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Heinz König

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Heinz König

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
hkoenig@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

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NEW VERSIONS OF THE RADON-NIKODÝM THEOREM

HEINZ KÖNIG

Abstract. On the basis of recent developments in measure theory the present note obtains a few new versions of the classical Radon-Nikodým theorem, with the aim to combine simple formulations with wide domains of application.

0. Introduction

After Fremlin [4] section 232 the Radon-Nikodým (=:RN) theorem must be on any list of the half-dozen most important theorems of measure theory. Yet there is a lack of versions which combine simple formulations with wide domains of application. The present note wants to contribute to this issue. Thus we do not intend to concentrate on RN theorems of conceptual sophistication, as it is the task for example in the handbook article of Candeloro and Volčič [2].

The entire paper assumes a pair of measures $\alpha, \beta : \mathfrak{A} \to [0,\infty]$ on a σ algebra $\mathfrak{A}$ in a nonvoid set X. As usual we define $\beta \ll \alpha$ to mean that $\alpha(A) = 0 \Rightarrow \beta(A) = 0$ for all $A \in \mathfrak{A}$.

The RN theorems as circumscribed above can in a sense be subdivided into two kinds. The theorems of the first kind impose *finiteness conditions* on α and/or β (and likewise the related σ finiteness conditions, which as a rule produce immediate consequences and therefore will not be considered in the sequel). The simplest and most obvious result is for $\alpha < \infty$ and arbitrary β, for example in Bauer [1] 17.10, Elstrodt [3] VII.2.3, and Leinert [12] 8.10.

0.1 Theorem. *Assume that $\alpha < \infty$. Then $\beta \ll \alpha$ is equivalent to the existence of an $F : X \to [0,\infty]$ measurable $\mathfrak{A}$ such that $\beta(A) = \int_A F d\alpha$ for all $A \in \mathfrak{A}$. The function F is unique modulo α* (that is unique up to its values on some subset $N \in \mathfrak{A}$ with $\alpha(N) = 0$).

However, in the applications the condition $\beta < \infty$ with arbitrary α seems to be more important, because one often faces an α which comes from far. Here it is well-known that $\beta \ll \alpha$ need not furnish an $F : X \to [0,\infty]$ measurable $\mathfrak{A}$ such that $\beta(A) = \int_A F d\alpha$ for all $A \in \mathfrak{A}$: for example let as in Halmos [5] section 31.(8) $\mathfrak{A}$ consist of the countable and the cocountable subsets of an uncountable X, and $\alpha(A) = \beta(A) = 0$ for countable and $\alpha(A) = \infty$ and $\beta(A) = 1$ for cocountable $A \in \mathfrak{A}$.

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In the situation $\beta < \infty$ we know of no definitive result so far. Our present first contribution will be such a result in section 1. It extends the partial result of the present author [8] 24.10 which has no uniqueness involved. But this partial result implies the remarkable previous result [8] 24.12 = Fremlin [4] 232E, which asserts that the equivalence in the above 0.1 persists, and hence in view of $\beta < \infty$ the full assertion persists, provided that one replaces $\beta \ll \alpha$ with a certain fortification in direction to $\alpha < \infty$, in [4] section 232 named β *truly continuous* α. This result in its different formulations will be discussed in section 2.

Our next step will be for the formation $\alpha \wedge \beta : \mathfrak{A} \to [0, \infty]$ defined to be

$$\alpha \wedge \beta(A) = \inf\{\alpha(A \setminus P) + \beta(P) : P \in \mathfrak{A} \text{ with } P \subset A\} \quad \text{for } A \in \mathfrak{A},$$

which is the lattice minimum of α and β in the space of all measures $\mathfrak{A} \to [0, \infty]$ equipped with the pointwise order $\leqq$. This formation, and likewise the lattice maximum $\alpha \vee \beta : \mathfrak{A} \to [0, \infty]$ defined to be

$$\alpha \vee \beta(A) = \sup\{\alpha(A \setminus P) + \beta(P) : P \in \mathfrak{A} \text{ with } P \subset A\} \quad \text{for } A \in \mathfrak{A},$$

appear, to the present author's surprise, almost never in the usual textbooks of measure theory. The formations have been used with profit in his book [8] section 23 (in a more comprehensive context) and in [9] section 1. In the present section 3 it will be proved that $\beta \ll \alpha$ is equivalent to $(t\alpha) \wedge \beta \uparrow \beta$ for $t \uparrow \infty$. We note that the identical fact is required in Kölzow [7] in order to replace Hilfssatz 2 (which is for the pointwise minimum). Then in section 4 we shall deduce that the RN theorem for $\beta < \infty$ and arbitrary α in section 1 remains true for arbitrary α and β which fulfil $\alpha \wedge \beta < \infty$. Thus we have a common roof for both $\alpha < \infty$ and $\beta < \infty$.

So far the RN theorems based on finiteness conditions for α and/or β. The RN theorems of the other kind are based on so-called *decompositions* of X, this time in terms of α. The notion is due to Kölzow [6], subsequent to the Bourbaki decomposition theorem for Radon measures. It is a main achievement of his work [7] that the existence of a decomposition more than implies an RN assertion, but is even equivalent to certain fortified RN assertions. Actual updates of the notion are in [10] section 4 and Fremlin [4] 211E. They lead to comprehensive existence theorems for decompositions, in [10] 4.11 in terms of inner τ premeasures and in [4] 415A for quasi-Radon measures. The present final section 5 applies our former results to obtain an RN theorem of decomposition type, with the aim to be as simple and flexible as possible. In particular the theorem inherits from the previous sections an *addendum on extended uniqueness* in the spirit of the so-called *monotone* RN version in Kölzow [7].

1. The Case $\beta < \infty$

The main result reads as follows. The surprise in it is the occurence of the symbol $\geqq$. This $\geqq$ will in fact persist throughout the entire paper.

1.1 Theorem *Assume that $\beta < \infty$. Then $\beta \ll \alpha$ is equivalent to the existence of an $F : X \to [0,\infty]$ measurable $\mathfrak{A}$ such that*

$$\beta(A) = \int_A F d\alpha \qquad \text{for all } A \in \mathfrak{A} \text{ with } \alpha(A) < \infty, \text{ and}$$

$$\beta(A) \geqq \int_A F d\alpha \qquad \text{for all } A \in \mathfrak{A}.$$

The function F is unique modulo α.

The proof of existence can be dispensed with in toto, because it is identical with the proof of the previous [8] theorem 24.10 presented at that place. To be sure, the old assertion is the precise first half of the present one, but an obvious inspection of its proof reveals that it furnishes the second half as well. This fact is included in our list of Errata et Addenda [11] (note that the list also contains two little flaws in the old proof). As to this old proof, we also want to recall the basic rôle of the minimum theorem [8] 23.15.

It remains to prove the uniqueness assertion, which likewise comes as part of the surprise. The assertion will even be obtained in the much more powerful form which follows. It is in the spirit of the so-called *monotone* RN version in Kölzow [7]. This assertion will likewise persist throughout the entire paper.

1.2 Addendum on Extended Uniqueness. *Assume that $\vartheta : \mathfrak{A} \to [0,\infty]$ is a measure and $G : X \to [0,\infty]$ measurable $\mathfrak{A}$ such that $\vartheta(A) = \int_A G d\alpha$ for all $A \in \mathfrak{A}$ with $\alpha(A) < \infty$. Then $\beta \leqq \vartheta$ implies that $F \leqq G$ modulo α.*

Proof of 1.2. To be shown is $\alpha\big([G < F]\big) = 0$. Since $[G < F]$ is the union of the $[G \leqq a < b \leqq F] = [G \leqq a] \cap [F \geqq b] =: D(a,b) \in \mathfrak{A}$ for the obvious countable collections of pairs $0 < a < b < \infty$, it suffices to show that $\alpha(D(a,b)) = 0$ for $0 < a < b < \infty$. Now on the one hand it cannot happen that $\alpha(D(a,b)) = \infty$, because

$$\infty > \beta(D(a,b)) \geqq \int_{D(a,b)} F d\alpha \geqq b\alpha(D(a,b)).$$

On the other hand it cannot happen that $0 < \alpha(D(a,b)) < \infty$, because then

$$\vartheta(D(a,b)) = \int_{D(a,b)} G d\alpha \leqq a\alpha(D(a,b)) < b\alpha(D(a,b)) \leqq \int_{D(a,b)} F d\alpha = \beta(D(a,b)).$$

Thus it remains that $\alpha(D(a,b)) = 0$. $\square$

2. The Truly Continuous Special Case

Fremlin [4] 232A defines β to be *truly continuous* α iff for each $\varepsilon > 0$ there exist $\delta > 0$ and $D \in [\alpha < \infty]$ such that $\alpha(A \cap D) \leqq \delta \Rightarrow \beta(A) \leqq \varepsilon$ for all $A \in \mathfrak{A}$. Here of course $[\alpha < \infty] := \{A \in \mathfrak{A} : \alpha(A) < \infty\}$. We compare the notion with the two properties

(#) β is inner regular $[\alpha < \infty]$;

(# 0) if $A \in \mathfrak{A}$ has $\beta(T) = 0$ for all its subsets $T \in [\alpha < \infty]$ then $\beta(A) = 0$; the latter is from [8] 24.12. We collect the answers in a common lemma.

2.1 Lemma. *We have* (#) $\Leftrightarrow$ (# 0). *Under the assumption* $\beta < \infty$ *we have equivalence of the properties*

1) β *is truly continuous* α;
2) $\beta \ll \alpha$ *and* (#);
3) $\beta \ll \alpha$ *and* (# 0).

Proof of (#) $\Leftrightarrow$ (# 0). The implication $\Rightarrow$ is obvious. To prove $\Leftarrow$ assume that $A \in \mathfrak{A}$ has

$$R := \sup\{\beta(S) : S \in [\alpha < \infty] \text{ with } S \subset A\} < \beta(A).$$

Thus $0 \leqq R < \infty$. Fix a sequence $(S_l)_l$ in $[\alpha < \infty]$ with $S_l \subset A$ and $\beta(S_l) \to R$. We can assume that $S_l \uparrow$ and hence $S_l \uparrow E \in \mathfrak{A}$ with $E \subset A$ and $\beta(E) = R$. Thus $\beta(A \setminus E) = \beta(A) - R > 0$. Hence (# 0) furnishes $T \in [\alpha < \infty]$ with $T \subset A \setminus E$ and $\beta(T) > 0$. It follows that $S_l \cup T \in [\alpha < \infty]$ with $S_l \cup T \subset A$ has $\beta(S_l \cup T) = \beta(S_l) + \beta(T) \leqq R$ for all $l \in \mathbb{N}$. This furnishes the contradiction $R + \beta(T) \leqq R$. $\square$

Proof of 1)$\Rightarrow$2) (without the assumption $\beta < \infty$). $\beta \ll \alpha$ is obvious. To prove (#) fix $A \in \mathfrak{A}$ and real $c < \beta(A)$, and then $\varepsilon > 0$ with $c + \varepsilon < \beta(A)$. Let $\delta > 0$ and $D \in [\alpha < \infty]$ be as in the definition of truly continuous. From this definition applied to $A \cap D'$ we obtain $\beta(A \cap D') \leqq \varepsilon$. It follows that $\beta(A \cap D) = \beta(A) - \beta(A \cap D') \geqq \beta(A) - \varepsilon > c$. $\square$

Proof of 2)$\Rightarrow$1) (under the assumption $\beta < \infty$). Fix $\varepsilon > 0$. From an omnipresent equivalent to $\beta \ll \alpha$ in case $\beta < \infty$ we have $\delta > 0$ such that $\alpha(A) \leqq \delta \Rightarrow \beta(A) \leqq \varepsilon$ for all $A \in \mathfrak{A}$. Moreover (#) furnishes $D \in [\alpha < \infty]$ with $\beta(D) \geqq \beta(X) - \varepsilon$ or $\beta(D') \leqq \varepsilon$. Thus for $A \in \mathfrak{A}$ with $\alpha(A \cap D) \leqq \delta$ we obtain $\beta(A) \leqq \beta(A \cap D) + \beta(D') \leqq 2\varepsilon$. $\square$

After this the main result in the present case reads as follows.

2.2 Theorem. *Assume that* $\beta < \infty$. *Then properties* 1)2)3) *in the above* 2.1 *are equivalent to the existence of an* $F : X \to [0, \infty]$ *measurable* $\mathfrak{A}$ *such that* $\beta(A) = \int_A F d\alpha$ *for all* $A \in \mathfrak{A}$. *The function* F *is unique modulo* α.

This is a known theorem: The equivalence in it has been obtained with condition 3) in [8] 24.12 and with condition 1) in Fremlin [4] 232E. However, the author thinks that the most adequate and pleasant form of the equivalence is with condition 2), which says that $\beta \ll \alpha$ and β is inner regular $[\alpha < \infty]$. Also this form of the equivalence is a routine deduction from the previous theorem 1.1, of which we skip the proof. Of course the uniqueness assertion in 2.2 is an obvious consequence of that in 1.1.

3. The Lattice Minimum $\alpha \wedge \beta$

The basic properties of the formation $\alpha \wedge \beta : \mathfrak{A} \to [0, \infty]$ are listed in [9] section 1. Most important at present is [9] 1.1.5) that for each $A \in \mathfrak{A}$ there exists $P \in \mathfrak{A}$ with $P \subset A$ such that $\alpha \wedge \beta(A) = \alpha(A \setminus P) + \beta(P)$. It is a consequence of the minimum theorem [8] 23.15.

3.1 LEMMA. *Assume that* $A \in \mathfrak{A}$ *with* $\alpha \wedge \beta(A) < \infty$ *and* $1 < c < \infty$. *If* $P, Q \in \mathfrak{A}$ *with* $P, Q \subset A$ *fulfil*

$$\begin{array}{lll} (1) & \alpha \wedge \beta(A) & = \alpha(A \setminus P) + \beta(P), \\ (\mathrm{c}) & (c\alpha) \wedge \beta(A) & = c\alpha(A \setminus Q) + \beta(Q), \end{array}$$

then

$$\alpha \wedge \beta(A) = \alpha\big(A \setminus (P \cap Q)\big) + \beta(P \cap Q),$$
$$(c\alpha) \wedge \beta(A) = c\alpha\big(A \setminus (P \cup Q)\big) + \beta(P \cup Q).$$

Thus if one prescribes one of the two sets $P \in \mathfrak{A}$ *with* $P \subset A$ *and* (1) *and* $Q \in \mathfrak{A}$ *with* $Q \subset A$ *and* (c)*, then the other one can be chosen so that* $P \subset Q$.

Proof. Fix $P, Q \in \mathfrak{A}$ with $P, Q \subset A$ which fulfil (1)(c), and note that all terms in the two equations are $< \infty$. Define $M := P \cap (A \setminus Q) = P \cap Q' \in \mathfrak{A}$, so that $\alpha(M), \beta(M) < \infty$. Moreover

$$\begin{aligned} M \subset P &\text{ and hence } M \cap (A \setminus P) = \varnothing, \\ &\text{and } A \setminus (P \setminus M) = A \cap (P' \cup M) = (A \setminus P) \cup M, \\ M \subset A \setminus Q &\text{ and hence } M \cap Q = \varnothing, \\ &\text{and } A \setminus (Q \cup M) = A \cap (Q' \cap M') = (A \setminus Q) \setminus M. \end{aligned}$$

We obtain on the one hand from (1)

$$\begin{aligned} (1') \quad \alpha(A \setminus P) + \beta(P) = \alpha \wedge \beta(A) &\leqq \alpha\big(A \setminus (P \setminus M)\big) + \beta(P \setminus M) \\ &= \alpha\big((A \setminus P) \cup M\big) + \beta(P \setminus M), \end{aligned}$$

with all terms $< \infty$. Hence $\beta(M) \leqq \alpha(M)$, and we have equality in (1') iff $\beta(M) = \alpha(M)$. On the other hand we obtain from (c)

$$\begin{aligned} (\mathrm{c}') \quad c\alpha(A \setminus Q) + \beta(Q) = (c\alpha) \wedge \beta(A) &\leqq c\alpha\big(A \setminus (Q \cup M)\big) + \beta(Q \cup M) \\ &= c\alpha\big((A \setminus Q) \setminus M\big) + \beta(Q \cup M), \end{aligned}$$

with all terms $< \infty$. Hence $c\alpha(M) \leqq \beta(M)$, and we have equality in (c') iff $c\alpha(M) = \beta(M)$. Now the two inequalities combine with $1 < c < \infty$ and $\alpha(M), \beta(M) < \infty$ to furnish $\alpha(M) = \beta(M) = 0$. It follows that we have in fact equality in (1')(c'). In view of

$$P \setminus M = P \cap (P' \cup Q) = P \cap Q \text{ and } Q \cup M = Q \cup (P \cap Q') = P \cup Q$$

the two assertions follow. $\square$

We come to the main result announced in the introduction.

3.2 PROPOSITION. *We have* $\beta \ll \alpha \Longleftrightarrow (t\alpha) \wedge \beta \uparrow \beta$ *for* $t \uparrow \infty$.

Proof. $\Leftarrow$) is obvious since $(t\alpha) \wedge \beta \ll \alpha$ for $0 < t < \infty$. $\Rightarrow$) Fix $A \in \mathfrak{A}$. Then $C := \lim\limits_{t\uparrow\infty}(t\alpha) \wedge \beta(A) \leqq \beta(A)$. To be shown is $C \geqq \beta(A)$, so that we can assume that $C < \infty$. From lemma 3.1 we obtain a sequence $P_1 \subset \cdots \subset P_n \subset \cdots \subset A$ in $\mathfrak{A}$ such that

$$C \geqq (n\alpha) \wedge \beta(A) = n\alpha(A \setminus P_n) + \beta(P_n) \quad \text{for } n \in \mathbb{N}.$$

Thus $P_n \uparrow$ some $P \in \mathfrak{A}$ with $P \subset A$. We have on the one hand $\beta(P_n) \leqq C$ for $n \in \mathbb{N}$ and hence $\beta(P) \leqq C$, and on the other hand $\alpha(A \setminus P_n) \leqq C/n < \infty$ for $n \in \mathbb{N}$ and hence $\alpha(A \setminus P) = 0$. From $\beta \ll \alpha$ we obtain $\beta(A \setminus P) = 0$. It follows that $\beta(A) = \beta(P) \leqq C$. $\square$

In conclusion we note that the treatment of the lattice formations $\alpha \wedge \beta$ and $\alpha \vee \beta$ appears to be far more delicate for *arbitrary* measures than for *finite* ones. A serious example is the mistake in Schwartz [13] p.54 lines 18-21, which has been repaired not earlier than in [9] 1.5.3) on the basis of a new idea. The present author thinks that the first adequate treatment of the formations in question, also and in particular for *signed* measures and even contents, is in his work [8] section 23 and [9].

NOTE added 1 August 2005. The author owes to Zbigniew Lipecki the hint that the above proposition 3.2 appeared earlier as theorem 1.4.8 in N.Boboc and Gh.Bucur, Măsură şi Capacitate, Bucureşti 1985 (in Romanian). It seems that this book did not become widely known; in particular it did not appear in Math.Reviews nor in Zentralblatt Math.

4. THE CASE $\alpha \wedge \beta < \infty$

The present section extends the results of section 1 from $\beta < \infty$ to the situation $\alpha \wedge \beta < \infty$.

4.1 THEOREM *Assume that $\alpha \wedge \beta < \infty$. Then $\beta \ll \alpha$ is equivalent to the existence of an $F : X \to [0, \infty]$ measurable $\mathfrak{A}$ such that*

$$\beta(A) = \int_A F d\alpha \qquad \text{for all } A \in \mathfrak{A} \text{ with } \alpha(A) < \infty, \text{ and}$$

$$\beta(A) \geqq \int_A F d\alpha \qquad \text{for all } A \in \mathfrak{A}.$$

The function F is unique modulo α.

4.2 ADDENDUM ON EXTENDED UNIQUENESS. *Assume that $\vartheta : \mathfrak{A} \to [0, \infty]$ is a measure and $G : X \to [0, \infty]$ measurable $\mathfrak{A}$ such that $\vartheta(A) = \int_A G d\alpha$ for all $A \in \mathfrak{A}$ with $\alpha(A) < \infty$. Then $\beta \leqq \vartheta$ implies that $F \leqq G$ modulo α.*

Proof of 4.1. To be shown is that $\beta \ll \alpha$ implies the existence of an $F : X \to [0, \infty]$ measurable $\mathfrak{A}$ as claimed. The converse is obvious, and the uniqueness assertion will follow from 4.2 as before.

The measures $\beta_n := (n\alpha) \wedge \beta$ for $n \in \mathbb{N}$ fulfil $\beta_n \leqq (n\alpha) \wedge (n\beta) = n(\alpha \wedge \beta) < \infty$ and $\beta_n \ll \alpha$. From 1.1 we obtain functions $F_n : X \to [0, \infty]$ measurable $\mathfrak{A}$ such that

$$\beta_n(A) = \int_A F_n d\alpha \qquad \text{for all } A \in \mathfrak{A} \text{ with } \alpha(A) < \infty, \text{ and}$$

$$\beta_n(A) \geqq \int_A F_n d\alpha \qquad \text{for all } A \in \mathfrak{A}.$$

From $\beta_n \leqq \beta_{n+1}$ and 1.2 then $F_n \leqq F_{n+1}$ modulo α. Thus we can assume that $F_1 \leqq \cdots \leqq F_n \leqq \cdots$ pointwise on X, so that $F_n \uparrow$ some $F : X \to [0, \infty]$ measurable $\mathfrak{A}$. On the other side $\beta_n \uparrow \beta$ from 3.2. Thus the Beppo Levi theorem implies the assertion. $\square$

Proof of 4.2. To be shown is $\alpha\big([G < F]\big) = 0$. As before it suffices to show for fixed $0 < a < b < \infty$ that $[G \leqq a < b \leqq F] = [G \leqq a] \cap [F \geqq b] =: D(a, b) \in \mathfrak{A}$ has $\alpha(D(a, b)) = 0$. Now on the one hand $\alpha \wedge \beta(D(a, b)) < \infty$

furnishes $P \in \mathfrak{A}$ with $P \subset D(a,b)$ such that $\alpha(D(a,b) \setminus P) < \infty$ and $\beta(P) < \infty$, and then $\beta(P) \geqq \int_P F d\alpha \geqq b\alpha(P)$ implies that $\alpha(P) < \infty$. Therefore $\alpha(D(a,b)) < \infty$. On the other hand as before it cannot happen that $0 < \alpha(D(a,b)) < \infty$, because then

$$\vartheta(D(a,b)) = \int\limits_{D(a,b)} G d\alpha \leqq a\alpha(D(a,b)) < b\alpha(D(a,b)) \leqq \int\limits_{D(a,b)} F d\alpha = \beta(D(a,b)).$$

Thus it remains that $\alpha(D(a,b)) = 0$. $\square$

5. The RN Theorem of Decomposition Type

We start to describe our concept of decomposition. The basic datum is a nonvoid set system $\mathfrak{M} \subset \mathfrak{A}$ which is *disjoint*, that means has pairwise disjoint members, and *dominates* $\mathfrak{A}$ in the sense that

$$\mathfrak{M}\top\mathfrak{A} := \{A \subset X : A \cap M \in \mathfrak{A} \text{ for all } M \in \mathfrak{M}\} \subset \mathfrak{A},$$

in terms of the *transporter* from [8]. Then one defines $\mathfrak{H} = \mathfrak{H}(\mathfrak{M}, \alpha) \subset \mathfrak{A}$ to consist of the $H \in \mathfrak{A}$ such that

$$\alpha(A) = \sum_{M \in \mathfrak{M}} \alpha(A \cap M) \quad \text{for all } A \in \mathfrak{A} \text{ with } A \subset H,$$

where uncountable sums of terms $\in [0, \infty]$ are meant as usual. Thus $\mathfrak{H}$ is *hereditary* in $\mathfrak{A}$.

5.1 Examples. 1) The simplest example is $\mathfrak{M} = \{X\}$, where of course $\mathfrak{H} = \mathfrak{A}$. Further examples are the countable disjoint covers $\mathfrak{M} \subset \mathfrak{A}$ of X, where once more $\mathfrak{H} = \mathfrak{A}$.

2) The decompositions in the sense of [10] section 4 are the above $\mathfrak{M}$ with $0 < \alpha(M) < \infty \; \forall M \in \mathfrak{M}$ which have $\mathfrak{H} \supset [\alpha < \infty]$. The decompositions in the sense of [4] 211E are the above $\mathfrak{M}$ with $\alpha(M) < \infty \; \forall M \in \mathfrak{M}$ which cover X and have $\mathfrak{H} = \mathfrak{A}$. In the present context we shall weaken $\alpha|\mathfrak{M} < \infty$ after the model of section 4.

3) In any case $\mathfrak{H} = \mathfrak{H}(\mathfrak{M}, \alpha)$ contains the subsets $H \in \mathfrak{A}$ with $\alpha(H) = 0$.

5.2 Lemma. *For $F : X \to [0, \infty]$ measurable $\mathfrak{A}$ we have*

$$\int\limits_H F d\alpha = \sum_{M \in \mathfrak{M}} \int\limits_{H \cap M} F d\alpha \quad \textit{for all } H \in \mathfrak{H}.$$

Proof. To be shown is $\leqq$, thus we can assume that the second member is $< \infty$. Then for some nonvoid countable $\mathfrak{P} \subset \mathfrak{M}$ the $M \in \mathfrak{M} \setminus \mathfrak{P}$ fulfil

$$0 = \int\limits_{H \cap M} F d\alpha = \int\limits_{0\leftarrow}^{\to\infty} \alpha\big(H \cap M \cap [F \geqq t]\big) dt,$$

and hence $\alpha\big(H \cap M \cap [F > 0]\big) = 0$. Thus $H \in \mathfrak{H}$ implies for the union $P := \bigcup_{M \in \mathfrak{P}} M \in \mathfrak{A}$ that

$$\alpha\big(H \cap P' \cap [F > 0]\big) = \sum_{M \in \mathfrak{M} \setminus \mathfrak{P}} \alpha\big(H \cap M \cap [F > 0]\big) = 0,$$

and hence $\int\limits_{H\cap P'} Fd\alpha = 0$. Since $\mathfrak{P}$ is countable it follows that

$$\int\limits_H Fd\alpha = \int\limits_{H\cap P} Fd\alpha = \sum_{M\in\mathfrak{P}} \int\limits_{H\cap M} Fd\alpha = \sum_{M\in\mathfrak{M}} \int\limits_{H\cap M} Fd\alpha. \quad \square$$

We come to the main results. We emphasize that in the particular case $\mathfrak{H} = \mathfrak{A}$ all the assertions look as before.

5.3 Theorem. *Let $\mathfrak{M} \subset \mathfrak{A}$ be nonvoid disjoint with $\mathfrak{M}\top\mathfrak{A} \subset \mathfrak{A}$. Assume that $\alpha \wedge \beta|\mathfrak{M} < \infty$. Then $\beta \ll \alpha$ is equivalent to the existence of an $F : X \to [0,\infty]$ measurable $\mathfrak{A}$ such that*

$$\beta(H) = \int\limits_H Fd\alpha \qquad \textit{for all } H \in \mathfrak{H} \textit{ with } \alpha(H) < \infty, \textit{ and}$$

$$\beta(H) \geqq \int\limits_H Fd\alpha \qquad \textit{for all } H \in \mathfrak{H}.$$

The restrictions $F|H$ are unique modulo α for all $H \in \mathfrak{H}$.

5.4 Addendum on Extended Uniqueness. *Assume that $\vartheta : \mathfrak{A} \to [0,\infty]$ is a measure and $G : X \to [0,\infty]$ measurable $\mathfrak{A}$ such that $\vartheta(H) = \int\limits_H Gd\alpha$ for all $H \in \mathfrak{H}$ with $\alpha(H) < \infty$. Then $\beta \leqq \vartheta$ implies that $F|H \leqq G|H$ modulo α for all $H \in \mathfrak{H}$.*

Proof of 5.3 (except the uniqueness assertion). In view of 5.1.3) the existence of an $F : X \to [0,\infty]$ measurable $\mathfrak{A}$ as claimed implies that $\beta \ll \alpha$. Now for the converse assertion assume that $\beta \ll \alpha$.

1) For fixed $M \in \mathfrak{M}$ we have $\alpha \wedge \beta(M) < \infty$ and thus obtain from 4.1 an $F_M : X \to [0,\infty]$ measurable $\mathfrak{A}$ with $F_M|M' = 0$ such that

$$\beta(A) = \int\limits_A F_M d\alpha \qquad \text{for all } A \in \mathfrak{A} \text{ with } A \subset M \text{ and } \alpha(A) < \infty, \text{ and}$$

$$\beta(A) \geqq \int\limits_A F_M d\alpha \qquad \text{for all } A \in \mathfrak{A} \text{ with } A \subset M.$$

Then the function $F : X \to [0,\infty]$ defined to be $F = \sum\limits_{M\in\mathfrak{M}} F_M$ is measurable $\mathfrak{A}$, because for $0 < t < \infty$ we have $[F \geqq t] \cap M = [F_M \geqq t] \in \mathfrak{A}$ for all $M \in \mathfrak{M}$.

2) For $H \in \mathfrak{H}$ we obtain from 1) and 5.2

$$\beta(H) \geqq \sum_{M\in\mathfrak{M}} \beta(H \cap M) \geqq \sum_{M\in\mathfrak{M}} \int\limits_{H\cap M} Fd\alpha = \int\limits_H Fd\alpha.$$

3) Now let $H \in \mathfrak{H}$ with $\alpha(H) < \infty$. We see from $\alpha(H) = \sum\limits_{M\in\mathfrak{M}} \alpha(H \cap M)$ that for some nonvoid countable $\mathfrak{P} \subset \mathfrak{M}$ the $M \in \mathfrak{M}\backslash\mathfrak{P}$ fulfil $\alpha(H\cap M) = 0$. Thus $H \in \mathfrak{H}$ implies for the union $P := \bigcup\limits_{M\in\mathfrak{P}} M \in \mathfrak{A}$ that

$$\alpha(H \cap P') = \sum_{M\in\mathfrak{M}\backslash\mathfrak{P}} \alpha(H \cap M) = 0 \text{ and hence } \beta(H \cap P') = 0.$$

It follows that

$$\beta(H) = \beta(H \cap P) = \sum_{M \in \mathfrak{P}} \beta(H \cap M) = \sum_{M \in \mathfrak{P}} \int\limits_{H \cap M} F d\alpha = \int\limits_{H \cap P} F d\alpha = \int\limits_{H} F d\alpha. \ \square$$

Proof of 5.4. Fix $H \in \mathfrak{H}$. To be shown is

$$\alpha\big(H \cap [G < F]\big) = \sum_{M \in \mathfrak{M}} \alpha\big(H \cap [G < F] \cap M\big) = 0.$$

As before it suffices to show for fixed $0 < a < b < \infty$ that $[G \leqq a < b \leqq F] = [G \leqq a] \cap [F \geqq b] =: D(a,b) \in \mathfrak{A}$ has $\alpha(H \cap D(a,b) \cap M) = 0$ for all $M \in \mathfrak{M}$. This can be done as in the proof of 4.2: Put $H \cap D(a,b) \cap M =: D \in \mathfrak{H}$ for short. On the one hand then $\alpha \wedge \beta(D) < \infty$ furnishes $P \in \mathfrak{H}$ with $P \subset D$ such that $\alpha(D \setminus P) < \infty$ and $\beta(P) < \infty$, and then $\beta(P) \geqq \int\limits_P F d\alpha \geqq b\alpha(P)$ implies that $\alpha(P) < \infty$. Therefore $\alpha(D) < \infty$. On the other hand one concludes as before that $0 < \alpha(D) < \infty$ cannot happen, because then

$$\vartheta(D) = \int\limits_D G d\alpha \leqq a\alpha(D) < b\alpha(D) \leqq \int\limits_D F d\alpha = \beta(D).$$

Thus it remains that $\alpha(D) = 0$. $\square$

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Universität des Saarlandes, Fakultät für Mathematik und Informatik, D-66041 Saarbrücken, Germany

E-mail address: `hkoenig@math.uni-sb.de`