrm{d}x\leq 1$$

which is possible by Lemma 2.8. Then $\tau(k,\hat{r})\leq 1$ and therefore

$$\tau(h,r)\leq c(\hat{r}-r)^{-\gamma}(h-k)^{-\alpha}\tau(k,\hat{r})^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}+\frac{1}{m}\frac{1}{2s}\frac{n}{n-1}}. \tag{2.40}$$

Obviously

$$\beta:=\frac{1}{2}\frac{n}{n-1}\frac{1}{st}+\frac{1}{m}\frac{1}{2s}\frac{n}{n-1}=\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[\frac{1}{t}+\frac{1}{m}\right]>1$$

if the parameters m, s and t are close to 1. Thus we may apply a lemma of Stampacchia [St] to inequality (2.40) to get the claim of Lemma 2.9 (see also [Bi], p. 122, for further details).

It remains to prove (2.36) which means that we have to give a variant of Lemma 5.20 ii) of [Bi]. This time we test the differentiated Euler equation valid for v_ε with $\eta^2\partial_\gamma v_\varepsilon\max[\Gamma_\varepsilon-k,0]$ being admissible on account of Lemma 2.7. We get

$$\begin{aligned}&\int_{A(k,\hat{r})}\eta^2(\Gamma_\varepsilon-k)D_P^2f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma\nabla v_\varepsilon)\,\mathrm{d}x\\ &+\ 2\int_{A(k,\hat{r})}\eta(\Gamma_\varepsilon-k)D_P^2f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\nabla\eta\otimes\partial_\gamma v_\varepsilon)\,\mathrm{d}x\\ &+\ \int_{A(k,\hat{r})}\eta^2D_P^2f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma v_\varepsilon\otimes\nabla\Gamma_\varepsilon)\,\mathrm{d}x\\ &=:\ T_1+2T_2+T_3\\ &=\ -\int_{A(k,\hat{r})}D_{x_\gamma}D_Pf_\varepsilon(\cdot,\nabla v_\varepsilon):\nabla(\eta^2\partial_\gamma v_\varepsilon(\Gamma_\varepsilon-k))\,\mathrm{d}x.\end{aligned} \tag{2.41}$$

If $N>1$ we make use of the special structure and of (1.2) to see

$$\begin{aligned}T_3 &= \frac{1}{2}\int_{A(k,\hat{r})}\eta^2D_P^2f_\varepsilon(\cdot,\nabla v_\varepsilon)(e_\gamma\otimes\nabla\Gamma_\varepsilon,e_\gamma\otimes\nabla\Gamma_\varepsilon)\,\mathrm{d}x\\ &\geq c\int_{A(k,\hat{r})}\Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla\Gamma_\varepsilon|^2\eta^2\,\mathrm{d}x.\end{aligned} \tag{2.42}$$

Also by the special structure we find

$$T_2=\frac{1}{2}\int_{A(k,\hat{r})}\eta(\Gamma_\varepsilon-k)D_P^2f_\varepsilon(\cdot,\nabla v_\varepsilon)(e_\gamma\otimes\nabla\eta,e_\gamma\otimes\nabla\Gamma_\varepsilon)\,\mathrm{d}x,$$

hence

$$
\begin{aligned}
T_2 \;\le\; & \tau \int_{A(k,\hat r)} \eta^2 D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(e_\gamma\otimes\nabla\Gamma_\varepsilon, e_\gamma\otimes\nabla\Gamma_\varepsilon)\,\mathrm{d}x\\
& + c(\tau)\int_{A(k,\hat r)} |\nabla\eta|^2(\Gamma_\varepsilon-k)^2\Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x\,,
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality for $D_P^2 f_\varepsilon(x,\nabla v_\varepsilon)$, Young's inequality and (1.2). Note that the "τ-term" can be absorbed in T_3. Using the ellipticity for T_1, we deduce from (2.41), (2.42) and the latter estimates:

$$
\begin{aligned}
&\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla\Gamma_\varepsilon|^2\eta^2\,\mathrm{d}x + \int_{A(k,\hat r)} \eta^2(\Gamma_\varepsilon-k)\Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla^2 v_\varepsilon|^2\,\mathrm{d}x\\
&\le\; c\Bigg[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}|\nabla\eta|^2(\Gamma_\varepsilon-k)^2\,\mathrm{d}x + |\text{r.h.s. of (2.41)}|\Bigg]\,. \qquad (2.43)
\end{aligned}
$$

On account of (1.3) we have

$$
\begin{aligned}
|\text{r.h.s. of (2.41)}| \;\le\; & c\Bigg[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\eta^2(\Gamma_\varepsilon-k)|\nabla^2 v_\varepsilon|\,\mathrm{d}x\\
& + \int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\eta^2|\nabla v_\varepsilon||\nabla\Gamma_\varepsilon|\,\mathrm{d}x\\
& + \int_{A(k,\hat r)} \eta|\nabla\eta||\nabla v_\varepsilon|(\Gamma_\varepsilon-k)\Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\,\mathrm{d}x\Bigg]\\
=:\; & c[S_1+S_2+S_3]\,,
\end{aligned}
$$

and with Young's inequality we get $(0<\tau<1)$

$$
S_1 \le \tau\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}\eta^2(\Gamma_\varepsilon-k)|\nabla^2 v_\varepsilon|^2\,\mathrm{d}x + c(\tau)\int_{A(k,\hat r)} \Gamma_\varepsilon^{\overline{q}-1-\frac{p-2}{2}}\eta^2(\Gamma_\varepsilon-k)\,\mathrm{d}x\,,
$$

and for τ small the first integral on the r.h.s. can be absorbed in the second integral on the l.h.s. of (2.43). In the same way we handle S_2, i.e.

$$
S_2 \le \tau\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}\eta^2|\nabla\Gamma_\varepsilon|^2\,\mathrm{d}x + c(\tau)\int_{A(k,\hat r)} \Gamma_\varepsilon^{\overline{q}-1-\frac{p-2}{2}}\eta^2|\nabla v_\varepsilon|^2\,\mathrm{d}x\,.
$$

Finally we have

$$
S_3 \le c\int_{A(k,\hat r)} |\nabla\eta|^2(\Gamma_\varepsilon-k)^2\Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x + c\int_{A(k,\hat r)} \eta^2|\nabla v_\varepsilon|^2\Gamma_\varepsilon^{\overline{q}-1-\frac{q-2}{2}}\,\mathrm{d}x\,.
$$

Collecting terms and dropping the second term on the l.h.s. of (2.43) we end up with

$$
\begin{aligned}
\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla\Gamma_\varepsilon|^2\eta^2\,\mathrm{d}x \;\le\; & c\Bigg[\int_{A(k,\hat r)} \eta^2(\Gamma_\varepsilon-k)\Gamma_\varepsilon^{\overline{q}-1-\frac{p-2}{2}}\,\mathrm{d}x\\
& + \int_{A(k,\hat r)} \eta^2|\nabla v_\varepsilon|^2\Gamma_\varepsilon^{\overline{q}-1-\frac{q-2}{2}}\,\mathrm{d}x\\
& + \int_{A(k,\hat r)} |\nabla\eta|^2(\Gamma_\varepsilon-k)^2\Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x\Bigg]\,.
\end{aligned}
$$

Observing

$$(\Gamma_\varepsilon - k)\Gamma_\varepsilon^{\overline{q}-1-\frac{p-2}{2}} \le \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}$$

and

$$|\nabla v_\varepsilon|^2\Gamma_\varepsilon^{\overline{q}-1-\frac{q-2}{2}} \le \Gamma_\varepsilon^{\overline{q}-\frac{q}{2}+1} \le \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1},$$

inequality (2.36) is established. □

From Lemma 2.9 the claim of Theorem 1.1 follows as outlined at the end of Step 5 of Section 2.1.

3 Some remarks on $W^1_{\overline{q},loc}$-regularity

Here we are going to sketch two different ways to establish local $W^1_{\overline{q}}$-regularity of local minimizers under some particular assumptions. The first lemma is very much in the spirit of [ELM], Lemma 13, and the proof follows the ideas given there.

Lemma 3.2 is based on a regularization from below, the main idea is closely related to Lemma 4.1 of [CGM]. However, in [CGM] the energy density f is not supposed to be a smooth function. This is why the assumptions of [CGM], Lemma 4.1 and Lemma 4.2, are quite involved in comparison to Assumption 3.1 below. On the other hand, our explicit construction (see the proof of Proposition 3.1) is more technical in order to end up with a regularization of class C^2.

We finally remark that it remains on open problem to find a general approach to $W^1_{\overline{q},loc}$-regularity.

LEMMA 3.1 *Suppose that we have in addition to the assumptions of Theorem 1.1*

i) $|D_x f(x,P)| \le c_2(1+|P|^2)^{\frac{\overline{q}}{2}}$ *for all* $(x,P) \in \overline{\Omega}\times\mathbb{R}^{nN}$ *with some positive number* c_2*;*

ii) *for all* $\varepsilon > 0$ *and for all* $x \in \Omega$ *such that* $B_\varepsilon(x) \Subset \Omega$ *there exits* $\bar{x} = \bar{x}(x,\varepsilon)$ *such that with some function* $c(\varepsilon) \ge 1$*,* $c(\varepsilon) \to 1$ *as* $\varepsilon \to 0$*,*

$$f(\bar{x},P) \le c(\varepsilon) f(y,P) \quad \text{for all } (y,P) \in B_\varepsilon(x)\times\mathbb{R}^{nN}.$$

Then any local J*-minimizer is of class* $W^1_{\overline{q},loc}$*.*

Proof. The proof follows [ELM], Lemma 13, however, the setting is a little bit different: we are interested in the smooth case (see i) of Lemma 3.1) and we use argue via the uniqueness of minimizers instead of showing that the gap functional vanishes for all candidates of the energy class.

If u_ε denotes the mollification of a local J-minimizer u, then (as in [ELM]) we observe that the a priori W^1_p bounds give

$$|\nabla u_\varepsilon| \le c\varepsilon^{-\frac{n}{p}}.$$

Moreover, by the minimality of the functions v_ε introduced in Lemma 2.1 we have

$$\int_{B_{2R}} f(\cdot,\nabla v_\varepsilon)\,\mathrm{d}x \le \int_{B_{2R}} f_\varepsilon(\cdot,\nabla v_\varepsilon)\,\mathrm{d}x \le \int_{B_{2R}} f_\varepsilon(\cdot,\nabla u_\varepsilon)\,\mathrm{d}x. \tag{3.1}$$

Now, for fixed $x \in B_{2R}$ we choose some $\bar{x}$ according to the second assumption of Lemma 3.1. Linearization and the first assumption of Lemma 3.1 give

$$f(x, \nabla u_\varepsilon) \le f(\bar{x}, \nabla u_\varepsilon) + c(1 + |\nabla u_\varepsilon|^2)^{\frac{\overline{q}}{2}} \varepsilon \le f(\bar{x}, \nabla u_\varepsilon) + c|\varepsilon|^{-\frac{n}{p}\overline{q}+1}\, .$$

If ϕ_ε denotes the smoothing kernel, then Jensen's inequality and ii) of Lemma 3.1 imply

$$\int_{B_{2R}} f(\cdot, \nabla u_\varepsilon)\, \mathrm{d}x \le c(\varepsilon) \int_{B_{2R}} \int_{|x-y|<\varepsilon} f(y, \nabla u(y)) \phi_\varepsilon(x - y)\, \mathrm{d}y\, \mathrm{d}x + |\varepsilon|^{-\frac{n}{p}\overline{q}+1+n}\, .$$

Finally, passing to the limit $\varepsilon \to 0$ and recalling (3.1), the uniqueness of minimizers and condition (1.4) prove the lemma. □

Let us now discuss the regularization from below, where the main idea is the following: instead of adding a leading part of order q, which makes the energy larger, we consider a sequence of energy densities approximating f from below. This means that we assume (an explicit construction is discussed in Proposition 3.1)

ASSUMPTION 3.1 *For any fixed $M \gg 1$ there is an energy density $f_M(x, P)$ of class C^2 s.t. all the partial derivatives occurring below are continuous functions and s.t. for any $x \in \overline{\Omega}$*

i) *$f_M(x, P) \le f(x, P)$ for all $P \in \mathbb{R}^{nN}$;*

ii) *$f_M(x, P) = f(x, P)$ if $|P| \le M$;*

iii) *$f_M(x, P)$ is of isotropic p-growth in the sense that*

$$\bar{a}|P|^p - \bar{b} \le f_M(x, P) \le A_M |P|^p + B_M$$

holds for all $P \in \mathbb{R}^{nN}$, with universal constants $\bar{a} > 0$, $\bar{b} \in \mathbb{R}$ and with constants $A_M > 0$, $B_M \in \mathbb{R}$ depending on M.

iv) *$f_M(x, P)$ is $(p, \overline{q})$-elliptic uniform w.r.t. p and $\overline{q}$, i.e.*

$$\begin{aligned} \bar{\lambda}(1 + |P|^2)^{\frac{p-2}{2}} |U|^2 \ &\le\ D_P^2 f_M(x, P)(U, U)\ \le\ \bar{\Lambda}(1 + |P|^2)^{\frac{\overline{q}-2}{2}} |U|^2\, , \\ |D_x D_P f(x, P)| \ &\le\ \bar{c}(1 + |P|^2)^{\frac{\overline{q}-1}{2}} \end{aligned}$$

holds for all U, $P \in \mathbb{R}^{nN}$ and with universal positive constants $\bar{\lambda}$, $\bar{\Lambda}$, $\bar{c}$.

Here the constants occurring in ii), iii) *and* iv) *are supposed to be uniform in $x \in \overline{\Omega}$.*

Next we fix a local J-minimizer $u \in W^1_{1,loc}(\Omega; \mathbb{R}^N)$ and a smooth domain $\Omega' \Subset \Omega$. Given Assumption 3.1 we let u_M denote the unique solution of the regularized problem

$$J_M[w, \Omega'] := \int_{\Omega'} f_M(\cdot, \nabla w)\, \mathrm{d}x \to \min \quad \text{in } u + \overset{\circ}{W}{}^1_p(\Omega'; \mathbb{R}^N)\, .$$

Note that this problem clearly is well posed because f_M is of isotropic p-growth and because $u \in W^1_{p,loc}(\Omega; \mathbb{R}^N)$. Then, since u_M is minimal w.r.t. $J_M[\cdot, \Omega']$, since u is an

$J_M[\cdot,\Omega']$-admissible comparison function and since $f_M(x,P) \le f(x,P)$ one gets with a universal constant K

$$\begin{aligned}\int_{\Omega'} f_M(\cdot,\nabla u_M)\,\mathrm{d}x &\le \int_{\Omega'} f_M(\cdot,\nabla u)\,\mathrm{d}x \\ &\le \int_{\Omega'} f(\cdot,\nabla u)\,\mathrm{d}x \;\le\; K\,. \end{aligned}\tag{3.2}$$

Now let us assume that except for the $W^1_{\overline{q}}$-regularity of u we have the assumptions of Theorem 1.1 or of Theorem 1.2. Then we construct an u_M-regularizing sequence $\{v_M^{\varepsilon,\delta}\}$ in the same way as discussed in Section 2.1. This means that for a fixed ball $B_{2R} = B_{2R}(x_0) \Subset \Omega'$ we define u_M^ε as the mollification of u_M with parameter $\varepsilon > 0$ and let $v_M^{\varepsilon,\delta}$ denote the unique solution of the minimization problem

$$J_M^\delta[w,B_{2R}] := \int_{B_{2R}} \Big(f_M(\cdot,\nabla w) + \delta(1+|\nabla w|^2)^{\frac{q}{2}}\Big)\,\mathrm{d}x \to \min$$

in $u_M^\varepsilon + \overset{\circ}{W}{}^1_q(B_{2R},\mathbb{R}^N)$, $0 < \delta \le 1$. Here q is chosen as discussed in (2.1) or (2.25), respectively.

Now, on one hand, for fixed M, $f_M(x,P)$ is of isotropic p-growth which is due to Assumption 3.1, iii). In particular we have Lemma 2.1 where

- f is replaced by f_M;
- u is replaced by u_M, i.e. $\delta(\varepsilon) = \delta(\varepsilon,M) = (1+\varepsilon^{-1} + \|\nabla u_M^\varepsilon\|^{2q}_{L^q(B_{2R};\mathbb{R}^{nN})})^{-1}$;
- v_ε is replaced by $v_M^\varepsilon = v_M^{\varepsilon,\delta(\varepsilon,M)}$;
- $\overline{q}$ replaced by p.

In fact, with these changes we can follow the proof of Lemma 2.1 line by line and obtain the corresponding convergence results of v_M^ε to u_M.

On the other hand, as in Section 2 we obtain a priori bounds for v_M^ε which are uniform w.r.t. ε and M. To be more precise: of course the data λ, Λ, c_1, p and $\overline{q}$ of Assumption 1.1 enter the a priori bounds derived in Section 2 (see in particular Lemma 2.2 and 2.3). Here we observe that $f_M(x,P)$ is supposed to satisfy a $(p,\overline{q})$-ellipticity condition which is uniform w.r.t. p and $\overline{q}$ (see Assumption 3.1, iv)), i.e. these data do not depend on M. Moreover, the a priori bounds of Section 2 depend on the L^p-norm of the gradient of the regularization, i.e. in order to apply the arguments of Section 2 to v_M^ε we need to know that

$$\sup_M \sup_{\varepsilon\le\varepsilon(M)} \|\nabla v_M^\varepsilon\|_{L^p(B_{2R};\mathbb{R}^N)} \le L\,, \tag{3.3}$$

which means that for any $M \gg 1$ we have to find a small number $\varepsilon(M)$ such that for all $\varepsilon \le \varepsilon(M)$ $\|\nabla v_M^\varepsilon\|_{L^p(B_{2R};\mathbb{R}^{nN})}$ can be bounded independent of M with a universal constant L. For proving (3.3) we make use of the variant of Lemma 2.1, iii). This, together with the uniform left-hand side estimate of Assumption 3.1, iii), yields as $\varepsilon \downarrow 0$

$$\begin{aligned}\|\nabla v_M^\varepsilon\|^p_{L^p(B_{2R};\mathbb{R}^N)} &\le c\left(1+\int_{B_{2R}} f_M(\cdot,\nabla v_M^\varepsilon)\,\mathrm{d}x\right) \\ &\to c\left(1+\int_{B_{2R}} f_M(\cdot,\nabla u_M)\,\mathrm{d}x\right) \;\le\; \tilde{c}\,,\end{aligned}$$

where the universal constant $\tilde{c}$ can be found on account of (3.2). Thus we have (3.3) if ε is chosen sufficiently small depending on M.

We proceed by fixing a ball $\tilde{B} \Subset B_{2R}$ and a number $1 < t < 2$. As discussed above, Lemma 2.2 and Lemma 2.3 remain valid, thus we obtain ($\Gamma_M^\varepsilon := (1 + |\nabla v_M^\varepsilon|^2)$)

$$\begin{aligned}\int_{\tilde{B}} |\nabla^2 v_M^\varepsilon|^t \,\mathrm{d}x &\le \int_{\tilde{B}} (\Gamma_M^\varepsilon)^{\frac{p-2}{2}\frac{t}{2}} |\nabla^2 v_M^\varepsilon|^t (\Gamma_M^\varepsilon)^{\frac{2-p}{2}\frac{t}{2}} \,\mathrm{d}x \\ &\le \int_{\tilde{B}} (\Gamma_M^\varepsilon)^{\frac{p-2}{2}} |\nabla^2 v_M^\varepsilon|^2 \,\mathrm{d}x + \int_{\tilde{B}} (\Gamma_M^\varepsilon)^{\frac{2-p}{2}\frac{t}{t-2}} \,\mathrm{d}x \\ &\le c(\tilde{B}),\end{aligned}$$

provided that t is sufficiently close to 1. As a result, we have uniform local W_t^2 bounds for v_M^ε, thus together with Lemma 2.1 local W_t^2 bounds for u_M which are uniform w.r.t. M. That is, for any $\tilde{\Omega} \Subset B_{2R}$ there is a local constant $c(\tilde{\Omega})$ s.t. for some suitable $1 < t$

$$\sup_M \|u_M\|_{W_t^2(\tilde{\Omega};\mathbb{R}^N)} \le c(\tilde{\Omega}) . \tag{3.4}$$

Note that condition (1.5) is not needed to obtain this bound.

With (3.4) we now may pass to the limit $M \to \infty$ and define by considering a suitable subsequence

$$u_M \rightharpoondown: \bar{u} \quad \text{in } W_{t,loc}^2(\Omega';\mathbb{R}^N)$$

as $M \to \infty$. In particular we may assume w.l.o.g.

$$\nabla u_M \to \nabla \bar{u} \quad \text{almost everywhere on } \Omega' .$$

This finally implies by Fatou's lemma (we just need lower semicontinuity) and by recalling (3.2)

$$\begin{aligned}\int_{\Omega'} f(x, \nabla \bar{u}) \,\mathrm{d}x &\le \liminf_{M\to\infty} \int_{\Omega'} f_M(x, \nabla u_M) \,\mathrm{d}x \\ &\le \int_{\Omega'} f(x, \nabla u) \,\mathrm{d}x .\end{aligned}$$

(For applying Fatou's lemma we note: almost everywhere convergence of ∇u_M together with ii) of Assumption 3.1 in fact gives almost everywhere convergence of $f_M(x, \nabla u_M)$.) Moreover, iii) of Assumption 3.1 and (3.2) yield

$$\|u_M\|_{W_p^1(\Omega';\mathbb{R}^N)} \le c ,$$

i.e. $\bar{u}$ takes the boundary datum $u_{|\partial\Omega}$ in the trace sense of a W_p^1-function. Thus, $\bar{u} = u$ by the strict convexity of $f(x, \cdot)$ and the minimizing property of u. Summing up and once more emphasizing that the variant of Lemma 2.3 gives a priori estimates for v_M^ε which are uniform w.r.t. ε and M, we have proved

LEMMA 3.2 *Suppose that except for the $W_{\bar{q},loc}^1$-regularity hypothesis we either have the assumptions of Theorem 1.1 or of Theorem 1.2. Suppose further that we have Assumption 3.1. Then any local J-minimizer u satisfies*

$$\nabla u \in \begin{cases} L_{loc}^{pn/(n-2)}(\Omega;\mathbb{R}^{nN}) , & \text{if } n \ge 3 , \\ \text{any } L_{loc}^s(\Omega;\mathbb{R}^{nN}) , \quad s < \infty , & \text{if } n = 2 . \end{cases}$$

Now the main question of course deals with the existence of the regularization introduced in Assumption 3.1. As in [CGM] we consider energy densities of special structure. Note that even the counterexamples given in [ELM] and [FMM] satisfy this assumption.

PROPOSITION 3.1 *There exists a sequence of energy densities f_M as described in Assumption 3.1 provided that f is of special structure, i.e.*

$$f(x,P) = g(x,|P|) \tag{3.5}$$

for some suitable function $g\colon \overline{\Omega}\times[0,\infty)\to[0,\infty)$, and provided that we suppose

$$|D_x g''(x,t)| \le c_2(1+t^2)^{\frac{\overline{q}-2}{2}} \quad \text{for all } (x,t)\in\overline{\Omega}\times[0,\infty)\,. \tag{3.6}$$

Here and in the following g' and g'' denote the derivatives of g w.r.t. the second argument.

Proof. We first note that (3.5) gives

$$D_P^2 f(x,P)(U,U) = g''(x,|P|)\frac{|P:U|^2}{|P|^2} + \frac{g'(x,|P|)}{|P|}\Big[|U|^2 - \frac{|P:U|^2}{|P|^2}\Big]\,, \tag{3.7}$$

in particular the choice $U=P$ and $U\perp P$, respectively, in (3.7) implies recalling Assumption 1.1

$$\lambda(1+t^2)^{\frac{p-2}{2}} \le \frac{g'(x,t)}{t} \le \Lambda(1+t^2)^{\frac{\overline{q}-2}{2}}\,, \tag{3.8}$$

$$\lambda(1+t^2)^{\frac{p-2}{2}} \le g''(x,t) \le \Lambda(1+t^2)^{\frac{\overline{q}-2}{2}}\,. \tag{3.9}$$

As a consequence $g'(x,\cdot)$ is an increasing function. From

$$D_x D_P f(x,P) = D_x\Big[g'(x,|P|)\Big]\frac{P}{|P|}$$

we obtain again using Assumption 1.1

$$|D_x g'(x,t)| \le c_1(1+t^2)^{\frac{\overline{q}-1}{2}}\,. \tag{3.10}$$

With these preliminaries we now fix $M\gg 1$ and choose $\eta\in C^1([0,\infty))$ such that $\eta\equiv 1$ on $[0,3M/2]$, $\eta\equiv 0$ on $[2M,\infty)$, $0\le\eta\le 1$, $|\nabla\eta|\le c/M$. We then let on $\overline{\Omega}\times[0,\infty)$ (recall $g''>0$)

$$h(t) := \eta(t) + (1-\eta(t))\lambda\frac{(1+t^2)^{\frac{p-2}{2}}}{g''(x,t)}\,,$$

in particular h is a continuous function with the following properties.

i) $h(x,M) = \eta(M) = 1$ for all $x\in\overline{\Omega}$.

ii) $0\le h(x,t)\le 1$ for all $(x,t)\in\overline{\Omega}\times[0,\infty)$. In fact, the left-hand side is trivial, the inequality on the right-hand side follows from the left-hand side of (3.9).

iii) We have

$$\begin{aligned} g''(x,t)h(x,t) &= g''(x,t)\eta(t) + \lambda(1-\eta(t))(1+t^2)^{\frac{p-2}{2}} \\ &\ge \lambda(1+t^2)^{\frac{p-2}{2}}\eta(t) + \lambda(1-\eta(t))(1+t^2)^{\frac{p-2}{2}} \\ &= \lambda(1+t^2)^{\frac{p-2}{2}}\,. \end{aligned}$$

iv) $g''(x,t)h(x,t) \le c(M)(1+t^2)^{\frac{p-2}{2}}$, which is obvious by the choice of η. Here $c(M)$ denotes a positive constant depending on the data and on M.

Next we let

$$g_M(x,t) := \begin{cases} g(x,t)\,, & \text{if } 0 \le t \le M\,,\\ g(x,M) + g'(x,M)(t-M) + \int_M^t \int_M^\rho g''(x,\tau)h(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\rho\,, & \text{if } t > M\,. \end{cases}$$

and finally $f_M(x,P) = g_M(x,|P|)$. Property i) of h in particular implies that f_M is of class C^2, the second one yields $f_M \le f$, $f_M(x,P) = f(x,P)$ if $|P| \le M$ is trivial by construction.

The third claim of Assumption 3.1, i.e. the p-growth condition for f_M follows from the properties iii) and iv) of the function h which in fact imply that g'_M is of growth order $p-1$, where the upper bounds may depend on M. Note that the lower growth rate of f also is an immediate consequence of the ellipticity condition in Assumption 3.1, iv).

Let us proceed with the discussion of this ellipticity condition. We have

$$D_P^2 f_M(x,P)(U,U) = \begin{cases} D_P^2 f(x,P)(U,U)\,, & \text{if } |P| \le M\,,\\ T_1 + T_2 + T_3\,, & \text{if } |P| > M\,, \end{cases}$$

where

$$\begin{aligned} T_1 &= \frac{g'(x,M)}{|P|}\Big[|U|^2 - \frac{|P:U|^2}{|P|^2}\Big] \ \ge\ 0\,,\\ T_2 &= \int_M^{|P|} g''(x,\tau)h(x,\tau)\,\mathrm{d}\tau \frac{1}{|P|}\Big[|U|^2 - \frac{|P:U|^2}{|P|^2}\Big] \ \ge\ 0\,,\\ T_3 &= g''(x,|P|)h(x,|P|)\frac{|P:U|^2}{|P|^2} \ \ge\ 0\,. \end{aligned}$$

To establish the ellipticity condition it is of course sufficient to consider $|P| > M$. For the estimate from below we distinguish two cases.

Case 1. Suppose that

$$|U|^2 \ge 2\frac{|P:U|^2}{|P|^2}\,.$$

Then, if $M < |P| \le 2M$, (3.8) gives

$$\begin{aligned} T_1 &\ge c|U|^2\frac{g'(x,M)}{|P|} \ \ge\ c|U|^2\frac{g'(x,M)}{|M|} \ \ge\ c(1+|M|^2)^{\frac{p-2}{2}}|U|^2\\ &\ge c(1+|P|^2)^{\frac{p-2}{2}}|U|^2\,. \end{aligned}$$

In order to dicuss the case $2M < |P|$, we rely on the third property of h. Thus, if $2M < |P|$, we have

$$T_2 \ge c(1+|P|^2)^{\frac{p-2}{2}}|U|^2\,,$$

and it remains to discuss

Case 2. Suppose that

$$|U|^2 < 2\frac{|P:U|^2}{|P|^2}\,.$$

We then have once more recalling property iii)

$$T_3 \geq g''(x,|P|)h(x,|P|)\frac{|U|^2}{2} \geq c(1+|P|^2)^{\frac{p-2}{2}}|U|^2$$

and our claim is proved.

Next we are going to establish the ellipticity bound from above. With (3.8) and the monotonicity of $g'(x,\cdot)$ we immediately get a suitable bound for T_1. Discussing T_2 and T_3 we just have to recall (3.9) together with $0 \leq h \leq 1$. Altogether the first claim of Assumption 3.1, iv), is verified.

It remains to prove the Lipschitz condition for $D_P f_M$, which of course is satisfied if $|P| \leq M$. Thus we assume that $|P| > M$. This gives

$$|D_x D_P f_M(x,P)| \leq |D_x g'(x,M)| + \left| \int_M^{|P|} D_x(g''(x,\tau)h(x,\tau))\,\mathrm{d}\tau \right| =: I_1 + I_2\,.$$

hence we obtain from (3.10)

$$I_1 \leq c_1(1+M^2)^{\frac{\overline{q}-1}{2}} \leq c_1(1+|P|^2)^{\frac{\overline{q}-1}{2}}\,.$$

Estimating I_2 we observe

$$|D_x(g''(x,\tau)h(x,\tau))| = |\eta(\tau)D_x g''(x,\tau)|\,,$$

and last claim of the proposition follows from the assumption (3.6). □

REMARK 3.1 *Since we do not have global higher integrability results, it is not clear whether we can exclude*

$$\inf_{u\in u_0+\overset{\circ}{W}{}^1_p}\int_\Omega f(\cdot,\nabla u)\,\mathrm{d}x < \inf_{u\in u_0+\overset{\circ}{W}{}^1_q}\int_\Omega f(\cdot,\nabla u)\,\mathrm{d}x$$

for the Dirichlet boundary value problem with data u_0. Instead of the formulation as an energy-class problem as discussed above one may also consider a relaxed problem in this case. For the definition and further details we refer to [ELM] and the references quoted therein. Here we just like to mention that it is not hard to show that the global regularization $\{u_\delta\}$ defined w.r.t. f_δ and boundary values u_0 forms a minimizing sequence for the relaxed problem and that the limit is the unique minimizer of this relaxed problem. As a consequence, the higher regularity results of Section 2 apply to the relaxed problem. Moreover, the limit function solves the Euler-Lagrange equation

$$\int_\Omega D_P f(\cdot,\nabla u) : \nabla\varphi\,\mathrm{d}x = 0 \quad \textit{for all } \varphi \in C_0^\infty(\Omega;\mathbb{R}^N)\,,$$
$$u - u_0 \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)\,.$$

We leave the details to the reader (in fact they can be found in a preliminary version of this paper ([BF4])).

The reader should also note that Marcellini (see [Ma]) investigates the exisitence and the regularity of solutions of elliptic equations under a (p,q)-growth condition. If a weak solution is in the space $W^1_{q,loc}(\Omega)$ and if $q < pn/(n-2)$, then Marcellini proves Lipschitz regularity (and even higher regularity), whereas the existence of a weak solution of class $W^1_{q,loc}(\Omega)$ is established under the restriction that $q < p(n+2)/n$.

Here it is not possible to argue with the same relations between p and q as done in [Ma] since our hypothesis on $D_x D_P f$ are weaker.

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