Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 114

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Saarbrücken 2004

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 114
submitted: October 15, 2004

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Spherical isometries are reflexive

Michael Didas

Let H be a complex Hilbert space. A commuting system $T = (T_1, \cdots, T_n)$ of bounded linear operators on H is called a spherical isometry if $T_1^*T_1 + T_2^*T_2 + \cdots + T_n^*T_n = 1_H$. In this note we prove that each spherical isometry is reflexive.

Let H be a complex Hilbert space. For an arbitrary family $\mathcal{S} \subset L(H)$ of bounded linear operators on H, we define $\mathrm{Lat}(\mathcal{S})$ to be the lattice of all closed subspaces M of H satisfying $SM \subset M$ for every $S \in \mathcal{S}$. The set

$$\mathrm{AlgLat}(\mathcal{S}) = \{C \in L(H) : \mathrm{Lat}(C) \supset \mathrm{Lat}(\mathcal{S})\}$$

is easily seen to be a subalgebra of $L(H)$ which is closed in the weak operator topology and contains the WOT-closed operator algebra $\mathcal{W}_T$ generated by $\mathcal{S} \cup \{1_H\}$. The family of operators $\mathcal{S}$ is called *reflexive* if the equality

$$\mathrm{AlgLat}(\mathcal{S}) = \mathcal{W}_{\mathcal{S}}$$

holds. In his pioneering work [13] from 1966, Sarason showed that analytic Toeplitz operators on the Hardy space $H^2(\mathbb{D})$ over the unit disc and commuting families of normal operators are reflexive. Motivated by the fact that a commuting family $\mathcal{S} \subset L(H)$ which is reflexive necessarily possesses a non-trivial joint invariant subspace and the observation that the formula $\mathrm{AlgLat}(T) = \mathcal{W}_T$ may be interpreted as a non-selfadjoint version of von Neumann's double commutant theorem (see page 5 in [7] for more details), the reflexivity problem has been (and is still) intensively studied by many authors. We will briefly follow the development of just one branch of the theory which has its starting point in 1971 with the following discovery due to Deddens [6].

Theorem 1 (Deddens) *Each (single) isometry on a complex Hilbert space is reflexive.*

In the multi-variable context, there exist different natural possibilities to define isometric operator tuples $T = (T_1, \cdots, T_n) \in L(H)^n$. These multi-dimensional generalizations of the concept of a single isometry $T \in L(H)$

correspond in some sense to the different possible analogues of the unit disc $\mathbb{D} \subset \mathbb{C}$ in higher dimensions. One way is to consider commuting families $T \in L(H)^n$ whose components are isometric operators. In this situation (we may regard it as the polydisc-case), the following reflexivity result is well known (Theorem 2.4 in [3], note that $\mathcal{S}$ need not have finitely many members).

Theorem 2 (Bercovici) *Each commuting family $\mathcal{S} \subset L(H)$ of isometries is reflexive.*

Geometrically spoken we now turn to the ball case: A *spherical isometry* is by definition a commuting system $T = (T_1, \cdots, T_n) \in L(H)^n$ satisfying

$$T_1^*T_1 + T_2^*T_2 + \cdots + T_n^*T_n = 1_H.$$

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 1\}$ be the open Euclidean unit ball in $\mathbb{C}^n$. A result of Athavale (Proposition 2 in [2]) says that each spherical isometry is subnormal and its minimal normal extension is a *spherical unitary*, that is, a commuting tuple of normal operators with Taylor spectrum contained in the unit sphere $\partial\mathbb{B}$.

Towards the reflexivity of spherical isometries, Müller and Ptak proved in 1999 the following intermediate result (Theorem 5 in [12]).

Theorem 3 (Müller and Ptak) *Each spherical isometry $T \in L(H)^n$ is hyporeflexive, i.e. $AlgLat(T) \cap (T)' = \mathcal{W}_T$, where $(T)'$ denotes the commutant of T.*

Eschmeier's reflexivity results for special subnormal systems over the unit ball (see [9]) imply the reflexivity of spherical isometries in two particular cases.

Theorem 4 (Eschmeier) *Each spherical isometry with dominating Taylor spectrum in $\overline{\mathbb{B}}$ and each spherical isometry possessing an isometric and weak* continuous $H^\infty(\mathbb{B})$-functional calculus is reflexive.*

However, the methods used for the proof of the last result seem to be limited by the availability of analytic structure in the dual algebra generated by a spherical isometry. As concrete examples show (see Corollary 3.3 and Theorem 3.4 in Eschmeier [11]) this kind of structure is not present in general.

The aim of this note is to give a complete solution of the reflexivity problem for spherical isometries. Our approach is based on the result of Bercovici cited above and the existence of abstract inner functions which has been established by Aleksandrov [1] in 1984.

Theorem 5 *Each spherical isometry on a complex Hilbert space is reflexive.*

We first introduce some more notation and collect some well-known basic facts that will be used during the proof.

We write $A(\mathbb{B})$ for the *ball algebra*, that is, the Banach algebra of all continuous complex-valued functions on $\overline{\mathbb{B}}$ which are holomorphic on $\mathbb{B}$, equipped with the supremum norm. Let λ denote the restriction of the $2n$-dimensional Lebesgue measure to $\mathbb{B}$. Recall that the algebra $H^\infty(\mathbb{B})$ of all bounded holomorphic functions on $\mathbb{B}$ is a weak* closed subspace of $L^\infty(\lambda)$, and hence carries a natural weak* topology as the dual of $L^1(\lambda)/{}^\perp H^\infty(\mathbb{B})$.

A complex regular Borel measure μ on $\partial\mathbb{B}$ will be called a *Henkin measure*, if there exists a weak* continuous extension

$$r_\mu : H^\infty(\mathbb{B}) \to L^\infty(\mu)$$

of the map $\mathbb{C}[z] \to L^\infty(\mu)$, $p \mapsto [p|\partial\mathbb{B}]$. Since the polynomials $\mathbb{C}[z]$ in n-complex variables $z = (z_1, \cdots, z_n)$ are weak* dense in $H^\infty(\mathbb{B})$, such a map r_μ is unique, if it exists, and in this case the range of r_μ is contained in the space

$$H^\infty(\mu) = \overline{\{[p|\partial\mathbb{B}] : p \in \mathbb{C}[z]\}}^{w^*} \subset L^\infty(\mu).$$

By $M^+(\partial\mathbb{B})$ we denote the set of all finite positive regular Borel measures on $\partial\mathbb{B}$, while $M_1^+(\partial\mathbb{B})$ stands for the corresponding set of probability measures. Let $\mu \in M^+(\partial\mathbb{B})$. A *$\mu$-inner function* is by definition an equivalence class

$$\theta \in H^\infty(\mu) \quad \text{with} \quad |\theta| = 1 \quad (\text{in } L^\infty(\mu)).$$

As has been shown in the celebrated work of Aleksandrov [1], the existence of non-trivial inner functions in $H^\infty(\mu)$ results from the fact that the triple $(A(\mathbb{B}), \partial\mathbb{B}, \mu)$ is *regular* (in the sense of Aleksandrov) for each choice of a measure $\mu \in M^+(\partial\mathbb{B})$.

Let H be a complex Hilbert space. Remember the fact that the Banach space $L(H)$ of all bounded linear operators on H is in duality with the trace class $C^1(H)$. The corresponding weak* topology on $L(H)$ is obviously stronger than the weak operator topology. Given a commuting tuple $T \in L(H)^n$, the operator algebra

$$\mathcal{A}_T = \overline{\{p(T) : p \in \mathbb{C}[z]\}}^{w^*}$$

is therefore contained in $\mathcal{W}_T$, the WOT-closure of the polynomials in T.

Let $\mathcal{A} \subset L(H)$ be a weak* closed subalgebra. The set of all weak* continuous linear functionals on $\mathcal{A}$ can be identified isometrically with the quotient space

$$Q_{\mathcal{A}} = C^1(H)/{}^{\perp}\mathcal{A}.$$

Given two vectors $x, y \in H$, we write $[x \otimes y] \in Q_{\mathcal{A}}$ for the equivalence class of the rank-one operator $H \to H$, $\xi \mapsto \langle \xi, y\rangle x$, which represents the vector functional

$$[x \otimes y] : \mathcal{A} \to \mathbb{C}, \quad A \mapsto \langle Ax, y\rangle$$

induced by x and y. The dual operator algebra $\mathcal{A}$ is said to have *property* $(\mathbb{A}_1(r))$ if, for every real number $s > r$ and every element $L \in Q_{\mathcal{A}}$ there are vectors $x, y \in H$ with

$$L = [x \otimes y] \quad \text{and} \quad \|x\|, \|y\| \leq (s\|L\|)^{\frac{1}{2}}.$$

Proof of Theorem 5. The proof is divided into two steps.

Step (1): The absolutely continuous case. We assume that the spherical isometry T possesses a contractive and weak* continuous functional calculus $\Phi_T : H^\infty(\mathbb{B}) \to L(H)$.

Let $U \in L(K)^n$ denote a minimal normal extension of T (which, as a spherical unitary, satisfies the spectral inclusion $\sigma(U) \subset \partial\mathbb{B}$) and let $\mu \in M_1^+(\partial\mathbb{B})$ be the trivial extension of a scalar-valued spectral measure for U. The spherical unitary tuple U possesses an isometric and weak* continuous $L^\infty(\mu)$-functional calculus $\Psi_U : L^\infty(\mu) \to L(K)$, which induces an isometric isomorphism (see Conway [4]) and weak* homeomorphism

$$\gamma_T : H^\infty(\mu) \to \mathcal{A}_T, \quad f \mapsto \Psi_U(f)|H$$

extending the polynomial functional calculus of T. From the fact that the composition

$$r : H^\infty(\mathbb{B}) \xrightarrow{\Phi_T} \mathcal{A}_T \xrightarrow{\gamma_T^{-1}} H^\infty(\mu)$$

is a weak* continuous contraction satisfying $r(p) = [p|\partial\mathbb{B}]$ for each polynomial $p \in \mathbb{C}[z]$, we deduce that the measure μ is a Henkin measure (and $r = r_\mu$). By Lemma 2.2.3 in [7], this implies that one-point sets have μ-measure zero, which is equivalent to saying that μ is continuous in the sense of Aleksandrov [1].

Given a polynomial $p \in \mathbb{C}[z]$ with $\|p\|_{\infty,\mathbb{B}} \leq 1$, Corollary 29 in [1] therefore allows us to choose a sequence of μ-inner functions

$$(\theta_k) \quad \text{in} \quad I = \{\theta \in H^\infty(\mu) : \theta \text{ inner}\} \qquad \text{with} \qquad r_\mu(p) = w^* - \lim_k \theta_k.$$

By the weak* continuity of γ_T, this immediately implies that

$$p(T) = \gamma_T(r_\mu(p)) = w^* - \lim_k \gamma_T(\theta_k) \quad \text{in } L(H).$$

The invariant subspace lattice of T can therefore be expressed as

$$\mathrm{Lat}(T) = \mathrm{Lat}(\mathcal{A}_T) = \mathrm{Lat}(\gamma_T(I)),$$

where $\gamma_T(I) = \{\gamma_T(\theta) : \theta \in I\} \subset L(H)$. Since, for every $x \in H$ and every inner function $\theta \in I$, we have

$$\|\gamma_T(\theta)x\|^2 = \|\Psi_U(\theta)x\|^2 = \langle \Psi_U(|\theta|^2)x, x\rangle = \|x\|^2,$$

the family $\gamma_T(I) \subset L(H)$ is a set of commuting isometries, and hence is reflexive by the result of Bercovici cited above (Theorem 2.4 in [3]).

Thus we can finish the first step of the proof by the observation that

$$\mathrm{AlgLat}(T) = \mathrm{AlgLat}(\gamma_T(I)) = \mathcal{W}_{\gamma_T(I)} \subset \overline{\mathrm{ran}(\gamma_T)}^{WOT} = \overline{\mathcal{A}_T}^{WOT} = \mathcal{W}_T.$$

Step (2): The general case. Let $T \in L(H)^n$ now be an arbitrary spherical isometry and let

$$T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)^n$$

be the unique orthogonal decomposition of T into its spherical unitary part $T_1 \in L(H_1)^n$ and its completely non-unitary part $T_0 \in L(H_0)^n$. By Corollary 2.4 in [8], the tuple T_0 possesses a weak* continuous $H^\infty(\mathbb{B})$-functional calculus (which is even of class $C_{\cdot 0}$). To prove the reflexivity of T, it suffices to show that T_0 and T_1 are reflexive and that the dual algebras $\mathcal{A}_{T_0}$ and $\mathcal{A}_{T_1}$ generated by these tuples satisfy the factorization property $(\mathbb{A}_1(r))$ for some $r \geq 1$ (see, for instance, the proof of Theorem VII.8.5 in [5]).

By Corollary 1.10 in [10] the dual algebra $\mathcal{A}_T$ generated by a spherical isometry $T \in L(H)^n$ satisfies property $(\mathbb{A}_1(c))$ for some universal constant c. Indeed, the cited result implies that there is a universal constant $c > 0$ such that, for every $\varepsilon > 0$, every element $L \in Q_{\mathcal{A}_T}$ and any given vector $a \in H$, there are vectors $x, y \in H$ with $\|x-a\| < \varepsilon$, $L = [x \otimes y]$ and $\|y\| \leq (c/\varepsilon)\|L\|$. By applying this result with $a = 0$ and $\varepsilon = (c\|L\|)^{\frac{1}{2}}$, we find that $\mathcal{A}_T$ has property $(\mathbb{A}_1(c))$.

Hence the absolutely continuous spherical isometry T_0 is reflexive (by the first part) and satisfies property $(\mathbb{A}_1(c))$. On the other hand, it is well known that every normal tuple is reflexive and the dual algebra generated by a normal tuple has property $(\mathbb{A}_1(1))$. Applying this remark to the spherical unitary T_1 completes the proof. $\square$

All the methods and concepts used in the above proof are also available in the more general context of a strictly pseudoconvex open subset D of a Stein submanifold X in $\mathbb{C}^n$ (generalizing the open unit ball $\mathbb{B}$), see [7] for details. Hence, defining a commuting n-tuple $T \in L(H)^n$ to be a ∂D-isometry, if T is subnormal and its minimal normal extension is a commuting tuple of normal operators with Taylor spectrum contained in ∂D, one obtains analogously that each ∂D-isometry on a complex Hilbert space is reflexive.

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