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Abstract

We prescribe two Jordan curves in a Riemannian manifold of certain property. A minimal surface of annulus type bounded by these curves is described as the harmonic extension of a critical point of some functional (Dirichlet Integral) in a certain space of boundary parametrizations. The $H^{2,2}$-regularity of the minimal surface of annulus type will be proved by applying the critical points theory and Morrey growth condition.

1 Introduction

In 1983 ([St1], see also [St2] [St3]), by extending the Ljusternik-Schnirelmann Theory on convex sets in Banach Spaces, a general theory of critical points was developed, and an approach to unstable solutions and Morse theory for Plateau's Problem of disc or annulus type in $\mathbb{R}^n$ was given. Here a minimal surface is described as the harmonic extension of a critical point of the following functional, defined on a set of boundary parametrizations:

$$\mathcal{E}(x) := \frac{1}{2}\int |\mathcal{H}(x)|^2 d\omega,$$

where $\mathcal{H}$ denotes the harmonic extension in $\mathbb{R}^n$. And $H^{2,2}$-regularity of the above minimal surface was proved in the situation of a normalized setting by the integral condition (see [St1]). In [IS], further details were given and the regularity of the surface in a situation of three-points normalized setting was studied.

Recently in [Ho], the existence of unstable minimal surfaces of higher topological structure with one boundary in a nonpositively curved Riemannian manifold was studied by applying the method in [St2], and the regularity of minimal surfaces was discussed.

In this paper, we want to give a simialr regularity result for a minimal surface of annulus type in manifolds satisfying some appropriate conditions, namely we will consider two boundary curves Γ_1, Γ_2 in a Riemannian manifold (N, h) such that one of the following holds.

(C1) There exists $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p, r)$, where $B(p, r)$ lies within the normal range of all of its points. Here we assume $r < \pi/(2\sqrt{\kappa})$, where κ is an upper bound of the sectional curvature of (N, h).

(C2) N is compact with nonpositive sectional curvature.

These conditions are related to the existence and the uniqueness of the harmonic extension for a given boundary parametrization.

We first construct suitable spaces of functions, the sets of boundary parametrizations, where we have to distinguish the cases of (C1) and (C2). And then following some idea of Struwe, we introduce a convex set which in fact serves as a tangent space for the given boundary parametrization. Moreover, we consider the following functional:

$$\mathcal{E}(x) := \frac{1}{2}\int |d\mathcal{F}(x)|_h^2 d\omega,$$

where $\mathcal{F}(x)$ denotes the harmonic extension of annulus type in a manifold N with metric h.

We may then describe a minimal surface as the harmonic extension of a critical point of $\mathcal{E}$.

We will always use the situation that N can be embedded properly into some $\mathbb{R}^k$ as a closed submanifold (see [Gr]).

We then compute the $H^{2,2}$-regularity of our surfaces using the Morrey growth condition, see Section 3.2. We generalize the idea in [St1] to a minimal surface of annlus type in Riemannian maniflods of the above property.

2 Preliminaries

2.1 Some definitions

Let (M, g) be a manifold of dimension 2 with boundary ∂M, metric (g_{ij}) and (N, h) a connected, oriented, complete Riemannian manifold with metric $(h_{\alpha\beta})$ of dimension $n \geq 2$, embedded isometrically and properly into some $\mathbb{R}^k$ as a closed submanifold by η (see [Gr]). Moreover, ∇ resp. $\widetilde{\nabla}$ denotes the covariant derivative in (N, h) resp. $\mathbb{R}^k$.

We use the summation convention for indices and a colon denotes the ordinary derivative with $i = 1, 2$, $\alpha = 1, \cdots, n$. And $d\omega$ resp. d_0 denotes the area element in $\Omega \subset \mathbb{R}^2$ resp. in $\partial\Omega$.

- The energy of $f \in C^2((M, g), (N, h))$ is defined by

$$E(f) := \frac{1}{2}\int_M |df|^2 dM_g = \frac{1}{2}\int_M g^{ij} h_{\alpha\beta} \circ f f^\alpha_{,i} f^\beta_{,j} dM_g.$$

The Euler-Lagrange equation of E for $f \in C^2((M, g), (N, h))$, called the tension field along f, is as follows:

$$\tau_h(f) := \langle \nabla_{\frac{\partial}{\partial z^i}} df, dz^i\rangle = g^{ij}(\nabla df)^\alpha_{ij} = g^{ij}(f^\alpha_{,ij} - f^\alpha_{,k}\Gamma^k_{ij} + f^\beta_{,i} f^\gamma_{,j} \Gamma^\alpha_{\beta\gamma} \circ f)\frac{\partial}{\partial y^\alpha} \circ f.$$

And $f \in C^2((M, g), (N, h))$ is called harmonic if $\tau_h(f) = 0$.

For $f = (f^a)_{a=1,\cdots,k}$, the second fundamental form of η is :

$$II \circ f(df, df) := \langle \widetilde{\nabla}_{\frac{\partial}{\partial z^i}} df - \nabla_{\frac{\partial}{\partial z^i}} df, dz^i \rangle \in T^{\perp}_{f(\cdot)} \eta(N).$$

• A weak Jacobi field $\mathbf{J}$ with boundary ξ along a harmonic function f is a vector field along f as a weak solution of

$$\int_M \langle \nabla \mathbf{J}, \nabla X \rangle + \langle tr\, R(\mathbf{J}, df)df, X \rangle d\omega = 0$$

for all $X \in H^{1,2} \cap L^\infty(M, f^*TN)$ with $X|_{\partial M} = \xi$.

• For $B := \{w \in \mathbb{R}^2 \,||w| < 1\}$,

$$H^{1,2} \cap C^0(B, N) := \{f \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | f(B) \subset N\},$$

with norm, $\|f\|_{1,2;0} := \|\nabla f\|_{L^2} + \|X\|_{C^0}$.

Let Γ be a Jordan curve in N which is diffeomorphic to $S^1 := \partial B$, and observe that N can be equipped with another metric $\tilde{h}$ such that Γ is a geodesic in $(N, \tilde{h})$. Note that $H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)_{\tilde{h}}\big)$ and $H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)_h\big)$ coincide as sets. Using the exponential map in $(N, \tilde{h})$, we define the following spaces.

$$H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) \;\; := \;\; \{u \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k) | u(\partial B) = \Gamma\}$$

with norm, $\|u\|_{\frac{1}{2},2;0} := \|\nabla \mathcal{H}(u)\|_{L^2} + \|u\|_{C^0}$, here $\mathcal{H}(u)$ is the harmonic extension in $\mathbb{R}^k$. And

$$\begin{aligned} T_u H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) \;\; &:= \;\; \{\xi \in H^{\frac{1}{2},2} \cap C^0(\partial B, u^*TN) | \xi(z) \in T_{u(z)}\Gamma, \text{ for all } z \in \partial B\} \\ &= \;\; H^{\frac{1}{2},2} \cap C^0(\partial B, u^*T\Gamma). \end{aligned}$$

2.2 The setting

Let Γ_1, Γ_2 be two Jordan curves of class C^3 in N with diffeomprphisms $\gamma^i : \partial B \to \Gamma_i, i = 1, 2$, and $\mathrm{dist}(\Gamma_1, \Gamma_2) > 0$. Moreover, for $\rho \in (0, 1)$,

$$A_\rho = \{w \in B \mid \rho < |w| < 1\}, \;\; C_1 = \{w \mid |w| = 1\}, \;\; C_2 = \{w \mid |w| = \rho\}.$$

And let

$$\mathcal{X}^i_{\text{mon}} := \{x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i) \mid \text{weakly monotone onto } \Gamma_i\}.$$

I) We first consider the following condition for $(N, h)(\supset \Gamma_1, \Gamma_2)$.

(C1) There exists $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p, r)$, where $B(p, r)$ lies within the normal range of all of its points. Here we assume $r < \pi/(2\sqrt{\kappa})$, where κ is an upper bound of the sectional curvature of (N, h).

In this paper, $B(p,r)$ denotes a geodesic ball of $p \in N$ with the properties in the condition (C1).

Remark 2.1. *If* $\Gamma_1, \Gamma_2 \subset N$ *satisfy (C1), for each* $x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ *and* $\rho \in (0,1)$ *there exists* $g_\rho \in H^{1,2} \cap C^0(\overline{A_\rho}, B(p,r))$ *and* $g^i \in H^{1,2} \cap C^0(\overline{B}, B(p,r))$ *with* $g_\rho|_{C_1} = x^1$, $g_\rho|_{C_2}(\cdot) = x^2(\frac{\cdot}{\rho})$ *and* $g^i|_{\partial B} = x^i$, $i = 1,2$.

Proof. Let $\Omega := \exp^{-1}(B(p,r)) \subset B(0,\tilde{r})_{\mathbb{R}^n} \subset \mathbb{R}^n$ for some $\tilde{r} > 0$.
For $\widetilde{x}^i := \exp^{-1}(x^i)$, we have an Euclidean harmonic extension $h_\rho(\widetilde{x}^1, \widetilde{x}^2)$ of finite energy, whose image is in $B(0,\tilde{r})_{\mathbb{R}^n}$. The map exp is a diffeomorphism and Ω is star shape, so there exists a retraction $\delta : B(0,\tilde{r})_{\mathbb{R}^n} \to \Omega$ with $\delta|_\Omega = Id$ in the class of $H^{1,2}$. Then the map $g_\rho := \exp(\delta(h_\rho(\widetilde{x}^1, \widetilde{x}^2))) : A_\rho \to \Omega$ is an $H^{1,2} \cap C^0(\overline{A_\rho}, B(p,r))$-extension with boundary x^1 and $x^2(\frac{\cdot}{\rho})$. We may also find an $H^{1,2} \cap C^0(\overline{B}, B(p,r))$-extension. □

From the results in [HKW], [JK] and the above Remark, we have a unique harmonic map of annulus and of disc type in $B(p,r) \subset N$ for a given boundary mapping in the class of $H^{\frac{1}{2},2} \cap C^0$. Now we define,

$$M^i := \{x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i) \,|\, x^i \text{ is weakly monotone, orientation preserving}\}.$$

Then M^i is complete, since the C^0-norm preserves the monotonicity.

We now investigate another alternative condition for (N,h).

(C2) N is compact with nonpositive sectional curvature.

A compact Riemannian manifold is homogeneously regular and the condition of non-positive sectional curvature for N implies $\pi_2(N) = 0$.

In order to define M^i, we need some preparation. First, we consider for $\rho \in (0,1)$,

$$G_\rho := \{f \in H^{1,2} \cap C^0(\overline{A_\rho}, N) |\, f|_{C_i} \text{ is continuous and weakly monotone onto } \Gamma_i\}.$$

We may take a continuous homotopy class, denoted by $F_\rho \subset G_\rho$, so that every two elements f, g in F_ρ are continuous homotopic (not necessarily relative), denoted by $f \sim g$, more exactly:

$$f \sim g \quad \Leftrightarrow \quad \begin{array}{l} \text{there exists a continuous mapping } H : [0,1] \times \overline{A_\rho} \to N \\ \text{with } H(0,\cdot) = f(\cdot), H(1,\cdot) = g(\cdot). \end{array}$$

Now define

$$\begin{aligned} M^1 &:= \{f|_{C_1}(\cdot) \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_1) \,|\, \text{orientation preserving}, \;\; f \in \mathfrak{F}_\rho\}, \\ M^2 &:= \{f|_{C_2}(\cdot \rho) \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_2) \,|\, \text{orientation preserving}, \;\; f \in \mathfrak{F}_\rho\}. \end{aligned}$$

Then for $x^i \in M^i$, there exists a unique harmonic extension to A_ρ with $x^1(\cdot)$ on C_1 and $x^2(\frac{\cdot}{\rho})$ on C_2 by [Le], [ES], [Hm].

Definition For $x^i \in M^i, i = 1,2$ let $\mathcal{F}_\rho(x^1, x^2)$ be the unique solution of the following Dirichlet Problem:

$$
\begin{aligned}
\tau_h(\mathcal{F}_\rho(x^1, x^2)) &= 0 \text{ in } A_\rho \\
\mathcal{F}_\rho(x^1, x^2)(e^{i\theta}) &= x^1(e^{i\theta}) \text{ on } C_1 \\
\mathcal{F}_\rho(x^1, x^2)(\rho e^{i\theta}) &= x^2(e^{i\theta}) \text{ on } C_2 (= \partial B_\rho),
\end{aligned} \tag{1}
$$

and define $\mathcal{E} : \mathcal{M} \longrightarrow \mathbb{R}$ with

$$
x \longmapsto E(\mathcal{F}(x)) := \frac{1}{2} \int_{A_\rho} |d\mathcal{F}_\rho(x^1, x^2)|_h^2 d\omega.
$$

II) Now let (N, h) and $\Gamma_i, i = 1,2$ satisfy (C1) or (C2).

We will introduce a kind of tangent space of $x^i \in M^i$.

For a given oriented $y^i \in \mathcal{X}^i_{\text{mon}}$, there exists a weakly monotone map $w^i \in C^0(\mathbb{R}, \mathbb{R})$ with $w^i(\theta + 2\pi) = w^i(\theta) + 2\pi$ such that $y^i(\theta) = \gamma^i(\cos(w^i(\theta)), \sin(w^i(\theta))) =: \gamma^i \circ w^i(\theta)$. We note that $w^i = \tilde{w}^i + Id$ for some $\tilde{w}^i \in C^0(\partial B, \mathbb{R})$. Roughly speaking, w^i can be considered as a map in $C^0(\partial B, \partial B)$ and then w^i is unique for given y^i, whereas $w^i \in C^0(\mathbb{R}, \mathbb{R})$ is unique up to $2\pi l$, $l \in Z$. And whether w^i is in $C^0(\partial B, \partial B)$ or $C^0(\mathbb{R}, \mathbb{R})$, it will be determined according to a given situation, simply denoted by $y^i = \gamma^i \circ w^i$.

Denoting the Dirichlet -Integral by D and the $\mathbb{R}^k$-harmonic extension by $\mathcal{H}$, let

$$
W^i_{\mathbb{R}^k} := \{w^i \in C^0(\mathbb{R}, \mathbb{R}) \mid \text{weakly monotone}, w^i(\theta + 2\pi) = w^i(\theta) + 2\pi; D(\mathcal{H}(\gamma^i \circ w^i)) < \infty\}.
$$

Clearly, $W^i_{\mathbb{R}^k}$ is convex. For further details, we refer to [St1].

Definition For $x^i \in M^i$, considering $w - w^i$ as a tangent vector along $\tilde{w}^i$, let

$$
\mathcal{T}_{x^i} = \{d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w}^i) \mid w \in W^i_{\mathbb{R}^k} \text{ and } \gamma^i \circ w^i = x^i\}.
$$

And $\mathcal{T}_{x^i}$ is convex in $T_{x^i} H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$, since $W^i_{\mathbb{R}^k}$ is convex.

Let $\widetilde{\exp}$ denote the exponential map with resoect to the metric $\tilde{h}$. We note then the following.

Remark 2.2. *In case of (C1), $\widetilde{\exp}_{x^i}\xi \in M^i$ for $\xi \in \mathcal{T}_{x^i}, i = 1,2$.*
For the case (C2), there exists $l_i > 0$, depending on γ^i such that for any $x^i \in M^i$, $\widetilde{\exp}_{x^i}\xi \in M^i$, if $\|\xi\|_{\mathcal{T}_{x^i}} < l_i, i = 1,2$.

Proof For $(C1)$ it is clear. In the case of $(C2)$, for some small $\delta > 0$, there exists a retraction r from the δ-neighborhood of N in $\mathbb{R}^k$ onto N, since N is compact. Then, letting $\|x^i - x_0^i\|_{\frac{1}{2},2;0} < \delta$,

$$\int_{A_\rho} |d(r(f_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0)))|^2 d\omega$$
$$\leq C(\|f_\rho\|_{C^0}, \varepsilon, N)\Big(\int_{A_\rho} |df_\rho|^2 d\omega + \int_B |d\mathcal{H}(x^1 - x_0^1)|^2 d\omega\Big) \leq C(\|f_\rho\|_{1,2;0}, \delta, N).$$

We have then some $l_i > 0$ with the desired property, since $\widetilde{\exp}_{x^i}\xi = \gamma^i(w)$ for $\xi = d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w}^i) \in \mathcal{T}_{x^i}$. $\square$

Lemma 2.1. *$\mathcal{E}$ is continuously partially differentiable in x^1 resp. x^2 with respect to variations $\xi^1 \in \mathcal{T}_{x^1}$ resp. $\xi^2 \in \mathcal{T}_{x^2}$ with*

$$\langle \delta_{x^1}\mathcal{E}, \xi^1 \rangle = \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega.$$

A similar result is obtained for the second variation.
And the derivatives are continuous in $M^1 \times M^2$.

Proof See [Ki]. $\square$

3 $H^{2,2}$- Regularity of minimal surfaces

3.1 A result

Now we define for $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times (0,1)$,

$$g_i(x) := \sup_{\substack{\xi^i \in \mathcal{T}_{x^i} \\ \|\xi^i\| < l_i}} (-\langle \delta_{x^i}\mathcal{E}, \xi^i \rangle), \quad i = 1, 2. \tag{2}$$

We have then the following result.

Theorem 3.1. *Let $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times (0,1)$ with $g_i(x) = 0$, $i = 1, 2$. Then $\mathcal{F}_\rho(x^1, x^2)$ is in the class of $H^{2,2}(A_\rho, N)$.*

Remark 3.1. *In addition to the above conditions in Theorem 3.1 let us require that $g_3(x) := \rho \cdot \partial_\rho \mathcal{E} = 0$. Then, $x = (x^1, x^2, \rho)$ is defined as a critical point of $\mathcal{E}$ such that $\mathcal{F}_\rho(x^1, x^2)$ is a minimal surface of annulus type in N. For details we refer to [Ki].*

Lemma 3.1. *Let $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1,x^2) : A_\rho \to N \overset{\eta}{\hookrightarrow} \mathbb{R}^k$ and $\mathcal{F}_\rho \in H^{1,2}(A_\rho, \mathbb{R}^k)$. If $\int_{A_\rho} |\partial_\theta d\mathcal{F}_\rho|^2 d\omega \le C < \infty$, then $\mathcal{F}_\rho(x^1,x^2) \in H^{2,2}(A_\rho, N)$.*

Proof By the young's inequality: in polar coordinates with $\Delta\mathcal{F}_\rho{:=}\Delta_{\mathbb{R}^k}\mathcal{F}_\rho$,

$$\begin{aligned}
|\nabla^2\mathcal{F}_\rho|^2 &= |\partial_r d\mathcal{F}_\rho|^2 + \frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 \\
&= \left|\Delta\mathcal{F}_\rho - \frac{1}{r^2}\partial_{\theta\theta}\mathcal{F}_\rho - \frac{1}{r}\partial_r\mathcal{F}_\rho\right|^2 + \frac{1}{r^2}|\partial_{\theta r}\mathcal{F}_\rho|^2 + \frac{1}{r^4}|\partial_\theta\mathcal{F}_\rho|^2 - 2\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho\partial_\theta\mathcal{F}_\rho + \frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 \\
&\le C(\varepsilon)|\Delta\mathcal{F}_\rho|^2 + (1+\varepsilon)\left|\frac{1}{r^2}\partial_{\theta\theta}\mathcal{F}_\rho + \frac{1}{r}\partial_r\mathcal{F}_\rho\right|^2 + \frac{1}{r^2}|\partial_{\theta r}\mathcal{F}_\rho|^2 + \frac{1}{r^4}|\partial_\theta\mathcal{F}_\rho|^2 - 2\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho\partial_\theta\mathcal{F}_\rho \\
&\qquad -2\varepsilon\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho\partial_\theta\mathcal{F}_\rho + \varepsilon\frac{1}{r^2}|\partial_{\theta r}\mathcal{F}_\rho|^2 + C(\varepsilon)\frac{1}{r^4}|\partial_\theta\mathcal{F}_\rho|^2 + \frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 \\
&\le C(\varepsilon)|\Delta\mathcal{F}_\rho|^2 + (2+\varepsilon)\frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 + C(\varepsilon)\frac{1}{r^2}\frac{1}{r^2}|\partial_\theta\mathcal{F}_\rho|^2 \\
&\le C(\varepsilon,\eta,A_\rho)|d\mathcal{F}_\rho|^2 + C(\varepsilon,\rho)|\partial_\theta d\mathcal{F}_\rho|^2,
\end{aligned}$$

since $\mathcal{F}_\rho$ is harmonic in $N \overset{\eta}{\hookrightarrow} \mathbb{R}^k$, i.e. $\tau_h(f) = 0$. $\square$

3.2 Morrey growth condition

We introduce a Lemma from [Mo].

Lemma 3.2. *Let G be a bounded domain in $\mathbb{R}^2$. Suppose $\varphi \in H_0^{1,2}(G)$ and $\psi \in L^1(G)$ satisfies the Morrey growth condition*

$$\int_{B_r(z_0)} |\psi| d\omega \le C_0 r^\mu, \ \ \textit{for all } \ B_r(z_0).$$

Then $\psi\varphi^2 \in L^1(G)$ and for all $B_r(z_0)$ there holds

$$\int_{B_r(z_0)\cap G} |\psi\varphi^2| d\omega \le C_1 C_0 r^{\mu/2} \int_G |d\varphi|^2 d\omega$$

for some uniform constant C_1.

Let x^i be as in Theorem 3.1 with $x^i = \gamma^i \circ w^i$, and $w^i = \tilde{w}^i + Id$, $\tilde{w}^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R})$, $i = 1,2$ (recall the construction in section 2.2). Moreover for a given function f on $\mathbb{R}$, $f_+(\cdot)$ resp. $f_-(\cdot)$ denotes the function $f_+(\cdot + h)$ resp. $f_-(\cdot - h)$, for $h \in \mathbb{R}$.
For $x^i \in M^i$ let $\mathcal{H}_\rho(x^1,x^2)$ denote the unique $\mathbb{R}^k$-harmonic extension with boundary x^i on C_i, $i = 1,2$ and $\mathcal{H}(\cdot)$ the $\mathbb{R}^k$-harmonic extension of disc type.
Then we have the following growth condition.

Lemma 3.3. *for each $P_0 \in \partial A_\rho$ there exist $C_0, \mu, r_0 > 0$ such that for all $r \in [0, r_0]$ it holds*

$$(3) \qquad \int_{A_\rho \cap B_r(P_0)} (|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w^1}, 0)|^2) d\omega \le C_0 r^\mu \int_{A_\rho} (|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w^1}, 0)|^2) d\omega.$$

Remark 3.2. *We also get the same result as in Lemma 3.3 for $|d\mathcal{F}_{\rho+}|^2$ (resp. $|d\mathcal{F}_{\rho-}|^2$) and $|d\mathcal{H}_\rho(\tilde{w}^1_+, \tilde{w}^2_+)|^2$ (resp. $|d\mathcal{H}_\rho(\tilde{w}^1_-, \tilde{w}^2_-)|^2$).*

We observe the following as in [Ho].

Remark 3.3. *(i) Let $\mathcal{F}_\rho : A_\rho \to N$ be harmonic, we have then for $X \in H_0^{1,2}(A_\rho, \mathbb{R}^k)$,*

$$-\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), X\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, dX\rangle d\omega = 0.$$

(ii) This means, for $X \in H^{1,2}(A_\rho, \mathbb{R}^k)$ the above expression only depends on the boundary of X. Thus, for $\phi = (\phi^1, \phi^2) \in H^{\frac{1}{2},2} \times H^{\frac{1}{2},2}(\frac{\cdot}{\rho})$ we define

$$(4) \qquad \mathbf{A}(\mathcal{F}_\rho)(\phi) := -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), X\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, dX\rangle d\omega,$$

where X is any mapping in $H^{1,2}(A_\rho, \mathbb{R}^k)$ with $X|_{\partial A_\rho} = \phi$.

Specially for $\phi^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, (x^i)^* T\Gamma_i), i = 1, 2,$ we take $X := \mathbf{J}_{\mathcal{F}_\rho}(\phi^1, \phi^2)$ which is tangent to N along $\mathcal{F}_\rho$, then from the definition of the second fundamental form $\langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \mathbf{J}_\rho(\phi^1, \phi^2)\rangle \equiv 0$, so

$$(5) \qquad \begin{aligned} \mathbf{A}(\mathcal{F}_\rho)(\phi) &= \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(\phi^1, \phi^2)\rangle d\omega \\ &= \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(\phi^1, 0)\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(0, \phi^2)\rangle d\omega \\ &= \langle \partial_{x^1}\mathcal{E}, \phi^1\rangle + \langle \partial_{x^2}\mathcal{E}, \phi^2\rangle. \end{aligned}$$

Hence for a critical point $x = (x^1, x^2, \rho)$ of $\mathcal{E}$, $\mathbf{A}(\mathcal{F}_\rho)(\xi) \ge 0$, for all $\xi = (\xi^1, \xi^2) \in \mathcal{T}_{x^1} \times \mathcal{T}_{x^2}$.

Proof of Lemma 3.3 In several steps we will show (3).

I) Let $P_0 \in C_1$ fixed, $B_r := B_r(P_0)$, and

$$(6) \qquad \tilde{w}^1_0 := Q^{-1} \int_{(B_{2r} \setminus B_r) \cap \partial B} \tilde{w}^1 d_o, \;\; w^1_0 := \tilde{w}^1_0 + Id : \mathbb{R} \to \mathbb{R},$$

where $\int_{\partial B\cap(B_{2r}\setminus B_r)} d_o := Q$,

$$\tilde{\xi_\phi} := -\left[\phi(|e^{i\theta} - P_0|)\right]^2(\tilde{w^1} - \tilde{w_0^1})\frac{\partial}{\partial\theta}\circ \bar{w^1} \in H^{\frac{1}{2},2}\cap C^0(\partial B, \bar{w^1}^*T(\partial B)),$$

where $\bar{w^1}$ means the map from ∂B into itself, and $\phi \in C^\infty$ is a non-increasing function of $|z|$ satisfying the conditions $0 \le \phi(z) \le 1$, $\phi \equiv 1$ if $|z| \le 2r$, $\phi \equiv 0$ if $|z| \ge 3r$, $|d\phi| \le \frac{C}{r}$, $|d^2\phi| \le \frac{C}{r^2}$ for some C, fixed r.

Since $(1-\phi^2)w^1 + \phi^2 w_0^1 \in W^1_{\mathbb{R}^k}$, $d\gamma^1(\tilde{\xi_\phi}) \in \mathcal{T}_{x^1}$, hence

$$\mathbf{A}(\mathcal{F}_\rho)(-d\gamma^1(\tilde{\xi_\phi}), 0) \ge 0. \tag{7}$$

Letting $x_0^1 := \gamma^1(w_0^1)$

$$x^1 - x_0^1 \;=\; d\gamma^1(w^1 - w_0^1) - \int_{w_0^1}^{w^1}\int_{s'}^{w^1} d^2\gamma^1(s'')ds''ds' = d\gamma^1(w^1 - w_0^1) - \alpha(w^1),$$

and for small $r > 0$,

$$\begin{aligned}\mathbf{A}(\mathcal{F}_\rho)(\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{C_1}, 0) \;&=\; \mathbf{A}(\mathcal{F}_\rho)(\phi^2 d\gamma^1(w^1 - w_0^1), 0) - \mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0)\\ &\le\; -\mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0),\end{aligned}$$

where $\mathcal{F}_\rho^0(A_\rho) \equiv x_0^1 \in \Gamma_1$.

On the other hand, for small $r > 0$, $\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{C_2} \equiv 0$, so we can take $\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)$ in the definition of $\mathbf{A}(\mathcal{F}_\rho)$. Hence

$$\begin{aligned}&\mathbf{A}(\mathcal{F}_\rho)(\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{C_1}, 0)\\ &= \int_{A_\rho}\langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega + \int_{A_\rho}\langle 2\phi d\phi(\mathcal{F}_\rho - \mathcal{F}_\rho^0), d\mathcal{F}_\rho\rangle d\omega\\ &\qquad - \int_{A_\rho}\langle \phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0), II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)\rangle d\omega\\ &\le -\mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0),\end{aligned}$$

and

$$\begin{aligned}\int_{A_\rho}\langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega \le& \int_{A_\rho}\langle \phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0), II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)\rangle d\omega\\ &- \int_{A_\rho}\langle 2\phi d\phi(\mathcal{F}_\rho - \mathcal{F}_\rho^0), d\mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0).\end{aligned} \tag{8}$$

For the estimate of $-\mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0)$, consider

$$\widetilde{\star\star} := \phi^2\int_{w_0^1}^{T^1(w^1)}\int_{s'}^{T^1(w^1)} d^2\gamma^1(s'')ds''ds' \in H^{1,2}(A_\rho, \mathbb{R}^k)$$

with $\widetilde{\star\star}|_{C_1} = \phi^2\alpha(w^1), \widetilde{\star\star}|_{C_2} \equiv 0$, where $w_0^1(r,\theta) = \tilde{w_0^1} + Id(r,\theta) = \tilde{w_0^1} + \theta,\ \ (r,\theta) \in [\rho, 1] \times \mathbb{R}$.
By simple computation we get

$$\begin{aligned} |\widetilde{\star\star}| &\le C(\gamma^1, x^1)\phi^2|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2, \\ |d\widetilde{\star\star}| &\le C(\gamma^1, x^1)|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2\phi|d\phi| + C(\gamma^1, x^1)|d\mathcal{H}_\rho(\tilde{w^1},0)||\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2\phi^2, \end{aligned}$$

and from (8) by the young's inequality

$$\begin{aligned} \int_{A_\rho} \langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega \le & \int_{A_\rho} |d\mathcal{F}_\rho|^2|\mathcal{F}_\rho - \mathcal{F}_\rho^0|\phi^2 d\omega \\ & + \frac{\varepsilon}{5}\int_{A_\rho} |d\mathcal{F}_\rho|^2\phi^2 d\omega + C(\varepsilon)\int_{A_\rho} |\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2|d\phi|^2 d\omega \\ & + C\|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}\|_{L^\infty(B_{3r})}\int_{A_\rho} \big(|d\mathcal{F}_\rho|^2\phi^2 + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2|d\phi|^2\big)d\omega \\ & + C\|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}\|_{L^\infty(B_{3r})}\int_{A_\rho} \big(|d\mathcal{H}_\rho(\tilde{w^1},0)|^2 + |d\mathcal{F}_\rho|^2\big)\phi^2 d\omega \\ & + C\int_{A_\rho} |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2|d\mathcal{F}_\rho|^2\phi^2 d\omega. \end{aligned}$$

Thus, for $r \in (0, r_0)$, sufficiently small, dependent on ε, C, modulus of continuity of $\mathcal{F}_\rho - \mathcal{F}_\rho^0$ and $\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}$ we have the following estimate:

$$\begin{aligned} \int_{A_\rho} \langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega \le & \ \varepsilon\int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w^1},0)|^2\big)\phi^2 d\omega \\ & + C(\varepsilon)\int_{A_\rho} \big(|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2\big)|d\phi|^2 d\omega. \end{aligned} \tag{9}$$

II) We will estimate $\int_{A_\rho} |d\mathcal{H}_\rho(\tilde{w^1},0)|^2\phi^2 d\omega$.

- First,

$$\begin{aligned} D\big[(\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1})\phi\big] = \int_{A_\rho} & \big[|d\mathcal{H}_\rho(\tilde{w^1},0)|^2\phi^2 + |(\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1})|^2|d\phi|^2 \\ & + 2d\mathcal{H}_\rho(\tilde{w^1},0)(\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1})\phi d\phi\big]d\omega, \end{aligned}$$

and by the Young's inequality

$$\begin{aligned} \int_{A_\rho} |d\mathcal{H}_\rho(\tilde{w^1},0)|^2\phi^2 d\omega \le & \ D\big[(\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1})\phi\big] \\ & + \frac{\varepsilon}{4}\int_{A_\rho} |d\mathcal{H}_\rho(\tilde{w^1},0)|^2\phi^2 d\omega + C(\varepsilon)\int_{A_\rho} \big(|\mathcal{H}_\rho(\tilde{w^1},0)|^2 + |\tilde{w_0^1}|^2\big)|d\phi|^2 d\omega. \end{aligned} \tag{10}$$

• The estimate of $D\big[(\mathcal{H}_\rho(\tilde{w^1},0)-\tilde{w_0^1})\phi\big]$:

On C^1, $\mathcal{F}_\rho-\mathcal{F}_\rho^0=d\gamma^1(w^1-w_0^1)-\int_{w_0^1}^{w^1}\int_{s'}^{w^1}d^2\gamma^1(s'')ds''ds'$, and $\phi|_{\partial B_{3r}(P_0)}\equiv 0$, so on $\partial(A_\rho\cap B_{3r}(P_0))$,

$$
\begin{aligned}
(\mathcal{H}_\rho(\tilde{w^1},0)-\tilde{w_0^1})\phi=|d\gamma^1(T^1(w^1))|^{-2}\big[&d\gamma^1(T^1(w^1))\cdot(\mathcal{F}_\rho-\mathcal{F}_\rho^0)\\
&+d\gamma^1(T^1(w^1))\cdot\int_{w_0^1}^{T^1(w^1)}\int_{s'}^{T^1(w^1)}d^2\gamma^1(s'')ds'\big]\phi.
\end{aligned}
$$

We denote the latter map on A_ρ by Ψ.

And it holds,

$$
\Delta\big[(\mathcal{H}_\rho(\tilde{w^1},0)-\tilde{w_0^1})\phi\big]=2d\mathcal{H}_\rho(\tilde{w^1},0)\cdot d\phi+(\mathcal{H}_\rho(\tilde{w^1},0)-\tilde{w_0^1})\Delta\phi=:f. \tag{11}
$$

Note that: for a solution $\varphi\in C^2(\Omega,\mathbb{R})$ of $\Delta\varphi=f$, it holds, with a boundary data φ_0

$$
D\varphi\le D\psi-\int f(\varphi-\psi),\ \text{ for all }\ \psi\in\varphi_0+H_0^{1,2}(\Omega).
$$

Hence, by the variation characterization of the equation (11), we get

$$
D\big[(\mathcal{H}_\rho(\tilde{w^1},0)-\tilde{w_0^1})\phi\big]\le D(\Psi)-\int_{A_\rho\cap B_{3r}}f\big[(\mathcal{H}_\rho(\tilde{w^1},0)-\tilde{w_0^1})\phi-\Psi\big]d\omega. \tag{12}
$$

Letting

$$
\begin{aligned}
\Psi &:= \frac{d\gamma^1(T^1(w^1))\cdot(\mathcal{F}_\rho-\mathcal{F}_\rho^0)+d\gamma^1(T^1(w^1))\cdot\int_{w_0^1}^{T^1(w^1)}\int_{s'}^{T^1(w^1)}d^2\gamma^1(s'')ds'}{|d\gamma^1(T^1(w^1))|^2}\phi\\
&= \frac{\Theta}{|d\gamma^1(T^1(w^1))|^2}\phi,
\end{aligned}
$$

$$
d[d\gamma^1(T^1(w^1))\cdot(\mathcal{F}_\rho-\mathcal{F}_\rho^0)]=d^2\gamma^1(T^1(w^1))d(T^1(w^1))(\mathcal{F}_\rho-\mathcal{F}_\rho^0)+d\gamma^1(T^1(w^1))d\mathcal{F}_\rho=:a,
$$

$$
d\Big(\int_{w_0^1}^{T^1(w^1)}\int_{s'}^{T^1(w^1)}d^2\gamma^1(s'')ds'd\Big)=d^2\gamma^1(T^1(w^1))d\mathcal{H}_\rho(\tilde{w^1},0)(\mathcal{H}_\rho(\tilde{w^1},0)-\tilde{w_0^1})=:b,
$$

$$
d|d\gamma^1(T^1(w^1))|^{-2}=-2|d\gamma^1(T^1(w^1))|^{-4}\langle d^2\gamma^1(T^1(w^1)),d^1\gamma^1(T^1(w^1))\rangle d\mathcal{H}_\rho(\tilde{w^1},0)=:c,
$$

we have

$$
|d\Psi|^2=\frac{|a+b|^2\phi^2+\Theta^2\phi^2c^2+\Theta^2|d\phi|^2+(a+b)c\phi^2\Theta+(a+b)\phi\Theta d\phi+\Theta^2\phi cd\phi}{|d\gamma^1(T^1(w^1))|^2},
$$

and we compute further from the property of ϕ

$$
\begin{aligned}
&\int_{A_\rho} |d\Psi|^2 d\omega \\
&\leq \; C\int_{A_\rho} |d\mathcal{F}_\rho|^2\phi^2 d\omega + C\int_{A_\rho} \left[|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2\right]|d\phi|^2 d\omega \\
&\qquad\qquad + C\delta\int_{A_\rho} \left[|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2|d\phi|^2 + |d\mathcal{H}_\rho(\tilde{w^1},0)|^2\phi^2\right] d\omega,
\end{aligned}
$$

where $\delta = \big\| |\mathcal{F}_\rho - \mathcal{F}_\rho^0| + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}| \big\|_{L^\infty(A_\rho \cap B_{3r})}$.

We can also compute that

$$
\begin{aligned}
&-\int_{A_\rho\cap B_{3r}} f\left[(\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1})\phi - \Psi\right] d\omega \\
&\leq \int_{A_\rho\cap B_{3r}} \left[2|d\mathcal{H}_\rho(\tilde{w^1},0)d\phi|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}||d\phi| + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2|\Delta\phi|\phi\right. \\
&+ C|d\mathcal{H}_\rho(\tilde{w^1},0)|\phi|\mathcal{F}_\rho - \mathcal{F}_\rho^0||d\phi| + C|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}||\mathcal{F}_\rho - \mathcal{F}_\rho^0||\Delta\phi|\phi \\
&+ C\|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}\|(|d\mathcal{H}_\rho(\tilde{w^1},0)|\phi|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}||d\phi| + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2|\Delta\phi|\phi)\left.\right] d\omega \\
&\leq \int_{A_\rho\cap B_{3r}} \left[C(|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|)^2(|d\phi|^2 + |\Delta\phi|)\right. \\
&\qquad\qquad + \left.(\frac{\varepsilon}{2} + C\|\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}\|_{L^\infty(A_\rho\cap B_{3r})})|d\mathcal{H}_\rho(\tilde{w^1},0)|^2\phi^2\right] d\omega.
\end{aligned}
$$

Now the estimate of $D\left[(\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1})\phi\right]$ follows from (12).

• From (10) and the above estimates, we get

$$
\begin{aligned}
&\int_{A_\rho} |d\mathcal{H}_\rho(\tilde{w^1},0)|^2\phi^2 d\omega \leq C\int_{A_\rho} |d\mathcal{F}_\rho|^2\phi^2 d\omega \\
&\qquad + C(\varepsilon)\int_{A_\rho} (|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2)(|d\phi|^2 + |\Delta\phi|) d\omega \\
&\qquad + (\frac{3\varepsilon}{4} + C\big\| |\mathcal{F}_\rho - \mathcal{F}_\rho^0| + |\mathcal{H}_\rho(\tilde{w^1},0) - \tilde{w_0^1}| \big\|_{L^\infty(A_\rho\cap B_{3r})})\int_{A_\rho} |d\mathcal{H}_\rho(\tilde{w^1},0)|^2\phi^2 d\omega. \qquad (13)
\end{aligned}
$$

III) From (9), (13), for $r \leq r_0$, where r_0 is dependent on ε, $C(x^1, \rho)$ and the modulus of continuity of $\mathcal{F}_\rho - \mathcal{F}_\rho^0$ and $\mathcal{H}_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}$, we get from the definition of ϕ:

$$\int_{A_\rho \cap B_{3r}} (|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w^1}, 0)|^2) d\omega \leq Cr^{-2} \int_{A_\rho \cap B_{3r} \setminus B_{2r}} (|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}|^2) d\omega$$

$$\begin{aligned} &\leq Cr^{-2} \int_{A_\rho \cap B_{3r} \setminus B_r} (|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}|^2) d\omega \\ \text{(Poincaré inequality)} \quad &\leq C \int_{A_\rho \cap B_{3r} \setminus B_r} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w^1}, 0)|^2\big) d\omega \\ &+Cr^{-2}\Big(\int_{\partial B \cap B_{2r} \setminus B_r} (\mathcal{F}_\rho - \mathcal{F}_\rho^0) d_o\Big)^2 + Cr^{-2}\Big(\int_{\partial B \cap B_{2r} \setminus B_r} \big(\mathcal{H}_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}\big) d_o\Big)^2, \end{aligned}$$

where the last term is 0 from the definition of $\tilde{w_0^1}$.

On ∂B, we have

$$\mathcal{F}_\rho - \mathcal{F}_\rho^0 = d\gamma^1(w_0^1)(\tilde{w^1} - \tilde{w^1}_0) + \int_{\partial B \cap B_{2r} \setminus B_r} \int_{w_0^1}^{w^1} \int_{w_0^1}^{s'} d^2\gamma^1(s'') ds'' ds',$$

so, from the estimate in integration and by the second inequality in Lemma 3.4,

$$\begin{aligned} &\int_{\partial B \cap B_{2r} \setminus B_r} (\mathcal{F}_\rho - \mathcal{F}_\rho^0) d_o \\ &= \int_{\partial B \cap B_{2r} \setminus B_r} d\gamma^1(w_0^1)(\tilde{w^1} - \tilde{w^1}_0) d_o + \int_{\partial B \cap B_{2r} \setminus B_r} \int_{w_0^1}^{w^1} \int_{w_0^1}^{s'} d^2\gamma^1(s'') ds'' ds' \\ &\leq C \int_{\partial B \cap (B_{2r} \setminus B_r)} |w^1 - w_0^1|^2 d_o \\ &\leq Cr \int_{B \cap (B_{2r} \setminus B_r)} |d\mathcal{H}_\rho(\tilde{w^1}, o)|^2 d\omega + \frac{C}{r}\Big(\int_{\partial B \cap B_{2r} \setminus B_r} (\tilde{w^1} - \tilde{w^1}_0) d_o\Big)^2. \end{aligned}$$

Here the last term is again zero fromt he definition of $\tilde{w_0^1}$.

Thus,

$$\begin{aligned} &Cr^{-2}\Big(\int_{\partial B \cap B_{2r} \setminus B_r} (\mathcal{F}_\rho - \mathcal{F}_\rho^0) d_o\Big)^2 \\ &\leq C\Big(\int_{B \cap (B_{2r} \setminus B_r)} |d\mathcal{H}_\rho(\tilde{w^1}, 0)|^2 d\omega\Big)^2 \leq C(x^1, \rho) \int_{B \cap (B_{2r} \setminus B_r)} |d\mathcal{H}_\rho(\tilde{w^1}, 0)|^2 d\omega, \end{aligned}$$

hence

$$\int_{A_\rho \cap B_r} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w^1}, 0)|^2\big) d\omega \leq C \int_{A_\rho \cap B_{3r} \setminus B_r} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w^1}, 0)|^2\big) d\omega.$$

Letting $\Upsilon(r) := \int_{A_\rho \cap B_r(P_0)} \big(|d\mathcal{F}_\rho|^2 + |\mathcal{H}_\rho(\tilde{w^1}, 0)|^2\big) d\omega$, the above inequality means that

$$\Upsilon(r) \le C\big(\Upsilon(3r) - \Upsilon(r)\big),$$

where C is independent of $r \le r_0$, for some small r_0.

Then the inequality (3) follows from the Iteration-lemma. $\square$

3.3 The proof of the main theorem

We will give here the proof of Theorem 3.1. We begin with Poincaré inequality as follows (see [St1] Lemma 5.5):

Lemma 3.4. *Let $z_0 \in \partial A_\rho$, $B_r := B_r(z_0)$, $G_r := A_\rho \cap (B_{3r} \backslash B_r)$, $K_r := A_\rho \cap (B_{2r} \backslash B_r)$ and $S_r := \partial A_\rho \cap B_{2r} \backslash B_r$. Then, for some small $r_0 > 0$, there exists a uniform constant C independent of z_0 such that for all $r \le r_0$ and for each $\varphi \in H^{1,2}(G_r)$:*

$$\begin{aligned} \int_{G_r} |\varphi|^2 d\omega &\le Cr^2 \int_{G_r} |d\varphi|^2 d\omega + C\left(\int_{S_r} \varphi \, d_o\right)^2, \quad \textit{and} \\ \int_{S_r} |\varphi|^2 d_o &\le Cr \int_{K_r} |d\varphi|^2 d\omega + \frac{C}{r}\left(\int_{S_r} \varphi \, d_o\right)^2, \end{aligned}$$

where d_o is the one-dimensional area element.

Proof. Let z_0, r fixed. Suppose by contradiction that for a sequence $\varphi_m \in H^{1,2}(G_r)$

$$1 \equiv \int_{G_r} |\varphi_m|^2 d\omega \ge mr^2 \int_{G_r} |d\varphi_m|^2 d\omega + m\left(\int_{S_r} \varphi_m \, d_o\right)^2.$$

Then $\{\varphi_m\}$ in bounded in $H^{1,2}(G_r)$ and some subsequence, denoted again by $\{\varphi_m\}$, converges weakly to some φ in $H^{1,2}(G_r)$ but strongly in $L^2(G_r)$ by Rellich-Kondrakov. From the above assumption, $d\varphi_m \to 0$ strongly.
Thus, $\{\varphi_m\}$ converges strongly to some constant C in $H^{1,2}(G_r)$ and $\varphi_m \to C$ in $L^2(S_r)$. On the other hand, $\int_{S_r} \varphi_m \, d_o \to 0$, so $\varphi \equiv 0$ in G_r, contradicting the assumption, since $\varphi_m \to \varphi$ in L^2.

The second inequality can be proved similarly, supposing by contradiction that

$$1 \equiv \int_{S_r} |\varphi_m|^2 d_o \ge mr \int_{K_r} |d\varphi_m|^2 d\omega + \frac{m}{r}\left(\int_{S_r} \varphi_m \, d_o\right)^2$$

and applying the above result for $\int_{K_r} |\varphi_m|^2 d\omega$.

By scaling, one can see that C is independenf of z_0,r. $\square$

Proof of Theorem 3.1

From Lemma 3.1 and by a well known result in [GT] it suffices to show that

$$\int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega \leq C < \infty, \tag{14}$$

where $\Delta_h d\mathfrak{F}_\rho := \frac{d\mathfrak{F}_\rho(r,\theta+h)-d\mathfrak{F}_\rho(r,\theta)}{h}$, $h \neq 0$ and C is independent of h.

We show (14) in several steps. The same notations as in the preceding sections will be used.

(I) With $\Delta_{-h}\Delta_h \mathfrak{F}_\rho|_{\partial B} = \Delta_{-h}\Delta_h\gamma^1 \circ e^{iw^1}$ and $\Delta_{-h}\Delta_h\mathfrak{F}_\rho|_{\partial B_\rho}(\cdot\rho) = \Delta_{-h}\Delta_h\gamma^2 \circ e^{iw^2(\cdot)}$,

$$\begin{aligned}\int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega &= -\int_{A_\rho} \langle d\mathfrak{F}_\rho, d\Delta_{-h}\Delta_h\mathfrak{F}_\rho\rangle d\omega \\ &= -\int_{A_\rho} \langle II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho), \Delta_{-h}\Delta_h\mathfrak{F}_\rho\rangle d\omega - \mathbf{A}(\mathfrak{F}_\rho)(\Delta_{-h}\Delta_h\mathfrak{F}_\rho|_{\partial A_\rho}).\end{aligned}$$

Denoting $\gamma^1 \circ e^{iw^1}$ and $\gamma^2 \circ e^{iw^2}$ by $\gamma^i(w^i(\theta))$ and $w^i(\cdot + h)$ resp. $w^i(\cdot - h)$ by w^i_+ resp. w^i_-, we have:

$$\begin{aligned}\Delta_{-h}\Delta_h\gamma^i(w^i) &= \Delta_{-h}\left[\frac{\gamma^i(w^i_+) - \gamma^i(w^i_-)}{h}\right] \\ &= \Delta_{-h}\left[d\gamma^i(w^i)\left(\frac{w^i_+ - w^i_-}{h}\right) + \frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\right] \\ &= d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) - \frac{1}{h}\int_{w^i}^{w^i_-} d^2\gamma^i(s')ds' \cdot \Delta_h w^i_- + \Delta_{-h}\left(\frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\right) \\ &= d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) + P^i.\end{aligned}$$

Since γ^i is smooth, clearly $d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) \in H^{\frac{1}{2},2} \cap C^0(\partial B, (x^i)^*T\Gamma_i)$.

Writing $w^i = \tilde{w}^i + Id$ for some $\tilde{w}^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R})$ and define a real valued map of $(r,\theta) \in [\rho, 1] \times \mathbb{R}$ as follows: for $i = 1$

$$T^1(w^1)(r,\theta) := H_\rho(\tilde{w}, 0)(r,\theta) + Id(r,\theta) \text{ with } Id(r,\theta) = \theta,$$

where $H_\rho(\tilde{w}, 0)$ is the harmonic extension to $A_\rho \approx [\rho, 1] \times \mathbb{R}/2\pi$ with $\tilde{w}$ on ∂B and 0 on ∂B_ρ. Then it holds that

$$T^1(w^1)(r, \theta + 2\pi) = T^1(w^1)(r,\theta) + 2\pi, \text{ for } (r,\theta) \in [\rho, 1] \times \mathbb{R},$$

and $e^{iT^1(w^1)}$ can be considered as a map from ∂B into itself.

Now define a map $S(P^1,0)(\cdot) : A_\rho \to \mathbb{R}^k$ with the boundary P^1 (resp. 0) on C_1 (resp. C_2) as follows:

$$
\begin{aligned}
S(P^1,0)(\cdot) \; &:= \; -\frac{1}{h}\int_{T^1(w^1)(\cdot)}^{T^1(w^1_-)(\cdot)} d^2\gamma^1(s')ds' \cdot H_\rho(\Delta_h w^1_-,0)(\cdot) \\
&\qquad +\Delta_{-h}\left(\frac{1}{h}\int_{T^1(w^1)(\cdot)}^{T^1(w^1_+)(\cdot)}\int_{T^1(w^1)(\cdot)}^{s'} d^2\gamma^1(s'')ds''ds'\right).
\end{aligned}
$$

Similarly, a map $S(0,P^2)(\cdot) : A_\rho \to \mathbb{R}^k$ with the boundary 0 (resp. P^2) on C_1 (resp. C_2):

$$
\begin{aligned}
S(0,P^2)(\cdot) \; &:= \; -\frac{1}{h}\int_{T^2(w^2)(\cdot)}^{T^2(w^2_-)(\cdot)} d^2\gamma^2(s')ds' \cdot H_\rho(0,\Delta_h w^2_-)(\cdot) \\
&\qquad +\Delta_{-h}\left(\frac{1}{h}\int_{T^2(w^2)(\cdot)}^{T^2(w^2_+)(\cdot)}\int_{T^2(w^2)(\cdot)}^{s'} d^2\gamma^2(s'')ds''ds'\right),
\end{aligned}
$$

where $T^2(w^2_-)(\cdot) = H_\rho(0,\tilde{w})(\cdot) + Id(\cdot)$, and $S(0,P^2)|_{C_1} \equiv 0, S(0,P^2)|_{C_2}(\cdot\rho) = P^2(\cdot)$.

Clearly $S(P^1,0), S(0,P^2) \in H^{1,2}(A_\rho,\mathbb{R}^k)$, so letting $S(P^1,P^2) := S(P^1,0) + S(0,P^2)$, we have a map in $H^{1,2}(A_\rho,\mathbb{R}^k)$ with boundary (P^1,P^2).

By computation, $\frac{h^2}{2}\Delta_{-h}\Delta_h w^i = \frac{1}{2}(w^i_- + w^i_+) - w^i$. And $\frac{1}{2}(w^i_- + w^i_+) \in W^i_{\mathbb{R}^k}$ which is convex. Thus, by the definition of $\mathcal{T}_{x^i}$,

$$
\frac{h^2}{2} d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) \in \mathcal{T}_{x^i}.
$$

And $\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i)$ is in $H^{\frac{1}{2},2}$ for which $A(\mathcal{F}_\rho)$ is well defined, recall Remark 3.3.

From (4) and Remark 3.3, since $g^1(x) = g^2(x) = 0$,

$$
\frac{h^2}{2} A(\mathcal{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1),0\right) = A(\mathcal{F}_\rho)\left(\frac{h^2}{2}d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1),0\right) \geq 0,
$$

so $A(\mathcal{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1),0\right) \geq 0$.

Similarly, for the second variation, $A(\mathcal{F}_\rho)\left(0, d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)(\frac{\cdot}{\rho})\right) \geq 0$.

From now on we will omit the scaling term $(\frac{\cdot}{\rho})$ for the second variation.

Moreover, from the definition of $A(\mathcal{F}_\rho)$, clearly

$$
\mathbf{A}(\mathcal{F}_\rho)(\phi^1+\xi^1,\phi^2+\xi^2) = \mathbf{A}(\mathcal{F}_\rho)(\phi^1,\phi^2) + \mathbf{A}(\mathcal{F}_\rho)(\xi^1,\xi^2),
$$

if there exist $H^{1,2}$ extension of (ϕ^1, ϕ^2) and (ξ^1, ξ^2).

Hence we have that

$$\begin{aligned}
&\mathbf{A}(\mathfrak{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^i), d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)\right) \\
(15) \qquad &= \mathbf{A}(\mathfrak{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1), 0\right) + \mathbf{A}(\mathfrak{F}_\rho)\left(0, d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)\right) \geq 0.
\end{aligned}$$

Now we can compute:

$$\begin{aligned}
\int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega &= -\int_{A_\rho} \langle II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho), \Delta_{-h}\Delta_h \mathfrak{F}_\rho\rangle d\omega - \mathbf{A}(\mathfrak{F}_\rho)(\Delta_{-h}\Delta_h \mathfrak{F}_\rho|_{\partial A_\rho}) \\
&= -\int_{A_\rho} \langle II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho), \Delta_{-h}\Delta_h \mathfrak{F}_\rho\rangle d\omega \\
&\qquad -\mathbf{A}(\mathfrak{F}_\rho)(P^1, P^2) - \mathbf{A}(\mathfrak{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1), d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)\right) \\
&\leq -\int_{A_\rho} \langle II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho), \Delta_{-h}\Delta_h \mathfrak{F}_\rho\rangle d\omega - \mathbf{A}(\mathfrak{F}_\rho)(P^1, P^2) \\
(16) \qquad &= -\int_{A_\rho} \langle II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho), \Delta_{-h}\Delta_h \mathfrak{F}_\rho\rangle d\omega \\
(17) \qquad &\quad +\int_{A_\rho} \langle II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho), S(P^1, 0)\rangle d\omega + \int_{A_\rho} \langle II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho), S(0, P^2)\rangle d\omega \\
(18) \qquad &\quad -\int_{A_\rho} \langle d\mathfrak{F}_\rho, dS(P^1, 0)\rangle d\omega - \int_{A_\rho} \langle d\mathfrak{F}_\rho, dS(0, P^2)\rangle d\omega.
\end{aligned}$$

(II) For the estimstes of the above terms we need some preparation.

First, letting $s(\tau) := \tau\mathfrak{F}_{\rho,+} + (1-\tau)\mathfrak{F}_\rho$, $0 \leq \tau \leq 1$,

$$\begin{aligned}
&|\Delta_h II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho)| = |\frac{1}{h}\{II \circ \mathfrak{F}_{\rho,+}(\mathfrak{F}_{\rho,+}, \mathfrak{F}_{\rho,+}) - II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho)\}| \\
&= |\frac{1}{h}\{II \circ \mathfrak{F}_{\rho,+}(d\mathfrak{F}_{\rho,+}, d\mathfrak{F}_{\rho,+}) - II \circ \mathfrak{F}_\rho(d\mathfrak{F}_{\rho,+}, d\mathfrak{F}_{\rho,+}) \\
&\qquad + II \circ \mathfrak{F}_\rho(d\mathfrak{F}_{\rho,+}, d\mathfrak{F}_{\rho,+}) - II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho)\}| \\
&= |\frac{1}{h}\{(dII(\mathfrak{F}_\rho) \cdot (\mathfrak{F}_{\rho,+} - \mathfrak{F}_\rho) + \int_0^1 \int_0^t d^2 II(s(\tau))|\mathfrak{F}_{\rho,+} - \mathfrak{F}_\rho|^2 d\tau dt)(d\mathfrak{F}_{\rho,+}, d\mathfrak{F}_{\rho,+}) \\
&\qquad + II \circ \mathfrak{F}_\rho(d\mathfrak{F}_{\rho,+} - d\mathfrak{F}_\rho, d\mathfrak{F}_{\rho,+}) + II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_{\rho,+} - d\mathfrak{F}_\rho)\}| \\
&= |dII(\mathfrak{F}_\rho) \cdot \Delta_h \mathfrak{F}_\rho(d\mathfrak{F}_{\rho,+}, d\mathfrak{F}_{\rho,+}) + \frac{1}{h}\int_0^1 \int_0^t d^2 II(s(\tau))|\mathfrak{F}_{\rho,+} - \mathfrak{F}_\rho|^2 d\tau dt(d\mathfrak{F}_{\rho,+}, d\mathfrak{F}_{\rho,+}) \\
&\qquad + II \circ \mathfrak{F}_\rho(\Delta_h d\mathfrak{F}_\rho, d\mathfrak{F}_{\rho,+}) + II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, \Delta_h d\mathfrak{F}_\rho)| \\
&\leq C(\|\mathfrak{F}_\rho\|_{C^0(A_\rho)})[|\Delta_h \mathfrak{F}_\rho||d\mathfrak{F}_{\rho,+}|^2 + |\Delta_h d\mathfrak{F}_\rho|(|d\mathfrak{F}_{\rho,+}| + |d\mathfrak{F}_\rho|)].
\end{aligned}$$

Now letting

$$-\frac{1}{h}\int_{T^1(w^1)}^{T^1(w^1_-)} d^2\gamma^1(s')ds' := \star \quad \text{and} \quad \frac{1}{h}\int_{T^1(w^1)}^{T^1(w^1_-)}\int_{T^1(w^1)}^{s'} d^2\gamma^1(s'')ds''ds' := \star\star,$$

we have

$$|\star| \leq C(\gamma^1)|H_\rho(\Delta_{-h}w^1,0)|, \quad |\star\star| \leq C(\gamma^1)|H_\rho(\Delta_h w^1,0)|,$$

and

$$\begin{aligned}
|d\star| &= \Big| -\frac{1}{h}\big[d^2\gamma^1(T^1(w^1_-))dT^1(w^1_-) - d^2\gamma^1(T^1(w^1))dT^1(w^1)\big]\Big| \\
&= \Big| -\frac{1}{h}\big[\frac{d^2\gamma^1(T^1(w^1_-)) - d^2\gamma^1(T^1(w^1))}{T^1(w^1_-)-T^1(w^1)}\big(T^1(w^1_-)-T^1(w^1)\big)dT^1(w^1_-) \\
&\qquad + d^2\gamma^1(T^1(w^1))\big(dT^1(w^1_-)-dT^1(w^1)\big)\big]\Big| \\
&\leq C(\|\gamma^1\|_{C^3})\big(|H_\rho(\Delta_{-h}w^1,0)||dH_\rho(w^1_-,0)| + |dH_\rho(\Delta_{-h}w^1,0)|\big),
\end{aligned}$$

$$\begin{aligned}
|d\star\star| &= \Big|d\big[\frac{1}{h}\big(\int_{T^1(w^1)}^{T^1(w^1_+)} d\gamma^1(s')ds' - \int_{T^1(w^1)}^{T^1(w^1_+)} d\gamma^1(T^1(w^1))ds'\big)\big]\Big| \\
&= \Big|\frac{1}{h}\big[\frac{d\gamma^1(T^1(w^1_+)) - d\gamma^1(T^1(w^1))}{T^1(w^1_+)-T^1(w^1)}\big(T^1(w^1_+)-T^1(w^1)\big)dT^1(w^1_+) \\
&\qquad - d^2\gamma^1(T^1(w^1))dT^1(w^1)\big(dT^1(w^1_+)-dT^1(w^1)\big)\big]\Big| \\
&\leq C(\|\gamma^1\|_{C^2})|H_\rho(\Delta_h w^1,0)|\big(|dH_\rho(\tilde{w^1}_+,0)| + |dH_\rho(\tilde{w^1},0)|\big).
\end{aligned}$$

Using the above results, we estimate (16), (17),(18) for some $C \in \mathbb{R}$, independent of h. First,

$$\begin{aligned}
(16) &\leq \int_{A_\rho} \big|\langle \Delta_h II \circ \mathfrak{F}_\rho(d\mathfrak{F}_\rho, d\mathfrak{F}_\rho), \Delta_h\mathfrak{F}_\rho\rangle\big| d\omega \\
&\leq C\int_{A_\rho} \big(|\Delta_h\mathfrak{F}_\rho|^2|d\mathfrak{F}_{\rho,+}|^2 + |\Delta_h d\mathfrak{F}_\rho|(|d\mathfrak{F}_{\rho,+}| + |d\mathfrak{F}_\rho|)|\Delta_h\mathfrak{F}_\rho|\big)d\omega \\
&\leq C\int_{A_\rho} |d\mathfrak{F}_{\rho,+}|^2|\Delta_h\mathfrak{F}_\rho|^2 d\omega + \varepsilon\int_{A_\rho}|\Delta_h d\mathfrak{F}_\rho|^2 d\omega + C(\varepsilon)\int_{A_\rho}(|d\mathfrak{F}_{\rho,+}|^2 + |d\mathfrak{F}_\rho|^2)|\Delta_h\mathfrak{F}_\rho|^2 d\omega.
\end{aligned}$$

For the estimate of (17),

$$
\begin{aligned}
&\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), S(P^1,0)\rangle d\omega \\
\le\ & \int_{A_\rho} \{|\langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), (\star) H_\rho(\Delta_h w^1_-, 0)\rangle| + |\langle \Delta_h II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), (\star\star)\rangle|\} d\omega \\
\le\ & C\int_{A_\rho} |d\mathcal{F}_\rho|^2 |\mathcal{H}_\rho(\Delta_{-h} w^1, 0)|^2 d\omega \\
& + C\int_{A_\rho} \{|\Delta_h \mathcal{F}_\rho||d\mathcal{F}_{\rho,+}|^2 |\mathcal{H}_\rho(\Delta_h w^1, 0)| + |\Delta_h d\mathcal{F}_\rho|(|d\mathcal{F}_{\rho,+}| + |d\mathcal{F}_\rho|)|\mathcal{H}_\rho(\Delta_h w^1, 0)|\} d\omega \\
\le\ & C\int_{A_\rho} |d\mathcal{F}_\rho|^2 |\mathcal{H}_\rho(\Delta_{-h} w^1, 0)|^2 d\omega + C\int_{A_\rho} |d\mathcal{F}_{\rho,+}|^2 (|\Delta_h \mathcal{F}_\rho|^2 + |\mathcal{H}_\rho(\Delta_h w^1, 0)|^2) d\omega \\
& + \varepsilon \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + C(\varepsilon)\int_{A_\rho} (|d\mathcal{F}_{\rho,+}|^2 |\mathcal{H}_\rho(\Delta_h w^1, 0)|^2 + |d\mathcal{F}_\rho|^2 |\mathcal{H}_\rho(\Delta_h w^1, 0)|^2) d\omega,
\end{aligned}
$$

note that $\Delta_h w^1_- = \Delta_{-h} w^1$, and we obtain a similar estimate for the second term of (17).
Thus, we have that

$$
\begin{aligned}
(17) \ \le\ & \varepsilon C \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + C(\varepsilon)\int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_{\rho,+}|^2\big) \cdot \\
& \big(|\Delta_h \mathcal{F}_\rho|^2 + |\mathcal{H}_\rho(\Delta_{-h} w^1, 0)|^2 + |\mathcal{H}_\rho(0, \Delta_{-h} w^2)|^2 + \mathcal{H}_\rho(\Delta_h w^1, 0)|^2 + |\mathcal{H}_\rho(0, \Delta_h w^2)|^2\big) d\omega.
\end{aligned}
$$

For the estimate of (18),

$$
\begin{aligned}
-\int_{A_\rho} \langle d\mathcal{F}_\rho, dS(P^1, 0)\rangle d\omega \le\ & \int_{A_\rho} \big|\langle d\mathcal{F}_\rho, d(\star)\mathcal{H}_\rho(\Delta_{-h} w^1, 0)\rangle\big| d\omega \\
& + \int_{A_\rho} \big|\langle d\mathcal{F}_\rho, (\star) d\mathcal{H}_\rho(\Delta_{-h} w^1, 0)\rangle\big| d\omega + \int_{A_\rho} \big|\langle \Delta_h d\mathcal{F}_\rho, d(\star\star)\rangle\big| d\omega \\
\le\ & \varepsilon C \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1, 0)|^2 d\omega \\
& + C(\varepsilon)\int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w}^1_-, 0)|^2 + |d\mathcal{H}_\rho(\tilde{w}^1_+, 0)|^2 + |d\mathcal{H}_\rho(\tilde{w}^1, 0)|^2\big) \cdot \\
& \qquad \big(|\mathcal{H}_\rho(\Delta_{-h} w^1, 0)|^2 + |\mathcal{H}_\rho(\Delta_h w^1, 0)|^2\big) d\omega.
\end{aligned}
$$

We get a similar estimate for the second term of (18):

$$
\begin{aligned}
(18) \quad \le\ & \varepsilon C \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1, 0)|^2 d\omega \\
& + C(\varepsilon) \int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\tilde{w}^1_-, 0)|^2 + |d\mathcal{H}_\rho(\tilde{w}^1_+, 0)|^2 + |d\mathcal{H}_\rho(\tilde{w}^1, 0)|^2 \\
& \qquad + |d\mathcal{H}_\rho(0, \tilde{w}^2_-)|^2 + |d\mathcal{H}_\rho(0, \tilde{w}^2_+)|^2 + |d\mathcal{H}_\rho(0, \tilde{w}^2)|^2 \big) \cdot \\
& \quad \big(|\mathcal{H}_\rho(\Delta_{-h} w^1, 0)|^2 + |\mathcal{H}_\rho(\Delta_h w^1, 0)|^2 + |\mathcal{H}_\rho(0, \Delta_{-h} w^2)|^2 + |\mathcal{H}_\rho(0, \Delta_h w^2)|^2 \big) d\omega.
\end{aligned}
$$

Now gathering all the above results:

$$
\begin{aligned}
(19) \quad & \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega \\
\le\ & \varepsilon C \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1, 0)|^2 d\omega \\
& + C(\varepsilon) \int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_\rho|^2 \\
& \qquad + |d\mathcal{H}_\rho(\tilde{w}^1_-, 0)|^2 + |d\mathcal{H}_\rho(\tilde{w}^1_+, 0)|^2 + |d\mathcal{H}_\rho(\tilde{w}^1, 0)|^2 \\
& \qquad + |d\mathcal{H}_\rho(0, \tilde{w}^2_-)|^2 + |d\mathcal{H}_\rho(0, \tilde{w}^2_+)|^2 + |d\mathcal{H}_\rho(0, \tilde{w}^2)|^2 \big) \cdot \\
& \quad \big(|\Delta_h \mathcal{F}_\rho|^2 + |\mathcal{H}_\rho(\Delta_{-h} w^1, 0)|^2 + |\mathcal{H}_\rho(\Delta_h w^1, 0)|^2 \\
& \qquad + |\mathcal{H}_\rho(0, \Delta_{-h} w^2)|^2 + |\mathcal{H}_\rho(0, \Delta_h w^2)|^2 \big) d\omega \\
=\ & \varepsilon C \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1, 0)|^2 d\omega + C(\varepsilon)\Xi\,.
\end{aligned}
$$

(III) On ∂B, it holds that $\Delta_h(\gamma^i \circ w^i) = d\gamma^i(w^i)\Delta_h w^i + \frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'$, so

$$
(20) \qquad \Delta_h w^i = |d\gamma^i(w^i)|^{-2}\big[d\gamma^i(w^i) \cdot \Delta_h \mathcal{F}_\rho - d\gamma^i(w^i) \cdot \frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\big].
$$

Using $T^i(w^i)$ at the right side of (20), we get a $H^{1,2}(A_\rho, \mathbb{R}^k)$- extension with boundary $\Delta_h w^i$ on C^1 and 0 on C_2, and by the D-minimality of the harmonic extension between the maps with the same boundary, we have

$$
\begin{aligned}
&\int_{A\rho} |d\mathcal{H}_\rho(\Delta_h w^1, 0)|^2 d\omega \\
&\quad \le \; C\int_{A_\rho} \big[|d\mathcal{H}_\rho(w^1,0)|\big(|\Delta_h\mathcal{F}_\rho| + |\star\star|\big) + |d\Delta_h\mathcal{F}_\rho| + |d\star\star|\big]^2 d\omega \\
&\quad \le \; C\int_{A_\rho} \big\{|d\mathcal{H}_\rho(w^1,0)|^2|\Delta_h\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(\Delta_h w^1,0)|^2|\mathcal{H}_\rho(\Delta_h w^1,0)|^2 + |d\Delta_h\mathcal{F}_\rho|^2 \\
&\qquad\qquad +|\mathcal{H}_\rho(\Delta_h w^1,0)|^2(|d\mathcal{H}_\rho(\tilde{w^1}_+,0)| + |d\mathcal{H}_\rho(\tilde{w^1},0)|)^2 \\
&\qquad\qquad +|d\mathcal{H}_\rho(\tilde{w^1},0)|^2|\Delta_h w^1,0)| + |d\mathcal{H}_\rho(\tilde{w^1},0)||\Delta_h\mathcal{F}_\rho||d\Delta_h\mathcal{F}_\rho| \\
&\qquad\qquad +|d\mathcal{H}_\rho(\tilde{w^1},0)||\mathcal{H}_\rho(\Delta_h w^1,0)|(|d\mathcal{H}_\rho(\tilde{w^1}_+,0)| + |d\mathcal{H}_\rho(\tilde{w^1},0)|)|\Delta_h\mathcal{F}_\rho| \\
&\qquad\qquad +|d\mathcal{H}_\rho(\tilde{w^1},0)||\mathcal{H}_\rho(\Delta_h w^1,0)||d\Delta_h\mathcal{F}_\rho| \\
&\qquad\qquad +|d\mathcal{H}_\rho(\tilde{w^1},0)||\mathcal{H}_\rho(\Delta_h w^1,0)||\mathcal{H}_\rho(\Delta_h w^1,0)|(|d\mathcal{H}_\rho(\tilde{w^1}_+,0)| + |d\mathcal{H}_\rho(\tilde{w^1},0)|) \\
&\qquad\qquad +|d\Delta_h\mathcal{F}_\rho||\mathcal{H}_\rho(\Delta_h w^1,0)|(|d\mathcal{H}_\rho(\tilde{w^1}_+,0)| + |d\mathcal{H}_\rho(\tilde{w^1},0)|)\big\}d\omega \\
(21)&\quad \le \; C\int_{A_\rho} |d\Delta_h\mathcal{F}_\rho|^2 d\omega + C\Xi
\end{aligned}
$$

by the young's inequality, and Ξ is from (20). Similarly, we get an estimate

$$
\int_{A_\rho} |d\mathcal{H}_\rho(0,\Delta_h w^2)|^2 d\omega \le C\int_{A_\rho} |d\Delta_h\mathcal{F}_\rho|^2 d\omega + C\Xi. \tag{22}
$$

Using the estimate (20) for $\int_{A_\rho} |d\Delta_h\mathcal{F}_\rho|^2 d\omega$ and from (21), (22),

$$
\begin{aligned}
&\int_{A_\rho} |d\Delta_h\mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1,0)|^2 d\omega + \int_{A_\rho} |d\mathcal{H}_\rho(0,\Delta_h w^2)|^2 d\omega \\
&\quad \le \; \varepsilon C\int_{A_\rho} |d\Delta_h\mathcal{F}_\rho|^2 d\omega + \varepsilon C\int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1,0)|^2 d\omega + \varepsilon C\int_{A_\rho} |d\mathcal{H}_\rho(0,\Delta_h w^2)|^2 d\omega \\
&\qquad + C(\varepsilon)\Xi\,.
\end{aligned}
$$

Since $\frac{1}{2}(a^2+b^2) \le (a+b)^2 \le \frac{3}{2}(a^2+b^2), a,b\in\mathbb{R}$ and $H_\rho(f,g) = H_\rho(f,0) + H_\rho(0,g)$, for some small $\varepsilon > 0$ in the above estimate we get finally the following inequality:

$$
\begin{aligned}
&\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1, \Delta_h w^2)|^2 d\omega \\
&\quad \le \; C(\varepsilon)\int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_{\rho+}|^2 + |d\mathcal{F}_{\rho-}|^2 \\
&\qquad\qquad +|d\mathcal{H}_\rho(\tilde{w^1},\tilde{w^2}|)^2 + |d\mathcal{H}_\rho(\tilde{w^1_+},\tilde{w^2_+})|^2 + |d\mathcal{H}_\rho(\tilde{w^1_-},\tilde{w^2_-}|^2\big)\cdot \\
(23)&\qquad\qquad \big(|\Delta_h\mathcal{F}_\rho|^2 + |H(\Delta_{-h}w^1,\Delta_{-h}w^2)|^2 + |H(\Delta_h w^1,\Delta_h w^2)|^2\big)d\omega.
\end{aligned}
$$

(IV) Now extend $\mathcal{F}_\rho$ to $\mathbb{R}^2\backslash B_{\rho^2}$ by conformal reflection as follows

$$\begin{aligned}\mathcal{F}_\rho(z) &= \mathcal{F}_\rho\big(\frac{z}{|z|^2}\big), \text{ if } 1 \leq |z| \\ \mathcal{F}_\rho(z) &= \mathcal{F}_\rho\big(\frac{z}{|z|^2}\rho^2\big), \text{ if } \rho^2 \leq |z| \leq \rho.\end{aligned}$$

Choose $r \in \big(0, \min\{\frac{\rho-\rho^2}{2}, r_0\}\big)$, and $\varphi \in C_0^\infty\big(B_{2r}(0)\big)$ with $\varphi \equiv 1$ on $B_r(0)$.
We may cover A_ρ with balls of radius r in such a way that at most k balls of the covering intersect at any point $p \in A_\rho$, for any r as above ($\mathbb{R}^2$ is metrizable). Let B^i denote the balls of the covering with centers p_i and $\varphi_i(p) := \varphi(p - p_i)$.
Then from (23),

$$\begin{aligned}&\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1, \Delta_h w^2)|^2 d\omega \\ &\leq \; C\,\Sigma_i \int_{\mathbb{R}^2\backslash A_{\rho^2}} \big(|\Delta_h\mathcal{F}_\rho|^2 + |H(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big)\varphi_i^2 \cdot \\ &\quad \underbrace{\big(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_{\rho+}|^2 + |d\mathcal{F}_{\rho-}|^2 + |d\mathcal{H}_\rho(\tilde{w^1}, \tilde{w^2})|^2 + |d\mathcal{H}_\rho(\tilde{w_+^1}, \tilde{w_+^2})|^2 + |d\mathcal{H}_\rho(\tilde{w_-^1}, \tilde{w_-^2})|^2\big)}_{=:\chi}\, d\omega.\end{aligned}$$

From Lemma 3.3 and Remark 3.2, χ satisfies the Morrey growth condition, so apply the Morrey Lemma with χ and $(\Delta_h\mathcal{F}_\rho)\varphi_i$ resp. $H(\Delta_{-h}w^1, \Delta_{-h}w^2)\varphi_i$ resp. $H(\Delta_h w^1, \Delta_h w^2)\varphi_i$.
Then we obtain

$$\begin{aligned}&\int_{B_{2r}(p_i)} \chi\big(|\Delta_h\mathcal{F}_\rho|^2 + |H(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big)\varphi_i^2 d\omega \\ &\quad \leq Cr^{\frac{\mu}{2}} \int_{B_2\backslash B_{\rho^2}} \chi d\omega \int_{B_{2r}(P_i)} \big(|d\Delta_h\mathcal{F}_\rho|^2 + |dH(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |dH(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega \\ &\quad +Cr^{\frac{\mu}{2}} \int_{B_2\backslash B_{\rho^2}} \chi d\omega \int_{B_{2r}(P_i)} \big(|\Delta_h\mathcal{F}_\rho|^2 + |H(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega.\end{aligned}$$

Summing over i we get a constant C, independent of r such that

$$\begin{aligned}&\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |d\mathcal{H}_\rho(\Delta_h w^1, \Delta_h w^2)|^2 d\omega \\ &\quad \leq Cr^{\frac{\mu}{2}} \int_{B_2\backslash B_{\rho^2}} \big(|d\Delta_h\mathcal{F}_\rho|^2 + |dH(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |dH(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega \\ &\quad +Cr^{\frac{\mu}{2}} \int_{B_2\backslash B_{\rho^2}} \big(|\Delta_h\mathcal{F}_\rho|^2 + |H(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega.\end{aligned}$$

Since $d\mathfrak{F}_\rho$, $dH(w^1, w^2) \in L^2$, choosing small $r > 0$, we obtain $C \in \mathbb{R}$, independent of $|h| \leq h_0$ with

$$\int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega \leq C.$$

□

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