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Abstract

In this paper we discuss a system of partial differential equations describing the steady flow of an incompressible fluid and prove the existence of a strong solution under suitable assumptions on the data. In the 2D-case this solution turns out to be of class $C^{1,\alpha}$.

1 Introduction

Suppose that we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, together with a system g: $\Omega \to \mathbb{R}^n$ of volume forces which – for technical simplicity – is assumed to be of class $L^\infty(\Omega; \mathbb{R}^n)$. Then we are looking for a velocity field u: $\Omega \to \mathbb{R}^n$ together with a pressure function π: $\Omega \to \mathbb{R}$ such that the following system of partial differential equations is satisfied

$$\left.\begin{aligned} &-\mathrm{div}\,\{T(\cdot, \varepsilon(u))\} + \nabla\pi + [\nabla u]u = g \;\text{ in } \Omega\,, \\ &\mathrm{div}\, u = 0 \;\text{ in } \Omega\,, \qquad u = 0 \;\text{ on } \partial\Omega\,. \end{aligned}\right\} \tag{1.1}$$

Here $\varepsilon(u)$ denotes the symmetric gradient of u, i.e. $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, and $[\nabla u]u$ is the convective term $u^k \frac{\partial u}{\partial x_k}$ (summation w.r.t. k). The tensor-valued function $T = T(x, \varepsilon)$ is defined for all $x \in \overline{\Omega}$ and all matrices $\varepsilon \in \mathbb{S}^n$ (:= space of symmetric $n \times n$-matrices) and arises as the gradient w.r.t. the second argument of a smooth convex potential $f = f(x, \varepsilon)$. More precisely, let us impose the following conditions on the potential f. The energy density f: $\overline{\Omega} \times \mathbb{S}^n \to [0, \infty)$ satisfies with exponents $1 < p \le \bar{q} < \infty$ and constants λ, Λ, $c_1 > 0$

$$\lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \;\le\; D^2_\varepsilon f(x, \varepsilon)(\sigma, \sigma) \;\le\; \Lambda(1 + |\varepsilon|^2)^{\frac{\bar{q}-2}{2}}|\sigma|^2\,, \tag{1.2}$$

$$|D_x D_\varepsilon f(x, \varepsilon)| \;\le\; c_1(1 + |\varepsilon|^2)^{\frac{\bar{q}-1}{2}} \tag{1.3}$$

for all $x \in \overline{\Omega}$ and all ε, $\sigma \in \mathbb{S}^n$. Here we assume that all the partial derivatives occurring in (1.2) and (1.3) are at least continuous functions. The reader should note that (1.2) implies the anisotropic growth condition

$$a|\varepsilon|^p - b \le f(x, \varepsilon) \le A|\varepsilon|^{\bar{q}} + B$$

with suitable positive constants a, A, b and B. Now we can state our existence and regularity result concerning the system (1.1).

THEOREM 1.1 *Suppose that (1.2) and (1.3) hold together with*

$$p > \begin{cases} 6/5\,, & \text{if } n = 2\,, \\ 9/5\,, & \text{if } n = 3\,. \end{cases} \tag{1.4}$$

Suppose further that

$$\bar{q} < p\Big(1 + \frac{1}{n}\Big)\,. \tag{1.5}$$

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i) *Then there exists a velocity field* u *of class* $\overset{\circ}{W}{}^1_p \cap W^2_{t,loc}(\Omega;\mathbb{R}^n)$ *for some* $t > 1$ *and a pressure function* π *of class* $W^1_{s,loc}(\Omega)$ *for some* $s > 1$ *such that (1.1) is satisfied almost everywhere.*

ii) *If* $n = 2$ *and if in addition* $\bar{q} < p + 2$*, then the first derivatives of* u *are Hölder continuous functions in* Ω*.*

REMARK 1.1 *At the end of Section 2 we will show that we can choose* $t = 2$ *provided that* $p \geq 2$*. If* $p < 2$*, then any number*

$$t < \begin{cases} 2 & \text{if} \ \ n = 2\,, \\ \frac{3p}{p+1} & \text{if} \ \ n = 3\,, \end{cases}$$

is admissible. Moreover, we will establish that $D_\varepsilon f(\cdot, \varepsilon(u))$ *is in the class* $W^1_{q/(q-1),loc}(\Omega;\mathbb{S}^n)$*. Therefore some bound for the exponent* s *can be calculated with the help of equation (1.1).*

Let us compare Theorem 1.1 to the known results.

i) A very general existence result for the non-autonomous isotropic case, i.e. $p = \bar{q}$, was obtained by Frehse, Málek and Steinhauer ([FMS]), even without the assumption that the tensor $T(x,\varepsilon)$ is generated by a potential f. Moreover, they replace (1.4) by the requirement that $p > 2n/(n+2)$. But the solution they obtain is just a weak solution, i.e. it belongs to the space $\overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n)$ and satisfies (1.1) in the distributional sense.

ii) The isotropic autonomous case, i.e. $T = T(\varepsilon)$ and $T = Df$ with potential f of p-growth was discussed in the paper [KMS] by Kaplický, Málek and Stará. They consider planar flows and prove the existence of a smooth velocity field u under condition (1.4).

iii) In the papers [BF1], [ABF], [BF3] and [BFZ] the autonomous anisotropic case was investigated with the result that Theorem 1.1 holds under weaker conditions relating the exponents p and $\bar{q}$. At this stage we like to remark that the occurrence of the variable x in the potential f is not only a technicality. As shown by Esposito, Leonetti and Mingione ([ELM]) in the setting of variational problems a Lavrentiev phenomenon has to be expected which means that even in dimension 2 singular solutions can occur if $\bar{q} < 3p/2$ is violated, whereas for the autonomous 2d-case the condition $\bar{q} < 2p$ is sufficient for regularity. For a further discussion we refer to the paper [BF4].

iv) A particular form of x-dependent problems arises in the theory of electrorheological fluids, we refer to [R], [E], [ER], [AM], [BF2], [BFZ], [DER] and the references quoted therein. Roughly speaking the potential f here is of the principal form $f(x,\varepsilon) = (1 + |\varepsilon|^2)^{p(x)/2}$ for a smooth function such that $1 < p_\infty \leq p(x) \leq p_0 < \infty$ with constants p_∞ and p_0. The existence of a strong solution to the problem (1.1) can be established by working in appropriate function spaces (see [R], [E] and [ER]), and for the 2D-case the regularity of this solution follows from the observation that locally we have the bound $p \leq p(x) \leq \bar{q}$ with "local exponents" p and $\bar{q}$ such that (1.5) together with $\bar{q} < p + 2$ holds.

2 Existence of a strong solution

Throughout this section we assume that all the hypotheses of the first part of Theorem 1.1 are satisfied.

We introduce the regularization ($0 < \delta < 1$)

$$f_\delta(x, \varepsilon) := \delta(1 + |\varepsilon|^2)^{\frac{q}{2}} + f(x, \varepsilon)$$

with exponent q satisfying (compare [ABF], Remark 1.5)

$$q > \bar{q}\,, \qquad n < q < p\Big(1 + \frac{2}{n}\Big) \tag{2.1}$$

which is possible by our assumptions on the data. Let u_δ denote a solution in $\overset{\circ}{W}{}^1_q(\Omega; \mathbb{R}^n) \cap$ Ker(div) of the problem

$$\begin{cases} \displaystyle\int_\Omega D_\varepsilon f_\delta(\cdot, \varepsilon(w)) : \varepsilon(\varphi)\,\mathrm{d}x - \int_\Omega w \otimes w : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega g \cdot \varphi\,\mathrm{d}x \\ \text{for all } \ \varphi \in C_0^\infty(\Omega; \mathbb{R}^n)\,, \quad \operatorname{div}\varphi = 0\,, \end{cases} \tag{1.1$_\delta$}$$

whose existence follows from a familiar fixed point argument which is applied for example in [BF3], Appendix, and which can be used in our setting without changes. Choosing $\varphi = u_\delta$ we immediately get that

$$\sup_{0<\delta<1} \|u_\delta\|_{W^1_p(\Omega;\mathbb{R}^n)} < \infty\,,$$

and we will show that any weak W^1_p-cluster point v of the sequence $\{u_\delta\}$ satisfies our claim. To this purpose we follow the ideas of [BF1] and [ABF]: we first establish some weak differentiability properties of the functions u_δ (see Lemma 2.1) which in turn enable us to prove a preliminary variant of a Caccioppoli-type inequality (see Lemma 2.2). From this inequality we deduce some initial local higher integrability result for ∇u_δ (being uniform in δ; see Lemma 2.3), which gives an improved version of the first Caccioppoli-type inequality (see Lemma 2.4). In the final step we will increase the exponent of the uniform local higher integrability (at least for the case that $n = 3$ together with $p < 2$; see Lemma 2.5). Putting all these ingredients together part i) of Theorem 1.1 will follow. Let us start with

LEMMA 2.1 *We have the following initial regularity properties of u_δ:*

i) $u_\delta \in W^2_{2,loc}(\Omega; \mathbb{R}^n)$;

ii) $D_\varepsilon f_\delta(\cdot, \varepsilon(u_\delta)) \in W^1_{q/(q-1),loc}(\Omega; \mathbb{S}^n)$ *together with*

$$\partial_k\{D_\varepsilon f_\delta(\cdot, \varepsilon(u_\delta))\} = D^2_\varepsilon f_\delta(\cdot, \varepsilon(u_\delta))(\partial_k \varepsilon(u_\delta), \cdot) + (\partial_k D_\varepsilon f_\delta)(\cdot, \varepsilon(u_\delta))\,, \quad k = 1, \dots, n\,.$$

iii) $h_\delta := (1 + |\varepsilon(u_\delta)|^2)^{p/4} \in W^1_{2,loc}(\Omega)$ *together with*

$$\nabla h_\delta = \frac{p}{2}(1 + |\varepsilon(u_\delta)|^2)^{\frac{p}{4}-1}|\varepsilon(u_\delta)|\nabla|\varepsilon(u_\delta)|\,.$$

Proof The idea for the proof is the same as in [BF1], Lemma 3.1, and [ABF], Lemma 2.2, we just indicate the minor adjustments. We fix a ball $B_R = B_R(x_0) \Subset \Omega$ and choose $\eta \in C_0^\infty(B_R)$ such that $(0 < r < r' < R)$ $\eta \equiv 1$ on B_r, $\eta \equiv 0$ outside of $B_{r'}$ and $|\nabla\eta| \leq c/(r'-r)$. Let Δ_h denote the difference quotient in direction e_k, $k = 1, \dots, n$, and consider a function $\psi \in \overset{\circ}{W}{}^1_q(B_{r'};\mathbb{R}^n)$ such that

$$\operatorname{div}\psi = \frac{1}{h}\nabla\eta^2\Delta_h u_\delta$$

together with

$$\|\nabla\psi\|_{L^q(B_{r'})} \leq c\|h^{-1}\nabla\eta^2\Delta_h u_\delta\|_{L^q(B_{r'})}\,.$$

Note that such a function ψ exists according to [La], [Pi] or [Ga], III, Theorem 3.2. Then

$$\varphi := h^{-1}\eta^2\Delta_h u_\delta - \psi$$

is admissible in (1.1_δ) which means that we get (2.5) of [ABF] with $f_\delta(\cdot,\varepsilon(u_\delta))$ replacing $f_\delta(\varepsilon(u_\delta))$ on both sides of the equation. We have

$$\begin{aligned}
\Delta_h(D_\varepsilon f_\delta(\cdot,\varepsilon(u_\delta)))(x) &= \frac{1}{h}\Big[D_\varepsilon f_\delta(x+he_k,\varepsilon(u_\delta)(x+he_k)) - D_\varepsilon f_\delta(x+he_k,\varepsilon(u_\delta)(x))\Big]\\
&\quad +\frac{1}{h}\Big[D_\varepsilon f_\delta(x+he_k,\varepsilon(u_\delta)(x)) - D_\varepsilon f_\delta(x,\varepsilon(u_\delta)(x))\Big]\\
&= \int_0^1 D^2_\varepsilon f_\delta(x+he_k,\varepsilon(u_\delta)(x)+th\varepsilon(\Delta_h u_\delta)(x))\,\mathrm{d}t\,(\varepsilon(\Delta_h u_\delta),\cdot)\\
&\quad +\frac{1}{h}\Big[D_\varepsilon f_\delta(x+he_k,\varepsilon(u_\delta)(x)) - D_\varepsilon f_\delta(x,\varepsilon(u_\delta)(x))\Big] =: I + I\!I\,.
\end{aligned}$$

Note that in $I\!I$ we may replace f_δ by f since the regularizing δ-part is not depending on x. If we introduce the parameter-dependent bilinear form

$$\tilde{\mathcal{B}}_x := \int_0^1 D^2_\varepsilon f_\delta(x+he_k,\varepsilon(u_\delta)(x)+th\varepsilon(\Delta_h u_\delta)(x))\,\mathrm{d}t\,,$$

then we obtain (2.6) of [ABF] with $\mathcal{B}_x$ being replaced by $\tilde{\mathcal{B}}_x$ and with some extra terms on the r.h.s. which originate from the new expression $I\!I$. Moreover, on the l.h.s. of the above mentioned inequality we may write

$$\int_{B_{r'}(x_0)} \tilde{\mathcal{B}}_x(\varepsilon(\Delta_h u_\delta),\varepsilon(\Delta_h u_\delta))\eta^2\,\mathrm{d}x$$

which is helpful for absorbing terms. To be precise we use the growth condition (1.3) in order to estimate

$$\begin{aligned}
\int_{B_{r'}(x_0)} |I\!I|\,|\varepsilon(\eta^2\Delta_h u_\delta - h\psi)|\,\mathrm{d}x &\leq c\Bigg[\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{\bar{q}-1}{2}}\eta^2|\varepsilon(\Delta_h u_\delta)|\,\mathrm{d}x\\
&\quad + \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{\bar{q}-1}{2}}\eta|\nabla\eta||\Delta_h u_\delta|\,\mathrm{d}x\\
&\quad + \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{\bar{q}-1}{2}}|h||\varepsilon(\psi)|\,\mathrm{d}x\Bigg] =: c[T_1+T_2+T_3]\,,
\end{aligned}$$

$\Gamma_\delta := 1+|\varepsilon(u_\delta)|^2$. Let $\sigma \in \mathbb{S}^n$. Then we have (using (1.2) and the definition of f_δ)

$$\begin{aligned}\tilde{\mathcal{B}}_x(\sigma,\sigma) &\geq \lambda(\delta)\int_0^1 \Big(1+|\varepsilon(u_\delta)(x)+th\varepsilon(\Delta_h u_\delta)(x)|^2\Big)^{\frac{q-2}{2}}\,\mathrm{d}t\,|\sigma|^2\\ &\geq c(\delta)\Gamma_\delta^{\frac{q-2}{2}}|\sigma|^2\,,\end{aligned}$$

where the last inequality follows from well-known estimates being valid since $q>2$. This gives

$$\begin{aligned}T_1 &\leq c(\delta)\int_{B_{r'}(x_0)} \eta^2 \Gamma_\delta^{\frac{\bar{q}-1}{2}}\Gamma_\delta^{\frac{2-q}{4}}\big[\tilde{\mathcal{B}}_x(\varepsilon(\Delta_h u_\delta),\varepsilon(\Delta_h u_\delta))\big]^{\frac{1}{2}}\,\mathrm{d}x\\ &\leq \rho\int_{B_{r'}(x_0)} \eta^2\tilde{\mathcal{B}}_x(\varepsilon(\Delta_h u_\delta),\varepsilon(\Delta_h u_\delta))\,\mathrm{d}x + c(\delta,\rho)\int_{B_{r'}(x_0)}\eta^2\Gamma_\delta^{\bar{q}-1+\frac{2-q}{2}}\,\mathrm{d}x\,,\end{aligned}$$

and for $\rho \ll 1$ the first term can be absorbed into the l.h.s. of the starting inequality, whereas the second integral is dominated by $\int_{B_{r'}(x_0)}\Gamma_\delta^{q/2}\,\mathrm{d}x$ and in conclusion by the r.h.s. of the inequality stated before (3.9) of [BF1] (choosing $s=q$ there). In the sequel we refer to this inequality as inequality $(*)$. Obviously

$$\begin{aligned}T_2 &\leq c(r'-r)^{-1}\left[\int_{B_{r'}(x_0)}|\Delta_h u_\delta|^q\,\mathrm{d}x + \int_{B_{r'}(x_0)}\Gamma_\delta^{\frac{\bar{q}-1}{2}\frac{q}{q-1}}\,\mathrm{d}x\right]\\ &\leq \text{r.h.s. of } (*)\,,\end{aligned}$$

which is a consequence of $\frac{\bar{q}-1}{2}\frac{q}{q-1}\leq\frac{q}{2}$.

For T_3 we argue in a similar way recalling the bound for $\int_{B_{r'}(x_0)}|\nabla\psi|^q\,\mathrm{d}x$.

All other calculations remain the same as in [ABF] which means that finally inequality $(*)$ holds in the situation at hand, thus (3.9) of [BF1] follows and part i) of Lemma 2.1 is a consequence of the fact that $q\geq 2$. The remaining statements of Lemma 2.1 can now be adjusted along the lines of [BF1], pages 373 and 374. $\square$

LEMMA 2.2 *Consider a ball $B_R = B_R(x_0) \Subset \Omega$ and choose radii $0<r<r'<R$. Then there exists a local constant $c(r,r') = c(r'-r)^{-\beta}$, $\beta>0$, such that for any ball $B_{r'}(\bar{x}) \Subset B_R$ and for any $\eta \in C_0^\infty(B_{r'}(\bar{x}))$, $0\leq\eta\leq 1$, $\eta\equiv 1$ on $B_r(\bar{x})$, $|\nabla\eta|\leq c(r'-r)^{-1}$ the following estimate holds*

$$\begin{aligned}\int_{B_{r'}(\bar{x})}\eta^2\Gamma_\delta^{\frac{p-2}{2}}|\nabla\varepsilon(u_\delta)|^2\,\mathrm{d}x &\leq c(r,r')\left[1+\int_{B_{r'}(\bar{x})}\Gamma_\delta^{\frac{2\bar{q}-p}{2}}\,\mathrm{d}x+\int_{B_{r'}(\bar{x})}\Gamma_\delta^{\frac{q}{2}}\,\mathrm{d}x\right.\\ &\qquad\left.+\int_{B_{r'}(\bar{x})}|u_\delta||\nabla u_\delta|^2\,\mathrm{d}x+\left|\int_{B_{r'}(\bar{x})}\partial_k u_\delta^j\partial_j u_\delta^i\partial_k u_\delta^i\eta^2\,\mathrm{d}x\right|\right],\end{aligned}$$

where the last integral on the r.h.s. vanishes if $n=2$. Again we have set $\Gamma_\delta := 1+|\varepsilon(u_\delta)|^2$.

REMARK 2.1 *Note that by (1.5)*

$$2\bar{q}-p < 2p\Big(1+\frac{1}{n}\Big)-p = p\Big(1+\frac{2}{n}\Big)\,,$$

hence we may choose q in addition to (2.1) such that

$$2\bar{q} - p < q\,.$$

Then the claim of Lemma 2.2 exactly takes the form of [ABF], Lemma 3.1 which means that we can drop the integral $\int_{B_{r'}(\bar{x})} \Gamma_\delta^{(2\bar{q}-p)/2}\,\mathrm{d}x$ in the inequality stated in Lemma 2.2.

Proof of Lemma 2.2. Using the notation from [ABF], proof of Lemma 3.1, we obtain with exactly the same calculations inequality (3.5) from this paper, i.e. we have ($\sigma_\delta := D_\varepsilon f_\delta(\cdot, \varepsilon(u_\delta))$)

$$\begin{aligned}
\int_{B_{r'}(\bar{x})} \eta^2 \partial_k \sigma_\delta : \partial_k \varepsilon(u_\delta)\,\mathrm{d}x \quad &\le \quad -2\int_{B_{r'}(\bar{x})} \eta \partial_k \sigma_\delta : (\nabla\eta \odot \partial_k u_\delta)\,\mathrm{d}x \\
&\quad + \int_{B_{r'}(\bar{x})} \partial_k(u_\delta \otimes u_\delta) : \varepsilon(\eta^2 \partial_k u_\delta)\,\mathrm{d}x \\
&\quad - \int_{B_{r'}(\bar{x})} g \partial_k(\eta^2 \partial_k u_\delta)\,\mathrm{d}x \\
&\quad -2\int_{B_{r'}(\bar{x})} \eta \partial_k \pi_\delta \mathbf{1} : (\nabla\eta \odot \partial_k u_\delta)\,\mathrm{d}x\,, \qquad (2.2)
\end{aligned}$$

π_δ being the pressure function defined in (3.1) of [ABF]. To estimate the l.h.s. of (2.2), we use (compare the formula in ii) of Lemma 2.1)

$$\begin{aligned}
\partial_k \sigma_\delta : \varepsilon(\partial_k u_\delta) \quad &= \quad \underbrace{D_\varepsilon^2 f_\delta(\cdot, \varepsilon(u_\delta))(\partial_k \varepsilon(u_\delta), \partial_k \varepsilon(u_\delta))}_{=:H_\delta^2} + (\partial_k D_\varepsilon f_\delta)(\cdot, \varepsilon(u_\delta)) : \varepsilon(\partial_k u_\delta) \\
&\ge \quad H_\delta^2 - c\Gamma_\delta^{\frac{\bar{q}-1}{2}} |\nabla \varepsilon(u_\delta)|\,,
\end{aligned}$$

where the last inequality follows from (1.3). Moreover we observe ($0 < \tau < 1$)

$$\Gamma_\delta^{\frac{\bar{q}-1}{2}} |\nabla\varepsilon(u_\delta)| \le \tau \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2 + c(\tau)\Gamma_\delta^{\frac{2\bar{q}-p}{2}}\,.$$

Thus, after appropriate choice of τ (recall the ellipticity estimate (1.2)) we get the bound

$$\partial_k \sigma_\delta : \varepsilon(\partial_k u_\delta) \ge \frac{1}{2} H_\delta^2 - c\Gamma_\delta^{\frac{2\bar{q}-p}{2}}\,, \qquad (2.3)$$

the constant c being uniform w.r.t. δ. For estimating the first integral on the r.h.s. of (2.2) we observe

$$\begin{aligned}
|\nabla \sigma_\delta|^2 \quad &= \quad \partial_k \sigma_\delta : \partial_k \sigma_\delta \\
&= \quad D_\varepsilon^2 f_\delta(\cdot, \varepsilon(u_\delta))(\partial_k \varepsilon(u_\delta), \partial_k \sigma_\delta) + (\partial_k D_\varepsilon f_\delta)(\cdot, \varepsilon(u_\delta)) : \partial_k \sigma_\delta \\
&\le \quad H_\delta D_\varepsilon^2 f_\delta(\cdot, \varepsilon(u_\delta))(\partial_k \sigma_\delta, \partial_k \sigma_\delta)^{\frac{1}{2}} + c\Gamma_\delta^{\frac{\bar{q}-1}{2}} |\nabla \sigma_\delta|\,,
\end{aligned}$$

which gives

$$|\nabla \sigma_\delta| \le c H_\delta \Gamma_\delta^{\frac{q-2}{4}} + c\Gamma_\delta^{\frac{\bar{q}-1}{2}}\,. \qquad (2.4)$$

Now (2.4) implies

$$
\begin{aligned}
2\int_{B_{r'}(\bar x)} \eta|\nabla\sigma_\delta||\nabla\eta||\nabla u_\delta|\,\mathrm{d}x &\le c\int_{B_{r'}(\bar x)} \eta H_\delta \Gamma_\delta^{\frac{q-2}{4}}|\nabla\eta||\nabla u_\delta|\,\mathrm{d}x \\
&\quad + c\int_{B_{r'}(\bar x)} \eta|\nabla\eta||\nabla u_\delta|\Gamma_\delta^{\frac{\bar q-1}{2}}\,\mathrm{d}x \\
&\le \tau\int_{B_{r'}(\bar x)} \eta^2 H_\delta^2\,\mathrm{d}x + c(\tau)\int_{B_{r'}(\bar x)} |\nabla\eta|^2\Gamma_\delta^{\frac{q-2}{2}}|\nabla u_\delta|^2\,\mathrm{d}x \\
&\quad + c\int_{B_{r'}(\bar x)} \eta|\nabla\eta||\nabla u_\delta|\Gamma_\delta^{\frac{\bar q-1}{2}}\,\mathrm{d}x\,,
\end{aligned}
$$

and the first two terms on the r.h.s. can be handled in a standard way (see [ABF], the calculations after (3.6) for the second term). The last integral can be discussed in a similar way as the second one with the result that

$$
\int_{B_{r'}(\bar x)} \eta|\nabla\eta||\nabla u_\delta|\Gamma_\delta^{\frac{\bar q-1}{2}}\,\mathrm{d}x \le c(r,r')\left[1+\int_{B_{r'}(\bar x)} \Gamma_\delta^{\frac{q}{2}}\,\mathrm{d}x\right],
$$

where we also used the assumption that $\bar q < q$.

The second and the third integral on the r.h.s. of (2.2) can be estimated exactly as in [ABF], for the pressure integral we observe the equation

$$
g + \operatorname{div}(\sigma_\delta - u_\delta\otimes u_\delta) = \nabla\pi_\delta\,,
$$

thus

$$
\begin{aligned}
&\left| -2\int_{B_{r'}(\bar x)} \eta\partial_k\pi_\delta\mathbf{1} : (\nabla\eta\odot\partial_k u_\delta)\,\mathrm{d}x\right| \\
&\quad\le c\Bigg[\int_{B_{r'}(\bar x)} \eta|\nabla\eta||\nabla\sigma_\delta||\nabla u_\delta|\,\mathrm{d}x + \int_{B_{r'}(\bar x)} \eta|\nabla\eta||\nabla(u_\delta\otimes u_\delta)||\nabla u_\delta|\,\mathrm{d}x \\
&\qquad + \int_{B_{r'}(\bar x)} \eta|\nabla\eta||g||\nabla u_\delta|\,\mathrm{d}x\Bigg] =: c[T_1+T_2+T_3]\,.
\end{aligned}
$$

The integrals T_2, T_3 already occurred, and T_1 has been discussed after inequality (2.4). The rest of the proof is now the same as in [ABF]. □

LEMMA 2.3 *For any subdomain $\Omega' \Subset \Omega$ there is a constant $c(\Omega')$ independent of δ such that*

$$
\int_{\Omega'} |\nabla u_\delta|^{p^*}\,\mathrm{d}x \le c(\Omega')\,,
$$

where

$$
p^* = \begin{cases} \dfrac{np}{n-p} & \text{if } p < n\,, \\ \text{any number} & \text{if } p \ge n\,. \end{cases}
$$

REMARK 2.2 *Our assumptions (1.4) and (2.1) in particular imply that $p^* > q$ which follows for $p < n$ from the fact that $p > 2n/(n+2)$. In case $p \ge n$ we just choose $p^* > q$.*

Proof of Lemma 2.3. Recalling Remark 2.1, the proof of Lemma 2.3 is a verbatim repetition of [ABF], proof of Lemma 3.2. $\square$

As in [ABF] we use Lemma 2.3 to improve the Caccioppoli-type inequality stated in Lemma 2.2.

LEMMA 2.4 *We use the notation from Lemma 2.2. There exists an exponent $\gamma > 0$ and uniform local constants c_1, c_2, $c_3 > 0$ such that for any matrix $Q \in \mathbb{R}^{n\times n}$ we have*

$$\begin{aligned}\int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{p-2}{2}} |\nabla\varepsilon(u_\delta)|^2 \,\mathrm{d}x \;\leq\; & c_1 \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{2\bar{q}-p}{2}} \,\mathrm{d}x \\ & + \frac{c_2}{(r'-r)^2} \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2 \,\mathrm{d}x + c_3 (r')^\gamma .\end{aligned}$$

Proof. If we replace u_δ by $u_\delta - Qx$ in the test-function used in the proof of Lemma 3.1 of [ABF], then one obtains a version of inequality (2.2) where on the r.h.s. the quantity $u_\delta - Qx$ occurs in place of u_δ on appropriate places. Using the estimates stated after (2.2), taking also care of the uniform local boundedness of u_δ (following from Lemma 2.3) and using again the equation $g + \operatorname{div}(\sigma_\delta - u_\delta \otimes u_\delta) = \nabla\pi_\delta$ for handling the pressure term, we arrive at the following result (compare (3.15) of [ABF])

$$\begin{aligned}\int_{B_{r'}(\bar{x})} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla\varepsilon(u_\delta)|^2 \,\mathrm{d}x \;\leq\; & c\Bigg[\int_{B_{r'}(\bar{x})} |\nabla\eta|^2 \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2 \,\mathrm{d}x \\ & + \int_{B_{r'}(\bar{x})} |\nabla\eta|\eta \Gamma_\delta^{\frac{\bar{q}-1}{2}} |\nabla u_\delta - Q| \,\mathrm{d}x \\ & + \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{2\bar{q}-p}{2}} \,\mathrm{d}x + \int_{B_{r'}(\bar{x})} \eta |\nabla\eta| |\nabla u_\delta - Q| |\nabla u_\delta| \,\mathrm{d}x \\ & + \left| \int_{B_{r'}(\bar{x})} \partial_k (u_\delta \otimes u_\delta) : \varepsilon(\eta \partial_k (u_\delta - Qx)) \,\mathrm{d}x \right| \\ & + \left| \int_{B_{r'}(\bar{x})} g \partial_k (\eta^2 \partial_k (u_\delta - Qx)) \,\mathrm{d}x \right| \Bigg] \\ =: \; & c \sum_{i=1}^{6} T_i .\end{aligned}$$

The quantities T_1, T_5 and T_6 already occur on the r.h.s. of (3.15) in [ABF] and can be treated as demonstrated there. The term T_3 occurs on the r.h.s. of the desired inequality, and clearly

$$\begin{aligned} T_2 \;\leq\; & c \int_{B_{r'}(\bar{x})} |\nabla\eta|^2 |\nabla u_\delta - Q|^2 \Gamma_\delta^{\frac{\bar{q}-2}{2}} \,\mathrm{d}x + c \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{\bar{q}}{2}} \,\mathrm{d}x \\ \leq\; & c \int_{B_{r'}(\bar{x})} |\nabla\eta|^2 |\nabla u_\delta - Q|^2 \Gamma_\delta^{\frac{q-2}{2}} \,\mathrm{d}x + c \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{2\bar{q}-p}{2}} \,\mathrm{d}x .\end{aligned}$$

For T_4 we get

$$\begin{aligned} T_4 &\le c(r'-r)^{-2}\int_{B_{r'}(\bar{x})}|\nabla u_\delta - Q|^2\,\mathrm{d}x + \int_{B_{r'}(\bar{x})}|\nabla u_\delta|^2\,\mathrm{d}x \\ &\le c\Bigg[(r'-r)^{-2}\int_{B_{r'}(\bar{x})}\Gamma_\delta^{\frac{q-2}{2}}|\nabla u_\delta - Q|^2\,\mathrm{d}x + (r')^\gamma\Bigg], \end{aligned}$$

which follows from Lemma 2.3. This completes the proof of Lemma 2.4. □

Finally, we use the foregoing results to extend Theorem 1.2 of [ABF] to the situation studied here.

LEMMA 2.5 *The functions u_δ are of class $W^1_{\tilde{q},loc}(\Omega;\mathbb{R}^n)$ uniformly w.r.t. δ, where $\tilde{q} = 3p$ in case that $n = 3$ and where we may choose any finite number $\tilde{q}$ in case that $n = 2$.*

REMARK 2.3 *Note that for example in the 3D-case we have $3p > p^*$ provided that $p < 2$. Thus Lemma 2.5 is an improvement of Lemma 2.3.*

Proof of Lemma 2.5. We follow [BF1], proof of Lemma 4.4, and consider the case that $n = 3$, the case $n = 2$ is left to the reader. From (2.1) we deduce that $p < q < \tilde{q}$, thus there exists $\theta \in (0,1)$ such that

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{\tilde{q}}\,. \tag{2.5}$$

Note that from (2.5) and (2.1) it follows that

$$\frac{q}{p}(1-\theta) < 1\,. \tag{2.6}$$

Fix a ball $B_{2R}(x_0) \Subset \Omega$ and consider $0 < r < r'$ and $\bar{x}$ such that $B_{r'}(\bar{x}) \Subset B_R = B_R(x_0)$. Finally, let $0 \le \eta \in C_0^\infty(B_{2R})$ such that $\eta \equiv 1$ on $B_r(\bar{x})$. From Sobolev's inequality we deduce

$$\begin{aligned} \int_{B_r(\bar{x})}\Gamma_\delta^{\frac{3}{2}p}\,\mathrm{d}x &\le \int_{B_{2R}}(\eta h_\delta)^6\,\mathrm{d}x \\ &\le c\Bigg[\int_{B_{2R}}|\nabla(\eta h_\delta)|^2\,\mathrm{d}x\Bigg]^3 \\ &\le c\Bigg[\int_{B_{2R}}|\nabla\eta|^2 h_\delta^2\,\mathrm{d}x + \int_{B_{2R}}\eta^2|\nabla h_\delta|^2\,\mathrm{d}x\Bigg]^3 \quad =: \ c[T_1+T_2]^3\,, \end{aligned}$$

where h_δ is defined in Lemma 2.1. Clearly

$$T_1 \le c\|\nabla\eta\|^2_{L^\infty(B_{2R})}\int_{B_{2R}}\Gamma_\delta^{\frac{p}{2}}\,\mathrm{d}x$$

and

$$T_2 \le c\int_{B_{2R}}\eta^2 D_\varepsilon^2 f_\delta(\cdot,\varepsilon(u_\delta))(\partial_k\varepsilon(u_\delta),\partial_k\varepsilon(u_\delta))\,\mathrm{d}x$$

by the ellipticity estimate (1.2). Note again that all constants are uniform w.r.t. δ. Assume now that $\eta \equiv 0$ outside of $B_{(r+r')/2}(\bar{x})$ and that $|\nabla \eta| \leq c/(r'-r)$. Then obviously

$$\begin{aligned} T_2 &\leq c \int_{B_{(r+r')/2}(\bar{x})} D_\varepsilon^2 f_\delta(\cdot, \varepsilon(u_\delta))(\partial_k \varepsilon(u_\delta), \partial_k \varepsilon(u_\delta))\, \mathrm{d}x \\ &\leq c \left[\int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{2\bar{q}-p}{2}}\, \mathrm{d}x + (r'-r)^{-2} \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2\, \mathrm{d}x + (r')^\gamma \right], \end{aligned}$$

where we used Lemma 2.4. As outlined in the proof of Corollary 4.1 in [BF1] it is possible to choose the matrix Q in such a way that

$$\int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2\, \mathrm{d}x \leq c \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{q}{2}}\, \mathrm{d}x\,.$$

Thus we have shown

$$\begin{aligned} \left[\int_{B_r(\bar{x})} \Gamma_\delta^{\frac{3}{2}p}\, \mathrm{d}x \right]^{\frac{1}{3}} &\leq c(r'-r)^{-2} \left[\int_{B_{2R}} \Gamma_\delta^{\frac{p}{2}}\, \mathrm{d}x + \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{q}{2}}\, \mathrm{d}x \right] \\ &\quad + c\left[(r')^\gamma + \int_{B_{r'}(\bar{x})} \Gamma_\delta^{\frac{2\bar{q}-p}{2}}\, \mathrm{d}x \right]. \end{aligned} \tag{2.7}$$

Recalling Remark 2.1 and Remark 2.2, (2.7) differs from inequality (4.17) from [BF1] merely by a uniform local constant, and the lemma follows precisely as outlined in [BF1] via an interpolation argument leading to (4.18) of [BF1] with exponent θ according to (2.5). Since we have (2.6), the arguments after (4.18) can be copied without changes. $\square$

After these preparations we will show that any weak W^1_p-cluster point of the sequence $\{u_\delta\}$ (whose existence follows from $\sup_{0<\delta<1} \|u_\delta\|_{W^1_p(\Omega;\mathbb{R}^n)} < \infty$) actually is a strong solution of (1.1). So let $u_\delta \rightharpoondown: u$ in $W^1_p(\Omega;\mathbb{R}^n)$ as $\delta \to 0$. Similar to [ABF], Section 4, we get from Lemma 2.5 that

$$D_\varepsilon f_\delta(\cdot, \varepsilon(u_\delta)) \in W^1_{q/(q-1),loc}(\Omega;\mathbb{S}^n)$$

uniformly w.r.t. δ, and as in [ABF] we can show that

$$D_\varepsilon f_\delta(\cdot, \varepsilon(u_\delta)) \overset{\delta\to 0}{\to} D_\varepsilon f(\cdot, \varepsilon(u))$$

strongly in $L^{q/(q-1)}_{loc}(\Omega;\mathbb{S}^n)$ and a.e. This implies

$$\int_\Omega D_\varepsilon f(\cdot, \varepsilon(u)) : \varepsilon(\varphi)\, \mathrm{d}x = \int_\Omega u \otimes u : \varepsilon(\varphi)\, \mathrm{d}x + \int_\Omega g \cdot \varphi\, \mathrm{d}x$$

for any $\varphi \in C^\infty_0(\Omega;\mathbb{R}^n)$ s.t. $\operatorname{div} \varphi = 0$. Combining Lemma 2.3 and Lemma 2.4 we find

$$\int_{\Omega'} \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2\, \mathrm{d}x \leq c(\Omega') < \infty$$

for any $\Omega' \Subset \Omega$ uniform w.r.t. δ. If $p \geq 2$, then obviously we get

$$\|u_\delta\|_{W^2_2(\Omega';\mathbb{R}^n)} \leq c(\Omega') < \infty\,.$$

If $p < 2$, then Lemma 2.5 together with Young's inequality implies the bound

$$\|u_\delta\|_{W^2_\alpha(\Omega';\mathbb{R}^n)} \leq c(\Omega') < \infty$$

for any exponent $\alpha < 3p/(p+1)$ in the 3D-case and for any number $\alpha < 2$ if $n = 2$. This shows that $u \in W^2_{t,loc}(\Omega;\mathbb{R}^n)$ for a suitable exponent t, hence u is a strong solution of the equation (1.1). This proves i) of Theorem 1.1. □

3 Planar flows

In this section we give the proof of Theorem 1.1, ii), supposing that from now on all the hypotheses of this theorem are valid.

We recall the following lemma on the higher integrability of functions which has been demonstrated in [BFZ], Lemma 1.2.

LEMMA 3.1 *Let $d > 1$, $\beta > 0$ be two constants. With a slight abuse of notation let f, g, h denote any non-negative functions in $\Omega \subset \mathbb{R}^n$ satisfying $f \in L^d_{loc}(\Omega)$, $\exp(\beta g^d) \in L^1_{loc}(\Omega)$, $h \in L^d_{loc}(\Omega)$. Suppose that there is a constant $C > 0$ such that*

$$\left(\fint_B f^d dx\right)^{\frac{1}{d}} \leq C \fint_{2B} fg \,\mathrm{d}x + C\left(\fint_{2B} h^d \,\mathrm{d}x\right)^{\frac{1}{d}}$$

holds for all balls $B = B_r(x)$ with $2B = B_{2r}(x) \Subset \Omega$. Then there is a real number $c_0 = c_0(n, d, C) > 0$ such that if $h^d \log^{c_0\beta}(e + h) \in L^1_{loc}(\Omega)$, then the same is true for f. Moreover, for all balls B as above we have

$$\begin{aligned}\fint_B f^d \log^{c_0\beta}\left(e + \frac{f}{\|f\|_{d,2B}}\right) \mathrm{d}x \leq & c\left(\fint_{2B} \exp(\beta g^d)\,\mathrm{d}x\right)\left(\fint_{2B} f^d \,dx\right)\\ & + c\fint_{2B} h^d \log^{c_0\beta}\left(e + \frac{h}{\|f\|_{d,2B}}\right) \mathrm{d}x,\end{aligned}$$

where $c = c(n, d, \beta, C) > 0$ and $\|f\|_{d,2B} = (\fint_{2B} f^d \,dx)^{1/d}$.

REMARK 3.1 *Lemma 3.1 is not limited to the case $n = 2$ or $n = 3$.*

We will apply Lemma 3.1 to suitable powers of the functions

$$\begin{aligned} H &:= \left[D^2_\varepsilon f(\cdot, \varepsilon(u))(\varepsilon(\partial_k u), \varepsilon(\partial_k u))\right]^{\frac{1}{2}}, \\ \Gamma &:= 1 + |\varepsilon(u)|^2, \end{aligned}$$

where u denotes a strong solution to the system (1.1) which has been constructed in Section 2. We recall that

$$u \in W^2_{r,loc}(\Omega;\mathbb{R}^2) \quad \text{for all } r < 2\,. \tag{3.1}$$

(compare Theorem 1.1, i), and Remark 1.1) and use the weak form of (1.1) in the same way as done in [BF2] (compare the calculations starting from (5) and ending up with (9) of this paper, where we have to take $E(x) = x$ and the tensorfield S has to be replaced by $T(x,\varepsilon)$; note also that due to (3.1) the arguments of [BF2] are valid under the present hypotheses (1.2) and (1.3)) in order to get

$$\begin{aligned}\int_\Omega \eta^2\partial_k\sigma:\varepsilon(\partial_k u)\,\mathrm{d}x &\le -2\int_\Omega \eta\partial_k\sigma_{ij}\partial_i\eta\partial_k[u-Qx]^j\,\mathrm{d}x\\ &\quad+\int_\Omega [\nabla u]u\cdot\partial_k\big(\eta^2\partial_k[u-Qx]\big)\,\mathrm{d}x \qquad (3.2)\\ &\quad-\int_\Omega g\cdot\partial_k\big(\eta^2\partial_k[u-Qx]\big)\,\mathrm{d}x+2\int_\Omega\partial_k\pi\eta\nabla\eta\cdot\partial_k[u-Qx]\,\mathrm{d}x\,,\end{aligned}$$

where $\eta\in C_0^\infty(\Omega)$, $0\le\eta\le 1$, $Q\in\mathbb{R}^{2\times 2}$ and $\sigma:=D_\varepsilon f(\cdot,\varepsilon(u))$. Note that (3.2) exactly corresponds to inequality (9) of [BF2]. Of course we may also use the "δ-version" of (3.2) (see inequality (2.2) and the beginning of the proof of Lemma 2.4) together with the a priori estimates of Section 2 (see Lemma 2.3 and 2.5) to obtain (3.2) after passing to the limit $\delta\to 0$. Let us fix some subdomain $\Omega'\Subset\Omega$ and consider a disc $B_{2r}\subset\Omega'$. Moreover, assume that $\operatorname{spt}\eta\subset B_{2r}$, $\eta\equiv 1$ on B_r and $|\nabla\eta|\le c/r$. Using $u\in L^\infty(\Omega';\mathbb{R}^2)$ (see (3.1)) we obtain the following estimate from (3.2) with a constant c independent of B_{2r}

$$\begin{aligned}\fint_{B_{2r}}\eta^2\partial_k\sigma:\varepsilon(\partial_k u)\,\mathrm{d}x &\le c\Bigg[\fint_{B_{2r}}\eta|\nabla\sigma||\nabla\eta||\nabla u-Q|\,\mathrm{d}x+\fint_{B_{2r}}|\nabla u|\eta^2|\nabla^2u|\,\mathrm{d}x\\ &\quad+\fint_{B_{2r}}\eta|\nabla\eta||\nabla u||\nabla u-Q|\,\mathrm{d}x+\fint_{B_{2r}}|g|\eta^2|\nabla^2u|\,\mathrm{d}x\\ &\quad+\fint_{B_{2r}}|g|\eta|\nabla\eta||\nabla u-Q|\,\mathrm{d}x+\fint_{B_{2r}}|\nabla\pi|\eta|\nabla\eta||\nabla u-Q|\,\mathrm{d}x\Bigg]\,,\end{aligned}$$

where $|\nabla\pi|$ can be estimated via (1.1) which implies as usual that

$$|\nabla\pi|\le|g|+|u||\nabla u|+|\nabla\sigma|\,.$$

Thus we get (using also $g\in L^\infty$)

$$\begin{aligned}\fint_{B_{2r}}\eta^2\partial_k\sigma:\varepsilon(\partial_k u)\,\mathrm{d}x &\le c\Bigg[\fint_{B_{2r}}\eta|\nabla\sigma||\nabla\eta||\nabla u-Q|\,\mathrm{d}x+\fint_{B_{2r}}|\nabla u|\eta^2|\nabla^2u|\,\mathrm{d}x\\ &\quad+\fint_{B_{2r}}\eta|\nabla\eta||\nabla u||\nabla u-Q|\,\mathrm{d}x+\fint_{B_{2r}}\eta^2|\nabla^2u|\,\mathrm{d}x\\ &\quad+\fint_{B_{2r}}\eta|\nabla\eta||\nabla u-Q|\,\mathrm{d}x\Bigg]\,. \qquad (3.3)\end{aligned}$$

As outlined in the proof of Lemma 2.2 we have the counterpart of (2.3):

$$\varepsilon(\partial_k u):\partial_k\sigma\ge\frac{1}{2}H^2-c\Gamma^{\frac{2\bar{q}-p}{2}}\,. \qquad (3.4)$$

We further observe that the inequality before (2.4) now reads as

$$|\nabla\sigma|^2\le H\Big[D_\varepsilon^2 f(\cdot,\varepsilon(u))(\partial_k\sigma,\partial_k\sigma)\Big]^{\frac{1}{2}}+c\Gamma^{\frac{\bar{q}-1}{2}}|\nabla\sigma|$$

which implies by the second inequality of (1.2) the estimate

$$|\nabla\sigma| \le cH\tilde{h} + c\Gamma^{\frac{\bar{q}-1}{2}}\,, \tag{3.5}$$

where $\tilde{h} := \max\{\Gamma^{(\bar{q}-2)/4}, \Gamma^{(2-p)/4}\}$. We return to (3.3) and make use of (3.4) and (3.5) in order to get the following inequality

$$\begin{aligned}
\fint_{B_{2r}} \eta^2 H^2 \,\mathrm{d}x \quad &\le \quad c\Bigg[\fint_{B_{2r}} \eta^2 \Gamma^{\frac{2\bar{q}-p}{2}}\,\mathrm{d}x + \fint_{B_{2r}} \eta|\nabla\eta| H\tilde{h}|\nabla u - Q|\,\mathrm{d}x \\
&\qquad + \fint_{B_{2r}} \Gamma^{\frac{\bar{q}-1}{2}}\eta|\nabla\eta||\nabla u - Q|\,\mathrm{d}x + \fint_{B_{2r}} \eta^2|\nabla u||\nabla^2 u|\,\mathrm{d}x \\
&\qquad + \fint_{B_{2r}} \eta|\nabla\eta||\nabla u||\nabla u - Q|\,\mathrm{d}x + \fint_{B_{2r}} \eta^2|\nabla^2 u|\,\mathrm{d}x + \fint_{B_{2r}} \eta|\nabla\eta||\nabla u - Q|\,\mathrm{d}x\Bigg] \\
&=: \quad c\sum_{i=1}^{7} I_i
\end{aligned} \tag{3.6}$$

We estimate the terms on the r.h.s. of (3.6) following ideas presented in [BFZ] using $|\nabla^2 u| \le c|\nabla\varepsilon(u)| \le cH\tilde{h}$ and Sobolev-Poincaré's inequality. We obtain for $\gamma := 4/3$

$$\begin{aligned}
I_2 \quad &\le \quad cr^{-1}\fint_{B_{2r}} H\tilde{h}|\nabla u - Q|\,\mathrm{d}x \\
&\le \quad c\Bigg[\fint_{B_{2r}} (Hh)^{\gamma}\,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \frac{1}{r}\Bigg[\fint_{B_{2r}} |\nabla u - Q|^4\,\mathrm{d}x\Bigg]^{\frac{1}{4}} \\
&\le \quad c\Bigg[\fint_{B_{2r}} (H\tilde{h})^{\gamma}\,\mathrm{d}x\Bigg]^{\frac{2}{\gamma}}.
\end{aligned}$$

Hölder's and Sobolev-Poincaré's inequality give

$$\begin{aligned}
I_3 \quad &\le \quad \frac{c}{r}\fint_{B_{2r}} |\nabla u - Q|\Gamma^{\frac{\bar{q}-1}{2}} \quad \le \quad \frac{c}{r}\Bigg[\fint_{B_{2r}} |\nabla u - Q|^4\,\mathrm{d}x\Bigg]^{\frac{1}{4}}\Bigg[\fint_{B_{2r}} \Gamma^{\gamma\frac{\bar{q}-1}{2}}\,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \\
&\le \quad c\Bigg[\fint_{B_{2r}} |\nabla^2 u|^{\gamma}\,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}}\Bigg[\fint_{B_{2r}} \Gamma^{\gamma\frac{\bar{q}-1}{2}}\,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \quad \le \quad c\Bigg[\fint_{B_{2r}} (H\tilde{h})^{\gamma}\,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}}\Bigg[\fint_{B_{2r}} \Gamma^{\bar{q}-1}\,\mathrm{d}x\Bigg]^{\frac{1}{2}} \\
&\le \quad c\Bigg[\fint_{B_{2r}} (H\tilde{h})^{\gamma}\,\mathrm{d}x\Bigg]^{\frac{2}{\gamma}} + c\fint_{B_{2r}} \Gamma^{\bar{q}-1}\,\mathrm{d}x\,.
\end{aligned}$$

I_4 is estimated via

$$
\begin{aligned}
I_4 &\le c \fint_{B_{2r}} H\tilde{h}|\nabla u| \,\mathrm{d}x \;\le\; c\Bigg[\fint_{B_{2r}} (H\tilde{h})^\gamma \,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \Bigg[\fint_{B_{2r}} |\nabla u|^4 \,\mathrm{d}x\Bigg]^{\frac{1}{4}} \\
&\le c\Bigg[\fint_{B_{2r}} (H\tilde{h})^\gamma \,\mathrm{d}x\Bigg]^{\frac{2}{\gamma}} + c\Bigg[\fint_{B_{2r}} |\nabla u|^4 \,\mathrm{d}x\Bigg]^{\frac{1}{2}},
\end{aligned}
$$

where the last term is bounded from above by $\fint_{B_{2r}} \tilde{\Gamma}^2 \,\mathrm{d}x$, $\tilde{\Gamma} := 1 + |\nabla u|^2$. We further observe

$$
\begin{aligned}
I_5 &\le \frac{c}{r}\Bigg[\fint_{B_{2r}} |\nabla u - Q|^4 \,\mathrm{d}x\Bigg]^{\frac{1}{4}} \Bigg[\fint_{B_{2r}} |\nabla u|^\gamma \,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \\
&\le c\Bigg[\fint_{B_{2r}} |\nabla^2 u|^\gamma \,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \Bigg[\fint_{B_{2r}} |\nabla u|^\gamma \,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \\
&\le c\Bigg[\fint_{B_{2r}} (H\tilde{h})^\gamma \,\mathrm{d}x\Bigg]^{\frac{2}{\gamma}} + c\fint_{B_{2r}} \tilde{\Gamma} \,\mathrm{d}x\,,
\end{aligned}
$$

for I_6 we have

$$
\begin{aligned}
I_6 &\le c \fint_{B_{2r}} H\tilde{h} \,\mathrm{d}x \;\le\; c\Bigg[\fint_{B_{2r}} (H\tilde{h})^\gamma \,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \Bigg[\fint_{B_{2r}} \,\mathrm{d}x\Bigg]^{\frac{1}{4}} \\
&\le c\Bigg[\fint_{B_{2r}} (H\tilde{h})^\gamma \,\mathrm{d}x\Bigg]^{\frac{2}{\gamma}} + c\fint_{B_{2r}} \Gamma \,\mathrm{d}x\,,
\end{aligned}
$$

and finally we see that

$$
\begin{aligned}
I_7 &\le \frac{c}{r} \fint_{B_{2r}} |\nabla u - Q| \,\mathrm{d}x \;\le\; c\Bigg[\fint_{B_{2r}} |\nabla^2 u|^\gamma \,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \Bigg[\fint_{B_{2r}} \,\mathrm{d}x\Bigg]^{\frac{1}{\gamma}} \\
&\le c\Bigg[\fint_{B_{2r}} (H\tilde{h})^\gamma \,\mathrm{d}x\Bigg]^{\frac{2}{\gamma}} + \fint_{B_{2r}} \Gamma \,\mathrm{d}x\,.
\end{aligned}
$$

Collecting terms, (3.6) implies

$$
\begin{aligned}
\fint_{B_r} H^2 \,\mathrm{d}x &\le c\Bigg[\fint_{B_{2r}} G \,\mathrm{d}x + \Bigg[\fint_{B_{2r}} (H\tilde{h})^\gamma \,\mathrm{d}x\Bigg]^{\frac{2}{\gamma}}\Bigg], \\
G &:= \max\big\{\tilde{\Gamma}^2, \tilde{\Gamma}^{\bar{q}-1}, \tilde{\Gamma}^{\frac{2\bar{q}-p}{2}}\big\},
\end{aligned}
$$

thus

$$\left[\fint_{B_r} H^2 \,\mathrm{d}x\right]^{\frac{\gamma}{2}} \le c \fint_{B_{2r}} (H\tilde{h})^\gamma \,\mathrm{d}x + c\left[\fint_{B_{2r}} G \,\mathrm{d}x\right]^{\frac{\gamma}{2}}.$$

Now we like to apply Lemma 3.1 with the choices

$$d = \frac{2}{\gamma} = \frac{3}{2}\,, \quad f = H^\gamma\,, \quad g = \tilde{h}^\gamma\,, \quad h = G^{\frac{2}{3}}\,.$$

To this purpose we claim that

$$\exp(\beta g^d) = \exp(\beta\tilde{h}^2) \in L^1_{loc}(\Omega) \tag{3.7}$$

holds for any number $\beta > 0$. Indeed, the calculations in the proof of Lemma 2.5 (combine the estimate for T_2 with the information that $\nabla u_\delta \in L^r_{loc}(\Omega;\mathbb{R}^{2\times 2})$ for any $r < \infty$) imply $\Phi := \Gamma^{p/4} \in W^1_{2,loc}(\Omega)$, hence by Trudinger's inequality (see Theorem 7.16 of [GT])

$$\int_{B_\rho} \exp(\beta_0 \Phi^2)\,\mathrm{d}x \le c(\rho) < \infty$$

for any disc $B_\rho \Subset \Omega$. Here β_0 depends on the W^1_2-bound of Φ on B_ρ. In particular we find that

$$\int_{B_\rho} \exp(\beta\Phi^{2-\kappa})\,\mathrm{d}x \le c(\rho,\beta,\kappa) < \infty \tag{3.8}$$

for any $0 < \kappa < 1$ and all $\beta > 0$. We have on account of $\bar{q} < p+2$ the inequality

$$\Gamma^{\frac{\bar{q}-2}{2}} \le \Gamma^{\frac{p}{4}(2-\kappa)} = \Phi^{2-\kappa}$$

for some small $\kappa > 0$, and clearly (since $p > 1$)

$$\Gamma^{\frac{2-p}{2}} \le \Phi^{2-\kappa}\,,$$

so that $\tilde{h}^2 \le \Phi^{2-\kappa}$. Thus (3.7) follows from (3.8).

Moreover, since $f^d = D^2_\varepsilon f(\cdot,\varepsilon(u))(\partial_k\varepsilon(u),\partial_k\varepsilon(u))$, we deduce $f^d \in L^1_{loc}(\Omega)$ from Lemma 2.4 together with the uniform local estimates for the r.h.s. The statement $h \in L^d_{loc}(\Omega)$ is immediate by the definition of G. From the lemma we now deduce

$$\int_{B_\rho} H^2 \log^{c_0\beta}(e+H)\,\mathrm{d}x \le c(\beta,\rho)$$

for all discs $B_\rho \subset \Omega'$ and all $\beta > 0$. Recall that (see (3.5))

$$|\nabla\sigma| \le cH\tilde{h} + c\Gamma^{\frac{\bar{q}-1}{2}}$$

and that

$$\int_{B_\rho} \left(\Gamma^{\frac{\bar{q}-1}{2}}\right)^2 \log^{c_0\beta}\left(e + \Gamma^{\frac{\bar{q}-1}{2}}\right)\mathrm{d}x \le c(\beta,\rho)\,.$$

The same is true for the function $H\tilde{h}$: to verify this statement we recall the following elementary inequality (see (2.12) of [BFZ]). Let a, $b \ge 0$. Then, for any $\alpha > 0$, there is a constant $c(\alpha) > 0$ such that

$$(ab)^2 \log^\alpha(e+ab) \le 2^\alpha a^2 \log^{\alpha+2}(e+a) + c(\alpha)\exp(6b)\,. \tag{3.9}$$

Applying (3.9) to $a = H$, $b = \tilde{h}$ we obtain the desired estimate

$$\int_{B_\rho} \Big((H\tilde{h})^{\frac{\bar{q}-1}{2}} \Big)^2 \log^{c_0\beta} \big(e + (H\tilde{h})^{\frac{\bar{q}-1}{2}} \big) \, dx \le c(\beta, \rho) \, .$$

Now, using the estimate for $|\nabla\sigma|$ combined with the observation that

$$(a+b)^2 \log^\alpha(e+a+b) \le c \Big[a^2 \log^\alpha(e+a) + b^2 \log^\alpha(e+b) \Big] \, ,$$

we finally arrive at

$$\int_{B_\rho} |\nabla\sigma|^2 \log^{c_0\beta}(1 + |\nabla\sigma|) \, dx \le c(\beta, \rho) < \infty$$

for any $\beta > 0$. This shows $\sigma \in C^0(\Omega; \mathbb{S}^n)$ by quoting Example 5.3 of [KKM]. Now $u \in C^{1,\alpha}(\Omega; \mathbb{R}^2)$ follows as outlined in [BF2], p. 1616. □

References

[AM] Acerbi, E., Mingione, G., Regularity results for stationary electrorheological fluids. Arch. Rat. Mech. Anal. 164 (2002), 213–259.

[ABF] Apushkinskaya, D., Bildhauer, M., Fuchs, M., Steady states of anisotropic generalized Newtonian fluids. To appear in J. Math. Fluid Mech.

[BF1] Bildhauer, M., Fuchs, M., Variants of the Stokes problem: the case of anisotropic potentials. J. Math. Fluid Mech. 5 (2003), 364–402.

[BF2] Bildhauer, M., Fuchs, M., A regularity result for stationary electrorheological fluids in two dimensions. Math. Meth. Appl. Sci. 27 (2004), 1607–1617.

[BF3] Bildhauer, M., Fuchs, M., Regularization of convex variational problems with applications to generalized Newtonian fluids. To appear in Archiv der Mathematik.

[BF4] Bildhauer, M., Fuchs, M., $C^{1,\alpha}$-solutions to non-autonomous anisotropic variational problems. Submitted.

[BFZ] Bildhauer, M., Fuchs, M., Zhong, X., A lemma on the higher integrability of functions with applications to the regularity theory of two-dimensional generalized Newtonian fluids. Submitted.

[DER] Diening, Ettwein, F., Růžička, M., $C^{1,\alpha}$-regularity for electrorheological fluids in two dimensions. Preprint University of Freiburg (2004).

[ELM] Esposito, L., Leonetti, F., Mingione, G., Sharp regularity for functionals with (p,q) growth. J. Diff. Eq. 204 (2004), 5–55.

[E] Ettwein, F., Elektrorheologische Flüssigkeiten: Existenz starker Lösungen in zweidimensionalen Gebieten. Diplomarbeit, Universität Freiburg, 2002.

[ER] Ettwein, F., Růžička, M., Existence of strong solutions for electrorheological fluids in two dimensions: steady Dirichlet problem. In: Geometric Analysis and Nonlinear Partial Differential Equations. By S. Hildebrandt and H. Karcher, Springer-Verlag, Berlin-Heidelberg-New York, 2003, 591–602.

[FMS] Frehse J., Málek, J., Steinhauer, M., On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. To appear in SIAM J. Math. Anal.

[Ga] Galdi, G., An introduction to the mathematical theory of the Navier-Stokes equations, vol. 1. Springer Tracts in Natural Philosophy vol. 38, Springer, New York, 1994.

[Gi] Giaquinta, M., Multiple integrals in the calculus of variations and nonlinear elliptic systems. Ann. Math. Studies 105, Princeton University Press, Princeton 1983.

[GT] Gilbarg, D., Trudinger, N., Elliptic partial differential equations of the second order. Second Edition, Springer-Verlag, Berlin-Heidelberg, 1983.

[KMS] Kaplický, P., Málek, J., Stará, J., $C^{1,\alpha}$-solutions to a class of nonlinear fluids in two dimensions – stationary Dirichlet problem. Zap. Nauchn. Sem. St.-Petersburg Odtel. Math. Inst. Steklov (POMI) 259 (1999), 89–121.

[KKM] Kauhanen, J., Koskela, P., Malý, J., On functions with derivatives in a Lorentz spaces. Manuscripta Math. 100 (1999), 87–101.

[La] Ladyzhenskaya, O.A., The mathematical theory of viscous incompressible flow, Gordon and Breach, 1969.

[Pi] Pileskas, K.I., On spaces of solenoidal vectors. Zap. Nauchn. Sem. Leningrad Odtel. Mat. Inst. Steklov (LOMI) 96 (1980), 237–239 (in Russian). English translation: J. Soviet Math. 21, no. 5 (1983).

[R] Růžička, M., Electrorheological fluids: modeling and mathematical theory. Lecture Notes in Mathematics Vol. 1748, Springer, Berlin-Heidelberg-New York 2000.

The references [ABF], [BF3], [BFZ] and [BF4] which are not yet published are available online as preprint version: www.math.uni-sb.de/preprint.html (no. 88, 93, 102, 108).