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ON THE TORSION OF OPTIMAL ELLIPTIC CURVES OVER FUNCTION FIELDS

MIHRAN PAPIKIAN

ABSTRACT. For an optimal elliptic curve E over $\mathbb{F}_q(t)$ of conductor $\mathfrak{p}\cdot\infty$, where $\mathfrak{p}$ is prime, we show that $E(F)_{\mathrm{tor}}$ is generated by the image of the cuspidal divisor group.

1. Introduction

Let $\mathbb{F}_q$ denote the finite field of q elements. We let p be the characteristic of $\mathbb{F}_q$ (so q is a power of p). Let $F=\mathbb{F}_q(t)$ be the field of rational functions on $\mathbb{P}^1_{\mathbb{F}_q}$, and let $A=\mathbb{F}_q[t]$ be the subring of F consisting of functions which are regular away from $\infty := 1/t$.

Let $\mathfrak{p}$ be a fixed prime ideal of A. Denote by $Y_0(\mathfrak{p})$ the coarse moduli scheme of pairs (D,Z), where D is a rank-2 Drinfeld A-module of general characteristic, and Z is a $\mathfrak{p}$-cyclic subgroup of D; for the definitions see, for example, [3]. The scheme $Y_0(\mathfrak{p})$ is a smooth affine geometrically irreducible curve defined over F. Denote by $X_0(\mathfrak{p})$ the unique smooth compactification of $Y_0(\mathfrak{p})$ over F. Let J be the Jacobian variety of $X_0(\mathfrak{p})$. The complement of $Y_0(\mathfrak{p})$ in $X_0(\mathfrak{p})$ consists of two F-rational points; these are called the *cusps* of $X_0(\mathfrak{p})$. The divisor on $X_0(\mathfrak{p})$ which is the difference of the two cusps generates a finite cyclic subgroup $\mathcal{C}$ of $J(F)$ called the *cuspidal divisor group.* It is known that $\mathcal{C}$ has order $N(\mathfrak{p})$, where $N(\mathfrak{p})=\frac{q^d-1}{q-1}$, if $d:=\deg(\mathfrak{p})$ is odd, and $N(\mathfrak{p})=\frac{q^d-1}{q^2-1}$ if d is even.

Let $\mathcal{J}$ be the Néron model of J over $\mathbb{P}^1_{\mathbb{F}_q}$. It is known that J has bad reduction only at two places of $\mathbb{P}^1_{\mathbb{F}_q}$, namely at $\mathfrak{p}$ and ∞. In other words, the v-fibre $\mathcal{J}_{\mathbb{F}_v}$ of $\mathcal{J}$ is not an abelian variety over $\mathbb{F}_v$ only when $v=\mathfrak{p}$ or $v=\infty$; here we denote by $\mathbb{F}_v$ the residue field at the place v. Moreover, it is known that the reduction of J at $\mathfrak{p}$ and ∞ is toric, i.e., the connected component of the identity $\mathcal{J}^0_{\mathbb{F}_v}$ is an algebraic torus over $\mathbb{F}_v$ when $v=\mathfrak{p},\infty$. We denote $\mathcal{J}_{\mathbb{F}_v}/\mathcal{J}^0_{\mathbb{F}_v}$ by $\Phi_{J,v}$; this is a finite abelian group called the *group of connected components of* $\mathcal{J}$ *at* v. By what was said, the groups $\Phi_{J,v}$ are trivial if v is not $\mathfrak{p}$ or ∞. Taking the schematic closure of $\mathcal{C}$ in $\mathcal{J}$ and then specializing to the $\mathfrak{p}$-fibre, we get a natural homomorphism $\mathcal{C}\to\Phi_{J,\mathfrak{p}}$. Gekeler proved [2] that this is an isomorphism. More recently, Pál proved [7] that the inclusion $\mathcal{C}\subset J(F)_{\mathrm{tor}}$ is in fact an equality. These results are the function field analogues of some of the results of Mazur in his celebrated paper [5].

The aim of the present article is to show that for certain one-dimensional quotients of J the F-rational torsion is again cuspidal, i.e., is generated by the image of $\mathcal{C}$. Let E be an elliptic curve over F. We say that E is *optimal* if there is a

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homomorphism $J \to E$ with connected and smooth kernel (i.e., the kernel is an abelian subvariety of J). An equivalent condition is that E is isomorphic to an abelian subvariety of J. If E is optimal then it has conductor $\mathfrak{p} \cdot \infty$ and the reduction of E at $\mathfrak{p}$ (resp. ∞) is multiplicative (resp. split multiplicative). For E we adopt notations similar to that for J, so, for example, $\mathcal{E}$ will be the Néron model of E over $\mathbb{P}^1_{\mathbb{F}_q}$ and $\Phi_{E,v}$ will be the v-fibre component group of $\mathcal{E}$. The main result is the following:

Theorem 1.1. *Let E be an optimal elliptic curve.*

(1) *The specialization map $E(F)_{\mathrm{tor}} \to \Phi_{E,\mathfrak{p}}$ is an isomorphism. In particular, $\mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p})$ acts trivially on $\Phi_{E,\mathfrak{p}}$.*
(2) *The homomorphism $J(F)_{\mathrm{tor}} \to E(F)_{\mathrm{tor}}$, induced from the quotient map $J \to E$, is surjective. In particular, $E(F)_{\mathrm{tor}}$ is generated by the image of the cuspidal divisor group $\mathcal{C}$ in E.*
(3) *$E(F)_{\mathrm{tor}} = \mathbb{Z}/n\mathbb{Z}$ for some $1 \leq n \leq 5$ coprime to p.*
(4) *The order of $\Phi_{E,\mathfrak{p}}$ divides the order of $\Phi_{E,\infty}$.*

This theorem is the function field analogue of a result over $\mathbb{Q}$ due to Mestre and Oesterlé [6]. Emerton [1] generalized Mestre-Oesterlé theorem from elliptic curves to arbitrary abelian subvarieties of the classical modular Jacobians. Both [6] and [1] extensively use in their proofs the results of Mazur [5] and Ribet [10]. One feature which is significantly different in our proof is that we completely avoid using any "level-lowering" results, and from the Eisenstein ideal theory essentially only need Pál's theorem $J(F)_{\mathrm{tor}} = \mathcal{C}$.

Example 1.2. It is remarkable that the possibilities for $E(F)_{\mathrm{tor}}$ in Theorem 1.1(3) exactly match the possibilities for the rational torsion of optimal elliptic curves over $\mathbb{Q}$ of prime conductor (aside from the requirement that n is coprime to p, of course). In fact, Mestre and Oesterlé show that $E(\mathbb{Q})_{\mathrm{tor}} = \mathbb{Z}/n\mathbb{Z}$ with $1 \leq n \leq 5$, and moreover give examples where all the cases occur. Unfortunately, at present we are unable to show that all the possibilities for $E(F)_{\mathrm{tor}}$ actually occur. This is due to lack of many examples of optimal elliptic curves over F. But at least there is an example due to Gekeler, which shows that $E(F)_{\mathrm{tor}}$ is not always trivial. Let $F = \mathbb{F}_7(t)$, and $E/F:\ y^2 = x^3+ax+b$, where $a = -3t(t^3+2)$ and $b = -2t^6+3t^3+1$. Then E is an optimal elliptic curve of conductor $(t^3-2)\cdot\infty$. One easily shows that $\#E(F)_{\mathrm{tor}} = \#\Phi_{E,\mathfrak{p}} = \#\Phi_{E,\infty} = 3$.

The quotient $\#\Phi_{E,\infty}/\#\Phi_{E,\mathfrak{p}}$ is an integer by Theorem 1.1(4), but it can be strictly larger than 1. Here is another example due to Gekeler. Let $F = \mathbb{F}_2(t)$ and $E/F:\ y^2 + txy + y = x^3 + x^2$. Then E is an optimal curve of conductor $(t^4+t^3+1)\cdot\infty$. By computing the j-invariant $j_E = t^{12}/(t^4+t^3+1)$ we conclude that $\Phi_{E,\mathfrak{p}} = 1$ but $\Phi_{E,\infty} = \mathbb{Z}/8\mathbb{Z}$.

The requirement on E being optimal in Theorem 1.1 is necessary. One way to see this is to recall that the rational torsion of elliptic curves over F is universally bounded, whereas the orders of component groups can be made arbitrarily large by taking the Frobenius conjugates of an elliptic curve.

In [9] we gave a formula for the variation of the orders of Tate-Shafarevich groups $Ш(E/K)$ of E over certain quadratic extensions K of F, under the assumption that E/K has analytic rank 0. One of the factors which appears in that formula is the fraction $\#E(F)_{\mathrm{tor}}/\#\Phi_{E,\mathfrak{p}}$. By Theorem 1.1 this fraction is always equal to 1, and hence can be omitted from the formula for $\#Ш(E/K)$. This was our initial motivation for considering the problem of the present article.

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2. Proof of the main theorem

Aside from the notation used in the introduction, we will also use the following notation and terminology: For a field L we will denote its algebraic closure by $\bar{L}$, and the separable closure by L^{sep}. By a finite flat group scheme over the base scheme S we always mean a finite flat *commutative* S-group scheme. We say that the finite flat group scheme G over $\mathbb{P}^1_{\mathbb{F}_q}$ is *constant* if it is étale and the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on $G_F(\overline{F})$ is trivial. We say that G is *μ-type* if its Cartier dual $G^\vee$ is constant. Given an abelian variety B, its dual abelian variety will be denoted by $\hat{B}$. As in [7], let $\mathfrak{E}$ be the *Eisenstein ideal* of the Hecke algebra $\mathbb{T}$, i.e., the ideal generated by the elements $T_\mathfrak{q} - q^{\deg(\mathfrak{q})} - 1$, where $\mathfrak{q} \neq \mathfrak{p}$ is any prime in A. We write $J[\mathfrak{E}]$ for the group of points in $J(\overline{F})$ which are killed by all elements of $\mathfrak{E}$. Let $\widetilde{F} := \overline{\mathbb{F}}_q(t)$. This is the maximal unramified extension of F.

Before giving the proof of Theorem 1.1, we need a preliminary lemma.

Lemma 2.1. *There is an inclusion $J(\widetilde{F})_{\mathrm{tor}} \subset J[\mathfrak{E}]$.*

Proof. Let $G = J(\widetilde{F})_{\mathrm{tor}}$. Since J is not isotrivial, G is finite. We claim that G has order coprime to p. To see this, fix a prime $\mathfrak{P}$ in $\widetilde{A} := \overline{\mathbb{F}}_q[t]$ over $\mathfrak{p}$. Let $k = \widetilde{A}/\mathfrak{P}$. Since $\widetilde{F}/F$ is unramified at $\mathfrak{p}$, the Néron model $\widetilde{\mathcal{J}}$ of $\widetilde{J} := J_{\widetilde{F}}$ over $\widetilde{A}_\mathfrak{P}$ is isomorphic to the base change of $\mathcal{J}$ to the strict henselization of $A_\mathfrak{p}$. In particular, $\Phi_{J,\mathfrak{p}} = \Phi_{\widetilde{J},\mathfrak{P}}$. Suppose G has non-trivial p-torsion. Fix a subgroup $G' \subset G$ of order p. By taking the schematic closure of G' in $\widetilde{\mathcal{J}}_{\widetilde{A}_\mathfrak{P}}$, we get a finite flat group scheme $\mathcal{G}'$ extending G over $\widetilde{A}_\mathfrak{P}$. If $\widetilde{\mathcal{J}}^0_k \cap \mathcal{G}'_k$ is non-trivial, then $\mathcal{G}'_k = \mu_p$ (as $\widetilde{\mathcal{J}}^0_k$ is a torus). This is impossible, since otherwise $(\mathcal{G}')^\vee$ has étale closed fibre but connected generic fibre (μ_p is connected in characteristic p). Hence we get a natural injection $G' \hookrightarrow \Phi_{J,\mathfrak{p}}$. This latter group is known to have no p-torsion, and we get a contradiction.

Next, we claim that G is an extension of a constant group scheme by a μ-type étale group scheme. Since G has order coprime to the characteristic of F (and hence also coprime to the characteristics all residue fields) and is unramified at all places, it extends to a finite étale group scheme $\mathcal{G}$ over $\mathbb{P}^1_{\mathbb{F}_q}$, cf. [4, §2]. It is easy to see that $\mathcal{G}$ is the schematic closure of G in $\mathcal{J}$. So we are reduced to studying the $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-structure of G. The action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on G factors through $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. By fixing a decomposition subgroup D_∞ of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ at ∞, we get a canonical inclusion $\mathrm{Gal}(\overline{\mathbb{F}}_\infty/\mathbb{F}_\infty) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. This latter map is an isomorphism as $\deg(\infty) = 1$. The specialization map $\mathcal{G} \to \mathcal{G}_{\mathbb{F}_\infty}$ commutes with the action of $\mathrm{Gal}(\overline{\mathbb{F}}_\infty/\mathbb{F}_\infty)$, so we are reduced to showing that $\mathcal{G}_{\mathbb{F}_\infty}$ is an extension of a constant group scheme over $\mathbb{F}_\infty$ by a μ-type étale group scheme. On the one hand, Drinfeld modular curves are totally degenerate at infinity, so $\mathcal{J}^0_{\mathbb{F}_\infty}$ is a split torus and by [4, §11] $\Phi_{J,\infty}$ is constant. On the other hand, $\mathcal{G}_{\mathbb{F}_\infty} \hookrightarrow \mathcal{J}_{\mathbb{F}_\infty}$. The claim follows.

Let $\mathcal{S}$ be the maximal μ-type étale subgroup of J. It is clear that $\mathcal{S} \subset G$. We claim that $\mathcal{L} := G/\mathcal{S}$ is a constant group scheme. Indeed, by what we have proved, we can write G as an extension of a constant group scheme by a μ-type étale group scheme. Since $\mathcal{S}$ is the maximal μ-type étale subgroup scheme of J, the group $\mathcal{L}$ must be constant.

It is clear that $\mathcal{S}$ and G are $\mathbb{T}$-modules, and they are also $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-invariant. Hence $\mathcal{L}$ is equipped with a commuting actions of $\mathbb{T}$ and the absolute Galois group, which satisfy the Eichler-Shimura congruence relations. We claim that the extension of $\mathbb{T}$-modules

$$0 \to \mathcal{S} \to G \to \mathcal{L} \to 0$$

in fact splits. The action of $\mathbb{T}$ uniquely extends by the universal property of Néron models to $\mathcal{J}$. Hence we get a natural map $\mathbb{T} \to \mathrm{End}_{\mathbb{F}_\mathfrak{p}}(\mathcal{J}_{\mathbb{F}_\mathfrak{p}})$, which is injective since $\mathcal{J}$ has toric reduction at $\mathfrak{p}$. Since the action of $\mathbb{T}$ on $\mathcal{J}_{\mathbb{F}_\mathfrak{p}}$ is continuous, it preserves $\mathcal{J}^0_{\mathbb{F}_\mathfrak{p}}$. Thus, $\Phi_{J,\mathfrak{p}}$ is naturally a $\mathbb{T}$-module. It is enough to show that the specialization of G to the $\mathfrak{p}$-fibre splits. This specialization provides a map $G \to \Phi_{J,\mathfrak{p}}$, and by restricting to $\mathcal{S}$, a map $\mathcal{S} \to \Phi_{J,\mathfrak{p}}$. This latter homomorphism is an isomorphism by Proposition 8.18 in [7], so the sequence splits as we have produced a $\mathbb{T}$-equivariant retraction $G \to \mathcal{S}$, cf. [5, p.142].

Now it is easy to see that G is annihilated by the Eisenstein ideal. Indeed, as a $\mathbb{T}$-module $G = \mathcal{S} \oplus \mathcal{L}$, and both summands are killed by $T_\mathfrak{q} - q^{\deg(\mathfrak{q})} - 1$, $\mathfrak{q} \neq \mathfrak{p}$, according to the Eichler-Shimura congruence relations. $\square$

Proof of Theorem 1.1. The dual of the optimal quotient map $\pi : J \to E$ is the closed immersion $\hat{\pi} : \hat{E} \hookrightarrow \hat{J}$, which, using the canonical self-duality of E and J, can be identified with a closed immersion $E \hookrightarrow J$. First, we claim that the functorially-induced homomorphism on component groups $\hat{\pi}_\Phi : \Phi_{E,\mathfrak{p}} \to \Phi_{J,\mathfrak{p}}$ is injective. It is enough to show that $\Phi_{E,\mathfrak{p}}[\ell] \to \Phi_{J,\mathfrak{p}}[\ell]$ is injective for any prime ℓ. Moreover, by Corollary 3.4 in [8] we can assume $\ell \neq p$. Suppose $\Phi_{E,\mathfrak{p}}[\ell]$ is non-trivial. Then by [4, §2] the ℓ-torsion of E is unramified at $\mathfrak{p}$. Indeed, according to *loc. cit.* $E[\ell]^{I_\mathfrak{p}}$ is isomorphic to $\mathcal{E}_{\mathbb{F}_\mathfrak{p}}[\ell]$, where $I_\mathfrak{p}$ is the inertia group at $\mathfrak{p}$. Our assumption implies that $\dim_{\mathbb{F}_\ell}(\mathcal{E}_{\mathbb{F}_\mathfrak{p}}[\ell]) = 2$, hence $E[\ell]^{I_\mathfrak{p}} = E[\ell]$ as $\dim_{\mathbb{F}_\ell}(E[\ell]) = 2$. We claim that $E[\ell]$ is in fact everywhere unramified. Since E has good reduction away from $\mathfrak{p}$ and ∞, using the Néron-Ogg-Shafarevich criterion, it is enough to show that $E[\ell]$ is unramified at ∞. It is even enough to show that $E[\ell]$ is at most tamely ramified at ∞. Indeed, $E[\ell]$ is a $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-module as E is defined over F. On the other hand, using Hurwitz genus formula, it is easy to see that F has no extensions ramified exactly at ∞ such that the ramification is tame. Since E has split multiplicative reduction at ∞, we can use Tate's uniformization of E to conclude that $F_\infty(E[\ell]) \subseteq F_\infty(\mu_\ell, \wp_E^{1/\ell})$, where $\wp_E$ is the Tate period of E at ∞. This latter extension of F_∞ is clearly at most tamely ramified. Hence $E[\ell] \subset E(\widetilde{F})$. Since E is an abelian subvariety of J, we have the inclusion $E(\widetilde{F}) \subset J(\widetilde{F})$. Using Lemma 2.1, we get $E[\ell] \subseteq D[\ell]$, where $D[\ell]$ denotes the ℓ-torsion subgroup of $J[\mathfrak{C}]$. According to [7, §§10-11], $\dim_{\mathbb{F}_\ell}(D[\ell]) = 2$, so $E[\ell] = D[\ell]$. There results a commutative functorial diagram

$$\begin{array}{ccc} E[\ell] & \longrightarrow & \Phi_{E,\mathfrak{p}}[\ell] \\ \| & & \downarrow \\ D[\ell] & \longrightarrow & \Phi_{J,\mathfrak{p}}[\ell]. \end{array}$$

The image of $D[\ell]$ in $\Phi_{J,\mathfrak{p}}[\ell]$ is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ by Proposition 8.18 in [7] and the description of $D[\ell]$ in Sections 10 and 11 of *loc. cit.* The elliptic curve E has multiplicative reduction at $\mathfrak{p}$, so $\Phi_{E,\mathfrak{p}}$ is cyclic. In particular, $\Phi_{E,\mathfrak{p}}[\ell] \cong \mathbb{Z}/\ell\mathbb{Z}$. Now it is easy to see from the above diagram that $\Phi_{E,\mathfrak{p}}[\ell] \to \Phi_{J,\mathfrak{p}}[\ell]$ must be injective, as we claimed.

Next, we claim that the functorially-induced homomorphism $\pi_\Phi : \Phi_{J,\mathfrak{p}} \to \Phi_{E,\mathfrak{p}}$ is surjective. For an abelian variety B over a local field Grothendieck defined a bifunctorial pairing [4, §1.2]: $\Phi_B \times \Phi_{\hat{B}} \to \mathbb{Q}/\mathbb{Z}$, which is perfect when B is semi-stable; see [4, §11]. Applied to our situation, this pairing induces a canonical isomorphism between $\mathrm{coker}(\pi_\Phi)$ and the Pontrjagin dual of $\ker(\hat{\pi}_\Phi)$. We showed that this latter group is trivial, so π_Φ is indeed surjective.

Consider the functorial commutative diagram arising from the immersion $E \to J$:

$$\begin{tikzcd} E(F)_{\mathrm{tor}} \arrow[r] \arrow[d, hook] & \Phi_{E,\mathfrak{p}} \arrow[d] \\ J(F)_{\mathrm{tor}} \arrow[r, "\sim"] & \Phi_{J,\mathfrak{p}}. \end{tikzcd}$$

The left vertical arrow is obviously injective, and we know that the lower horizontal arrow is an isomorphism. Hence the homomorphism $E(F)_{\mathrm{tor}} \to \Phi_{E,\mathfrak{p}}$ is injective. There is a similar commutative diagram arising from the quotient map $J \to E$:

$$\begin{tikzcd} J(F)_{\mathrm{tor}} \arrow[r, "\sim"] \arrow[d] & \Phi_{J,\mathfrak{p}} \arrow[d, two heads] \\ E(F)_{\mathrm{tor}} \arrow[r] & \Phi_{E,\mathfrak{p}}. \end{tikzcd}$$

We showed that the left vertical arrow is surjective. Hence $E(F)_{\mathrm{tor}} \to \Phi_{E,\mathfrak{p}}$ must be surjective, and since it is also an injection, we get the isomorphism $E(F)_{\mathrm{tor}} \cong \Phi_{E,\mathfrak{p}}$ of part (1). Now the same diagram also implies that $J(F)_{\mathrm{tor}} \to E(E)_{\mathrm{tor}}$ is surjective. This proves (2).

We turn to the proof of (3). Suppose $E(F)$ has an element of order n. We know that n is coprime to p, and also $E(F)[n] \cong \Phi_{E,\mathfrak{p}}[n] \cong \mathbb{Z}/n\mathbb{Z}$. The argument at the beginning of the proof can be used to show that $E[n]$ is everywhere unramified, so $E[n] \subset E(\widetilde{F})$. Thus $E/\widetilde{F}$ is a non-isotrivial elliptic curve over $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$ with constant n-torsion. This implies that there is a non-constant morphism $\mathbb{P}^1_{\overline{\mathbb{F}}_q} \to X(n)_{\overline{\mathbb{F}}_q}$, where $X(n)$ is the moduli scheme of elliptic curves with full n-torsion. Since n is coprime to p, $X(n)_{\overline{\mathbb{F}}_q}$ is an irreducible smooth projective curve over $\overline{\mathbb{F}}_q$. We conclude that the genus of $X(n)_{\overline{\mathbb{F}}_q}$ must be 0. On the other hand, the genus of $X(n)_{\overline{\mathbb{F}}_q}$ is equal to the genus of $X(n)_{\mathbb{C}}$. Using the formula for the genus of $X(n)_{\mathbb{C}}$, we see that $n \leq 5$.

The proof of (4) is implicit in the previous paragraph. Indeed, suppose $\Phi_{E,\mathfrak{p}} = \mathbb{Z}/n\mathbb{Z}$. Then $E[n]$ is everywhere unramified, so $\mathcal{E}_{\mathbb{F}_\infty}[n]$ must be of rank two over $\mathbb{Z}/n\mathbb{Z}$. Since $\mathcal{E}^0_{\mathbb{F}_\infty}[n]$ has order n, this forces $\Phi_{E,\infty}[n] = \mathbb{Z}/n\mathbb{Z}$. □

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