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1 Introduction and main results

Let H be a complex Hilbert space. Given an arbitrary set $\mathcal{S} \subset L(H)$ of bounded linear operators on H, let $\mathcal{W}_{\mathcal{S}} \subset L(H)$ be the smallest WOT-closed subalgebra of $L(H)$ containing $\mathcal{S} \cup \{1_H\}$, and let $\mathrm{Lat}(\mathcal{S})$ denote the lattice of all closed linear subspaces of H that are invariant under each operator $S \in \mathcal{S}$. We write $\mathrm{AlgLat}(\mathcal{S}) = \{C \in L(H) : \mathrm{Lat}(C) \supset \mathrm{Lat}(\mathcal{S})\}$ for the set of all operators leaving invariant each $\mathcal{S}$-invariant subspace. Note that $\mathrm{AlgLat}(\mathcal{S})$ is a subalgebra of $L(H)$ which is closed in the weak operator topology and always contains $\mathcal{W}_{\mathcal{S}}$.

A subset $\mathcal{S} \subset L(H)$ is called reflexive if $\mathrm{AlgLat}(\mathcal{S}) = \mathcal{W}_{\mathcal{S}}$. The concept of reflexivity was introduced by Sarason [29] who showed in 1966 that each normal operator $N \in L(H)$ and each analytic Toeplitz operator on the Hardy space $H^2(\mathbb{D})$ over the unit disc is reflexive. By a result of Deddens [7] from 1971, each isometry on a complex Hilbert space is reflexive. In 1979 it was shown by Wogen [32] that all quasinormal operators are reflexive. All these reflexivity results are special cases of the following theorem obtained by Olin and Thomson in 1980.

1.1 Theorem. (Olin–Thomson) *Every subnormal operator is reflexive.*

Though many concrete subnormal multiplication tuples on Bergman- or Hardy-type function spaces are known to be reflexive (see Bercovici [3], Eschmeier [9], Ptak [25], McCarthy [23]), a multi-operator analogue of the Olin-Thomson result seems to be out of reach at the moment. Recall that a multi-operator $T \in L(H)^n$ is called subnormal if there exist a larger Hilbert space $K \supset H$ and a commuting tuple $N \in L(K)^n$ of normal operators such that $T = N|H$.

Olin and Thomson's proof of the single-operator result is based on Sarason's decomposition theorem for compactly supported measures on $\mathbb{C}$ and corresponding decomposition theorems for subnormal operators which, together with the Riemann mapping theorem, allow one to reduce the reflexivity problem for a general subnormal operator $T \in L(H)$ to the special case where T is in addition of the class $\mathbb{A}$, that is, possesses an isometric and weak* continuous $H^\infty(\mathbb{D})$-functional calculus. Since Sarason's decomposition theorem and the Riemann mapping theorem are known to fail in higher dimensions, it seems to be impossible to obtain a general reflexivity result for subnormal tuples along this way. The most general reflexivity results for subnormal systems known so far apply to subnormal tuples related to multi-variable analogues of the class $\mathbb{A}$.

To be more precise, let $X \subset \mathbb{C}^n$ be a Stein submanifold and let $D \subset X$ be a relatively compact open subset. We say that a subnormal n-tuple $T \in L(H)^n$ is of class $\mathbb{A}$ over D if T possesses an isometric and weak* continuous $H^\infty(D)$-functional calculus. The following are generalizations of the classical case $D = \mathbb{D}$ to the multi-variable case (see [10] and [11]).

1.2 Theorem. (Eschmeier) *Every subnormal n-tuple T of class $\mathbb{A}$ over the unit ball $\mathbb{B}_n$ and every completely non-unitary subnormal n-tuple of class $\mathbb{A}$ over the unit polydisc $\mathbb{D}^n$ is reflexive.*

As a consequence one obtains reflexivity results based on richness conditions of the Taylor spectrum.

1.3 Corollary. (Eschmeier) *Every subnormal n-tuple T with dominating Taylor spectrum in $\mathbb{B}_n$ and every n-tuple of completely non-unitary subnormal operators with dominating Taylor spectrum in $\mathbb{D}^n$ is reflexive.*

In [8] the first-named author extended the methods of the ball case to strictly pseudoconvex open subsets $D \subset X$ of Stein submanifolds X in $\mathbb{C}^n$.

It is the aim of this paper to present a unified approach to the reflexivity problem for subnormal tuples of class $\mathbb{A}$ which contains the various sets D from above and at the same time applies to bounded symmetric domains and suitable products $D = D_1 \times D_2$. Let $X \subset \mathbb{C}^n$ be a, not necessarily closed, complex submanifold and let $D \subset X$ be a relatively compact open suset. Our approach is based on the following four rather general conditions (F1) to (F4) concerning the function theory on the set D.

(F1) The closure $\bar{D}$ of D is a Stein compactum in $\mathbb{C}^n$.

Note that in the above setting the closure of D in $\mathbb{C}^n$ coincides with the closure of D relative to X. We denote by λ_X the canonical volume measure of X as a submanifold of $\mathbb{C}^n$. Let λ be the trivial extension of the measure $\lambda_X|D$ to $\bar{D}$. Recall that the set $H^\infty(D)$ of all bounded analytic functions on D is a weak* closed subalgebra of $L^\infty(\lambda)$ and hence carries a natural weak* topology turning it into a dual algebra with separable predual (see e.g. Sections 2.1 and 2.2 in [8]). Our second condition ensures that the algebra $\mathcal{O}(\bar{D})$ of all functions analytic on some open neighbourhood of $\bar{D}$ in $\mathbb{C}^n$ is large enough.

(F2) The space of holomorphic germs $\mathcal{O}(\bar{D})$ is weak* dense in $H^\infty(D)$.

By demanding that D can be suitably embedded into a Euclidean ball of sufficiently large dimension we can achieve that Aleksandrov's work on the abstract inner function problem can be applied to the function algebra

$$A_0(D) = \overline{\mathcal{O}(\bar{D})}^{\|\cdot\|_{\infty,\bar{D}}} \subset C(\bar{D})$$

and its Shilov boundary $\partial_0 D = \partial_{A_0(D)} \subset \bar{D}$.

(F3) There exists a natural number N and an injective mapping $f \in A_0(D)^N$ satisfying $f(\partial_0 D) \subset \partial \mathbb{B}_N$.

Since the equality $\|f\|_{\infty,\bar{D}} = \|f\|_{\infty,\partial_0 D}$ also holds for the vector-valued function $f \in A_0(D)^N$, it follows that $f(\bar{D}) \subset \bar{\mathbb{B}}_N$ in the setting of condition (F3).

The final condition ensures that functions in $H^\infty(D)$ can be identified with their boundary values with respect to a suitable Henkin measure supported by $\partial_0 D$. Let us denote by $M^+(\bar{D})$ the set of all positive regular Borel measures on $\bar{D}$. In our context, a measure $\mu \in M^+(\bar{D})$ is called a Henkin measure if there exists a contractive and weak* continuous algebra homomorphism

$$r_\mu : H^\infty(D) \to L^\infty(\mu)$$

extending the canonical map $\mathcal{O}(\bar{D}) \to L^\infty(\mu)$, $f \mapsto [f|\bar{D}]$. According to property (F2) such an extension is unique. If the induced map r_μ is even isometric, then we say that μ is a faithful Henkin measure. In this case r_μ induces a dual algebra isomorphism onto its range.

(F4) There is a faithful Henkin probability measure σ supported by $\partial_0 D$.

Natural candidates for σ are the normalized surface measure on the topological boundary ∂D when D is smoothly bounded or, in the case of symmetric domains, the invariant measure on the Shilov boundary of D.

From now on we assume that D satisfies the above conditions (F1) – (F4), and we fix a separable complex Hilbert space H. A commuting n-tuple $T \in L(H)^n$ will be called absolutely continuous (over D) if T possesses a contractive weak* continuous functional calculus $\Phi_T : H^\infty(D) \to L(H)$. As will become clear later, the conditions (F1) and (F2) guarantee the uniqueness of Φ_T. Via Φ_T any pair of vectors $x, y \in H$ induces a weak* continuous linear form

$$x \otimes y : H^\infty(D) \to \mathbb{C}, \quad f \mapsto \langle \Phi_T(f)x, y \rangle$$

on $H^\infty(D)$, which can be regarded as an element in the predual $Q(D) = L^1(\lambda)/{}^\perp H^\infty(D)$ of the dual algebra $H^\infty(D)$.

An absolutely continuous commuting n-tuple $T \in L(H)^n$ over D is said to possess the factorization property $(\mathbb{A}_1)^+$ if there exists a constant $R > 0$ such that, for any given vectors $a, b \in H$ and any given functional $L \in Q(D)$, there are vectors $x, y \in H$ satisfying $L = x \otimes y$ as well as

$$\|x - a\| \leq R\|L - a \otimes b\|^{\frac{1}{2}}, \quad \|y\| \leq R(\|L - a \otimes b\|^{\frac{1}{2}} + \|b\|).$$

If, for every $\varepsilon > 0$, every $a \in H$ and every sequence $(L_k)_{k \geq 1}$ in $Q(D)$, there are vectors $x \in H$, $y_k \in H$ $(k \geq 1)$ and constants $C, d > 0$ such that $L_k = x \otimes y_k \quad (k \geq 1)$ and

$$\|x - a\| < \varepsilon, \quad \|y_k\| \leq Ck^d\|L_k\| \quad (k \geq 1),$$

then T is said to satisfy the factorization property $(\mathbb{A}_{1,\aleph_0})^+$.

By definition a commuting tuple $T \in L(H)^n$ is of class $\mathbb{A}$ (over D) if it is absolutely continuous and its weak* continuous $H^\infty(D)$-functional calculus Φ_T is isometric.

Using the above terminology, our main result can be formulated as follows.

1.4 Theorem. *Let D be a relatively compact open subset of a complex submanifold $X \subset \mathbb{C}^n$ satisfying conditions (F1) to (F4). Then each subnormal n-tuple $T \in L(H)^n$ of class $\mathbb{A}$ over D possesses the factorization properties $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$. In particular, the weak* and the weak operator topology coincide on the dual algebra $\mathcal{H}_T = \Phi_T(H^\infty(D))$, and $\mathcal{H}_T$ is super-reflexive, that is, each unital weak* closed subalgebra of $\mathcal{H}_T$ is reflexive.*

This solves the reflexivity problem for subnormal n-tuples of class $\mathbb{A}$ over the following different types of underlying sets.

1.5 Corollary. *Suppose that D satisfies one of the following conditions:*

(a) $D \subset X$ is a relatively compact, strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$;

(b) $D \subset \mathbb{C}^n$ is a circular bounded symmetric domain;

(c) $D = D_1 \times \cdots \times D_k$ where each of the factors D_i $(i = 1, \ldots, k)$ is a set as described in (a) or (b), not necessarily all of the same type.

Then every subnormal n-tuple of class $\mathbb{A}$ over D is reflexive.

In Section 2 we recall and slightly modify Aleksandrov's construction of abstract inner functions for regular triples. In Section 3 we apply the results on Aleksandrov regular triples to solve measure theoretic factorization problems of type $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$. The spectral theorem for normal tuples is used in Section 4 to reduce property $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$ for subnormal tuples of class $\mathbb{A}$ to the measure theoretic versions of these properties obtained before. In this way we complete the proof of Theorem 1.4. The final section is devoted to the proof of Corollary 1.5.

2 Abstract inner functions

Let K be a compact Hausdorff space. We denote by $M^+(K)$ the set of all positive regular Borel measures on K. Let $A \subset C(K)$ be a norm closed subspace and let $\mu \in M^+(K)$ be a given measure on K. For any strictly positive function $\varphi \in C(K)$, $\varphi > 0$, we define

$$A_\varphi = \{f \in A : |f| < \varphi\}.$$

Aleksandrov's approach to the inner function problem is based on the notion of a regular triple (A, K, μ) which can be defined as follows (see [1], Proposition 9).

2.1 Proposition. *Let A, K, μ be as above. Then the following conditions on the triple (A, K, μ) are equivalent:*

(a) there exists a real number $\tau > 0$ such that

$$\sup_{f \in A_\varphi} \int_K |f|^2 d\mu \geq \tau \int_K \varphi^2 d\mu \qquad (\varphi \in C(K), \varphi > 0);$$

(b) for each strictly positive function $\varphi \in C(K)$, $\varphi > 0$, there exists a sequence (f_n) in A_φ such that $\lim_{n\to\infty} |f_n| = \varphi$ μ-almost everywhere.

□

If one of these two conditions is satisfied, then we call the triple (A, K, μ) *regular* (in the sense of Aleksandrov).

Observe that part (b) implies that in part (a) one can choose $\tau = 1$. In this case equality holds instead of the claimed inequality.

Using a result of Ryll-Wojtaszczyk on the existence of certain weak* zero sequences of homogeneous polynomials with additional properties, Aleksandrov proved the regularity of the triple $(A(\mathbb{B}_n), \bar{\mathbb{B}}_n, \sigma)$, where σ denotes the surface measure on $\partial\mathbb{B}_n$ (Proposition 2 in [1]). Making use of the special nature of σ as the invariant measure induced by the unitary group acting on $\partial\mathbb{B}_n$, he was even able to replace the measure σ in the latter regularity result by an arbitrary measure $\mu \in M^+(\bar{\mathbb{B}}_n)$ carried by the boundary $\partial\mathbb{B}_n$ (cf. Proposition 17 in [1]).

2.2 Theorem. (Aleksandrov) *For each measure $\mu \in M^+(\bar{\mathbb{B}}_n)$ satisfying supp(μ) $\subset \partial\mathbb{B}_n$, the triple $(A(\mathbb{B}_n), \bar{\mathbb{B}}_n, \mu)$ is regular.* □

Most of the regularity results known so far are deduced from this by applying the following elementary embedding technique. To formulate the result we denote, for any Borel measurable function $f : X \to Y$ between topological spaces X and Y and any Borel measure μ on X, by μ^f the Borel measure on Y defined by $\mu^f(A) = \mu\big(\overset{-1}{f}(A)\big)$.

2.3 Lemma. *Let K_1, K_2 be compact Hausdorff spaces, $A_1 \subset C(K_1)$, $A_2 \subset C(K_2)$ closed subspaces and $\mu \in M^+(K_2)$ a positive measure. Suppose that there exists a topological embedding $F : K_2 \hookrightarrow K_1$ satisfying $A_1 \circ F \subset A_2$. Then the regularity of the triple (A_1, K_1, μ^F) implies that of (A_2, K_2, μ).*

Proof. By hypothesis, the map $\tilde{F} : K_2 \to F(K_2), x \mapsto F(x)$, is a homeomorphism.

Given $\varphi_2 \in C(K_2)$, $\varphi_2 > 0$, the mapping $\varphi_2 \circ \tilde{F}^{-1} : F(K_2) \to (0, \infty)$ has a continuous and strictly positive extension $\varphi_1 \in C(K_1)$. The regularity of (A_1, K_1, μ^F) implies the existence of a sequence $(f_n)_{n\geq 1}$ in $(A_1)_{\varphi_1}$ and a μ^F-zero set $N \subset K_1$ such that $|f_n| \xrightarrow{n} \varphi_1$ pointwise on $K_1 \setminus N$. By construction we have $\varphi_1 \circ F = \varphi_2$. This implies that $f_n \circ F \in (A_2)_{\varphi_2}$ for all $n \geq 1$ and that $|f_n \circ F| \xrightarrow{n} \varphi_2$ pointwise on the complement of the μ-zero set $\overset{-1}{F}(N)$. □

To apply the preceding lemma in the situation described in the introduction we need one more elementary observation. Let $D \subset X$ be a relatively compact open subset of a complex submanifold $X \subset \mathbb{C}^n$

2.4 Lemma. *Let $U \subset \mathbb{C}^N$ be a bounded open subset. Then, for each mapping $f \in A_0(D)^N$ with $f(\bar{D}) \subset \bar{U}$, the composition operator*

$$A_0(\bar{U}) \to A_0(D), \quad g \mapsto g \circ f$$

is a well-defined contraction.

Proof. Obviously, the map $C_f : C(\bar{U}) \to C(\bar{D})$, $g \mapsto g \circ f$, is a contractive linear operator. It suffices to show that $C_f\mathcal{O}(\bar{U}) \subset A_0(D)$. Towards this end, fix a bounded open set $V \supset \bar{U}$ and a function $g \in \mathcal{O}(\bar{V})$. For given

$\varepsilon > 0$, we can choose a $\delta > 0$ such that $|g(z) - g(w)| < \varepsilon$ for all $z, w \in \bar{V}$ with $|z - w| < \delta$. It suffices to choose a mapping $F \in \mathcal{O}(\bar{D}, \mathbb{C}^N)$ such that

$$|F(z) - f(z)| < \delta \qquad (z \in \bar{D})$$

and $F(\bar{D}) \subset V$. Then $g \circ F \in \mathcal{O}(\bar{D})$ and $\|g \circ F - g \circ f\|_{\infty, \bar{D}} < \varepsilon$. □

Now it is obvious that condition (F3) implies regularity results in our setting.

2.5 Proposition. *Suppose that D satisfies condition (F3). Then, for each measure $\mu \in M^+(\bar{D})$ with $\mathrm{supp}(\mu) \subset \partial_0 D$, the triple $(A_0(D), \bar{D}, \mu)$ is regular.*

Proof. We apply Lemma 2.3 with $A_1 = A(\mathbb{B}_N)$, $K_1 = \bar{\mathbb{B}}_N$, $A_2 = A_0(D)$, $K_2 = \bar{D}$. The role of the embedding F is played by the mapping $f : \bar{D} \to \mathbb{C}^N$ given by property (F3). For a measure $\mu \in M^+(\bar{D})$ with $\mathrm{supp}(\mu) \subset \partial_0 D$, we have $\mathrm{supp}(\mu^f) = f(\mathrm{supp}(\mu)) \subset \partial \mathbb{B}_N$, implying the regularity of the triple (A_1, K_1, μ^f). To finish the proof note that, according to Lemma 2.4, the inclusion $A(\mathbb{B}_N) \circ f \subset A_0(D)$ holds. □

One of the main reasons why Aleksandrov's concept of a regular triple posseses such a high flexibility seems to be the fact that regularity is inherited by finite co-dimensional subspaces – at least if the underlying measure has no atoms (see Proposition 13 in [1]). In particular the space A may be replaced by a finite intersection of kernels of continuous linear functionals on A without loss of regularity.

2.6 Proposition. (Aleksandrov) *If (A, K, μ) is a regular triple, $A_0 \subset A$ is a norm closed and finite co-dimensional subspace and μ has no atoms, then the triple (A_0, K, μ) is also regular.* □

The following result is a slight improvement of Aleksandrov's solution of the abstract inner function problem. Although the proof differs only by minor changes from the original one (see Section 1 of [1]), we present all details, hoping to demonstrate the strength of the regularity concept as well as its applicability to the problem of prescribing abstract boundary values with additional constraints. To formulate the result we introduce the notation

$$H^\infty_A(\mu) = \overline{A}^{w^*} \subset L^\infty(\mu)$$

for any subspace $A \subset C(K)$ and any measure $\mu \in M^+(K)$. For $\mu \in M^+(K)$ and an arbitrary Borel set $S \subset K$, let $\mu_S \in M^+(K)$ be the measure defined by $\mu_S(A) = \mu(A \cap S)$.

2.7 Theorem. *Let K be a compact Hausdorff space, $S \subset K$ an arbitrary Borel set and $\mu \in M^+(K)$ a positive regular Borel measure.*

Suppose that μ_S has no atoms and that the triple (A, K, μ_S) is regular. Then, for each $\varphi \in C(K)$ with $\varphi > 0$ and each $f \in C(K)$ with $|f| < \varphi$, there exists an element $g \in H_A^\infty(\mu)$ satisfying $|f+g| = \varphi$ μ_S-almost everywhere, $|f+g| \leq \varphi$ μ-almost everywhere on K and

$$\int_K g d\mu = 0.$$

Proof. Replacing A by the kernel of the functional $A \to \mathbb{C}$, $g \mapsto \int_K g d\mu$, we may assume that the condition $\int_K g d\mu = 0$ is satisfied for all $g \in H_A^\infty(\mu)$. Again using the fact that regularity is inherited by closed and finite co-dimensional subspaces, we can inductively choose a sequence $(f_n)_{n\geq 1}$ of functions in A such that

(a) $f_n \in \bigcap_{k=1}^{n-1} \ker\left(\langle \cdot, f_k\rangle_{L^2(\mu)} : A \to \mathbb{C}\right) \subset A$,

(b) $|f_n| < \varphi - |f + \sum_{k=1}^{n-1} f_k|$ pointwise on K,

(c) $\int_K |f_n|^2 d\mu_S \geq \frac{1}{2}\int_K (\varphi - |f + \sum_{k=1}^{n-1} f_k|)^2 d\mu_S$

for each $n \geq 1$. Condition (a) guarantees that the chosen sequence $(f_n)_{n\geq 1}$ consists of pairwise orthogonal functions in $L^2(\mu)$. We consider the associated partial sums

$$F_n = \sum_{k=1}^{n} f_k \in A \qquad (n \geq 1).$$

Using (b) we immediately obtain the estimate $|f+F_n| < \varphi$ and consequently $|F_n| < \varphi + |f|$ on K $(n \geq 1)$. Thus the sequence $(F_n)_{n\geq 1}$ is bounded in $C(K)$ and hence also in $L^2(\mu)$. This implies that the orthogonal series $\sum_{k=1}^\infty f_k$ converges in $L^2(\mu)$, forcing $\|f_k\|_{2,\mu} \to 0$ as $k \to \infty$. Together with condition (c) it follows that

$$\int_K (\varphi - |f + F_n|)^2 d\mu_S \leq 2\|f_{n+1}\|_{2,\mu}^2 \xrightarrow{n} 0.$$

Now we define $g = \sum_{k=1}^\infty f_k = \lim_{n\to\infty} F_n \in L^2(\mu)$ and choose a subsequence $(g_n)_{n\geq 1}$ of $(F_n)_{\geq 1}$ in such a way that $\varphi - |f+g_n| \xrightarrow{n} 0$ μ_S-almost everywhere and $g_n \xrightarrow{n} g$ μ-almost everywhere.

The corresponding limit $g \in L^2(\mu)$ clearly satisfies $|f+g| = \varphi$ μ_S-almost everywhere. Since $g_n \to g$ μ-almost everywhere on K and since the estimate $|f+g_n| < \varphi$ holds on K $(n \geq 1)$, we deduce that $|f+g| \leq \varphi$ μ-almost everywhere on K.

As an immediate consequence we obtain that $g \in L^\infty(\mu)$. To prove that $g \in H_A^\infty(\mu)$ it therefore suffices to observe that, for each $\psi \in L^1(\mu)$, we have $\int_K \psi g_n d\mu \xrightarrow{n} \int_K \psi g d\mu$ by Lebesgue's dominated convergence theorem. □

As a consequence of the above theorem we obtain the existence of a weak* zero sequence of inner functions satisfying some additional convergence property.

2.8 Corollary. *Let K be a compact metric space and $S \subset K$ a closed subset. Fix $\mu \in M^+(K)$ such that the measure $\mu_S \in M^+(K)$ has no atoms and gives rise to a regular triple (A, K, μ_S). Then there exists a weak* zero sequence (g_k) in $H_A^\infty(\mu)$ such that $|g_k| = 1$ holds μ_S-almost everywhere, $\|g_k\|_{\infty,\mu} \leq 1$ and $g_k \xrightarrow{k} 0$ μ-almost everywhere on $K \setminus S$.*

Proof. The function $\varphi \in C(K)$ defined by

$$\varphi(z) = \max\{\frac{1}{2}, 1 - \operatorname{dist}(z, S)\} \qquad (z \in K)$$

satisfies $\frac{1}{2} \leq \varphi \leq 1$ and $\overset{-1}{\varphi}(\{1\}) = S$.

Let $(w_n)_{n\geq 1}$ be a countable dense subset of $L^1(\mu)$. For each $n \in \mathbb{N}$, the space

$$A_n = \bigcap_{k=1}^{n} \ker(\langle \cdot, w_k \rangle_{L^\infty(\mu)-L^1(\mu)} : A \to \mathbb{C}) \subset A$$

is norm closed and finite co-dimensional, implying that, for every $n \geq 1$, the triple (A_n, K, μ_S) is regular. The last theorem allows us to choose a sequence of functions $g_n \in H_{A_n}^\infty(\mu) \subset H_A^\infty(\mu)$ with $|g_n| = \varphi^n$ μ_S–almost everywhere and $|g_n| \leq \varphi^n$ μ–almost everywhere.

Since the identity $\varphi^n = 1$ holds on S whereas $\varphi < 1$ on $K \setminus S$, the functions g_n satisfy all the required conditions. Note that by construction we have

$$\langle g_n, w_k \rangle_{L^\infty(\mu)-L^1(\mu)} \quad \xrightarrow{n\to\infty} 0$$

for each fixed $k \geq 1$. Since $(g_n)_{n\geq 1}$ is bounded in $H_A^\infty(\mu)$, it follows that $(g_n)_{n\geq 1}$ is a weak* zero sequence in $H_A^\infty(\mu)$. □

Besides the above abstract inner function results, the following Lusin-type theorem for regular triples (cf. [1], Theorem 37) will be useful for us.

2.9 Theorem. (Aleksandrov) *Let (A, K, μ) be a regular triple. Then, for each $\varphi \in C(K)$, $\varphi > 0$, and each $\varepsilon > 0$, there exists a function $g \in A$ satisfying $|g| \leq \varphi$ on K and $\mu(\{|g| \neq \varphi\}) < \varepsilon$.* □

3 Factorizations based on inner functions

Let $\mu \in M_1^+(K)$ be a regular Borel probability measure on a compact Hausdorff space K and let $A \subset L^\infty(\mu)$ be an arbitrary subalgebra. In this situation we define $H^p_A(\mu) = \overline{A}^{\|\cdot\|_{p,\mu}} \subset L^p(\mu) \quad (1 \leq p < \infty)$ and as before $H^\infty_A(\mu) = \overline{A}^{w^*} \subset L^\infty(\mu)$.

Since μ is a finite measure, we obtain, for $1 \leq p < r < \infty$, the inclusions

$$H^\infty_A(\mu) \subset H^r_A(\mu) \subset H^p_A(\mu).$$

Note that the space of all weak* continuous linear forms on $H^\infty_A(\mu)$ can be identified isometrically with the quotient space

$$Q_A(\mu) = L^1(\mu)/{}^\perp H^\infty_A(\mu).$$

In what follows we shall use this identification without further comment. For any choice of vectors $x, y \in L^2(\mu)$, the formula

$$x \otimes y : H^\infty_A(\mu) \to \mathbb{C}, \quad f \mapsto \int_K f x \bar{y} d\mu = \langle fx, y \rangle$$

defines an element $x \otimes y \in Q_A(\mu)$ satisfying $\|x \otimes y\| \leq \|xy\|_{1,\mu} \leq \|x\|\|y\|$. On the right-hand side of this inequality we used the abbreviation $\|\cdot\|$ for $\|\cdot\|_{2,\mu}$ as we shall do in the rest of this section.

3.1 Definition. *The space $H^2_A(\mu)$ is said to have:*

(a) *property $(\mathbb{A}_1)^+$ if there exists a constant $R > 0$ such that, for any given functional $L \in Q_A(\mu)$ and any given vectors $a, b \in H^2_A(\mu)$, there are vectors $x, y \in H^2_A(\mu)$ with $L = x \otimes y$ and*

$$\|x - a\| \leq R\|L - a \otimes b\|^{\frac{1}{2}}, \qquad \|y\| \leq R(\|L - a \otimes b\|^{\frac{1}{2}} + \|b\|);$$

(b) *property $(\mathbb{A}_{1,\aleph_0})^+$ if, for any given $\varepsilon > 0$, any given sequence $(L_k)_{k\geq 1}$ in $Q_A(\mu)$ and any vector $a \in H^2_A(\mu)$, there are constants $C, d > 0$ and vectors $x, y_k \in H^2_A(\mu)$ $(k \geq 1)$ with*

$$\|x - a\| < \varepsilon, \quad L_k = x \otimes y_k, \quad \|y_k\| \leq Ck^d\|L_k\| \quad (k \geq 1).$$

Unless otherwise stated we shall from now on assume that A is a norm closed subalgebra of $C(K)$ containing the constant function 1. The aim of this section is to establish a sufficient condition on the triple (A, K, μ) guaranteeing that $H^2_A(\mu)$ possesses both of the above factorization properties. To formulate the result, let us say that a set $S \subset K$ is a *regular boundary* of A if S is a closed subset of K and if, for each measure $\mu \in M^+(K)$ with support contained in S, the triple (A, K, μ) is regular.

3.2 Theorem. *Let K be a compact metric space. Suppose that $A \subset C(K)$ is a unital closed subalgebra, $S \subset K$ is a regular boundary of A and $\mu \in M^+_1(K)$ is a probability measure on K satisfying the following conditions:*

(a) *the restriction $\mu|S$ has no atoms;*

(b) *there is a measure $\sigma \in M_1^+(K)$ with no atoms and supp$(\sigma) \subset S$ such that the canonical mapping $A \to H_A^\infty(\mu)$ extends to a dual algebra isomorphism $r : H_A^\infty(\sigma) \to H_A^\infty(\mu)$.*

Then the space $H_A^2(\mu)$ possesses both property $(\mathbb{A}_1)^+$ and property $(\mathbb{A}_{1,\aleph_0})^+$. Furthermore, there are universal constants $R, C_0, d > 0$ such that $H_A^2(\mu)$ satisfies property $(\mathbb{A}_1)^+$ with constant R and such that $H_A^2(\mu)$ satisfies property $(\mathbb{A}_{1,\aleph_0})^+$ with constants d and $C = C_0/\varepsilon$. □

The proof of this theorem is divided into several parts. Most of them are patterned after standard factorization lemmas from the theory of dual algebras. The basic new ideas are contained in the proofs of Lemma 3.5 and Lemma 3.6 below. The following factorization lemma is well known (cf. Lemma 1.4 in [10]). Its proof will therefore be omitted.

3.3 Lemma. *Let $\mu \in M_1^+(K)$ be a probability measure, $A \subset L^\infty(\mu)$ a subalgebra and $k \geq 1$ a natural number. Define $p = 2k+1$ and $q = \frac{2k+1}{2k}$ (which is the conjugate exponent of p). Then, for each $h \in L^q(\mu)$ satisfying*

$$\|h\|_{q,\mu} = \sup \left\{ |\int_K fhd\mu| : f \in H_A^p(\mu), \|f\|_{p,\mu} \leq 1 \right\},$$

there exists a function $v \in H_A^{2+\frac{1}{k}}(\mu) \subset H_A^2(\mu)$ such that $|v|^2 = |h|$ μ-almost everwhere. □

As can be seen by a repetition of the proof of Lemma 1.3 in [12], the above lemma immediately yields approximate factorizations of elements in $Q_A(\mu)$.

3.4 Lemma. *Let $\mu \in M_1^+(K)$ be an arbitrary probability measure and let $A \subset L^\infty(\mu)$ be a subalgebra. Then, for $L \in Q_A(\mu)$ and $\varepsilon > 0$, there exist vectors $x, y \in H_A^2(\mu)$ satisfying*

$$\|L - x \otimes y\| < \varepsilon, \qquad \|x\|, \|y\| \leq \|L\|^{\frac{1}{2}}.$$

□

The improvement of such approximate factorizations relies on the existence of suitable sequences consisting of abstract inner functions.

3.5 Lemma. *Under the hypotheses of Theorem 3.2 there exists a sequence $(\theta_k)_{k\geq 1}$ in the closed unit ball of $H_A^\infty(\mu)$ such that:*

(a) each of the multiplication operators

$$M_{\theta_k} : H_A^\infty(\mu) \to H_A^\infty(\mu), \quad f \mapsto \theta_k f \qquad (k \geq 1)$$

is an isometry;

(b) the sequence $(\theta_k)_{k\geq 1}$ tends to zero μ-almost everywhere on $K \setminus S$ and forms a weak* zero sequence in $H_A^\infty(\mu)$.

Proof. According to Corollary 2.8 there exists a weak* zero sequence $(g_k)_{k\geq 1}$ in the closed unit ball of $H_A^\infty(\mu+\sigma)$ such that $|g_k| = 1 \quad (\mu+\sigma)$-almost everywhere on S and $g_k \xrightarrow{k} 0 \quad (\mu+\sigma)$-almost everywhere on $K \setminus S$.

Making use of the canonical dual algebra homomorphisms

$$r_\sigma^{\mu+\sigma} : H_A^\infty(\mu+\sigma) \to H_A^\infty(\sigma) \quad \text{and} \quad r_\mu^{\mu+\sigma} : H_A^\infty(\mu+\sigma) \to H_A^\infty(\mu)$$

which act by mapping the equivalence class of a function in the first space to the equivalence class of the same function in the second space, we define

$$g_k^\sigma = r_\sigma^{\mu+\sigma}(g_k) \in H_A^\infty(\sigma) \qquad \text{and} \qquad \theta_k = r_\mu^{\mu+\sigma}(g_k) \in H_A^\infty(\mu) \qquad (k \geq 1).$$

By construction it follows that $|g_k^\sigma| = 1$ σ-almost everywhere on K implying that each of the multiplication operators $M_{g_k^\sigma} : H_A^\infty(\sigma) \to H_A^\infty(\sigma)$ is an isometry ($k \geq 1$). Furthermore, $(\theta_k)_{k\geq 1}$ fulfills the convergence conditions described in part (b) of the assertion. To conclude the proof it remains to verify part (a). As can be checked on elements of A (and then carries over to $H_A^\infty(\mu+\sigma)$ by weak* continuity), the mapping $r_\mu^{\mu+\sigma}$ can be factorized as

$$r_\mu^{\mu+\sigma} : H_A^\infty(\mu+\sigma) \xrightarrow{r_\sigma^{\mu+\sigma}} H_A^\infty(\sigma) \xrightarrow{r} H_A^\infty(\mu)$$

where r is the map explained in Theorem 3.2. Therefore, given any $k \geq 1$, we have $\theta_k = r(g_k^\sigma)$ and hence the diagram

$$\begin{tikzcd} H_A^\infty(\sigma) \arrow[r, "r"] \arrow[d, "M_{g_k^\sigma}"'] & H_A^\infty(\mu) \arrow[d, "M_{\theta_k}"] \\ H_A^\infty(\sigma) \arrow[r, "r"] & H_A^\infty(\mu) \end{tikzcd}$$

commutes. Since the operators r and $M_{g_k^\sigma}$ are isometric, so is the multiplication operator M_{θ_k} on $H_A^\infty(\mu)$. $\square$

3.6 Lemma. *Let $\varepsilon > 0$, $\delta > 0$ be real numbers. Let $g_1, \cdots, g_l, h_1, \cdots, h_l$ in $L^2(\mu)$ $(l \geq 1)$ be arbitrary and let $L \in Q_A(\mu)$ be a functional satisfying $\|L\| < \delta^2$. Then there exist vectors $x \in H_A^2(\mu)$ and $y \in L^2(\mu)$ having the following properties:*

(a) $\|x\| < \delta, \quad \|y\| < \delta$;

(b) $\|L - x \otimes y\| < \varepsilon$;

(c) $\max_{i=1,\cdots,l} \|xh_i\chi_{K\setminus S}\|_{1,\mu} < \varepsilon, \quad \max_{i=1,\cdots,l} \|yg_i\chi_{K\setminus S}\|_{1,\mu} < \varepsilon$.

Proof. Fix a sequence $(\theta_k)_{k\geq 1}$ having the properties described in the previous lemma. Since, for each $k \geq 1$, the preadjoint of the weak* continuous isometry $(M_{\theta_k})^2$ is surjective and allows the lifting with constant $(1+\eta)$ for every $\eta > 0$, we find a sequence $(L_k)_{k\geq 1}$ in $Q_A(\mu)$ with $\|L_k\| < \delta^2$ and

$$\langle L_k, \theta_k^2 f\rangle = \langle L, f\rangle \quad (f \in H_A^\infty(\mu),\ k \geq 1).$$

According to Lemma 3.4 there exist vectors u_k, $v_k \in H_A^2(\mu)$ with $\|u_k\|$, $\|v_k\| < \delta$ and

$$\|L_k - u_k \otimes v_k\| < \frac{1}{k} \quad (k \geq 1).$$

Obviously, each of the vectors defined by

$$x_k = \theta_k u_k \in H_A^2(\mu), \qquad y_k = \bar{\theta}_k v_k \in L^2(\mu) \qquad (k \geq 1)$$

has norm strictly less than δ. For any $f \in H_A^\infty(\mu)$ with $\|f\|_{\infty,\mu} \leq 1$, we have the estimate

$$\begin{aligned} |(L - x_k \otimes y_k)(f)| &= |L_k(\theta_k^2 f) - \int_K \theta_k u_k \theta_k \bar{v}_k f \, d\mu| \\ &\leq \|L_k - u_k \otimes v_k\| \|\theta_k^2 f\|_{\infty,\mu} < \frac{1}{k} \stackrel{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

Since $\theta_k \stackrel{k}{\to} 0$ μ-almost everywhere on $K \setminus S$, it follows as an application of the dominated convergence theorem that

$$\|x_k h_i \chi_{K\setminus S}\|_{1,\mu} = \int_{K\setminus S} |\theta_k u_k h_i| d\mu \leq \delta \left(\int_{K\setminus S} |\theta_k|^2 |h_i|^2 d\mu \right)^{\frac{1}{2}} \stackrel{k\to\infty}{\longrightarrow} 0$$

for $i = 1, \cdots, l$. By the same arguments, $\|y_k g_i \chi_{K\setminus S}\|_{1,\mu} \stackrel{k\to\infty}{\longrightarrow} 0$ for every $i = 1, \cdots, l$. To conclude the proof, it therefore suffices to choose $k \geq 1$ large enough and to define $x = x_k \in H_A^2(\mu)$ and $y = y_k \in L^2(\mu)$. $\square$

A result of Aleksandrov (see Theorem 2.9) and a standard Lusin-type argument allow us to prescribe the boundary values of functions in A up to a set of arbitrary small measure.

3.7 Lemma. *Let $\varepsilon > 0$ and suppose that $0 < c \leq d$ are given real numbers and that $\kappa : S \to \mathbb{R}$ is a Borel measurable function with $c \leq \kappa \leq d$. Then, under the hypotheses of Theorem 3.2, there exists a function $g \in A$ satisfying*

$$|g| \leq d \quad \text{on} \quad K \qquad \text{and} \qquad \mu(\{z \in S : \kappa(z) \neq |g(z)|\}) < \varepsilon.$$

Proof. By Lusin's theorem we can choose a continuous function $p : S \to \mathbb{R}$ with $c \le p \le d$ such that the set $Z_1 = \{z \in S : \kappa(z) \ne p(z)\}$ is small in the sense that $\mu(Z_1) < \varepsilon/2$. By Tietze's extension theorem we may extend p to a continuous function q on K in such a way that $c \le q \le d$ holds on all of K. From the regularity of the triple (A, K, μ_S) it follows that there exists a function $g \in A$, $|g| \le q$ on K, such that $\mu(Z_2) < \varepsilon/2$ where $Z_2 = \{z \in S : |g(z)| \ne q(z)\}$ (see Theorem 2.9). Our choices guarantee that $|g(z)| = \kappa(z)$ for all $z \in S \setminus (Z_1 \cup Z_2)$ and that $|g| \le d$ on K. $\square$

3.8 Lemma. *Given $0 < \delta < 1/3$, $\varepsilon > 0$, $h_1, \cdots, h_r \in L^2(\mu)$, $L \in Q_A(\mu)$ and $a, b \in L^2(\mu)$ such that $\|L - a \otimes b\| < \delta^4$, there exist*

$$x \in H^2_A(\mu), \quad y \in L^2(\mu)$$

and a Borel seet $Z \subset S$ with $\mu(Z) < \varepsilon$ such that

$$\|L - (a+x) \otimes (b+y)\| < \varepsilon, \qquad \|x\| < 3\delta, \qquad \|y\chi_{K \setminus S}\| < \delta^2,$$

$$\|(b+y)\chi_S\| < \delta^2 + \frac{\|b\chi_S\|}{1-2\delta}, \qquad \|b+y\| < \delta^2 + \frac{\|b\|}{1-2\delta},$$
$$|a+x| \ge (1-2\delta)|a| \quad \text{on} \quad S \setminus Z,$$
$$\|x \otimes (h_j \chi_{K\setminus S})\| < \varepsilon \quad (j = 1, \cdots, r).$$

Proof. The proof is divided into several steps.

(1) We first show how to construct x. Lemma 3.6 allows us to choose functions $u \in H^2_A(\mu)$ and $v \in L^2(\mu)$ satisfying

$$\|u\| < \delta^2, \quad \|v\| < \delta^2, \quad \|L - a \otimes b - u \otimes v\| < \varepsilon/6,$$

$$\|u \otimes (b\chi_{K\setminus S})\| < \varepsilon/6, \quad \|a \otimes (v\chi_{K\setminus S})\| < \varepsilon/6, \quad \|u \otimes (h_j\chi_{K\setminus S})\| < \varepsilon/2$$

for $j = 1, \cdots, r$. Fix a constant $\eta \in (0, \varepsilon)$ such that

$$\int_Z (|uv| + (1 + 2/\delta)|ub|) d\mu < \varepsilon/6$$

for each Borel set $Z \subset S$ with $\mu(Z) < \eta$. By Lemma 3.7 there exists a function $g \in A$ with $\|g\|_{\infty,K} \le 2/\delta$ such that on S the function $|g|$ is close to the measurable function

$$\kappa : S \to \mathbb{R}, \quad z \mapsto \begin{cases} 2/\delta\,; & \text{if} \quad |a(z)| \le |u(z)|/\delta \\ 1\,; & \qquad \text{otherwise} \end{cases}$$

in the sense that the Borel set

$$Z = \{z \in S : |g(z)| \ne \kappa(z)\} \subset S$$

has measure $\mu(Z) < \eta$. Using Corollary 2.8, we find a sequence $(p_i)_{i\geq 1}$ in the unit ball of $H^\infty_A(\mu)$ with $|p_i| = 1$ μ-almost everywhere on S such that $p_i \xrightarrow{i} 0$ μ-almost everywhere on $K \setminus S$. Enlarging Z by a μ-zero set we may assume in addition that $|p_i| = 1$ on $S \setminus Z$. The dominated convergence theorem allows us to choose an index $i \geq 1$ in such a way that the function $f = p_i g \in H^\infty_A(\mu)$ satisfies the estimate

$$\|uf\chi_{K\setminus S}\| = \left(\int_{K\setminus S} |up_i g|^2 d\mu\right)^{\frac{1}{2}} < (\varepsilon/6)\cdot 1/(\|h_j\| + \|b\| + \delta^2)$$

for $j = 1, \cdots, r$. Since $\|(1+f)u\| \leq \|u\| + \|p_i g\|_{\infty,\mu}\|u\| \leq \delta^2(1+2/\delta) < 3\delta$, the function

$$x = (1+f)u \in H^2_A(\mu)$$

satisfies $\|x\| < 3\delta$ as well as

$$\|x \otimes (h_j\chi_{K\setminus S})\| \leq \|u \otimes (h_j\chi_{K\setminus S})\| + \|(uf)\otimes(h_j\chi_{K\setminus S})\| < \varepsilon/2 + \varepsilon/6 < \varepsilon,$$

$$\|x \otimes (b\chi_{K\setminus S})\| \leq \|u \otimes b\chi_{K\setminus S}\| + \|(uf)\otimes(b\chi_{K\setminus S})\| < \varepsilon/6 + \varepsilon/6 = \varepsilon/3.$$

For later reference we remark that $\|(uf)\otimes(v\chi_{K\setminus S})\| < \varepsilon/6$.

(2) Secondly, we show that on $S \setminus Z$

$$|a+x| \geq |u| \qquad \text{and} \qquad |a+x| \geq (1-2\delta)|a|.$$

Note that on $S_1 = \{z \in S\setminus Z : |a(z)| \leq |u(z)|/\delta\}$ we have $|g| = \kappa = 2/\delta$ and $|p_i| = 1$, and hence $|a+u| \leq (1+1/\delta)|u|$ and $|uf| = |up_i g| = (2/\delta)|u|$. Therefore, on S_1 we obtain the estimate

$$|a+x| = |uf + a + u| \geq (2/\delta)|u| - (1+1/\delta)|u| = (1/\delta - 1)|u| \geq |u|.$$

On $S_1 \cap \{z \in S : a(z) \neq 0\}$ this yields $\left|\frac{a+x}{a}\right| = \left|\frac{a+x}{u}\right|\left|\frac{u}{a}\right| \geq (\frac{1}{\delta}-1)\delta = 1-\delta$, and hence on S_1 we find the estimate

$$|a+x| \geq (1-\delta)|a|.$$

On the set $S_2 = \{z \in S\setminus Z : |a(z)| > |u(z)|/\delta\}$ we have $|g| = 1 = |p_i|$, and therefore $|x| \leq 2|u|$, implying that

$$|a+x| \geq (1/\delta - 2)|u| \geq |u|.$$

Since we have $|a| \leq |a+x| + |x| \leq |a+x| + 2|u| \leq |a+x| + 2\delta|a|$ on S_2, we find that

$$|a+x| \geq (1-2\delta)|a|$$

on S_2. Combining the four main estimates on the sets S_1 and S_2, we deduce that the estimates stated at the beginning of step (2) hold on the set $S\setminus Z$.

(3) We show how to construct y. Define a function $w \in L^2(\mu)$ by setting

$$w = \frac{\bar{u}}{\bar{a} + \bar{x}}(v - (1 + \bar{f})b) = \frac{\bar{u}v}{\bar{a} + \bar{x}} - \frac{\bar{x}b}{\bar{a} + \bar{x}}$$

on $W = (S \setminus Z) \cap \{z \in S : a(z) + x(z) \neq 0\}$ and $w = 0$ elsewhere.

The function

$$y = v\chi_{K\setminus S} + w\chi_S \in L^2(\mu)$$

then obviously satisfies $\|y\chi_{K\setminus S}\| < \delta^2$. Moreover, we have

$$\begin{aligned}
& \|y + b\|^2 \\
= \;& \int_{K\setminus S} |v + b|^2 d\mu + \int_W \left| \frac{\bar{u}}{\bar{a} + \bar{x}} v - \frac{\bar{x}}{\bar{a} + \bar{x}} b + b \right|^2 d\mu + \int_{S\setminus W} |b|^2 d\mu \\
= \;& \int_{K\setminus S} |v + b|^2 d\mu + \int_W \left| \frac{\bar{u}}{\bar{a} + \bar{x}} v + \frac{\bar{a}}{\bar{a} + \bar{x}} b \right|^2 d\mu + \int_{S\setminus W} |b|^2 d\mu \\
= \;& \int_{K\setminus S} |v|^2 d\mu + \int_{K\setminus S} 2\mathrm{Re}\ (v\bar{b}) d\mu + \int_{K\setminus S} |b|^2 d\mu + \int_{S\setminus W} |b|^2 d\mu \\
& + \int_W |v|^2 \left| \frac{u}{a + x} \right|^2 d\mu + \int_W \frac{2\mathrm{Re}\ (v\bar{b}\bar{u}a)}{|a + x|^2} d\mu + \int_W \left| \frac{a}{a + x} \right|^2 |b|^2 d\mu \\
\leq \;& \int_K |v|^2 d\mu + 2 \int_K |v| \frac{|b|}{1 - 2\delta} d\mu + \int_K \frac{|b|^2}{(1 - 2\delta)^2} d\mu \\
\leq \;& \left(\|v\| + \frac{\|b\|}{1 - 2\delta} \right)^2 \quad < \quad \left(\delta^2 + \frac{\|b\|}{1 - 2\delta} \right)^2 .
\end{aligned}$$

An analogous calculation shows that $\|(y + b)\chi_S\| < \delta^2 + \frac{\|b\chi_S\|}{1-2\delta}$.

(4) To complete the proof, it suffices to estimate the norm of

$$L - (a + x) \otimes (b + y) = L - a \otimes b - x \otimes y - a \otimes y - x \otimes b.$$

Towards this aim, we write

$$\begin{aligned}
x \otimes y &= (u + uf) \otimes (v\chi_{K\setminus S} + w\chi_S) \\
&= u \otimes (v\chi_{K\setminus S} + w\chi_S) + (uf) \otimes (v\chi_{K\setminus S} + w\chi_S) \\
&= u \otimes v + (uf) \otimes (v\chi_{K\setminus S}) + [u \otimes (-v\chi_S + w\chi_S) + (uf) \otimes (w\chi_S)].
\end{aligned}$$

Abbreviating the term in square brackets by z, we obtain the decomposition

$$\begin{aligned}
L - (a + x) \otimes (b + y) &= L - a \otimes b - u \otimes v \\
& \quad -(uf) \otimes (v\chi_{K\setminus S}) - a \otimes (v\chi_{K\setminus S}) - x \otimes (b\chi_{K\setminus S}) \\
& \quad -\big(z + a \otimes (w\chi_S) + x \otimes (b\chi_S)\big).
\end{aligned}$$

Note that, for all but the last term in the above formula, we have derived suitable estimates during steps (1) – (3) of the proof. To estimate the last term from above, observe that, for $\varphi \in H_A^\infty(\mu)$, we have

$$\begin{aligned}
(z + a \otimes (w\chi_S) + x \otimes (b\chi_S))(\varphi) &= \int_S \varphi(-u\bar{v} + u\bar{w} + uf\bar{w} + a\bar{w} + x\bar{b})d\mu \\
&= \int_S \varphi((a+x)\bar{w} - u\bar{v} + x\bar{b})d\mu \\
&= \int_Z \varphi(-u\bar{v} + u\bar{b} + uf\bar{b})d\mu.
\end{aligned}$$

Since $|f| \leq 2/\delta$ μ-almost everywhere, we can conclude that

$$\|z + a \otimes (w\chi_S) + x \otimes (b\chi_S)\| \leq \int_Z (|uv| + (1 + \frac{2}{\delta})|ub|)d\mu < \varepsilon/6.$$

This finally leads to the desired estimate

$$\|L - (a+x) \otimes (b+y)\| < \varepsilon/6 + \varepsilon/6 + \varepsilon/6 + \varepsilon/3 + \varepsilon/6 = \varepsilon,$$

which finishes the proof. □

Exactly as in the case of the unit ball one can show that the preceding lemma allows one to factorize arbitrary elements in $Q_A(\mu)$ with additional control on the norms of the factors. Making the obvious replacements in the proof of Theorem 1.8 in [10] we arrive at the first of the claimed factorization properties of the space $H_A^2(\mu)$.

3.9 Proposition. *Under the hypotheses of Theorem 3.2 the space $H_A^2(\mu)$ satisfies property $(\mathbb{A}_1)^+$.* □

As another consequence of Lemma 3.8 one obtains a factorization result for finite families of functionals in $Q_A(\mu)$. The proof can be obtained by copying the ball case (see Lemma 2.1 in [10]).

3.10 Lemma. *Let $m \geq 1$ be an integer, let $L_1, \cdots, L_m \in Q_A(\mu)$ and let $\varepsilon > 0$, $0 < \delta < 1/3$, $\rho_1, \cdots, \rho_m > 0$ be given real numbers. Suppose that $a \in L^2(\mu)$ and $b_k \in L^2(\mu)$ $(k = 1, \cdots, m)$ are functions with*

$$\|L_k - a \otimes b_k\| < \rho_k \quad (k = 1, \cdots, m).$$

Then there are vectors $x \in H_A^2(\mu)$ and $y_k \in L^2(\mu)$ $(k = 1, \cdots, m)$ with

$$\|L_k - (a+x) \otimes y_k\| < \varepsilon,$$

$$\|x\| < \frac{3}{\delta} \sum_{i=1}^m \sqrt{\rho_i}, \quad \|(y_k - b_k)\chi_{K \setminus S}\| < \sqrt{\rho_k},$$

$$\|y_k \chi_S\| < \frac{\sqrt{\rho_k}}{(1-2\delta)^{m-k}} + \frac{\|b_k \chi_S\|}{(1-2\delta)^m}$$

for $k = 1, \cdots, m$. □

A standard approximation device now yields $A_{1,\aleph_0}$-factorizations. For details, we refer once again to the ball case (see Proposition 2.2 in [10]).

3.11 Proposition. *Under the hypotheses of Theorem 3.2 the space* $H^2_A(\mu)$ *has property* $(\mathbb{A}_{1,\aleph_0})^+$. □

3.12 Remark. The proofs of Theorem 1.8 and Proposition 2.2 in [9] show that in the setting of Theorem 3.2 the constants R and d the existence of which is required in the definition of properties $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$ can be chosen as universal constants that do not depend on A, K, S, μ or σ. Furthermore, the constant C required in the definition of property $(\mathbb{A}_{1,\aleph_0})^+$ depends only on ε and can be chosen as $C(\varepsilon) = C_0/\varepsilon$ with a suitable universal constant C_0.

4 Subnormal n-tuples

Let $X \subset \mathbb{C}^n$ be a complex submanifold, and let $D \subset X$ be a relatively compact open subset in X such that D satisfies the conditions (F1) to (F4).

As before, we denote by $A_0(D)$ the closure of the algebra $\mathcal{O}(\bar{D})$ in the Banach algebra $C(\bar{D})$.

4.1 Lemma. *The character space of* $A_0(D)$ *consists precisely of the point evaluations at points of* $\bar{D}$.

Proof. Choose a neighbourhood basis $(D_k)_{k\geq 1}$ of $\bar{D}$ consisting of Stein open sets in $\mathbb{C}^n$. Let $\varphi : A_0(D) \to \mathbb{C}$ be a character. For each k, the map φ induces a character $\varphi_k : \mathcal{O}(D_k) \xrightarrow{\text{rest}} A_0(D) \xrightarrow{\varphi} \mathbb{C}$ of $\mathcal{O}(D_k)$ which is a point evaluation at a suitable point z_k of D_k by the character theorem (Satz V.5.7 in [15]). Since the coordinate functions separate the points, we conclude that there is a point $z \in \bigcap_{k\geq 1} D_k = \bar{D}$ such that $z = z_k$ for all k. Hence φ coincides with the point evaluation at z on each of the spaces $\mathcal{O}(D_k)|\bar{D}$, and therefore on $A_0(D)$. □

Using general properties of the multivariable holomorphic functional calculus, one obtains the uniqueness of $A_0(D)$-functional calculi. For a commuting tuple $T \in L(H)^n$, we denote by $\sigma(T)$ its Taylor spectrum and write $\mathcal{O}(\sigma(T)) \to L(H)$, $f \mapsto f(T)$, for the analytic functional calculus of T (cf. Chapter 2 in [13]).

4.2 Lemma. *Suppose that $T \in L(H)^n$ is a commuting n-tuple possessing a continuous $A_0(D)$-functional calculus $\Phi : A_0(D) \to L(H)$. Then we have:*

(a) *$\sigma(T) \subset \bar{D}$ and $\Phi(f|\bar{D}) = f(T)$ for all $f \in \mathcal{O}(\bar{D})$;*

(b) *Φ is uniquely determined.*

Proof. By Lemma 4.1 the character space of $A_0(D)$ consists of the point evaluations at points $z \in \bar{D}$. Denoting by $(T)'$ the commutant of T and by $z = (z_1, \ldots, z_n) \in A_0(D)^n$ the tuple of coordinate functions, we find that

$$\sigma(T) \subset \sigma_{(T)'}(T) \subset \sigma_{A_0(D)}(z) = \bar{D}.$$

For each Stein open neighborhood U of $\bar{D}$, the map

$$\mathcal{O}(U) \to L(H), \quad f \mapsto \Phi(f|\bar{D})$$

defines a continuous extension of the canonical $\mathcal{O}(\mathbb{C}^n)$-functional calculus of T to a continuous $\mathcal{O}(U)$-functional calculus, and hence coincides with Taylor's analytic functional calculus, that is, $\Phi(f|\bar{D}) = f(T)$ for each f in $\mathcal{O}(U)$ (Lemma 5.1.1 in [13]). Since $\bar{D}$ is a Stein compactum , the proof of part (a) is complete. Part (b) follows, since $\mathcal{O}(\bar{D})$ is dense in $A_0(D)$. □

By Lemma 4.2 and hypothesis (F2) the contractive weak* continuous $H^\infty(D)$-functional calculus Φ_T of an absolutely continuous subnormal tuple $T \in L(H)^n$ over D is unique. Its range is a weak* dense subset of the dual algebra

$$\mathcal{H}_T(\bar{D}) = \overline{\{f(T) : f \in \mathcal{O}(\bar{D})\}}^{w^*} \subset L(H).$$

If in addition T is of class $\mathbb{A}$, that is, Φ_T is isometric, then $H^\infty(D)$ and $\mathcal{H}_T(\bar{D})$ are isomorphic as dual algebras via Φ_T.

For a positive measure $\mu \in M^+(\bar{D})$ and $1 \leq p < \infty$, we define

$$H^p(\mu) = \overline{\mathcal{O}(\bar{D})} \subset L^p(\mu) \quad \text{and} \quad H^\infty(\mu) = \overline{\mathcal{O}(\bar{D})}^{w^*} \subset L^\infty(\mu).$$

Note that if we replace $\mathcal{O}(\bar{D})$ by $A_0(D)$ in this definition, then we obtain the same spaces. Using the fact that the weak and the norm closure of $\mathcal{O}(\bar{D})$ in $L^p(\mu)$ coincide one easily deduces that $H^\infty(\mu) \cdot H^p(\mu) \subset H^p(\mu)$.

Now suppose that $\mu \in M^+(\bar{D})$ is a Henkin measure. By condition (F2) the induced map

$$r_\mu : H^\infty(D) \to L^\infty(\mu)$$

has values in $H^\infty(\mu)$ and therefore gives rise to a contractive and weak* continuous $H^\infty(D)$-functional calculus

$$\Phi_\mu : H^\infty(D) \to L(H^2(\mu)), \quad f \mapsto M_{r_\mu(f)}$$

for the tuple $M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(H^2(\mu))^n$. Here $M_g \in L(H^2(\mu))$ denotes the operator of multiplication with a function $g \in H^\infty(\mu)$.

4.3 Proposition. *Let $T \in L(H)^n$ be a subnormal n-tuple of class $\mathbb{A}$ over D. Then, for each vector $a \in H$ and each real number $\varepsilon > 0$, there exist a faithful Henkin probability measure $\mu \in M_1^+(\bar{D})$ and an isometric embedding $j : H^2(\mu) \hookrightarrow H$ with $\|j(1) - a\| < \varepsilon$ and*

$$j \circ \Phi_\mu(f) = \Phi_T(f) \circ j \qquad (f \in H^\infty(D)).$$

One can achieve that the support of μ is contained in the Taylor spectrum of the minimal normal extension of T.

Proof. Let $a \in H$ and $\varepsilon > 0$ be given. We denote by E the projection-valued spectral measure of the minimal normal extension $N \in L(K)^n$ of T. It is well known (Proposition 1.9 in [10]) that there is a separating unit vector $x \in H$ for N such that $\|x - a\| < \varepsilon$. The associated scalar-valued spectral measure $\mu_N = \langle E(\cdot)x, x\rangle$ is a probability measure. Since the support of E coincides with $\sigma(N)$ which is (by a result of Putinar [26]) contained in $\sigma(T)$, the measure μ_N can be trivially extended to a measure $\mu \in M_1^+(\bar{D})$. The spectral theorem for normal tuples implies the existence of an isometric, weak* continuous and involutive functional calculus $\Psi : L^\infty(\mu) \to L(K)$ for N. According to Lemma 4.2, and since the holomorphic functional calculus is compatible with restrictions (Lemma 2.5.8 in [13]), the restriction algebra of N, that is, the weak* closed algebra

$$\mathcal{W} = \{f \in L^\infty(\mu) : \Psi(f)H \subset H\} \subset L^\infty(\mu)$$

contains $\mathcal{O}(\bar{D})$ and hence $H^\infty(\mu)$. Using the well-known fact that the map $\mathcal{W} \to L(H), \quad f \mapsto \Psi(f)|H$ is isometric again (Proposition 1.1 in [6]), we see that Ψ induces a dual algebra isomorphism

$$\gamma_T : H^\infty(\mu) \to \mathcal{H}_T(\bar{D}), \qquad f \mapsto \Psi(f)|H$$

mapping $f \in \mathcal{O}(\bar{D})$ to $f(T)$. Therefore the composition

$$r : H^\infty(D) \xrightarrow{\Phi_T} \mathcal{H}_T(\bar{D}) \xrightarrow{\gamma_T^{-1}} H^\infty(\mu)$$

is a dual algebra isomorphism extending the canonical map $\mathcal{O}(\bar{D}) \to H^\infty(\mu)$. Hence μ is a faithful Henkin measure with $r_\mu = r$. The identity

$$\|\Phi_T(f)x\|^2 = \|\Psi(f)x\|^2 = \langle \Psi(|f|^2)x, x\rangle = \|f\|_{2,\mu}^2 \qquad (f \in \mathcal{O}(\bar{D}))$$

guarantees the existence of an isometry $j : H^2(\mu) \to H$ extending the map $\mathcal{O}(\bar{D}) \to H, \quad f \mapsto \Phi_T(f)x$. By construction, we have $\|j(1)-a\| = \|x-a\| < \varepsilon$. To complete the proof it suffices to observe that

$$j(\Phi_\mu(f)g) = \Phi_T(fg)x = \Phi_T(f)\Phi_T(g)x = \Phi_T(f)j(g)$$

holds for all functions $f, g \in \mathcal{O}(\bar{D})$. □

According to Proposition 2.5 our hypothesis that D satisfies condition (F3) implies that the Shilov boundary $S = \partial_0 D$ of the Banach algebra $A_0(D)$ is a regular boundary for $A_0(D)$. To apply Theorem 3.2, we still have to check that the occurring measures have no atoms in S.

A point $w \in \bar{D}$ is called a *peak point* for $A_0(D)$ if there is a function h in $A_0(D)$ such that $|h| < 1$ on $\bar{D} \setminus \{w\}$ and $h(w) = 1$. In this case we say that h is a *peaking function* for w. Obviously each peak point for $A_0(D)$ is contained in the Shilov boundary S of $A_0(D)$. Condition (F3) guarantees that also the converse holds.

4.4 Lemma. *For each point $w \in S$, there exists a function $h \in A_0(D)$ with $|h| < 1$ on $\bar{D} \setminus \{w\}$ and $h(w) = 1$.*

Proof. Let $f \in A_0(D)^N$ be an injective map with $f(S) \subset \partial\mathbb{B}_N$. Choose a function $g \in \mathcal{O}(\bar{\mathbb{B}}_N)$ with $g(f(w)) = 1$ and $|g| < 1$ on $\bar{\mathbb{B}}_N \setminus \{f(w)\}$. Then by Lemma 2.4 the function $h = g \circ f$ belongs to $A_0(D)$. Clearly, $h(w) = 1$ and the injectivity of f implies that $|h| < 1$ on $\bar{D} \setminus \{w\}$. □

Now it is easy to deduce that Henkin measures cannot have atoms in S.

4.5 Corollary. *Suppose that $\mu \in M^+(\bar{D})$ is a Henkin measure. Then $\mu(\{w\}) = 0$ for each point $w \in S$. Hence $\mu|S$ has no atoms.*

Proof. Fix a peaking function $h \in A_0(D)$ for w as in Lemma 4.4. Then the sequence $(r_\mu(h^j))_{j\geq 1}$ is a weak* zero sequence in $H^\infty(\mu)$ and therefore

$$\mu(\{w\}) = \int_{\bar{D}} \chi_{\{w\}} h^j \, d\mu = \langle [\chi_{\{w\}}], r_\mu(h^j) \rangle \xrightarrow{j} 0,$$

as we wanted to show. □

Recall that by condition (F4) there is a faithful Henkin probability measure σ on $\bar{D}$ with support contained in S. Since S is a regular boundary for $A_0(D)$ (Proposition 2.5) and since Henkin measures cannot possess atoms in S, we can apply Theorem 3.2 to see that, for each faithful Henkin probability measure $\mu \in M_1^+(\bar{D})$, the space $H^2(\mu)$ possesses the factorization properties $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$.

We use Proposition 4.3 to extend this result to every subnormal n-tuple of class $\mathbb{A}$ over D.

4.6 Theorem. *Let $T \in L(H)^n$ be a subnormal n-tuple of class $\mathbb{A}$ over D. Then T satisfies the factorization properties $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$.*

Proof. To prove property $(\mathbb{A}_1)^+$, let us fix a functional $L \in Q(D)$ and vectors $a, b \in H$. We may of course assume that $d = \|L - a \otimes b\| > 0$.

Let R be a constant such that, for each faithful Henkin probability measure $\mu \in M_1^+(\bar{D})$, the space $H^2(\mu)$ satisfies property $(\mathbb{A}_1)^+$ with constant R.

Choose a real number $\varepsilon > 0$ such that

$$R(d + \varepsilon\|b\|)^{\frac{1}{2}} + \varepsilon < (R+1)d^{\frac{1}{2}}.$$

By Proposition 4.3 there exist a faithful Henkin probability measure μ on $\bar{D}$ and an isometry $j : H^2(\mu) \to H$ with $\|j(1) - a\| < \varepsilon$ and

$$j\Phi_\mu(f) = \Phi_T(f)j \qquad (f \in H^\infty(D)).$$

The space $H^\infty(\mu)$ can isometrically be identified with the dual space of the quotient space $Q(\mu) = L^1(\mu)/{}^\perp H^\infty(\mu)$. We denote by $r_* : Q(\mu) \to Q(D)$ the predual of the dual algebra isomorphism $r = r_\mu : H^\infty(D) \to H^\infty(\mu)$. Then the above intertwining relation implies that

$$(jx) \otimes (jy) = r_*(x \otimes y)$$

for any pair of vectors $x, y \in H^2(\mu)$. We define $h = j(1)$ and write P for the orthogonal projection from H onto $jH^2(\mu)$. Choose functions $\tilde{a}, \tilde{b} \in H^2(\mu)$ with $j(\tilde{a}) = Pa$ and $j(\tilde{b}) = Pb$ and a functional $\tilde{L} \in Q(\mu)$ with $r_*(\tilde{L}) = L$. By Theorem 3.2 there are functions $x, y \in H^2(\mu)$ with $\tilde{L} = x \otimes y$ and

$$\|x - \tilde{a}\| \leq R\|\tilde{L} - \tilde{a} \otimes \tilde{b}\|^{\frac{1}{2}}, \qquad \|y\| \leq R(\|\tilde{L} - \tilde{a} \otimes \tilde{b}\|^{\frac{1}{2}} + \|\tilde{b}\|).$$

It follows that $L = (jx) \otimes (jy)$ and that

$$\begin{aligned}
\|jx - a\| &\leq \|x - \tilde{a}\| + \|Pa - a\| \\
&\leq R\|L - Pa \otimes Pb\|^{\frac{1}{2}} + \|(P - I)(a - h)\| \\
&\leq R(d + \varepsilon\|b\|)^{\frac{1}{2}} + \varepsilon \\
&< (R+1)d^{\frac{1}{2}}.
\end{aligned}$$

In the same way we obtain that $\|jy\| \leq (R+1)(d^{\frac{1}{2}} + \|b\|)$. Thus we have shown that T satisfies property $(\mathbb{A}_1)^+$ with constant $R+1$ instead of R. In the same way one obtains property $(\mathbb{A}_{1,\aleph_0})^+$ for T. □

Let $T \in L(H)^n$ be a subnormal tuple of class $\mathbb{A}$ over D. In the following we indicate how results from [8] can be used to deduce the remaining assertions of Theorem 1.4 from Theorem 4.6.

For a given vector $x \in H$, the space

$$H_x = \overline{\{\Phi_T(f)x : f \in H^\infty(D)\}}$$

is the smallest Φ_T-invariant subspace of H containing x. The vector x is called an *analytic factor* of the representation Φ_T if there exists a conjugate analytic function $e : D \to H_x$ such that

$$\mathcal{E}_\lambda = x \otimes e(\lambda) \qquad (\lambda \in D).$$

Here $\mathcal{E}_\lambda \in Q(D)$ denotes the point evaluation at λ, that is, $\mathcal{E}_\lambda(f) = f(\lambda)$ for $f \in H^\infty(D)$.

Suppose that $x \in H$ is an analytic factor of Φ_T. Then the relation

$$\langle \Phi_T(g)x, (\Phi_T(f)|H_x)^* e(\lambda)\rangle = f(\lambda)\langle \Phi_T(g)x, e(\lambda)\rangle,$$

valid for all $f, g \in H^\infty(D)$ and $\lambda \in D$, implies that $(\Phi_T(f)|H_x)^* e(\lambda) = \overline{f(\lambda)} e(\lambda)$ for all $f \in H^\infty(D)$ and $\lambda \in D$. In particular, it follows that

$$f(D) \subset \sigma(\Phi_T(f)|H_x) \qquad (f \in H^\infty(D)).$$

Consequently, for each analytic factor x of Φ_T, the restriction $T|H_x$ of T to H_x is a subnormal tuple of class $\mathbb{A}$ over D with $H^\infty(D)$-functional calculus given by

$$H^\infty(D) \to L(H_x), \quad f \mapsto \Phi_T(f)|H_x.$$

Proof of Theorem 1.4. Let $T \in L(H)^n$ be a subnormal tuple of class $\mathbb{A}$ over D. According to Theorem 4.6 the tuple T has property $(\mathbb{A}_{1,\aleph_0})^+$ and, for each analytic factor $x \in H$ of Φ_T, the restriction $T|H_x$ has property $(\mathbb{A}_1)^+$. Then Theorem 4.3.9 in [8] implies that the dual algebra $\mathcal{H}_T(\bar{D}) = \Phi_T(H^\infty(D))$ is reflexive. The careful reader will notice that Theorem 4.3.9 in [8] as formulated there only applies to the case where D is a relatively compact open subset of a Stein submanifold $X \subset \mathbb{C}^n$. But it is elementary to check that the condition that X is Stein is nowhere needed in the proof. To see that $\mathcal{H}_T(\bar{D})$ is even super-reflexive it suffices to show, by a well-known result of Loginov and Sulman (Theorem 2.3 in [21]) that, for each WOT-continuous linear functional L on $\mathcal{H}_T(\bar{D})$, there are vectors $x, y \in H$ with $L(A) = \langle Ax, y\rangle$ for $A \in \mathcal{H}_T(\bar{D})$. But this follows directly from Theorem 4.6 by using that $L \circ \Phi_T \in Q(D)$ has a factorization of the form $L \circ \Phi_T = x \otimes y$ with suitably chosen vectors $x, y \in H$. □

5 Strictly pseudoconvex and bounded symmetric domains

At the beginning of this section we briefly indicate how the validity of the conditions (F1) to (F4) can be verified for strictly pseudoconvex and bounded symmetric domains. In the final part we study products $D = D_1 \times D_2$ where one factor is strictly pseudoconvex and the other satisfies the conditions (F1) to (F4).

Let us start with the case where D is a relatively compact, strictly pseudoconvex open subset of a Stein submanifold X in $\mathbb{C}^n$. More precisely, we assume that there exist an open neighbourhood $U \subset X$ of ∂D and a strictly plurisubharmonic C^2-function $\rho : U \to \mathbb{R}$ such that $D \cap U = \{z \in U : \rho(z) < 0\}$. Note that we do neither assume that D is connected nor that the boundary of D is smooth. Using the hypothesis that X is a Stein submanifold in $\mathbb{C}^n$, one can show that conditions (F1) and (F2) hold for D and $\mathcal{O}(\bar{D})$ is dense in

$$A(D) = \{f : f \in C(\bar{D}) \text{ and } f|D \in \mathcal{O}(D)\}$$

(see Proposition 2.1.6 in [8]). It follows that $A_0(D) = A(D)$. Since the Shilov boundary of $A(D)$ coincides with the topological boundary ∂D for a strictly pseudoconvex open set D (Corollary 2.1.3 in [8]), we have $\partial_0 D = \partial D$. Therefore, the validity of condition (F3) can be checked using the classical embedding theorems of Fornæss and Løw which guarantee the existence of an injective map $f \in A(D)^N$ such that $f(\partial D) \subset \partial \mathbb{B}_N$ (for details, see the proof of Corollary 2.1.3 in [8]). Moreover, by Proposition 5.2.1 in [8] there is a faithful Henkin probability measure μ on $\bar{D}$ supported by the boundary of D as demanded in condition (F4). This proves part (a) of Corollary 1.5.

5.1 Corollary. *Each subnormal n-tuple of class $\mathbb{A}$ over a relatively compact strictly pseudoconvex open subset $D \subset X$ of a Stein submanifold $X \subset \mathbb{C}^n$ is reflexive.* □

Let $D \subset \mathbb{C}^n$ be a bounded domain which is symmetric. By definition this means that, for each $z \in D$, there exists a biholomorphic map $s_z : D \to D$ such that $s_z \circ s_z$ is the identity on D and s_z has z as isolated fixed point. We will further assume that D is circled around the origin, that is, $0 \in D$ and $e^{it} D \subset D$ for $t \in \mathbb{R}$. It is well known that every bounded symmetric domain is isomorphic to a bounded symmetric and circled one which is unique up to a linear isomorphism of $\mathbb{C}^n$ (see Section 1.6 in Loos [22]). By Corollary 4.6 in [22], a set D of this type is convex. Consequently, $\bar{D}$ is polynomially convex and hence a Stein compactum as demanded in condition (F1).

In this case the Minkowski functional of D defines a norm on $\mathbb{C}^n$ such that D is the open unit ball with respect to this norm. With any function f in $H^\infty(D)$ we can therefore associate a family $(f_r)_{0<r<1}$ in $\mathcal{O}(\bar{D})$ by setting $f_r(z) = f(rz)$. It follows that D satisfies condition (F2). If f belongs to $A(D)$, then the uniform continuity of f on $\bar{D}$ implies that $\|f_r - f\|_{\infty,\bar{D}} \xrightarrow{r\to 1} 0$. Hence $A(D) = A_0(D)$ and $\partial_0 D$ coincides with the Bergman-Shilov boundary S of D. By Theorem 6.5 in [22], the set S consists precisely of those points in $\bar{D}$ with maximal Euclidian distance from the origin. Hence, if $\delta > 0$ denotes the common distance of all these boundary points from 0, then we have $D \subset B_\delta(0)$ and $S = \bar{D} \cap \partial B_\delta(0)$. Obviously, the map $f : \bar{D} \to \mathbb{C}^n$, $z \mapsto z/\delta$, satisfies all the properties described in condition (F3).

The set $\mathrm{Aut}(D)$ of all conformal maps $f : D \to D$ is a topological group in the relative topology of $\mathcal{O}(D, \mathbb{C}^n)$, and $\mathrm{Aut}_0(D) = \{f \in \mathrm{Aut}(D);\ f(0) = 0\}$

is a closed subgroup. By a theorem of Cartan (Theorem 2.1.3 in [27]) the map

$$\{L \in GL(n, \mathbb{C});\ LD = D\} \to \mathrm{Aut}_0(D), \quad L \mapsto L|D$$

is a topological isomorphism between compact topological groups. Via this map we identify the elements of $\mathrm{Aut}_0(D)$ with invertible linear maps on $\mathbb{C}^n$. Let K be the intersection of $\mathrm{Aut}_0(D)$ with the connected component of the identity in $\mathrm{Aut}(D)$. It is well known that the elements of K map S into itself and that this K-action on S is transitive.

Let $\varkappa$ be the Haar measure of the compact topological group K, and let $\mu \in M_1^+(\bar{D})$ be the trivial extension of the unique K–invariant probability measure on S to all of $\bar{D}$. Then μ is the unique measure in $M^+(\bar{D})$ with support contained in S and

$$\int_S f d\mu = \int_K f(Lu) d\varkappa(L)$$

for all functions $f \in \mathcal{L}^1(\mu)$ and every point $u \in S$. We construct a map

$$r_\mu : H^\infty(D) \to L^\infty(\mu)$$

in the following way. Given $f \in H^\infty(D)$ and $0 < r < 1$, define f_r on $\overline{D}$ as before by $f_r(z) = f(rz)$. Since μ is circularly invariant in the sense of Bochner [4] and since the family $(f_r)_{0<r<1}$ is bounded in $L^2(\mu)$, by Theorem 3 in Bochner [4], the limit $F = \lim_{r \uparrow 1} f_r$ exists in $L^2(\mu)$. For a suitable increasing sequence (r_k) in $(0, 1)$, the sequence (f_{r_k}) converges to F pointwise μ-almost everywhere. Setting $r_\mu(f) = F$, we obtain a contractive linear map which maps each function $f \in \mathcal{O}(\overline{D})$ to its equivalence class in $L^\infty(\mu)$. Since

$$\int_S \varphi f_{r_k} d\mu \xrightarrow{k} \int_S \varphi r_\mu(f) d\mu,$$

we conclude that $r_\mu(H^\infty(D)) \subset H^\infty(\mu) = \overline{\mathcal{O}(\overline{D})}^{w^*} \subset L^\infty(\mu)$.

To prove that r_μ is a dual algebra isomorphism, one can use standard properties of the Poisson kernel P of D (see Koranyi [19] for details). The Poisson kernel P is a Borel measurable map $P : S \times D \to [0, \infty)$ which is separately continuous and satisfies:

(i) $P(Lu, Lz) = P(u, z)$ for $u \in S$, $z \in D$ and $L \in K$;

(ii) $\int_S P(u, z) d\mu(u) = 1$ for every $z \in D$;

(iii) $f(z) = \int_S P(u, z) f(u) d\mu(u)$ for every $f \in \mathcal{O}(\overline{D})$ and $z \in D$;

(iv) $\lim_{z\to u_0} \int_{\|u-u_0\|>\eta} P(u,z)d\mu(u) = 0$ for every $u_0 \in S$ and $\eta > 0$.

Obviously, the induced Poisson transformation

$$\mathcal{P} : L^\infty(\mu) \to L^\infty(D), \quad (\mathcal{P}f)(z) = \int_S P(u,z)f(u)d\mu(u)$$

is a well-defined contractive linear map. For $f \in H^\infty(D)$ and any sequence (f_{r_k}) as in the definition of $r_\mu(f)$, we have

$$(\mathcal{P} \circ r_\mu(f))(z) = \lim_{k\to\infty} \int_S P(u,z)f_{r_k}(u)d\mu(u) = \lim_{k\to\infty} f(r_k z) = f(z)$$

for all $z \in D$. It follows that $\mathcal{P} \circ r_\mu = \mathrm{id}_{H^\infty(D)}$ and that r_μ is isometric.

Since $H^\infty(D)$ has a separable predual, to show the weak* continuity of r_μ, it suffices to check that r_μ maps each weak* zero sequence $(h_k)_{k\geq 0}$ in $H^\infty(D)$ to a weak* zero sequence in $H^\infty(\mu)$. Note that any such sequence is norm bounded in $H^\infty(D)$ with

$$\int_S P(u,z)(r_\mu(h_k))(z)d\mu(u) = h_k(z) \xrightarrow{k} 0$$

for every point $z \in D$. It therefore suffices to show that $L^1(\mu)/{}^\perp H^\infty(\mu)$ is the closed linear span of the equivalence classes of the elements $P(\cdot, z)$ $(z \in D)$. By the theorem of Hahn-Banach this will follow if we can show that the only element $f \in H^\infty(\mu)$ with $(\mathcal{P}f)(z) = 0$ for all $z \in D$ is the trivial one $f = 0$.

Let $f \in H^\infty(\mu)$. To finish the argument we show that the norm-bounded family of functions $(\mathcal{P}f)_r : S \to \mathbb{C}$, $z \mapsto (\mathcal{P}f)(rz)$, in $L^\infty(\mu)$ is weak* convergent to f as $r \uparrow 1$. Let $g \in C(S)$ be a fixed continuous function. Since $C(S) \subset L^1(\mu)$ is dense, all that remains is to show that the integrals

$$I_r = \int_S g(z)\big((\mathcal{P}f)(rz) - f(z)\big)d\mu(z)$$

converge to zero as r tends to one from below.

Fix a point $u_0 \in S$. Using the relation between the invariant measure μ and the Haar measure $\varkappa$ of K explained above, the invariance of μ under the action of the group K, as well as Fubini's theorem, one obtains the representations

$$\begin{aligned} I_r &= \int_K g(Lu_0)\big((\mathcal{P}f)(rLu_0) - f(Lu_o)\big)d\varkappa(L) \\ &= \int_S \Big(\int_K f(Lu)\big((g(Lu_0) - g(Lu)\big)d\varkappa(L)\Big)P(u,ru_0)d\mu(u). \end{aligned}$$

We leave the details of the straightforward computations to the reader.

Let $\eta > 0$ be arbitrary. By writing S as the disjoint union

$$S = \{u \in S;\ \|u - u_0\| \le \eta\} \cup \{u \in S;\ \|u - u_0\| > \eta\}$$

we obtain a corresponding decomposition of I_r into integrals $I_r(1)$ and $I_r(2)$. By the uniform continuity of the function $g \in C(S)$, the integrals $I_r(1)$ tend to 0 as $\eta \downarrow 0$. Since

$$I_r(2) \ \le\ 2\|f\|_{L^\infty(\mu)}\|g\|_{\infty,S} \int\limits_{\|u-u_0\|>\eta} P(u, ru_0)d\mu(u),$$

condition (iv) implies that $\lim\limits_{r\uparrow 1} I_r = 0$.

Thus it follows that μ is a faithful Henkin probability measure supported by the Shilov boundary $S = \partial_0 D$ of $A(D) = A_0(D)$. Since conditions (F1) to (F4) hold for D, we obtain part (b) of Corollary 1.5.

5.2 Corollary. *Each subnormal n-tuple of class $\mathbb{A}$ over a bounded symmetric and circled domain $D \subset \mathbb{C}^n$ is reflexive.* □

The remaining part of this article is devoted to the study of product sets $D = D_1 \times D_2$, where $D_i \subset X_i$ are relatively compact open subsets of given complex submanifolds $X_i \subset \mathbb{C}^{n_i}$. We shall see that the product D satisfies conditions (F1), (F3) and (F4) whenever the sets D_1 and D_2 possess these properties. Whether the same result holds for property (F2) is not clear to us, but we shall indicate some positive results in this direction.

Our first aim is to show that the product of faithful Henkin measures is a faithful Henkin measure again. For this purpose, we need some results from the theory of vector-valued integration.

Let E be a separable complex Banach space and let μ be a finite positive measure on a compact set $K \subset \mathbb{C}^n$. By $L^1(\mu, E)$ we denote the Banach space of all equivalence classes of μ-Bochner-integrable functions $f : K \to E$. To describe the dual of $L^1(\mu, E)$, we call a function $f : K \to E'$ weak* μ-measurable if, for each $x \in E$, the function $\langle x, f(\cdot)\rangle : K \to \mathbb{C}$ is μ-measurable in the usual sense (see Chapter X in [20]). We denote the set of all bounded weak* μ-measurable functions $f : K \to E'$ by $\mathrm{BM}(\mu, E')$. The space $L^\infty(\mu, E')$ of all equivalence classes of functions $f \in \mathrm{BM}(\mu, E')$ modulo equality μ-almost everywhere becomes a Banach space under the essential supremum norm. By Proposition IV.7.16 in [28], the bilinear form $b_\mu : L^1(\mu, E) \times L^\infty(\mu, E') \to \mathbb{C},\ ([f], [g]) \mapsto \int_K \langle f(z), g(z)\rangle d\mu(z)$, induces an isometric isomorphism

$$\Delta_\mu : L^\infty(\mu, E') \longrightarrow (L^1(\mu, E))', \qquad g \mapsto b_\mu(\cdot, g).$$

By a lifting of $L^\infty(\mu)$ we mean an isometric C^*-algebra homomorphism $\rho : L^\infty(\mu) \to BM(\mu)$ such that $\rho([f]) = f$ μ-almost everywhere. For the existence of liftings, we refer the reader to [17].

Though we usually work with equivalence classes of measurable functions, some of our arguments will become clearer when we formulate them on the level of functions rather than equivalence classes. Therefore we introduce the abbreviations $\mathcal{L}^1(\mu)$ ($\mathcal{L}^\infty(\mu)$) for the spaces of all μ-integrable (bounded μ-measurable) functions $f : K \to \mathbb{C}$.

Let $K_i \subset \mathbb{C}^{n_i}$ $(i = 1, 2)$ be compact sets and let $\mu_i \in M^+(K_i)$ be finite positive Borel measures on K_i. Then the product measure $\mu_1 \times \mu_2$ is a finite positive Borel measure on $K_1 \times K_2$ (Proposition 7.6.2 in [5]). By using, for instance, the density of the space of all continuous functions one can easily show that the map $\pi = \pi_{\mu_1,\mu_2} : L^1(\mu_1 \times \mu_2) \to L^1(\mu_1, L^1(\mu_2))$,

$$[f] \mapsto [z_1 \mapsto [f(z_1, \cdot)]],$$

where $[f(z_1, \cdot)] = 0$ if $f(z_1, \cdot) \notin \mathcal{L}^1(\mu_2)$, is an isometric isomorphism between Banach spaces. By dualizing and passing to inverses we obtain a dual algebra isomorphism

$$L^\infty(\mu_1 \times \mu_2) \overset{S_{\mu_1,\mu_2}}{\longrightarrow} L^\infty(\mu_1, L^\infty(\mu_2)).$$

For a bounded operator $T : E \to F$ beween separable Banach spaces E and F, the induced operator $T' : L^\infty(\mu, F') \to L^\infty(\mu, E')$, $\quad f \mapsto T'f$, is the adjoint of the corresponding operator $T : L^1(\mu, E) \mapsto L^1(\mu, F)$. In particular, if $Z \subset E'$ is a weak* closed subspace, then we can identify $L^\infty(\mu, Z)$ with a weak* closed subspace of $L^\infty(\mu, E')$ via the inclusion map $j : Z \to E'$.

Let $D \subset X$ be a relatively compact open subset of a complex submanifold $X \subset \mathbb{C}^n$. We denote by $\lambda \in M^+(\bar{D})$ as before the trivial extension of the volume measure of D to $\bar{D}$.

5.3 Lemma. *Let E, F be separable Banach spaces, $T \in L(E, F)$ a continuous linear operator and $Z \subset E'$ a weak* closed subspace.*

(a) The space $H^\infty(D, E')$ is a weak closed subspace of $L^\infty(\lambda, E')$.*

(b) The map $T' : L^\infty(\lambda, F') \to L^\infty(\lambda, E')$ induces a weak continuous operator $T' : H^\infty(D, F') \to H^\infty(D, E')$.*

(c) The weak topology on $H^\infty(D, Z)$ coincides with the relative topology of the weak* topology of $H^\infty(D, E')$ and of $L^\infty(\lambda, E')$.*

Proof. We briefly indicate the reduction of part (a) to the well-known scalar case. Then (b) and (c) will easily follow.

Let $B \subset L^\infty(\lambda, E')$ be the closed unit ball. By Krein-Smulian it suffices to show that $H^\infty(D, E') \cap B \subset B$ is weak* closed. Since $L^1(\lambda, E)$ is separable, the relative weak* topology on B is metrizable. Let (f_k) be a sequence in $H^\infty(D, E')$ which is weak* convergent to some element $f \in L^\infty(\lambda, E')$.

Then, for each $h \in \mathcal{L}^1(\lambda)$ and each vector $a \in E$, we have

$$\int_D h\langle a, f_k\rangle d\lambda \xrightarrow{k} \int_D h\langle a, f\rangle d\lambda.$$

Since $H^\infty(D) \subset L^\infty(\lambda)$ is weak* closed, for each vector $a \in E$, there is a unique function $f_a \in H^\infty(D)$ with $f_a = \langle a, f\rangle$ in $L^\infty(\lambda)$. Since $\|f_a\|_{\infty,D} \leq \|a\|\|f\|_{L^\infty(\lambda,E')}$, the map

$$g : D \to E', \quad g(z)(a) = f_a(z)$$

is bounded and holomorphic with $\langle a, g\rangle = \langle a, f\rangle$ in $L^\infty(\lambda)$, that is, λ-almost everywhere for each vector $a \in E$. Since E is separable, it follows that $f = g$ in $L^\infty(\lambda, E')$. □

Let $D_i \subset X_i$ be relatively compact open subsets of complex submanifolds $X_i \subset \mathbb{C}^{n_i}$. Let $\lambda_1 \in M_1^+(\bar{D}_1)$ and $\lambda_2 \in M_1^+(\bar{D}_2)$ denote the trivial extensions of the volume measures. The space $H^\infty(D_1, H^\infty(D_2))$ can be regarded as a weak* closed subspace of each of the spaces $H^\infty(D_1, L^\infty(\lambda_2))$, $L^\infty(\lambda_1, H^\infty(D_2))$ and $L^\infty(\lambda_1, L^\infty(\lambda_2))$. The relative topologies of the corresponding weak* topologies of the latter spaces coincide on $H^\infty(D_1, H^\infty(D_2))$. All of these spaces are dual algebras with respect to pointwise multiplication.

5.4 Lemma. *The mapping*

$$S : H^\infty(D_1 \times D_2) \to H^\infty(D_1, H^\infty(D_2)), \quad (Sf)(z_1) = f(z_1, \cdot)$$

is an isomorphism of dual algebras.

Proof. Let $f \in H^\infty(D_1 \times D_2)$ be given. Then $F : D_1 \to H^\infty(D_2)$, $F(z_1) = f(z_1, \cdot)$ is a bounded function with $\|F\|_{\infty,D_1} = \|f\|_{\infty,D_1 \times D_2}$. We identify $H^\infty(D_2)$ isometrically with the dual space of the Banach space $Q(D_2) = L^1(D_2)/{}^\perp H^\infty(D_2)$. To show that F is analytic it suffices to observe that the functions

$$F_\varphi : D_1 \to \mathbb{C}, \quad z_1 \mapsto \langle [\varphi], F(z_1)\rangle = \int_{D_2} \varphi(z_2) f(z_1, z_2) d\lambda_2(z_2) \quad (\varphi \in \mathcal{L}^1(D_2))$$

are analytic. Thus the map S is well defined and isometric. Obviously, S is a surjective algebra homomorphism.

To see that S is weak* continuous, it suffices to check that the diagram

$$\begin{array}{ccc} L^\infty(\lambda_1, L^\infty(\lambda_2)) & \xrightarrow{\pi'_{\lambda_1,\lambda_2}} & L^\infty(\lambda_1, \times\lambda_2) \\ \cup\big\uparrow & & \big\uparrow\cup \\ H^\infty(D_1, H^\infty(D_2)) & \xrightarrow{S^{-1}} & H^\infty(D_1 \times D_2) \end{array}$$

commutes. $\square$

Let $\mu_i \in M^+(\bar{D}_i)$ $(i = 1, 2)$ be finite positive measures. Then the map

$$\tau = \tau_{\mu_1,\mu_2} : L^\infty(\mu_1 \times \mu_2) \to L^\infty(\mu_2 \times \mu_1), \quad \tau([f]) = [f^{\mathrm{op}}],$$

where $f^{\mathrm{op}}(z_2, z_1) = f(z_1, z_2)$, defines a dual algebra isomorphism which is the adjoint of the corresponding map

$$L^1(\mu_2 \times \mu_1) \to L^1(\mu_1 \times \mu_2), \quad [f] \mapsto [f^{\mathrm{op}}].$$

5.5 Proposition. *Let $\mu \in M^+(\bar{D}_2)$ be a finite positive measure. Then the dual algebra isomorphism Γ defined as the composition*

$$\Gamma : L^\infty(\lambda_1, L^\infty(\mu)) \overset{S^{-1}}{\to} L^\infty(\lambda_1 \times \mu) \overset{\tau}{\to} L^\infty(\mu \times \lambda_1) \overset{S}{\to} L^\infty(\mu, L^\infty(\lambda_1)),$$

where $S^{-1} = S^{-1}_{\lambda_1,\mu}$ and $S = S_{\mu,\lambda_1}$, maps the space $H^\infty(D_1, L^\infty(\mu))$ into $L^\infty(\mu, H^\infty(D_1))$.

Proof. Fix a lifting $\rho : L^\infty(\mu) \to BM(\mu)$ and a function $F \in H^\infty(D_1, L^\infty(\mu))$. We claim that

$$\tilde{F} : \bar{D}_2 \to H^\infty(D_1), \quad (\tilde{F}(z_2))(z_1) = \rho(F(z_1))(z_2)$$

defines a function $\tilde{F} \in \mathrm{BM}(\mu, H^\infty(D_1))$ with $\Gamma(F) = [\tilde{F}]$. Obviously, we have $\|\tilde{F}\|_{\infty,\bar{D}_2} = \|F\|_{\infty,D_1}$. Note that the function $\rho \circ F \in H^\infty(D_1, \mathrm{BM}(\mu))$ is a bounded λ_1-measurable map (see Chapter X, §1 in [20]). Hence, for any function $\varphi \in \mathcal{L}^1(\lambda_1)$, the product $\varphi \cdot (\rho \circ F)$ is a λ_1-integrable function with values in the Banach space $\mathrm{BM}(\mu)$. An elementary calculation shows that

$$\langle [\varphi] \ + \ {}^\perp H^\infty(D_1), \tilde{F}(\cdot)\rangle = \int_{D_1} \varphi(\rho \circ F) d\lambda_1 \in \mathrm{BM}(\mu).$$

Here $\langle \cdot, \cdot \rangle$ refers to the duality $\langle L^1(\lambda_1)/{}^\perp H^\infty(D_1), H^\infty(D_1)\rangle$. Thus we have shown that $\tilde{F} \in \mathrm{BM}(\mu, H^\infty(D_1))$.

Next the reader should observe that the function $f : \bar{D}_1 \times \bar{D}_2 \to \mathbb{C}$ defined by $f(z_1, z_2) = \rho(F(z_1))(z_2)$ $(= 0$ if $z_1 \notin D_1)$ is $(\lambda_1 \times \mu)$-measurable. Indeed, since the trivial extension of F to $\bar{D}_1$ is a λ_1-measurable function with values in the Banach space $L^\infty(\mu)$, there is a λ_1-zero set $N_1 \subset \bar{D}_1$ and a sequence (F_k) of $L^\infty(\mu)$-valued step functions such that

$$(F_k(z_1)) \xrightarrow{k} F(z_1) \qquad (z_1 \in \bar{D}_1 \setminus N_1).$$

Now it suffices to check that the functions

$$f_k : \bar{D}_1 \times \bar{D}_2 \to \mathbb{C}, \quad f_k(z_1, z_2) = \rho(F_k(z_1))(z_2)$$

are $(\lambda_1 \times \mu)$-measurable and converge pointwise to f on the complement of the $(\lambda_1 \times \mu)$-zero set $N_1 \times \bar{D}_2$.

Using Fubini's theorem one can easily deduce that $S^{-1}([F]) = [f]$. Indeed, for $g \in \mathcal{L}^1(\lambda_1 \times \mu)$, we find that

$$\begin{aligned}
\langle [g], S^{-1}([F]) \rangle &= \langle \pi_{\lambda_1,\mu}([g]), [F] \rangle \\
&= \int_{\bar{D}_1} \langle [g(z_1, \cdot)], F(z_1) \rangle d\lambda_1(z_1) \\
&= \int_{\bar{D}_1} \left(\int_{\bar{D}_2} g(z_1, z_2) \rho(F(z_1))(z_2) d\mu(z_2) \right) d\lambda_1(z_1) \\
&= \int_{\bar{D}_1 \times \bar{D}_2} g f d(\lambda_1 \times \mu) \\
&= \langle [g], [f] \rangle.
\end{aligned}$$

A repetition of the last argument, with the roles of λ_1 and μ exchanged, shows that $S^{-1}([\tilde{F}]) = [f^{\mathrm{op}}]$. Thus the proof is complete. □

In the setting of Proposition 5.5 the dual algebra isomorphism

$$\Gamma : L^\infty(\lambda_1, L^\infty(\mu)) \to L^\infty(\mu, L^\infty(\lambda_1))$$

induces an isometric weak* continuous algebra homomorphism

$$\gamma : H^\infty(D_1, L^\infty(\mu)) \to L^\infty(\mu, H^\infty(D_1)).$$

Fix a function $F \in H^\infty(D_1, L^\infty(\mu))$. Suppose that $\hat{F} : D_1 \to \mathrm{BM}(\mu)$ is a function with $[\hat{F}(z_1)] = F(z_1)$ in $L^\infty(\mu)$ for all $z_1 \in D_1$ such that

(i) $\tilde{F} : \bar{D}_2 \to H^\infty(D_1)$, $(\tilde{F}(z_2))(z_1) = (\hat{F}(z_1))(z_2)$ is a well-defined function in $\mathrm{BM}(\mu, H^\infty(D_1))$ and

(ii) $f : \bar{D}_1 \times \bar{D}_2 \to \mathbb{C}$, $f(z_1, z_2) = (\hat{F}(z_1))(z_2)$ $(= 0$ if $z_1 \notin D_1)$ is $(\lambda_1 \times \mu)$-measurable.

Then a careful analysis of the last proof shows that $\gamma(F) = [\tilde{F}]$.

5.6 Corollary. *Suppose that $\sigma_1 \in M^+(\bar{D}_1)$ and $\sigma_2 \in M^+(\bar{D}_2)$ are faithful Henkin measures. Then $\sigma_1 \times \sigma_2 \in M^+(\bar{D}_1 \times \bar{D}_2)$ is a faithful Henkin measure.*

Proof. By assumption there are isometric and weak* continuous algebra homomorphisms

$$r_i : H^\infty(D_i) \to L^\infty(\sigma_i) \qquad (i = 1, 2)$$

such that $r_i(f|D_i) = [f|\bar{D}_i]$ for $f \in \mathcal{O}(\bar{D}_i)$ and $i = 1, 2$. Let us denote by $r : H^\infty(D_1 \times D_2) \to L^\infty(\sigma_1 \times \sigma_2)$ the composition of the following isometric weak* continuous algebra homomorphisms

$$\begin{array}{rcccc} H^\infty(D_1 \times D_2) & \stackrel{S}{\to} & H^\infty(D_1, H^\infty(D_2)) & \stackrel{r_2}{\to} & H^\infty(D_1, L^\infty(\sigma_2)) \\ & \stackrel{\gamma}{\to} & L^\infty(\sigma_2, H^\infty(D_1)) & \stackrel{r_1}{\to} & L^\infty(\sigma_2, L^\infty(\sigma_1)) \\ & \stackrel{S^{-1}}{\to} & L^\infty(\sigma_2 \times \sigma_1) & \stackrel{\tau}{\to} & L^\infty(\sigma_1 \times \sigma_2). \end{array}$$

Consider an arbitrary holomorphic map $f : U_1 \times U_2 \to \mathbb{C}$ where $U_i \supset \bar{D}_i$ are open neighbourhoods of $\bar{D}_i$ in $\mathbb{C}^{n_i}$. Then $F = r_2 \circ S(f|D_1 \times D_2)$ acts as

$$F(z_1) = [f(z_1, \cdot)|\bar{D}_2] \qquad (z_1 \in D_1).$$

The remarks preceding Corollary 5.6 imply that $\gamma(F) = [\tilde{F}]$, where

$$\tilde{F} : \bar{D}_2 \to H^\infty(D_1), \quad \tilde{F}(z_2) = f(\cdot, z_2)|D_1.$$

It follows that $r_1 \circ \gamma \circ r_2 \circ S(f|D_1 \times D_2) = [G]$ where

$$G : \bar{D}_2 \to L^\infty(\sigma_1), \quad G(z_2) = [f(\cdot, z_2)|\bar{D}_1].$$

To deduce that $r(f|D_1 \times D_2) = [f|\bar{D}_1 \times \bar{D}_2]$, and thus to complete the proof, it suffices to observe that

$$\begin{aligned} \langle [g], S^{-1}([G]) \rangle &= \langle \pi_{\sigma_2,\sigma_1}([g]), [G] \rangle \\ &= \int_{\bar{D}_2} \langle [g(z_2, \cdot)], G(z_2) \rangle d\sigma_2(z_2) \\ &= \int_{\bar{D}_2} \left(\int_{\bar{D}_1} g(z_2, z_1) f(z_1, z_2) d\sigma_1(z_1) \right) d\sigma_2(z_2) \\ &= \int_{\bar{D}_2 \times \bar{D}_1} g(z_1, z_2) f^{\mathrm{op}}(z_2, z_1) d(\sigma_2 \times \sigma_1)(z_1, z_2) \\ &= \langle [g], [f^{\mathrm{op}}|\bar{D}_2 \times \bar{D}_1] \rangle \end{aligned}$$

holds for all functions $g \in \mathcal{L}^1(\sigma_2 \times \sigma_1)$. □

Let us denote by S_i $(i = 1, 2)$ the Shilov boundaries of the Banach algebras $A_0(D_i)$, and let S be the Shilov boundary of $A_0(D_1 \times D_2)$. Condition (F3) for D_1 and D_2 implies that the sets S_i are given by the peak points for $A_0(D_i)$ (Lemma 4.4). This condition in turn allows us to prove the identity $S = S_1 \times S_2$.

5.7 Lemma. *Let D_1 and D_2 satisfy condition (F3). Then we have:*

(a) *each point in $S_1 \times S_2$ is a peak point for $A_0(D_1 \times D_2)$;*

(b) *$S = S_1 \times S_2$;*

(c) *$D_1 \times D_2$ satisfies condition (F3).*

Proof. (a) Let $w = (w_1, w_2) \in S_1 \times S_2$. Lemma 4.4 allows us to choose peaking functions $h_i \in A_0(D_i)$ for w_i $(i = 1, 2)$. Then $h \in A_0(D_1 \times D_2)$ defined by $h(z_1, z_2) = h_1(z_1)h_2(z_2)$ is a peaking function for w.

(b) Consider a function $f \in A_0(D_1 \times D_2)$ and a point $z = (z_1, z_2)$ in $\bar{D}_1 \times \bar{D}_2$. Since $f(\cdot, z_2) \in A_0(D_1)$, there is a point $w_1 \in S_1$ with $|f(z)| \leq |f(w_1, z_2)|$. Repeating this argument, we find a point $w_2 \in S_2$ with the property $|f(w_1, z_2)| \leq |f(w_1, w_2)|$. Hence $S_1 \times S_2$ is a boundary for $A_0(D_1 \times D_2)$ and therefore contains S. The reverse inclusion follows from part (a).

(c) Let $F_i \in A_0(D_i)^{N_i}$ $(i = 1, 2)$ be injective maps with $F_i(S_i) \subset \partial\mathbb{B}_{N_i}$. Then the map $F \in A_0(D_1 \times D_2)^{N_1+N_2}$ defined by $F(z_1, z_2) = (F_1(z_1), F_2(z_2))/\sqrt{2}$ is injective and maps $S_1 \times S_2$ into $\partial\mathbb{B}_{N_1+N_2}$. □

Now it is easy to deduce that the product $D_1 \times D_2$ of domains both satisfying conditions (F3) and (F4) again satisfies these conditions.

5.8 Corollary. *Let $D_i \subset X_i$ be relatively compact open subsets of complex submanifolds $X_i \subset \mathbb{C}^{n_i}$ $(i = 1, 2)$. Suppose that D_1 and D_2 satisfy conditions (F1), (F3) and (F4). Then the product domain $D_1 \times D_2$ satisfies the same conditions.*

Proof. Since the product $U_1 \times U_2$ of any two Stein open subsets $U_i \subset X_i$ is a Stein open set in $X = X_1 \times X_2$, it is clear that condition (F1) is inherited by $D = D_1 \times D_2$. To see that D satisfies conditions (F3) and (F4) it suffices to apply Corollary 5.6 and Lemma 5.7. □

We do not know whether condition (F2) is inherited by product domains in general. Let $D_i \subset X_i$ be relatively compact open subsets of complex submanifolds $X_i \subset \mathbb{C}^{n_i}$ $(i = 1, 2)$ such that $\mathcal{O}(\bar{D}_i)$ is weak* dense in $H^\infty(D_i)$. A necessary condition for $\mathcal{O}(\bar{D}_1 \times \bar{D}_2)$ to be weak* dense in $H^\infty(D_1 \times D_2)$ is the weak* density of the set

$$\mathrm{LH}\{f \otimes g : f \in H^\infty(D_1) \text{ and } g \in H^\infty(D_2)\}$$

in $H^\infty(D_1 \times D_2)$. Our first elementary observation is that this condition is also sufficient.

5.9 Lemma. *Suppose that*

$$LH\{f \otimes g : f \in H^\infty(D_1) \text{ and } g \in H^\infty(D_2)\} \subset H^\infty(D_1 \times D_2)$$

is weak dense. If $\mathcal{O}(\bar{D}_1)|D_1 \subset H^\infty(D_1)$ and $\mathcal{O}(\bar{D}_2)|D_2 \subset H^\infty(D_2)$ are weak* dense, then $\mathcal{O}(\bar{D}_1 \times \bar{D}_2)|D_1 \times D_2 \subset H^\infty(D_1 \times D_2)$ is weak* dense.*

Proof. Let $f \in H^\infty(D_1)$ and $g \in H^\infty(D_2)$ be fixed. Choose a net (f_α) in $\mathcal{O}(\bar{D}_1)$ such that $(f_\alpha) \xrightarrow{\alpha} f$ weak* in $H^\infty(D_1)$. Then $(f_\alpha \otimes g) = ((f_\alpha \otimes 1)(1 \otimes g))$ is weak* convergent to $(f \otimes 1)(1 \otimes g) = f \otimes g$ in $H^\infty(D_1 \times D_2)$. Thus we have shown that

$$\mathrm{LH}\{f \otimes g : f \in \mathcal{O}(\bar{D}_1)|D_1 \text{ and } g \in H^\infty(D_2)\} \subset H^\infty(D_1 \times D_2)$$

is weak* dense. A repetition of the above argument, using this time the weak* density of $\mathcal{O}(\bar{D}_2)$ in $H^\infty(D_2)$, yields the assertion. □

By slightly strengthening condition (F2) for one of the factors, we obtain the desired result.

5.10 Lemma. *Suppose that*

$$\mathcal{O}(\bar{D}_1, H^\infty(D_2))|D_1 \subset H^\infty(D_1, H^\infty(D_2))$$

is weak dense. Then both*

$$LH\{f \otimes g : f \in H^\infty(D_1) \text{ and } g \in H^\infty(D_2)\} \subset H^\infty(D_1 \times D_2)$$

and $\mathcal{O}(\bar{D}_1)|D_1 \subset H^\infty(D_1)$ are weak dense.*

Proof. Let $f \in \mathcal{O}(U, H^\infty(D_2))$ be analytic on some open neighbourhood $U \supset \bar{D}_1$ in $\mathbb{C}^{n_1}$. Since $\mathcal{O}(U, H^\infty(D_2)) \cong \mathcal{O}(U)\widehat{\otimes} H^\infty(D_2)$, there is a sequence of functions

$$f^{(k)} = \sum_{i=1}^{r_k} f_i^{(k)} \otimes g_i^{(k)} \qquad (f_i^{(k)} \in \mathcal{O}(U), \quad g_i^{(k)} \in H^\infty(D_2))$$

such that $(f^{(k)}|D_1) \xrightarrow{k} f|D_1$ in $H^\infty(D_1, H^\infty(D_2))$. Thus we see that

$$\mathrm{LH}\{f \otimes g : f \in H^\infty(D_1) \text{ and } g \in H^\infty(D_2)\} \subset H^\infty(D_1, H^\infty(D_2))$$

is weak* dense. Using the dual algebra isomorphism $S : H^\infty(D_1 \times D_2) \to H^\infty(D_1, H^\infty(D_2))$, $(Sf)(z_1) = f(z_1, \cdot)$, one obtains the first assertion.

To prove the weak* density of $\mathcal{O}(\bar{D}_1)$ in $H^\infty(D_1)$, denote the point evaluation at an arbitrary, but fixed point $w \in D_2$ by $\mathcal{E}_w : H^\infty(D_2) \to \mathbb{C}$, $f \mapsto f(w)$, and observe that the map

$$H^\infty(D_1, H^\infty(D_2)) \to H^\infty(D_1), \quad f \mapsto \mathcal{E}_w \circ f$$

is surjective, weak* continuous and maps $\mathcal{O}(\bar{D}_1, H^\infty(D_2))$ into $\mathcal{O}(\bar{D}_1)$. $\square$

The hypothesis of the last lemma is satisfied, for instance, if $D_1 \subset X_1$ is a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X_1 \subset \mathbb{C}^{n_1}$.

5.11 Theorem. *Suppose that $X_1 \subset \mathbb{C}^{n_1}$ is a Stein submanifold and that $D_1 \subset X_1$ is a relatively compact strictly pseudoconvex open subset. Then*

$$\mathcal{O}(\bar{D}_1, H^\infty(D_2))|D_1 \subset H^\infty(D_1, H^\infty(D_2))$$

is weak dense. The same is true for every bounded convex open set D_1 in $\mathbb{C}^{n_1}$.*

Proof. To simplify the notation, we write $E = H^\infty(D_2)$. By a well-known embedding theorem of Fornæss (Theorem 10 in [14]), there are a Stein open neighbourhood V of $\bar{D}_1$ in X_1, a biholomorphic map $\psi : V \to Y$ onto a closed complex submanifold $Y \subset \mathbb{C}^m$ for a suitable m, and a C^2-strictly convex bounded open set $C \subset \mathbb{C}^m$ such that

$$\psi(D_1) = Y \cap C, \qquad \psi(\partial D_1) = Y \cap \partial C.$$

Let $\psi_0 : D_1 \to Y \cap C$, $z \mapsto \psi(z)$, be the biholomorphic map induced by ψ. Fix a function $f \in H^\infty(D_1, E)$. Then $f \circ \psi_0^{-1}$ belongs to $H^\infty(Y \cap C, E)$. By a theorem of Henkin and Leiterer (Theorem 4.11.1 and the subsequent remarks in [16]) the restriction map

$$r : H^\infty(C) \to H^\infty(Y \cap C), \quad g \mapsto g|Y \cap C$$

is onto and possesses a continuous linear right inverse. By Lemma 5.4 there are canonical isometric identifications

$$\begin{aligned} H^\infty(C, H^\infty(D_2)) &\cong H^\infty(D_2, H^\infty(C)), \\ H^\infty(Y \cap C, H^\infty(D_2)) &\cong H^\infty(D_2, H^\infty(Y \cap C)). \end{aligned}$$

Using these identifications and the fact that r has a bounded linear right inverse, we find that the restriction map

$$H^\infty(C, E) \to H^\infty(Y \cap C, E), \quad g \mapsto g|Y \cap C$$

remains surjective. Choose a function $G \in H^\infty(C, E)$ with $G|Y \cap C = f \circ \psi_0^{-1}$. Since $C \subset \mathbb{C}^m$ is a convex bounded open set, there is a sequence of holomorphic functions $G_k : U_k \to E$ defined on suitable open neighbourhoods U_k of $\bar{C}$ such that the sequence $(G_k|C)$ is bounded in $H^\infty(C, E)$ and converges pointwise on C to G.

Since $\bar{D}_1$ is a Stein compact subset of X_1 (Corollary 2.1.2 in [8]), there are Stein open neighbourhoods V_k of $\bar{D}_1$ in X_1 and holomorphic mappings

$F_k : V_k \to E$ such that the sequence $(F_k|D_1)_{k\in\mathbb{N}}$ is bounded in $H^\infty(D_1, E)$ and converges pointwise on D_1 to f. In fact, one can choose the functions F_k as suitable restrictions of the functions

$$\overset{-1}{\psi}(U_k) \to E, \quad z \mapsto G_k(\psi(z)).$$

As an application of a theorem of Siu ([30]) on the existence if Stein open neighbourhoods of Stein submanifolds, one can show that the sets V_k are closed complex submanifolds of suitable Stein open sets $W_k \subset \mathbb{C}^{n_1}$ (cf. the remark preceding Proposition 2.1.6 in [8]). Since the restriction maps

$$\mathcal{O}(W_k, E) \cong \mathcal{O}(W_k)\widehat{\otimes}_\pi E \to \mathcal{O}(V_k)\widehat{\otimes}_\pi E \cong \mathcal{O}(V_k, E)$$

are onto (Corollary 4.1.8 in [16]), we can choose holomorphic extensions $f_k : W_k \to E$ of the functions $F_k : V_k \to E$ constructed above. To complete the proof, it suffices to observe that $(f_k|D_1)$ is weak* convergent to f.

In the case where D_1 is a bounded convex open set in $\mathbb{C}^{n_1}$ the assertion is obviously true. □

Corollary 5.8, Lemma 5.9 and Theorem 5.11 imply that the class of sets satisfying the conditions (F1) to (F4) is closed under the formation of Cartesian products with strictly pseudoconvex sets and circled bounded symmetric domains. Thus we have shown that part (c) of Corollary 1.5 holds.

5.12 Corollary. *If $D = D_1 \times \cdots \times D_k$ is a product set where each factor D_i is either a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X_i \subset \mathbb{C}^{n_i}$ or a circled bounded symmetric domain in $\mathbb{C}^{n_i}$, then each subnormal n-tuple of class $\mathbb{A}$ over D is reflexive.* □

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