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Martin Fuchs and Sergey Repin

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Martin Fuchs

Saarland University
Dep. of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Sergey Repin

V.A. Steklov Mathematical Institute
St. Petersburg Branch
191011 St. Petersburg
Russia
repin@pdmi.ras.ru

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax:	+ 49 681 302 4443
e-Mail:	preprint@math.uni-sb.de
WWW:	http://www.math.uni-sb.de/

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Abstract

The paper is focused on functional type a posteriori estimates of the difference between the exact solution of a variational problem modeling certain types of generalized Newtonian fluids and any function from the admissible energy class. In contrast to the a posteriori estimates obtained for example by the finite element method our estimates do not contain any local (mesh dependent) constants, and therefore they can be used regardless of the way in which an approximation has been constructed.

1 Introduction

The purpose of this note is to establish explicit estimates for the quality of approximate solutions for a system of nonlinear partial differential equations modeling the stationary and also slow flow of certain generalized Newtonian fluids. To be precise, let us first discuss the fluid models we like to investigate. We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ occupied by a viscous incompressible fluid whose properties depend on a given convex dissipative potential Π acting on the space $\mathbb{S}^n$ of smooth, symmetric $(n \times n)$–matrices. If the velocity field u is independent of time and also small, then the following system of partial differential equations is satisfied by u and the pressure function p :

$$(1.1) \qquad -\operatorname{div}\sigma = f - \nabla p \quad \text{in } \Omega;$$

$$(1.2) \qquad \operatorname{div} u = 0 \quad \text{in } \Omega;$$

$$(1.3) \qquad \sigma \in \partial\,\Pi\,(\varepsilon(u)) \quad \text{in } \Omega;$$

$$(1.4) \qquad u = u_0 \quad \text{on } \partial\Omega.$$

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Here $\varepsilon(u)$ denotes the symmetric gradient of u, σ represents the deviatoric part of the stress–tensor, and $f : \Omega \to \mathbb{R}^n$ is a given system of volume forces, whereas u_0 is a fixed boundary datum such that $\operatorname{div} u_0 = 0$ in Ω. The notion of a generalized Newtonian fluid arises from the requirement that a constitutive relation like (1.3) holds, where $\partial \Pi$ denotes the subdifferential of the potential Π, which coincides with the derivative Π', if Π is Gateaux–differentiable. We refer to [6] for the definition and a discussion of the properties of the subdifferential. In the case that $\Pi(\varepsilon) = \frac{\nu}{2}|\varepsilon|^2$ with constant viscosity $\nu > 0$ we see that (1.1) – (1.4) correspond to the Stokes–problem for a Newtonian fluid, see e.g. [12] or [10], and in our paper we concentrate on lower–order perturbations of this quadratic potential, thus in what follows we assume that $u_0 \in H^1(\Omega, \mathbb{R}^n)$ together with $f \in L^2(\Omega, \mathbb{R}^n)$. Here $H^1(\Omega, \mathbb{R}^n)$ is the Sobolev–space of functions from $L^2(\Omega, \mathbb{R}^n)$ such that the first order weak derivatives are also square–integrable on Ω. So let for $\nu > 0$

$$\Pi(\varepsilon) = \frac{\nu}{2}|\varepsilon|^2 + \pi(\varepsilon),\ \varepsilon \in \mathbb{S}^n, \tag{1.5}$$

where $\pi : \mathbb{S}^n \to \mathbb{R}$ is of the following form:

a) $\pi(\varepsilon) = k|\varepsilon|$ with $k > 0$. In this case Π corresponds to the Bingham fluid model.

or
b) $\pi(\varepsilon) = k|\varepsilon|^\alpha$, where $k > 0$ and $\alpha \in (1, 2]$. This potential models so–called power–law fluids.

or
c) $\pi(\varepsilon) = k\big(1 + |\varepsilon|^2\big)^{\alpha/2}$ with α and k as in b). For this choice we also have some kind of power–law fluid showing a different behaviour as $\varepsilon \to 0$ and at the same time it serves as one of the simplest models arising in the theory of electrorheological fluids, where – in the realistic situation – k, ν and α are not constant but smooth functions of $x \in \overline{\Omega}$.

or
d) $\pi(\varepsilon) = k|\varepsilon| \ln\big(1 + |\varepsilon|\big)$, $k > 0$. Now Π corresponds to a fluid of Powell–Eyring type (, and reduces to the Prandtl–Eyring fluid, if we let $\nu \searrow 0$).

The mathematical background of generalized Newtonian fluids is explained for example in [13], [14], [16], [17] and [8], the physical relevance of the various models is extensively discussed for example in the monographs [1] and [4], for an introduction into the theory of electrorheological fluids we refer to [26].

We assume from now that Π from (1.5) is given by one of the cases a) – d). Then it is well–known (see, e.g. [6] or [8]) that (1.1) – (1.4) has the following generalized formulation: let $\mathcal{H}^1$ denote the closure of all smooth solenoidal vector–fields with compact support in Ω w.r.t. the norm of $H^1(\Omega, \mathbb{R}^n)$. Then $u \in u_0 + \mathcal{H}^1$ is termed a weak solution of (1.1) –

(1.4) if and only if

$$\int_\Omega \sigma\,(u) : \varepsilon\,(w)\,dx = \int_\Omega f\cdot w\,dx \quad \forall\, w\in\mathcal{H}^1, \tag{1.6}$$

where

$$\begin{aligned} &\sigma\,(u) := \sigma_1+\sigma_2,\\ &\sigma_1 := \nu\,\varepsilon\,(u),\;\; \sigma_2 := \pi'\big(\varepsilon\,(u)\big), \end{aligned} \tag{1.7}$$

and π' is the Gateaux–derivative of π (or an element of the respective subdifferential, if π is nondifferentiable). We remark that (1.6) is the Euler equation for the functional

$$J\,(v) = \int_\Omega \Big(\frac{\nu}{2}\,\big|\,\varepsilon(v)\big|^2 + \pi\big(\varepsilon(v)\big) - f\cdot v\Big)\;dx,$$

and since J is strictly convex, continuous and coercive on $u_0+\mathcal{H}^1$, the variational problem

$$(\mathcal{P}) \qquad J\,(v) \rightsquigarrow \min\;\text{ on } u_0+\mathcal{H}^1$$

admits a unique solution u whose smoothness is discussed for example in [8], [9], [27] and [28].

The main goal of this paper now is to give estimates of the difference between this exact solution u and any function v from the energy class $u_0+\mathcal{H}^1$. The general version of such an estimate takes the form

$$\Phi(u-v) \;\;\le\;\; \mathcal{M}(v,D), \tag{1.8}$$

where Φ is a nonnegative functional vanishing at zero, $\mathcal{M}$ is a nonnegative functional that vanishes if and only if $v=u$, and D is the set of problem data including for example the domain, the coefficients, etc. Estimates of the form (1.8) have a practical value provided that

i.) The functional $\mathcal{M}$ is explicitly computable for any admissible v;

ii.) $\mathcal{M}(v_k,D)\to 0$ as v_k tends to u in the energy space;

iii.) $\mathcal{M}(v,D)$ provides a realistic upper bound for the quantity $\Phi(u-v)$.

Estimates of the type (1.8) sharing the properties i.) – iii.) are called functional type a posteriori estimates. In contrast to the a posteriori estimates derived in the last decades for various numerical solutions (e.g., for those obtained by the finite element method), these estimates are derived on purely functional grounds by using the methods of the calculus of variations and PDE theory. Therefore, they contain no local (mesh–dependent) constants, and they are applicable for approximations that may not exactly satisfy the Galerkin orthogonality conditions arising in a particular numerical scheme. Having such

an estimate one can explicitly control the accuracy of an approximation regardless of the way in which it has been constructed.
We like to mention that functional type a posteriori estimates have already been established in the papers [22] – [25], for a posteriori error estimates for finite element approximations of the (Navier–) Stokes equation we refer to [3], [11], [19], [20] and [29].

Our paper is organized as follows: in Section 2 we describe and comment the main results of this paper, i.e. we give the principal estimates of the difference between the exact solution u of problem (1.6) and an arbitrary solenoidal approximation with correct boundary values. One estimate is convenient if the conjugate function of the dissipative potential can be explicitly calculated, while the second estimate is applicable also in those cases when the explicit form of the conjugate function is unknown. Section 3 contains the proofs of these basic results. In Section 4 we discuss the meaning of these estimates for our concrete models a) – d).
Finally, in Section 5, we present estimates for a more complicated case dealing with approximations v that may not exactly satisfy the divergence–free condition.

2 Statement of the main results

Let $u \in u_0 + \mathcal{H}^1$ denote the unique solution of (1.6) with $\sigma(u)$ defined in (1.7) and π satisfying one of the cases a) – d) stated after (1.5). We let $\sum := L^2(\Omega, \mathbb{R}^{n\times n})$ and

$$Q_f := \left\{ (\tau_1, \tau_2) \in \sum \times \sum \mid \int_\Omega (\tau_1 + \tau_2) : \varepsilon(w)\, dx = \int_\Omega f \cdot w\, dx \quad \forall\, w \in \mathcal{H}^1 \right\}.$$

Finally, we denote by π^* the conjugate function of π. Then we have:

THEOREM 2.1. *For any $v \in u_0 + \mathcal{H}^1$ and for arbitrary choices of $(\tau_1, \tau_2) \in Q_f$ the following estimate holds*

$$\int_\Omega \frac{\nu}{2} \left|\varepsilon(u-v)\right|^2 dx + G(u,v) \quad \leq \quad D_1\left(\varepsilon(v), \tau_1\right) + D_2\left(\varepsilon(v), \tau_2\right), \tag{2.1}$$

where the functionals G, D_1 and D_2 are given by

$$\begin{aligned} G(u,v) &:= \int_\Omega \left(\pi\left(\varepsilon(v)\right) + \pi^*(\sigma_2) - \sigma_2 : \varepsilon(v)\right) dx, \\ D_1\left(\varepsilon(v), \tau_1\right) &:= \int_\Omega \left(\frac{\nu}{2} \left|\varepsilon(v)\right|^2 + \frac{1}{2\nu} \left|\tau_1\right|^2 - \tau_1 : \varepsilon(v) \right) dx, \\ D_2\left(\varepsilon(v), \tau_2\right) &:= \int_\Omega \big(\pi\left(\varepsilon(v)\right) + \pi^*(\tau_2) - \tau_2 : \varepsilon(v)\big)\, dx. \end{aligned}$$

Let us give some comments on this result: since in the cases under consideration π is a convex function, we see that $G(u,v) \geq 0$ (recall $\sigma_2 = \pi'\big(\varepsilon(u)\big)$) with equality if and only if $u = v$. Clearly, by Korn's and Poincaré's inequality, the first term on the l.h.s. of (2.1) measures the distance from the approximation v to the exact solution w.r.t. the norm of the space $H^1(\Omega, \mathbb{R}^n)$. Note, that $G(u, v)$ measures how accurately the tensor $\tilde{\sigma}_2 := \pi'\big(\varepsilon(v)\big)$ (obtained with the help of the approximation v) represents the exact tensor σ_2.

Thus the deviation from the exact solution u is controlled by the sum of the functionals D_1 and D_2. These functionals represent certain measures of errors in the constitutive relations

$$\tau_1 = \nu\, \varepsilon(v), \quad \tau_2 = \pi'\big(\varepsilon(v)\big),$$

which means that if we take $\tau_1 = \sigma_1, \tau_2 = \sigma_2$ with σ_1, σ_2 defined according to (1.7), then

$$\begin{aligned} D_1\left(\varepsilon(v), \tau_1\right) &= \int_\Omega \left(\frac{\nu}{2} \,|\, \varepsilon(v)|^2 + \frac{1}{2\nu} \left|\nu\, \varepsilon(u)\right|^2 - \nu\, \varepsilon(u) : \varepsilon(v) \right) \, dx, \\ &= \tfrac{\nu}{2} \int_\Omega \left|\, \varepsilon(u - v)\right|^2 \, dx, \\ D_2\left(\varepsilon(v), \tau_2\right) &= \int_\Omega \big(\pi\big(\varepsilon(v)\big) + \pi^*(\sigma_2) - \sigma_2 : \varepsilon(v)\big) \, dx, \end{aligned}$$

and (2.1) holds with equality. Thus, by minimizing the right–hand side of (2.1) w.r.t. τ_1, τ_2 we can obtain the upper bound of the error as close to its exact value as it is required.

However, in practice, the condition

$$\int_\Omega (\tau_1 + \tau_2) : \varepsilon(w) \, dx \;=\; \int_\Omega f : w \, dx \quad \forall\, w \in \mathcal{H}^1 \tag{2.2}$$

required for the pair (τ_1, τ_2) of tensors from Q_f is difficult to satisfy which clearly reduces the applicability of (2.1). In order to have a practically computable upper bound of the deviation from the exact solution we modify (2.1) by introducing new variables in choosing τ_1 in a special way. The purpose of such a rearrangement is not only to avoid condition (2.2) but also giving the possibility of removing the conjugate function π^* from the estimate. To do so, we have to introduce some notation: given tensors $æ_1, æ_2 \in \sum$ such that $æ_1 + æ_2$ has square summable divergence and a function $q \in H^1(\Omega)$, we let $\overline{w} \in H^1_0(\Omega, \mathbb{R}^n)$ (:= the subspace of $H^1(\Omega, \mathbb{R}^n)$ consisting of functions with zero trace) denote the unique solution of

$$\operatorname{div} \varepsilon(\overline{w}) = -\operatorname{div}(æ_1 + æ_2) - f + \nabla q \quad \text{in } \Omega. \tag{2.3}$$

Then we have:

THEOREM 2.2. *For any $v \in u_0 + \mathcal{H}^1$, for arbitrary choices of $æ_1, æ_2, \tau_2 \in \sum$ s.t. $\operatorname{div}(æ_1 + æ_2) \in L^2(\Omega, \mathbb{R}^n)$, for any function $q \in H^1(\Omega)$ and for all numbers $\beta > 0$ the following estimate holds with a positive constant C_Ω*

$$
\begin{aligned}
&\int_\Omega \frac{\nu}{2}\left|\varepsilon(v-u)\right|^2 dx + G(u,v) \\
&\le \int_\Omega \frac{1}{2\nu}(1+\beta)\big|\nu\varepsilon(v) - æ_1\big|^2 dx \\
&\quad + \frac{1+\beta}{2\beta\nu}\left(C_\Omega \,\|\operatorname{div}(æ_1+æ_2) + f - \nabla q\|_{L^2} + \|\tau_2 - æ_2\|_{L^2}\right)^2 \\
&\quad + \int_\Omega \left(\pi\big(\varepsilon(v)\big) + \pi^*(\tau_2) - \tau_2 : \varepsilon(v)\right) dx.
\end{aligned} \tag{2.4}
$$

Let us draw some consequences from (2.4).

Estimate I. If we choose $\tau_2 = æ_2$ in (2.4), then (2.4) takes the form

$$
\int_\Omega \frac{\nu}{2}\left|\varepsilon(u-v)\right|^2 dx + G(u,v) \le \mathcal{M}_1(v, æ_1, æ_2, q, \beta) \tag{2.5}
$$

with

$$
\begin{aligned}
&\mathcal{M}_1(v, æ_1, æ_2, q, \beta) \\
&:= \int_\Omega \frac{1}{2\nu}(1+\beta)\big|\nu\varepsilon(v) - æ_1\big|^2 dx \\
&\quad + \int_\Omega \big(\pi(\varepsilon(v)) + \pi^*(æ_2) - æ_2 : \varepsilon(v)\big)\, dx \\
&\quad + \frac{1+\beta}{2\beta\nu} C_\Omega^2 \,\|\operatorname{div}(æ_1+æ_2) + f - \nabla q\|_{L^2}.
\end{aligned}
$$

Let us discuss the meaning of this estimate. First we observe that the functional $\mathcal{M}_1$ contains only known data (v, f, Ω, ν) or such data that are in our disposal like $(\beta, æ_1, æ_2, q)$. Therefore $\mathcal{M}_1$ is explicitly computable. $\mathcal{M}_1$ provides an upper bound for the deviation from the exact solution and it consists of three functionals that depend on v and the free tensor–valued functions $æ_1, æ_2$. The latter can be viewed as certain images of the parts of the stress tensor associated with the Newtonian and non–Newtonian dissipative potentials, respectively. The first two terms vanish if $\varepsilon(v), æ_1$ and $æ_2$ satisfy the constitutive relations associated with these potentials, while the third one is zero if $\operatorname{div}(æ_1 + æ_2) = \nabla q - f$. Secondly, it is easy to see that

$$
\inf_{v, æ_1, æ_2, q} \mathcal{M}_1(v, æ_1, æ_2, q, \beta) = 0
$$

if and only if $v = u, q = p$ (the true pressure), and $æ_1$ and $æ_2$ coincide with the respective parts of the true stress tensors defined in (1.7). Therefore one can use $\mathcal{M}_1$ as a variational functional whose values give a direct measure of the quality of the approximation.

Estimate II. The idea here is to shift the variable q into another term, where it appears without derivatives. If E denotes the unit matrix, we let $\text{æ}_1 := \tilde{\text{æ}}_1 + q\,E$ and get $\operatorname{div}\ \text{æ}_1 = \operatorname{div}\ \tilde{\text{æ}}_1 + \nabla q$, Then (2.5) gives

$$\int_\Omega \frac{\nu}{2}\left|\varepsilon(v-u)\right|^2\,dx + G\,(u,v) \quad\le\quad \mathcal{M}_2\,(v,\tilde{\text{æ}}_1,\text{æ}_2,q,\beta) \tag{2.6}$$

with $\mathcal{M}_2$ defined according to

$$\begin{aligned}
&\mathcal{M}_2\,(v,\tilde{\text{æ}}_1,\text{æ}_2,q,\beta)\\
:=\ &\int_\Omega \frac{1}{2\nu}\,(1+\beta)\big|\nu\varepsilon(v)-\tilde{\text{æ}}_1-q\,E\big|^2\,dx\\
&+\int_\Omega \big(\pi(\varepsilon(v))+\pi^*(\text{æ}_2)-\text{æ}_2:\varepsilon(v)\big)\,dx\\
&+\frac{1+\beta}{2\beta\nu}\,C_\Omega^2\,\|\operatorname{div}\,(\tilde{\text{æ}}_1+\text{æ}_2)+f\|_{L^2}.
\end{aligned}$$

The properties of the functional $\mathcal{M}_2$ are quite similar to the ones of $\mathcal{M}_1$. For example it is easy to see that $\mathcal{M}_2\,(v,\tilde{\text{æ}}_1,\text{æ}_2,q,\beta)\ge 0$ with equality if and only if

$$\begin{aligned}
&\tilde{\text{æ}}_1 = \nu\,\varepsilon(v)-q\,E,\\
&\text{æ}_2 = \pi'\big(\varepsilon(v)\big),\\
&\operatorname{div}\,(\tilde{\text{æ}}_1+\text{æ}_2)+f=0,
\end{aligned}$$

which means that $v=u$ and $q=p$. Therefore, the exact minimum of $\mathcal{M}_2\,(v,\tilde{\text{æ}}_1,\text{æ}_2,q,\beta)$ is also attained on the solution of the problem under consideration.

Estimate III. The estimates (2.5) and (2.6) are convenient if the explicit form of the conjugate function π^* is known. However, in some interesting cases it is not possible to give a formula for π^*. In order to treat these cases we rearrange the estimate (2.4) in a suitable way. Let $\eta\in\sum$ and define $\tau_2 := \pi'(\eta)$ $\big($note that for the models under consideration we have $\tau_2\in\sum\big)$. Since by elementary properties of π^* we have that

$$\int_\Omega \big(\pi(\eta)+\pi^*(\tau_2)-\tau_2:\eta\big)\,dx = 0, \tag{2.7}$$

we get for the quantity D_2 defined in Theorem 2.1 and which occurs as the third term on the r.h.s. of (2.4)

$$D_2\big(\varepsilon(v),\tau_2\big) = \int_\Omega \big(\pi(\varepsilon(v))-\pi(\eta)+\pi'(\eta):\big(\eta-\varepsilon(v)\big)\big)\,dx. \tag{2.8}$$

Inserting (2.8) into (2.4) we see that

$$\int_\Omega \frac{\nu}{2}\left|\varepsilon(v-u)\right|^2\,dx + G\left(u,v\right) \le \mathcal{M}_3\left(v,æ_1,æ_2,\eta,q,\beta\right) \tag{2.9}$$

with

$$\begin{aligned}
&\mathcal{M}_3\left(v,æ_1,æ_2,\eta,q,\beta\right)\\
:=\ &\int_\Omega \frac{1}{2\nu}\left(1+\beta\right)\left|\nu\varepsilon(v)-æ_1\right|^2\,dx\\
&+\frac{1+\beta}{2\nu\beta}\left(C_\Omega\left\|\operatorname{div}\left(æ_1+æ_2\right)+f-\nabla q\right\|_{L^2}+\left\|\pi'(\eta)-æ_2\right\|_{L^2}\right)^2\\
&+\int_\Omega\left(\pi(\varepsilon(v))-(\eta)+\pi'(\eta):\left(\eta-\varepsilon(v)\right)\right)\,dx.
\end{aligned}$$

Here $\beta>0$ is an arbitrary number and $æ_1,æ_2,\eta$ denote tensors from $\sum$ such that $\operatorname{div}(æ_1+æ_2)\in L^2(\Omega,\mathbb{R}^n)$. The majorant $\mathcal{M}_3$ has the same principal properties as $\mathcal{M}_1$ and $\mathcal{M}_2$. Indeed, it is easy to check that

$$\mathcal{M}_3\left(v,æ_1,æ_2,\eta,q,\beta\right)=0$$

if and only if $v=u$, $æ_1=\nu\,\varepsilon(u)$, $æ_2=\pi'(\varepsilon(u))$, $\eta=\varepsilon(u)$ and $q=p$. If $v\ne u$ and if we choose $æ_1=\sigma_1$, $æ_2=\sigma_2$, $\eta=\varepsilon(u)$ and $q=p$, then the r.h.s. of (2.9) tends towards the l.h.s. of (2.9) as $\beta\searrow 0$. Finally, we note that if $\pi'(\eta)$ has square summable divergence, then the number of variables in $\mathcal{M}_3$ can be reduced by setting $æ_2=\pi'(\eta)$.

In Section 4 we will apply the results described above to the specific potentials π defined in a) – d) of Section 1. Section 5 contains further results for the case that the approximation v does not exactly satisfy the condition div $v=0$.

3 Proofs of the main results

We follow the notation introduced before Theorem 2.1 and observe that for $v\in u_0+\mathcal{H}^1$ we have

$$\begin{aligned}
J(v)-J(u) &= \int_\Omega\Big(\frac{\nu}{2}\left|\varepsilon(v-u)\right|^2+\nu\,\varepsilon(u):\varepsilon(v-u)\\
&\quad+\pi\big(\varepsilon(v)\big)-\pi\big(\varepsilon(u)\big)-f\cdot(v-u)\Big)\,dx\\
&= \int_\Omega\frac{\nu}{2}\left|\varepsilon(v-u)\right|^2\,dx+\int_\Omega\Big(\pi\big(\varepsilon(v)\big)-\pi\big(\varepsilon(u)\big)-\pi'\big(\varepsilon(u)\big):\varepsilon(v-u)\Big)\,dx\\
&\quad+\int_\Omega\Big(\nu\,\varepsilon(u):\varepsilon(v-u)+\pi'\big(\varepsilon(u)\big):\varepsilon(v-u)-f\cdot(v-u)\Big)\,dx,
\end{aligned}$$

and with (1.6) we conclude that

$$J(v) - J(u) = \int_\Omega \frac{\nu}{2} \left|\varepsilon(v-u)\right|^2 \, dx + G\,(u,v), \tag{3.1}$$

G being defined through the relation

$$G\,(v,u) = \int_\Omega \Big(\pi\big(\varepsilon(v)\big) - \pi\big(\varepsilon(u)\big) - \pi'\big(\varepsilon(u)\big) : \varepsilon(v-u)\Big) \, dx.$$

Since $\sigma_2 = \pi'\big(\varepsilon(u)\big)$ and therefore

$$\int_\Omega \Big(\pi\big(\varepsilon(u)\big) + \pi^*(\sigma_2) - \sigma_2 : \varepsilon(u)\Big) \, dx = 0$$

we arrive at the representation for the functional G given in Theorem 2.1. In order to get a suitable lower bound for the functional J (i.e. a lower bound for $J(u)$) we introduce the Lagrangian

$$L(v, \tau_1, \tau_2) := \int_\Omega \Big(\varepsilon(v) : (\tau_1 + \tau_2) - \frac{1}{2\nu}\,|\tau_1|^2 - \pi^*(\tau_2) - f \cdot v\Big) \, dx$$

being defined on $(u_0 + \mathcal{H}^1) \times \sum \times \sum$. We note that

$$\sup_{\tau_1 \in \sum} \int_\Omega \Big(\varepsilon(v) : \tau_1 - \frac{1}{2\nu}\,|\tau_1|^2\Big) \, dx = \int_\Omega \frac{\nu}{2}\,|\,\varepsilon(v)|^2 \, dx.$$

Since $\pi^*(\tau)$ has a growth rate w.r.t. $|\tau|$ greater or equal to 2 (this follows from the growth behaviour of π), any tensor–valued function τ with the property that $\int_\Omega \pi^*(\tau)\, dx < \infty$ belongs to $\sum$.

This implies

$$\sup_{\tau_2 \in \sum} \int_\Omega \Big(\varepsilon(v) : \tau_2 - \pi^*(\tau_2)\Big) \, dx = \int_\Omega \pi\big(\varepsilon(v)\big) \, dx$$

and we arrive at

$$\begin{aligned}
\inf_{u_0 + \mathcal{H}^1} J &= \inf_{v \in u_0 + \mathcal{H}^1} \; \sup_{\tau_1, \tau_2 \in \sum} \; L(v, \tau_1, \tau_2) \\
&\geq \sup_{\tau_1, \tau_2 \in \sum} \; \inf_{v \in u_0 + \mathcal{H}^1} \; L(v, \tau_1, \tau_2) \\
&= \sup_{(\tau_1, \tau_2) \in Q_f} I(\tau_1, \tau_2),
\end{aligned}$$

where

$$I(\tau_1, \tau_2) := \int_\Omega \Big(\varepsilon(u_0) : (\tau_1 + \tau_2) - \frac{1}{2\nu}\,|\tau_1|^2 - \pi^*(\tau_2) - f \cdot u_0\Big) \, dx.$$

Thus we have the upper bound

$$J(v) - J(u) \leq J(v) - I(\tau_1, \tau_2) \tag{3.2}$$

valid for any pair $(\tau_1,\tau_2)\in Q_f$. Writing

$$J(v)-I(\tau_1,\tau_2)=D_1\big(\varepsilon(v),\tau_1\big)+D_2\big(\varepsilon(v),\tau_2\big)$$

$$+\int_\Omega \big(f\cdot(u_0-v)-(\tau_1+\tau_2):\varepsilon(u_0-v)\big)\,dx$$

with D_1, D_2 defined in Theorem 2.1 and recalling the definition of Q_f, the claim of Theorem 2.1 is a consequence of (3.1) and (3.2). $\square$

For the proof of Theorem 2.2 we take any $\tau_2, æ_1, æ_2\in\sum$ such that div $(æ_1+æ_2)\in L^2(\Omega,\mathbb{R}^n)$. Young's inequality implies for all $\beta>0$ the estimate

$$\begin{aligned}D_1\big(\varepsilon(v),\tau_1\big) &= \frac{1}{2\nu}\int_\Omega \big|\nu\varepsilon(v)-\tau_1\big|^2\,dx\\ &= \frac{1}{2\nu}\int_\Omega \big|\nu\varepsilon(v)-æ_1+æ_1-\tau_1\big|^2\,dx\\ &\le \int_\Omega \frac{1}{\nu}\left(\frac{1+\beta}{2}\big|\nu\varepsilon(v)-æ_1\big|^2+\frac{1+\beta}{2\beta}\big|æ_1-\tau_1\big|^2\right)dx.\end{aligned}$$

We further have

$$\|æ_1-\tau_1\|_{L^2}\le\|æ_1+æ_2-\tau_1-\tau_2\|_{L^2}+\|æ_2-\tau_2\|_{L^2}. \tag{3.3}$$

Define $\overline{w}\in H^1_0(\Omega,\mathbb{R}^n)$ as the unique solution of problem (2.3) and let $\eta_0:=æ_1+æ_2+\varepsilon(\overline{w})$. Then

$$\begin{aligned}\int_\Omega \eta_0:\varepsilon(w)\,dx &= \int_\Omega \big(æ_1+æ_2+\varepsilon(\overline{w})\big):\varepsilon(w)\,dx\\ &= \int_\Omega (f-\nabla q)\cdot w\,dx\\ &= \int_\Omega f\cdot w\,dx\end{aligned} \tag{3.4}$$

holds for all $w\in\mathcal{H}^1$ on account of (2.3). For $\varepsilon(\overline{w})$ we have the energy estimate

$$\big\|\varepsilon\left(\overline{w}\right)\|_{L^2} \;\le\; C_\Omega\|\operatorname{div}(æ_1+æ_2)+f-\nabla q\big\|_{L^2} \tag{3.5}$$

which is also an immediate consequence of (2.3). Let us finally set $\tau_1:=\eta_0-\tau_2$. By (3.4) the pair (τ_1,τ_2) is in Q_f and

$$\big\|æ_1+æ_2-(\tau_1+\tau_2)\|_{L^2} \;=\; \|æ_1+æ_2-\eta_0\big\|_{L^2}=\|\varepsilon\left(\overline{w}\right)\|_{L^2}. \tag{3.6}$$

Combining (3.5) and (3.6) we therefore get

$$\|æ_1 + æ_2 - (\tau_1 + \tau_2)\|_{L^2} \;\leq\; C_\Omega \| \operatorname{div}(æ_1 + æ_2) + f - \nabla q\|_{L^2}.$$

Inserting this result into (3.3) we see that

$$\|æ_1 - \tau_1\|_{L^2} \;\leq\; C_\Omega \| \operatorname{div}(æ_1 + æ_2) + f - \nabla q\|_{L^2}.$$

This implies the estimate

$$D_1\big(\varepsilon(v), \tau_1\big) \leq \int_\Omega \frac{1+\beta}{2\nu} \left|\nu\varepsilon(v) - æ_1\right|^2 dx$$

$$+ \quad \frac{1+\beta}{2\nu\beta} \Big(C_\Omega \| \operatorname{div}(æ_1 + æ_2) + f - \nabla q\|_{L^2} + \|\tau_2 - æ_2\big\|_{L^2}\Big).$$

Since $D_2\big(\varepsilon(v), \tau_2\big)$ remains unchanged, estimate (2.4) now follows from inequality (2.1), Theorem 2.2 is proved. □

4 Discussion of the examples

In this section we apply the results of Section 2 to the particular models discussed in the introduction.

4.1 Power–law models

4.1.1 Case $\pi(\varepsilon) = k|\varepsilon|^\alpha, \;\alpha \in (1, 2]$.

We have

$$\pi'(\varepsilon) = k\alpha|\varepsilon|^{\alpha-2}\varepsilon, \;\pi^*(\tau) = \big(\tfrac{1}{\alpha k}\big)^{\frac{1}{\alpha-1}} \tfrac{1}{\alpha^*}\; |\tau|^{\alpha^*}.$$

Therefore, for any $v \in \mathcal{H}^1 + u_0$ we obtain the estimate

$$(4.1)\qquad \begin{aligned} &\int_\Omega \frac{\nu}{2}|\varepsilon(v-u)|^2 dx + G(u, v) \\ &\leq \int_\Omega \frac{1}{2\nu}(1+\beta)|\nu\varepsilon(v) - æ_1|^2 dx + D_2(\varepsilon(v), æ_2) \\ &+ \tfrac{1+\beta}{2\beta\nu}\, C_\Omega^2 \|\operatorname{div}(æ_1 + æ_2) + f - \nabla q\|_{L^2}^2, \end{aligned}$$

where the term $D_2(\varepsilon(v), æ_2)$ has the form

$$D_2(\varepsilon(v), æ_2) = \int_\Omega \Big(k|\varepsilon(v)|^\alpha + \big(\tfrac{1}{\alpha k}\big)^{\frac{1}{\alpha-1}} \tfrac{1}{\alpha^*}|æ_2|^{\alpha^*} - æ_2 : \varepsilon(v)\Big) dx$$

and

$$G(u,v) = \int\limits_{\Omega} k\Big(|\varepsilon(v)|^{\alpha} - |\varepsilon(u)|^{\alpha} + \alpha|\varepsilon(u)|^{\alpha-2}\varepsilon(u) : \varepsilon(u-v)\Big)dx.$$

If the estimate (2.9) is used, then we obtain

$$\begin{aligned}
\int\limits_{\Omega} \frac{\nu}{2}|\varepsilon(v-u)|^2dx + G(u,v) \leq & \int\limits_{\Omega} \tfrac{1+\beta}{2\nu}|\nu\varepsilon(v) - æ_1|^2 dx \\
& + \tfrac{1+\beta}{2\beta\nu}\left(C_\Omega\|\mathrm{div}(æ_1+æ_2) + f - \nabla q\|_{L^2} + \|k\alpha|\eta|^{\alpha-2}\eta - æ_2\|_{L^2}\right)^2 \\
& + \int\limits_{\Omega} \Big(k|\varepsilon(v)|^{\alpha} - k|\eta|^{\alpha} + k\alpha|\eta|^{\alpha-2}|\eta : (\eta - \varepsilon(v))dx.
\end{aligned} \tag{4.2}$$

If $k=0$, then $G(u,v)=0$ and by setting in (4.2) $æ_2 = 0$ we obtain the following estimate for the Stokes problem (cf. [25])

$$\int\limits_{\Omega} \nu|\varepsilon(v-u)|^2dx \leq \int\limits_{\Omega} \tfrac{1+\beta}{\nu}|\nu\varepsilon(v) - æ_1|^2dx + \tfrac{1+\beta}{\beta\nu}C_\Omega^2\|\mathrm{div}æ_1 + f - \nabla q\|^2_{L^2}. \tag{4.3}$$

Since ν is a constant, estimate (4.3) implies

$$\nu\|\varepsilon(v-u)\|_{L^2} \leq \|\nu\varepsilon(v) - æ_1\|_{L^2} + C_\Omega\|\mathrm{div}æ_1 + f - \nabla q\|_{L^2} \tag{4.4}$$

which can be easily seen by taking for β the value for which $\frac{d}{d\beta}$ of the r.h.s. of (4.3) vanishes.

4.1.2 Case $\pi(\varepsilon) = k\left(\lambda + |\varepsilon|^2\right)^{\alpha/2}, \;\alpha \in (1,2],\; \lambda > 0.$

Here

$$\pi'(\varepsilon) = k\alpha(\lambda + |\varepsilon|^2)^{\alpha/2-1}\varepsilon.$$

and

$$\begin{aligned}
G(u,v) = \int\limits_{\Omega} k\Big(&\left(\lambda + |\varepsilon(v)|^2\right)^{\alpha/2} - \left(\lambda + |\varepsilon(u)|^2\right)^{\alpha/2} \\
&+ \alpha(\lambda + |\varepsilon(u)|^2)^{\alpha/2-1}\varepsilon(u) : \varepsilon(u-v)\Big)dx.
\end{aligned}$$

Since in this case the explicit form of π^* is unknown, we apply (2.9) and obtain

$$\begin{aligned}
&\int\limits_\Omega \frac{\nu}{2}|\varepsilon(v-u)|^2 dx + G(u,v) \\
&\quad \leq \int\limits_\Omega \tfrac{1+\beta}{2\nu}|\nu\varepsilon(v) - æ_1|^2 dx \\
&\quad + \tfrac{1+\beta}{2\beta\nu}\left(C_\Omega\|\mathrm{div}(æ_1+æ_2) + f - \nabla q\|_{L^2}\right. \\
&\quad + \left.\|k\alpha(\lambda+|\eta|^2)^{\alpha/2-1}\eta - æ_2\|_{L^2}\right)^2 \\
&\quad + \int\limits_\Omega k\Big((\lambda+|\varepsilon(v)|^2)^{\alpha/2} - (\lambda+|\eta)|^2)^{\alpha/2} \\
&\quad + \alpha(\lambda+|\eta|^2)^{\alpha/2-1}\eta : (\eta - \varepsilon(v)\Big)dx.
\end{aligned} \tag{4.5}$$

Note that if λ tends to zero, then (4.5) transforms to (4.2). Also we note that the method of deriving this estimate did not use the fact that λ, ν and α are constants. It was only assumed that $\nu > 0$, $\lambda \geq 0$, and $\alpha \in (1,2]$. Therefore, the case where λ, ν and α are smooth functions satisfying these conditions is encompassed in (4.5).

4.2 Bingham model

The Bingham model can be viewed as a special case of the power model with $\alpha = 1$. In this model, π^* is given by the relation

$$\pi^*(\tau) = \begin{cases} 0, & \text{if } |\ \tau\ | \leq k, \\ +\infty, & \text{if } |\ \tau\ | > k \end{cases}$$

and

$$\pi'(\varepsilon) = \begin{cases} k\frac{\varepsilon}{|\varepsilon|}, & \text{if } |\ \varepsilon\ | > 0, \\ \xi,\ |\xi| \leq 1, & \text{if } |\ \varepsilon\ | = 0. \end{cases}$$

Therefore, the term $D_2(\varepsilon(v), æ_2)$ is finite only if $|æ_2| \leq k$. In the latter case it has the form

$$D_2(\varepsilon(v), æ_2) = \int\limits_\Omega (k|\varepsilon(v)| - æ_2 : \varepsilon(v))dx.$$

Similarly,

$$G(\varepsilon(v), \sigma_2) = \int\limits_\Omega (k|\varepsilon(v)| - \sigma_2 : \varepsilon(v))dx$$

provided that $|\sigma_2| \le k$ at almost all points of Ω. We see that $G(u,v)=0$ if σ_2 and $\varepsilon(v)$ satisfy the relation $\sigma_2 = k\frac{\varepsilon(v)}{|\varepsilon(v)|}$.
Now (2.5) has the form

$$\begin{aligned}
&\int\limits_\Omega \frac{\nu}{2}|\varepsilon(v-u)|^2dx + G(u,v) \\
(4.6)\qquad &\le \int\limits_\Omega \tfrac{1+\beta}{2\nu}|\nu\varepsilon(v) - æ_1|^2dx + \int\limits_\Omega \Big(k|\varepsilon(v)| - æ_2 : \varepsilon(v)\Big)dx \\
&+ \tfrac{1+\beta}{2\beta\nu}C_\Omega^2\|\mathrm{div}(æ_1+æ_2) + f - \nabla q\|^2_{L^2},
\end{aligned}$$

where $æ_2$ must satisfy the condition

$$|æ_2| \le k \quad \text{for a.e. x} \in \Omega.$$

4.3 Powell Eyring model

Here

$$\pi'(\varepsilon) = k\left(\frac{1}{1+|\varepsilon|} + \frac{\ln(1+|\varepsilon|)}{|\varepsilon|}\right)\varepsilon$$

and

$$\begin{aligned}
G(u,v) = &\int\limits_\Omega k\Big(|\varepsilon(v)|\ln(1+|\varepsilon(v)|) - |\varepsilon(u)|\ln(1+|\varepsilon(u)|) \\
&+ \left(\frac{1}{1+|\varepsilon(u)|} + \frac{\ln(1+|\varepsilon(u)|)}{|\varepsilon(u)|}\right)\varepsilon(u) : \varepsilon(u-v)\Big)dx.
\end{aligned}$$

In this case, the explicit form of π^* is also unknown, so that we use the estimate (2.9) and obtain

$$\begin{aligned}
&\int\limits_\Omega \frac{\nu}{2}|\varepsilon(v-u)|^2dx + G(u,v) \le \int\limits_\Omega \tfrac{1+\beta}{2\nu}|\nu\varepsilon(v) - æ_1|^2dx \\
&+ \tfrac{1+\beta}{2\beta\nu}\Big(C_\Omega\|\mathrm{div}(æ_1+æ_2) + f - \nabla q\|_{L^2} \\
(4.7)\qquad &+ \|k(\frac{1}{1+|\eta|} + \frac{\ln(1+|\eta|)}{|\eta|})\varepsilon(\eta) - æ_2\|_{L^2}\Big)^2 \\
&+ \int\limits_\Omega k\Big(|\varepsilon(v)|\ln(1+|\varepsilon(v)|) - |\eta|\ln(1+|\eta|) \\
&+ \left(\frac{1}{1+|\eta|} + \frac{\ln(1+|\eta|)}{|\eta|}\right)\eta : (\eta - \varepsilon(v))\Big)dx.
\end{aligned}$$

5 Estimates for nonsolenoidal approximations

Up to now we considered approximations v from the energy space $u_0 + \mathcal{H}^1$, in particular div $v = 0$ is required. If we drop this condition, then the estimate of the deviation from the exact solution becomes more complicated: if v is taken from the class $u_0 + H_0^1(\Omega, \mathbb{R}^n)$, then it is necessary to estimate explicitly the distance from v to the set of solenoidal vectorfields and to transform our previous estimates in such a way that the additional error caused by the violation of solenoidality becomes transparent. To this purpose we make use of

LEMMA 5.1. *(see, e.g. [15], [21], or [10]) Let $G \subset \mathbb{R}^n$ denote a bounded Lipschitz domain. Then there exists a positive constant depending on G such that for any function $\phi \in L^2(G)$ such that $⨍_G \phi\, dx = 0$ there exists a function $\overline{u} \in H_0^1(G, \mathbb{R}^n)$ such that* $\operatorname{div} \overline{u} = \phi$ *and*

$$\|\nabla \overline{u}\|_{L^2} \le \mathbf{C}_G \|\Phi\|_{L^2}. \tag{5.1}$$

Now, if $\hat{v}$ is an arbitrary function from $H_0^1(\Omega, \mathbb{R}^n)$ we let $\phi := \operatorname{div} \hat{v}$ and apply Lemma 5.1 to get a field $u_\phi \in H_0^1(\Omega, \mathbb{R}^n)$ satisfying $\operatorname{div}(\hat{v} - u_\phi) = 0$ together with $\|\nabla u_\phi\|_{L^2} \le \mathbf{C}_\Omega \|\operatorname{div} \hat{v}\|_{L^2}$. This means that the field $w_0 := \hat{v} - u_\phi \in H_0^1(\Omega, \mathbb{R}^n)$ is solenoidal and satisfies

$$\|\nabla(\hat{v} - w_0)\|_{L^2} \le \mathbf{C}_\Omega \|\operatorname{div} \hat{v}\|_{L^2}. \tag{5.2}$$

Obviously (5.2) is the required measure of the distance from $\hat{v}$ to the set of solenoidal fields.

The lemma above also implies the following condition known in the literature as the Ladyzhenskaya - Babuska - Brezzi (LBB) condition: there exists a positive constant C_{LBB} such that

$$\inf_{\phi \in L_0^2, \phi \neq 0} \ \sup_{w \in H_0^1(\Omega, \mathbb{R}^n), w \neq 0} \frac{1}{\|\phi\|_{L^2}} \frac{1}{\|\nabla w\|_{L^2}} \int_\Omega \phi \operatorname{div} w\, dx \ \ge \ C_{\mathrm{LBB}}, \tag{5.3}$$

where $L_0^2 = \{\phi \in L^2(\Omega) : ⨍_\Omega \phi\, dx = 0\}$. In fact, for any $\phi \in L_0^2$ we can find $v_\phi \in H_0^1(\Omega, \mathbb{R}^n)$ such that

$$\operatorname{div} v_\phi = \phi, \ \|\nabla v_\phi\|_{L^2} \ \le \ \mathbf{C}_\Omega \|\phi\|_{L^2}, \tag{5.4}$$

thus

$$\begin{aligned} \sup_{w \in H_0^1(\Omega, \mathbb{R}^n), w \neq 0} \frac{\int_\Omega \phi \operatorname{div} w dx}{\|\nabla w\|_{L^2} \|\phi\|_{L^2}} \quad &\ge \quad \frac{1}{\|\nabla v_\phi\|_{L^2}} \frac{1}{\|\phi\|_{L^2}} \int_\Omega \phi \operatorname{div} v_\phi dx \\ &\overset{(5.4)}{=} \quad \frac{\|\phi\|_{L^2}}{\|\nabla v_\phi\|_{L^2}} \overset{(5.4)}{\ge} \frac{1}{\mathbf{C}_\Omega}, \end{aligned}$$

and (5.3) follows with $C_{\text{LBB}} := 1/\mathbf{C}_\Omega$.

Concerning estimates for the constant $\mathbf{C}_\Omega$ see, e.g., [18].

Now we are ready to derive estimates for nonsolenoidal approximations. We recall that u denotes the unique solution of the problem $(\mathcal{P})$ from Section 1 and – as in the previous section – we carry out our calculations for the specific models under consideration.

5.1 Power–law models

5.1.1 Case $\pi(\varepsilon) = k|\varepsilon|^\alpha,\ \alpha \in (1,2]$.

Consider a function $\overline{v} \in u_0 + H^1_0(\Omega, \mathbb{R}^n)$. From (4.1) it follows that

$$\begin{aligned}\nu\|\varepsilon(v-u)\|_{L^2} \quad \leq \quad & \Big(\big(\|\nu\varepsilon(v) - æ_1\|_{L^2} + C_\Omega\|\operatorname{div}(æ_1+æ_2) + f - \nabla q\|_{L^2}\big)^2 \\ & + 2\nu\, D_2\big(\varepsilon(v), æ_2\big)\Big)^{1/2},\end{aligned} \tag{5.5}$$

where $v \in \mathcal{H}^1 + u_0$. Since

$$\|\varepsilon(\overline{v}-u)\|_{L^2} \leq \|\varepsilon(\overline{v}-\overline{v}_0)\|_{L^2} + \|\varepsilon(\overline{v}_0 - u)\|_{L^2}, \tag{5.6}$$

we find that

$$\|\varepsilon(\overline{v}-u)\|_{L^2} \leq \rho(\overline{v}) + \|\varepsilon(\overline{v}_0 - u)\|_{L^2} \tag{5.7}$$

with $\rho\,(\overline{v})$ and $\overline{v}_0$ being defined by the lemma. Namely, there exists $\overline{v}_0 \in u_0 + \mathcal{H}^1$ such that

$$\|\varepsilon(\overline{v}-\overline{v}_0)\|_{L^2} \leq \mathbf{C}_\Omega\|\operatorname{div}\overline{v}\|_{L^2} := \rho(\overline{v}).$$

To $\|\varepsilon(\overline{v}_0 - u)\|_{L^2}$ we can apply (5.5): we have

$$\|\nu\varepsilon(\overline{v}_0) - æ_1\|_{L^2} \leq \|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2} + \nu\rho(\overline{v}). \tag{5.8}$$

To discuss $D_2\big(\varepsilon(\overline{v}_0), æ_2\big)$, we note that

$$\begin{aligned} & \int_\Omega \Big(\pi\big(\varepsilon(\overline{v}_0)\big) + \pi^*(æ_2) - \varepsilon(\overline{v}_0) : æ_2\Big)\, dx \\ \leq \quad & \int_\Omega \Big(\pi\big(\varepsilon(\overline{v})\big) + \pi^*(æ_2) - \varepsilon(\overline{v}) : æ_2\Big)\, dx \\ & + \int_\Omega \Big(\pi'\big(\varepsilon(\overline{v}_0)\big) - æ_2\Big) : \varepsilon(\overline{v}_0 - \overline{v})\, dx \\ = \quad & D_2\big(\varepsilon(\overline{v}), æ_2\big) + \int_\Omega \Big(\pi'\big(\varepsilon(\overline{v})\big) - æ_2\Big) : \varepsilon(\overline{v}_0 - \overline{v})\, dx \\ & + \int_\Omega \Big(\pi'\big(\varepsilon(\overline{v}_0)\big) - \pi'(\varepsilon(\overline{v}))\Big) : \varepsilon(\overline{v}_0 - \overline{v})\, dx.\end{aligned} \tag{5.9}$$

Using the inequality (the proof being presented in the Appendix)

$$\left(\frac{a}{|a|^{\Theta}} - \frac{b}{|b|^{\Theta}}\right) \cdot (a-b) \leq 2^{\Theta}(1+\Theta)|a-b|^{2-\Theta}, \tag{5.10}$$

valid for any $a, b \in \mathbb{R}^{\ell}, \ell \geq 1$, and any choice of $\Theta \in [0,1)$ we find that

$$\begin{aligned}
&\int_{\Omega} \big(\pi'(\varepsilon(\overline{v}_0)) - \pi'(\varepsilon(\overline{v}))\big) : \varepsilon(\overline{v}_0 - \overline{v})\; dx \\
= \;& \int_{\Omega} k\alpha \left(\frac{\varepsilon(\overline{v}_0)}{|\varepsilon(\overline{v}_0)|^{2-\alpha}} - \frac{\varepsilon(\overline{v})}{|\varepsilon(\overline{v})|^{2-\alpha}}\right) : \varepsilon(v_0 - v)\; dx \\
\leq \;& 2^{2-\alpha}(3-\alpha)k\alpha \int_{\Omega} |\varepsilon(\overline{v} - \overline{v}_0)|^{\alpha}\; dx \\
\leq \;& 2^{2-\alpha}(3-\alpha)k\alpha\, |\Omega|^{1-\frac{\alpha}{2}}\, \rho^{\alpha}(\overline{v}).
\end{aligned} \tag{5.11}$$

Therefore, combining (5.9) and (5.11), we arrive at

$$\begin{aligned}
D_2\big(\varepsilon(\overline{v}_0), æ_2\big) \;\leq\;& D_2\big(\varepsilon(\overline{v}), æ_2\big) + \rho\,(\overline{v})\|\pi'(\varepsilon(\overline{v})) - æ_2\|_{L^2} \\
+\;& 2^{2-\alpha}(3-\alpha)k\alpha\, |\Omega|^{1-\frac{\alpha}{2}}\rho^{\alpha}(\overline{v}).
\end{aligned} \tag{5.12}$$

Now, by (5.7), (5.8) and (5.12), we obtain the final estimate

$$\begin{aligned}
&\nu\|\varepsilon(\overline{v} - u)\|_{L^2} \leq \nu\rho\,(\overline{v}) + \Big[\big(\|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2} + \nu\rho\,(\overline{v}) \\
&\quad + C_{\Omega}\, \|\operatorname{div}(æ_1 + æ_2) + f - \nabla q\|_{L^2}\big)^2 + 2\nu\, D_2\big(\varepsilon(\overline{v}), æ_2\big) \\
&\quad + 2\nu\rho\,(\overline{v})\|\pi'(\varepsilon(\overline{v})) - æ_2\|_{L^2} + 2^{2-\alpha}(3-\alpha)k\alpha\, |\Omega|^{1-\frac{\alpha}{2}}\rho^{\alpha}(\overline{v})\Big]^{\alpha}.
\end{aligned} \tag{5.13}$$

We observe that apart of a more complicated form the principal structure of the estimate (5.13) is the same as for solenoidal fields. The right–hand side of (5.13) is a combination of the terms

$$\begin{aligned}
&\|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2},\; D_2\big(\varepsilon(\overline{v}),\; æ_2\big), \\
&\|\pi'(\varepsilon(\overline{v})) - æ_2\|_{L^2},\; \|\operatorname{div}(æ_1 + æ_2) + f - \nabla q\|_{L^2} \text{ and } \rho\,(\overline{v}).
\end{aligned}$$

All of them are nonnegative and their simultaneous vanishing means that

$$\nu\varepsilon(\overline{v}) - æ_1 = 0,\; æ_2 = \pi'(\varepsilon(\overline{v})),\; \operatorname{div}(æ_1 + æ_2) = f - \nabla q,\; \operatorname{div}\overline{v} = 0.$$

Thus, the right–hand side of (5.13) can be zero only on the exact solution of the problem in question. Moreover, it is continuous with respect to convergence $\overline{v}_k \to u$ and $æ_{1k} \to æ_1, æ_{2k} \to æ_2$ in the appropriate spaces.

5.1.2 Case $\pi(\varepsilon) = k(\lambda + |\varepsilon|^2)^{\alpha/2},\ \alpha \in (1,2],\ \lambda > 0.$

First, we deduce from (4.5) the estimate

$$
\begin{aligned}
\nu^2\|\varepsilon(v-u)\|^2_{L^2} \leq \big(&\|\nu\varepsilon\,(v) - æ_1\|_{L^2} + C_\Omega\,\|\operatorname{div}\,(æ_1 + æ_2) + f - \nabla q\|_{L^2} \\
&+ \|k\alpha\,(\lambda + |\eta|^2)^{\frac{\alpha}{2}-1}\,\eta - æ_2\|_{L^2}\big)^2 \\
&+ \int_\Omega \big((\pi(\varepsilon(v)) - \pi(\eta) + \pi'(\eta) : (\eta - \varepsilon(v))\big)\,dx.
\end{aligned}
\tag{5.14}
$$

Let $\overline{v} \in u_0 + H^1_0(\Omega, \mathbb{R}^n)$ and, as in the previous case, consider $\overline{v}_0 \in u_0 + \mathcal{H}^1$ such that

$$\|\varepsilon(\overline{v} - \overline{v}_0)\|_{L^2} \leq \mathbf{C}_\Omega\|\operatorname{div}\overline{v}\|_{L^2} := \rho\,(\overline{v}).$$

Therefore, from (5.14) we obtain

$$
\begin{aligned}
\nu\|\varepsilon(\overline{v} - u)\|_{L^2} \leq \nu\rho\,(\overline{v}) + \big[\big(&\|\nu\varepsilon\,(\overline{v}_0) - æ_1\|_{L^2} \\
&+ C_\Omega\,\|\operatorname{div}\,(æ_1 + æ_2) + f - \nabla q\|_{L^2} + \|k\alpha\,(\lambda + |\eta|^2)^{\alpha/2-1}\,\eta - æ_2\|_{L^2}\big)^2 \\
&+ \int_\Omega \big(\pi(\varepsilon(\overline{v}_0)) - \pi(\eta) + \pi'(\eta) : (\eta - \varepsilon(\overline{v}_0)\big)\,dx\big]^{1/2}.
\end{aligned}
\tag{5.15}
$$

We note that

$$|\pi'(\varepsilon)| \leq k\alpha\lambda^{\alpha/2-1}\,|\varepsilon| \tag{5.16}$$

and, consequently,

$$\pi(\varepsilon(\overline{v}_0)) - \pi(\varepsilon(\overline{v})) \leq |\pi'(\varepsilon(\overline{v}_0))|\,|\varepsilon(\overline{v}_0 - \overline{v})|. \tag{5.17}$$

We further rewrite the last term on the r.h.s. of (5.15) as follows:

$$
\begin{aligned}
&\int_\Omega \Big(\pi(\varepsilon(\overline{v}_0)) - \pi(\eta) + \pi'(\eta) : (\eta - \varepsilon(\overline{v}_0))\Big)\,dx \\
&= \int_\Omega \Big(\pi(\varepsilon(\overline{v})) - \pi(\eta) + \pi'(\eta) : (\eta - \varepsilon(\overline{v})\Big)\,dx \\
&\quad + \int_\Omega \Big(\pi(\varepsilon(\overline{v}_0)) - \pi(\varepsilon(\overline{v})) + \pi'(\eta) : \varepsilon(\overline{v} - \overline{v}_0)\Big)\,dx.
\end{aligned}
\tag{5.18}
$$

Note that the first integral in the right–hand side of (5.18) contains the function $\overline{v}$ and the tensor–valued function η (which is in our disposal), while the second one can be estimated from above by means of (5.16) and (5.17). Indeed,

$$
\begin{aligned}
\int_\Omega \Big(\pi(\varepsilon(\overline{v}_0)) - \pi(\varepsilon(\overline{v}))\Big)\,dx &\leq k\alpha\lambda^{\alpha/2-1}\|\varepsilon(\overline{v}_0)\|_{L^2}\,\|\varepsilon(\overline{v}_0 - \overline{v})\|_{L^2} \\
&\leq k\alpha\lambda^{\alpha/2-1}\big(\|\varepsilon(\overline{v})\|_{L^2} + \rho\,(\overline{v})\big)\rho(\overline{v})
\end{aligned}
\tag{5.19}
$$

and

$$\int_\Omega \pi'(\eta) : \varepsilon(\overline{v} - \overline{v}_0)\ dx \leq k\alpha\lambda^{\alpha/2-1}\ \|\eta\|_{L^2}\rho\,(\overline{v}). \tag{5.20}$$

Analogously

$$\|\nu\varepsilon(\overline{v}_0) - æ_1\|_{L^2} \leq \|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2} + \nu\rho\,(\overline{v}). \tag{5.21}$$

Now, by (5.15), (5.19), (5.20) and (5.21), we deduce the desired estimate

$$\begin{aligned} \nu\|\varepsilon(\overline{v} - u)\|_{L^2} \leq \nu\rho\,(\overline{v}) + \Big[\big(&\|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2} + \nu\rho\,(\overline{v}) \\ &+ C_\Omega\,\|\operatorname{div}(æ_1 + æ_2) + f - \nabla q\|_{L^2} + \|k\alpha\,(\lambda + |\eta|^2)^{\alpha/2-1} - æ_2\|_{L^2}\big)^2 \\ &+ k\alpha\lambda^{\alpha/2-1}\big(\|\varepsilon(\overline{v})\|_{L^2} + \|\eta\|_{L^2} + \rho\,(\overline{v})\big)\rho\,(\overline{v}) \\ &+ \int_\Omega \big(\pi(\varepsilon(\overline{v})) - \pi(\eta) + \pi'(\eta) : (\eta - \varepsilon(\overline{v})\big)\ dx\Big]^{1/2}. \end{aligned} \tag{5.22}$$

It is easy to see that if $\overline{v}$ is a solenoidal field (i.e. $\rho\,(\overline{v}) = 0$), then (5.22) is equivalent to (5.14).

5.2 Bingham model

First, we use (4.6) and obtain

$$\begin{aligned} \nu\|\varepsilon(v - u)\|_{L^2} \ \leq\ & \Big[\big(\|\nu\varepsilon(v) - æ_1\|_{L^2} + C_\Omega\,\|\operatorname{div}(æ_1 + æ_2) + f - \nabla q\|_{L^2}\big)^2 \\ & + \int_\Omega \big(k|\varepsilon(v)| - æ_2 : \varepsilon(v)\big)\ dx\Big]^{1/2}. \end{aligned} \tag{5.23}$$

Let $\overline{v} \in u_0 + H_0^1(\Omega, \mathbb{R}^n)$ and let $\overline{v}_0 \in u_0 + \mathcal{H}^1$ denote the solenoidal field defined as in the previous cases. We have (recall taht $|æ_2| \leq k$)

$$\begin{aligned} \|\nu\varepsilon(\overline{v}_0) - æ_1\|_{L^2} &\leq \|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2} + \nu\rho\,(\overline{v}), \\ \int_\Omega \big(k|\varepsilon(\overline{v}_0)| - æ_2 : \varepsilon(\overline{v}_0)\big)\ dx &\leq \int_\Omega \big(k|\varepsilon(\overline{v})| - æ_2 : \varepsilon(\overline{v})\big)\ dx \\ &\quad + \int_\Omega \big(k|\varepsilon(\overline{v}_0 - \overline{v})| - æ_2 : \varepsilon(\overline{v}_0 - \overline{v})\big)\ dx \\ &\leq \int_\Omega \big(k|\varepsilon(\overline{v})| - æ_2 : \varepsilon(\overline{v})\big)\ dx + 2k\int_\Omega |\varepsilon(\overline{v}_0 - \overline{v})|\ dx \\ &\leq \int_\Omega \big(k|\varepsilon(\overline{v})| - æ_2 : \varepsilon(\overline{v})\big)\ dx + 2k|\Omega|^{1/2}\rho\,(\overline{v}). \end{aligned}$$

By these estimates we obtain

$$\nu\|\varepsilon(\overline{v}-u)\|_{L^2} \leq \nu\rho\,(\overline{v}) + \Big[\big(\|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2} + \nu\rho\,(\overline{v}) \tag{5.24}$$
$$+\, C_\Omega\,\|\,\mathrm{div}\,(æ_1 + æ_2) + f - \nabla q\|_{L^2}\big)^2$$
$$+ \int_\Omega \big(k|\varepsilon(\overline{v})| - æ_2 : \varepsilon(\overline{v})\big)\; dx + 2k|\Omega|^{1/2}\rho\,(\overline{v})\Big]^{1/2}.$$

As in (4.6), this estimate is applicable only if $|æ_2| \leq k$ for a.e. $x \in \Omega$.

5.3 Powell-Eyring model

Here, $\pi(\varepsilon) = k|\varepsilon|\ell n(1+|\varepsilon|)$ and an estimate for $\overline{v}$ can be derived along the same way as for the model 5.1.2. Indeed, it is easy to see that

$$|\pi'(\varepsilon)| \leq 2k\,|\varepsilon|. \tag{5.25}$$

We have the estimate (which follows from (4.7))

$$\nu^2\|\varepsilon(v-u)\|^2_{L^2} \;\leq\; \big(\|\nu\varepsilon(v) - æ_1\|_{L^2} + C_\Omega\,\|\,\mathrm{div}\,(æ_1+æ_2) + f - \nabla q\|_{L^2} \tag{5.26}$$
$$+\|\pi'(\eta) - æ_2\|_{L^2}\big)^2$$
$$+\int_\Omega \big(\pi(\varepsilon(v)) - \pi(\eta) + \pi'(\eta) : (\eta - \varepsilon(v)\big)\,dx.$$

¿From (5.26) we easily get

$$\nu\|\varepsilon(\overline{v}-u)\|_{L^2} \leq \nu\rho\,(\overline{v}) + \Big[\big(\|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2} + \nu\rho\,(\overline{v}) \tag{5.27}$$
$$+\, C_\Omega\,\|\,\mathrm{div}\,(æ_1+æ_2) + f - \nabla q\|_{L^2} + \|\pi'(\eta) - æ_2\|_{L^2}\big)^2$$
$$+ \int_\Omega \big(\pi(\varepsilon(\overline{v})) - \pi(\eta) + \pi'(\eta) : (\eta - \varepsilon(\overline{v})\big)\; dx$$
$$+ \int_\Omega \big(\pi(\varepsilon(\overline{v}_0)) - \pi(\varepsilon(\overline{v})) + \pi'(\eta) : (\varepsilon(\overline{v} - \overline{v}_0))\big)\; dx\Big]^{1/2}.$$

Since by (5.25)

$$\int_\Omega \big(\pi(\varepsilon(\overline{v}_0)) - \pi(\varepsilon(\overline{v})\big)\; dx \leq \int_\Omega 2k\,|\varepsilon(\overline{v}_0)|\;|\varepsilon(\overline{v}_0 - \overline{v}|\; dx$$
$$\leq\;\; 2k\,\big(\|\varepsilon(\overline{v})\|_{L^2} + \rho(\overline{v})\big)\rho\,(\overline{v})$$

and

$$\int_\Omega \pi'(\eta) : \varepsilon(\overline{v} - \overline{v}_0)\; dx \leq 2k\,\|\eta\|_{L^2}\,\rho\,(\overline{v}),$$

we finally obtain from (5.27) the desired estimate

$$
\begin{aligned}
\nu\|\varepsilon(\overline{v}-u)\|_{L^2} \le \nu\rho\,(\overline{v}) + \Big[&\big(\|\nu\varepsilon(\overline{v}) - æ_1\|_{L^2} + \rho\,(\overline{v}) \\
&+ C_\Omega \,\|\operatorname{div}(æ_1+æ_2) + f - \nabla q\|_{L^2} + \|\pi'(\eta) - æ_2\|_{L^2}\big)^2 \\
&+ 2k\,\big(\|\varepsilon(\overline{v})\|_{L^2} + \|\eta\|_{L^2} + \rho(\overline{v})\big)\rho\,(\overline{v}) \\
&+ \int_\Omega \big(\pi(\varepsilon(\overline{v})) - \pi(\eta) + \pi'(\eta) : (\eta - \varepsilon(\overline{v})\big)\; dx\Big]^{1/2}.
\end{aligned}
\tag{5.28}
$$

6 Appendix: proof of inequality (5.10)

Estimate (5.10) is a consequence of the following inequality: given $s \in (0,1)$ and numbers $\xi_1, \xi_2 > 0$, then we have that

$$
\big|\, \xi_2^s - \xi_1^s \big| \;\le\; \frac{s}{2}\Big[\xi_1^{s-1} + \xi_2^{s-1}\Big]\big|\, \xi_2 - \xi_1\big|. \tag{6.1}
$$

Suppose for the moment that (6.1) is correct and consider $a,\, b \in \mathbb{R}^\ell$ and $\Theta \in [0,1)$. W.l.o.g. we may assume that $0 < |a| \le |b|$. Let

$$
T(a,b) := \left| \frac{a\big(|b|^\Theta - |a|^\Theta\big) + |a|^\Theta(a-b)}{|a|^\Theta |b|^\Theta} \right| \; |b-a|.
$$

¿From (6.1) we get

$$
\big||b|^\Theta - |a|^\Theta\big| \;\le\; \frac{\Theta}{2}\,\big(|a|^{\Theta-1} + |b|^{\Theta-1}\big)|b-a|,
$$

hence

$$
\begin{aligned}
T(a,b) \;&\le\; \tfrac{\Theta}{2}\,|a|^{1-\Theta}\big(|a|^{\Theta-1} + |b|^{\Theta-1}\big)\tfrac{|\,b-a|^2}{|b|^\Theta} \\
&+\; \tfrac{|\,b-a|^2}{|b|^\Theta} = \Big\{\tfrac{\Theta}{2}\,\big(1 + \big(\tfrac{|b|}{|a|}\big)^{\Theta-1}\big) + 1\Big\}\,\tfrac{|\,b-a|^2}{|b|^\Theta}\;.
\end{aligned}
$$

Recalling $|a| \le |b|$ and $\Theta - 1 \in [-1,0)$ we arrive at

$$
T(a,b) \le (\Theta+1)\frac{|\,b-a|^2}{|b|^\Theta} \;=\; (\Theta+1)\frac{|\,b-a|^\Theta}{|b|^\Theta}\;|\,b-a|^{2-\Theta}.
$$

Since $\frac{|\,b-a|^\Theta}{|b|^\Theta} \le 2^\Theta$, (5.10) is established.

For proving (6.1) we assume w.l.o.g. that $\xi_1 < \xi_2$. Since the function $x \mapsto x^{s-1}, x > 0$, is convex, the secant through the points $(\xi_1, \xi_1^{s-1}), (\xi_2, \xi_2^{s-1})$ lies above the graph of $x^{s-1}, \xi_1 \le x \le \xi_2$. This gives for $x \in [\xi_1, \xi_2]$

$$
x^{s-1} \le \xi_1^{s-1} + \frac{\xi_2^{s-1} - \xi_1^{s-1}}{\xi_2 - \xi_1}\;\big(x - \xi_1\big),
$$

and if we integrate this inequality w.r.t. $x \in [\xi_1, \xi_2]$ we get

$$\frac{1}{s}\left(\xi_2^s - \xi_1^s\right) \le \frac{1}{2}\left(\xi_1^{s-1} + \xi_2^{s-1}\right)(\xi_2 - \xi_1),$$

thus (6.1) is established. $\square$

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Martin Fuchs
Universität des Saarlandes
Fachbereich 6.1 Mathematik
Postfach 15 11 50
D–66041 Saarbrücken
Germany
e–mail: fuchs@math.uni-sb.de

Sergey Repin
V.A. Steklov Mathematical Institute
St. Petersburg Branch
Fontanka 27
191011 St. Petersburg
Russia
e–mail: repin@pdmi.ras.ru