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GLOBAL DISTRIBUTIONS AND SPECIAL ZETA VALUES*

ABSTRACT. In this paper, we develop the theory of global distributions (i.e. distributions of global fields) and apply it to the study of special values of abelian L-functions of a number field and division points of rank one Drinfeld modules. We introduce the concept of ϵ-distributions and give examples by ϵ-partial zeta functions. We determine the ranks of level groups of various kinds of universal distributions of a global field k, such as universal ϵ-, punctured, punctured even and odd distributions of k. We show the universality of several distributions derived from special values of the ϵ-partial zeta functions by studying $\mathbb{Q}$-linear independence of some special values. We also propose a conjecture and a question about the universality of ϵ-distributions of special values of ϵ-partial zeta functions.

Introduction.

In classical number theory, ordinary distributions on $\mathbb{Q}/\mathbb{Z}$, as well as higher dimensional versions or p-adic versions, were studied extensively and were used to describe the relations of various special values such as Dirichlet L-functions, modular functions and Γ-functions, see [Chaps.2 and 6, La], [Chap.12, Wa], [LK] and [Ko], and some references there. These special values are of great arithmetic interest in the classical cyclotomic or the complex multiplication theory. Many distributions, e.g., those derived from the cyclotomic numbers or the Siegel (modular) functions, are universal up to torsion, and some, e.g., special Γ-values, are conjectured to be universal. Recently, the study of the universal ordinary distribution on $\mathbb{Q}/\mathbb{Z}$ has become active again, see [Da] and [Ou], since Anderson [An] invented a new method to compute its $\pm$-cohomology and constructed a canonical $\mathbb{F}_2$-basis for the cohomology. Thus the concept of distributions on $\mathbb{Q}/\mathbb{Z}$ is an important device in the study of some arithmetic quantities. In [Yi1], the author extended the concept of the classical distributions to global fields and determined the ranks of the universal level groups. Quite recently, Belliard and Oukhaba [BO] gave detailed information about the torsion of the level groups. We [BGY] have also applied the distribution method to the study of the classical and characteristic-p Γ-monomials. In this paper, we further develop the theory of global distributions and apply it to the study of special values of abelian L-functions over global fields.

Let k be a number field with r_1 real places and r_2 pairs of conjugate complex places. Let $\epsilon = (\epsilon_1, \cdots, \epsilon_{r_1})$, where $\epsilon_i = 0$ or 1. We first classify the ordinary distributions of k into ϵ-distributions. The case $\epsilon = 0$ (i.e. all $\epsilon_i = 0$) corresponds to the concept of even (or real) distributions defined in [Yi1]. We then introduce ϵ-partial zeta functions $Z^{\epsilon}_{\mathfrak{m}}(s, \mathfrak{a})$, where $\mathfrak{a}$ and $\mathfrak{m}$ are non-zero integral ideals of k. They generalize the classical partial zeta functions corresponding to wide (i.e. $\epsilon = 0$) or narrow (i.e. $\epsilon = 1$) ray ideal classes. Let $|\epsilon| = \sum_i \epsilon_i$ and let $\bar{\epsilon} = r_1 + r_2 - |\epsilon|$. We show that the order of vanishing of $Z^{\epsilon}_{\mathfrak{m}}(s, \mathfrak{a})$ at $s = 0$ is greater than or equal to $\bar{\epsilon}$ (we assume $\mathfrak{m} \nmid \mathfrak{a}$ when $\epsilon = 0$). Let $f^{\epsilon}(\mathfrak{a}\mathfrak{m}^{-1})$ be the coefficient of $s^{\bar{\epsilon}}$ in the Taylor series expansion of $Z^{\epsilon}_{\mathfrak{m}}(s, \mathfrak{a})$ at $s = 0$. Then f^{ϵ} is a function from the set of non-zero fractional ideals (non-integral if $\epsilon = 0$) of k to $\mathbb{C}$. According to the conjecture of Stark (cf. [St] and [Ta]) or its generalization by Rubin [Ru], these functions, especially the one denoted by f^+ in the case $\epsilon = 0$, encode important information about the class fields of k. Also the values at $s = 1$ of idele class L-functions of k can be expressed as finite sums by these functions. Thus they are of great arithmetic interest. We show that f^{ϵ} is an ϵ-distribution (f^+ is punctured) and that the Stickelberger distribution F^{ϵ} (with some modification in the case $|\epsilon| \le 1$) associated to f^{ϵ} is a universal ϵ-distributions up to torsion. For the distributions f^{ϵ} (where $|\epsilon| \ge 1$) themselves, we propose the question that for which number field k and for which ϵ they are universal.

We consider punctured distributions in section 3. We first give a sufficient and necessary condition when a punctured distribution can be completed to a non-punctured distribution, using which we get an upper bound for the rank of level $\mathfrak{m}$ group of a universal punctured distribution. We prove that the rank of level $\mathfrak{m}$ group of the distribution $F^0 = \sum_{\epsilon} F^{0\epsilon}$ is equal to the upper bound, where $F^{0\epsilon}$ is the punctured reduction of F^{ϵ}. To get this result, we need to study $\mathbb{Q}$-linear independence of some special

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values of f^+ (we show more). We have conjectured that f^+ is a universal punctured even distribution up to torsion [Conj.3.1, BGY]. It is equivalent to a conjecture on the Galois action of some special values of f^+ by the results we get in this paper. The latter conjecture also implies that the complex numbers $f^+(\mathfrak{a}\mathfrak{p}^{-1})$ for all primes $\mathfrak{p}$ and all integral ideals $\mathfrak{a}$ with $\mathfrak{p} \nmid \mathfrak{a}$ are $\mathbb{Q}$-linear independent. In the cases when $k = \mathbb{Q}$ and when k is imaginary quadratic, the conjecture can be proved easily by the explicit reciprocity law via the obvious expression of f^+. Our conjecture could be interpreted as a weak analytic version of the (unkown) explicit reciprocity law of k. There are many conjectures on special values of L-functions, see, for example, [Hu] and [Za], but only few of them have satisfactory answers. Our conjecture is the first towards distributions. Could it be proved directly by analytic method? We expect an answer.

In the last section of the paper, we consider the case when k is a function field. In this case, the even distribution f^+ can be expressed in the obvious manner as the valuations at a fixed place of the division points of sign-normalized rank one Drinfeld modules [Ha2, Sect.6]. It takes rational values up to a constant factor, and thus we have no the result on $\mathbb{Q}$-linear independence. Also the conjecture is not meaningful in this case. But the division points themselves form a multiplicative punctured even distribution, which satisfies the multiplicative version of $\mathbb{Q}$-linear independence and of the conjecture. Thus we can determine the rank of the level $\mathfrak{m}$ group of the universal punctured even distribution and show the universality of the multiplicative distribution. The results here describe almost all $\mathbb{Z}$-linear relations satisfied by these division points and by the Stickelberger elements introduced in [Yi3]. This is the principal motivation for the author to introduce the concept of global distributions. In the rational number field case, similar results were obtained by several people, see [Ba], [Mi] and [Sc]. Finally, we mention that the two parts about number fields and about function fields in the paper are relatively independent.

1. ϵ-distributions.

In this section, we further perfect the definition of distributions of a number field given in [Sect.1, Yi1]. We also introduce the concept of ϵ-distributions.

Let k be a number field. We think of k as our base field. Let $\mathbb{A}$ be the integral closure of $\mathbb{Z}$ in k. Let $v_1, \cdots, v_{r_1}$ be all the real places of k. In the paper, we always fix this order of the real places. Let $\epsilon = (\epsilon_1, \cdots, \epsilon_{r_1})$ be an r_1-vector with entries in the field $\mathbb{F}_2$ of two elements. Let T_0 (resp. $\bar{T}_0$) be the set of non-zero integral (resp. fractional) ideals of $\mathbb{A}$. Fix $\mathfrak{m} \in T_0$. We call integral ideals $\mathfrak{a}, \mathfrak{b}$ ϵ-equivalent modulo $\mathfrak{m}$ and write $\mathfrak{a} \sim^\epsilon_\mathfrak{m} \mathfrak{b}$ if there exists $x \in 1 + \mathfrak{a}^{-1}\mathfrak{m}$, $v_i(x) > 0$ when $\epsilon_i = 1$ such that $\mathfrak{b} = x\mathfrak{a}$. Let $T^\epsilon_\mathfrak{m}$ be the set of equivalence classes of T_0 under $\sim^\epsilon_\mathfrak{m}$. Let $G^\epsilon_\mathfrak{m}$ be the set of equivalence classes of the ideals coprime to $\mathfrak{m}$. We call it the ϵ-ray class group of k of conductor $\mathfrak{m}$. The equivalence relation $\sim^\epsilon_\mathfrak{m}$ has the following properties.

Lemma 1.1. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{m}$ be non-zero integral ideals of k. We have*
1) If $\mathfrak{a} \sim^\epsilon_\mathfrak{m} \mathfrak{b}$, then the gcds of $\mathfrak{a}$ with $\mathfrak{m}$ and $\mathfrak{b}$ with $\mathfrak{m}$ are equal, i.e. $(\mathfrak{a}, \mathfrak{m}) = (\mathfrak{b}, \mathfrak{m})$.
2) Assume $\mathfrak{c}$ and $\mathfrak{m}$ are coprime. Then $\mathfrak{a}\mathfrak{c} \sim^\epsilon_\mathfrak{m} \mathfrak{b}\mathfrak{c}$ if and only if $\mathfrak{a} \sim^\epsilon_\mathfrak{m} \mathfrak{b}$.
3) $T^\epsilon_\mathfrak{m}$ is a disjiont union of copies of $G^\epsilon_\mathfrak{n}$, where $\mathfrak{n}$ runs over all divisors of $\mathfrak{m}$.

Proof. Assume $\mathfrak{b} = x\mathfrak{a}$ for some $x \in 1 + \mathfrak{a}^{-1}\mathfrak{m}$. Let integral ideal $\mathfrak{d} \supseteq \mathfrak{m}$. Then $(x-1)\mathfrak{a} \subseteq \mathfrak{m} \subseteq \mathfrak{d}$. If $\mathfrak{d} \supseteq \mathfrak{a}$, then $\mathfrak{b} = x\mathfrak{a} \subseteq \mathfrak{d}$. The inverse is also valid since $\sim^\epsilon_\mathfrak{m}$ is an equivalence relation. We get 1). Assume $\mathfrak{a}\mathfrak{c} \sim^\epsilon_\mathfrak{m} \mathfrak{b}\mathfrak{c}$. There exists $x \in 1 + (\mathfrak{a}\mathfrak{c})^{-1}\mathfrak{m}$ such that $\mathfrak{b}\mathfrak{c} = x\mathfrak{a}\mathfrak{c}$, which implies $\mathfrak{b} = x\mathfrak{a}$ and $x \in \mathfrak{a}^{-1}$. Thus $(x-1)\mathfrak{a}$ is an integral ideal. Since $\mathfrak{m} \mid (x-1)\mathfrak{a}\mathfrak{c}$ and $\mathfrak{c}, \mathfrak{m}$ are coprime, we get $x \in 1 + \mathfrak{a}^{-1}\mathfrak{m}$. This shows 2). The proof of 3) is easy.

In the case $\epsilon = 1$ (i.e. all $\epsilon_i = 1$), we denote $\sim^\epsilon_\mathfrak{m}$, $T^\epsilon_\mathfrak{m}$ and $G^\epsilon_\mathfrak{m}$ by $\sim_\mathfrak{m}$, $T_\mathfrak{m}$ and $G_\mathfrak{m}$ respectively. We also interpret $G_\mathfrak{m}$ as the Galois group over k of the narrow ray class field $K_\mathfrak{m}$ of conductor $\mathfrak{m}$ via Artin morphism. In the obvious way, the set $\{T_\mathfrak{m} \mid \mathfrak{m} \in T_0\}$ is a projective system of finite sets ordered by divisibility. In the case $\epsilon = 0$ we write $\sim^+_\mathfrak{m} = \sim^\epsilon_\mathfrak{m}$ and $G^+_\mathfrak{m} = G^\epsilon_\mathfrak{m}$, which is the wide ray class group of $\mathbb{A}$ modulo $\mathfrak{m}$.

Let $\mathfrak{u}, \mathfrak{v} \in \bar{T}_0$. Assume $\mathfrak{v} \subset \mathfrak{u}$. Let U and U^+ be the unit group and totally positive unit group of k respectively. Let $w(\mathfrak{u}/\mathfrak{v}) \subset 1 + \mathfrak{u}$ be a complete set of representatives of $(1 + (\mathfrak{u}/\mathfrak{v}))/U^+$. We may assume that each x in $w(\mathfrak{u}/\mathfrak{v})$ is totally positive. In this paper, we always make this assumption.

By Mazur's abstract definition [Sect.7.1, MSd] of distributions on a projective system of finite sets, a distribution on $\{T_\mathfrak{m}\}$ is a family of functions $\{f_\mathfrak{m} : T_0 \longrightarrow V \mid \mathfrak{m} \in T_0\}$, where V is an abelian group and

$f_{\mathfrak{m}}$ factors through $T_0 \to T_0/\sim_{\mathfrak{m}}$, and subject to the conditions whenever $\mathfrak{n} \mid \mathfrak{m}$ we have

$$f_{\mathfrak{n}}(\mathfrak{a}) = \sum_{\substack{\mathfrak{b}\sim_{\mathfrak{n}}\mathfrak{a} \\ \mathfrak{b} \bmod \sim_{\mathfrak{m}}}} f_{\mathfrak{m}}(\mathfrak{b}) = \sum_{x\in w} f_{\mathfrak{m}}(x\mathfrak{a}) \tag{1.1}$$

for all $\mathfrak{a} \in T_0$, where $w = w(\mathfrak{a}^{-1}\mathfrak{n}/\mathfrak{a}^{-1}\mathfrak{m})$. There are several equivalent definitions of distributions on a projective system, see [Chap.2, La] or [Chap.12, Wa].

One can also use the projective system $\{T^{\epsilon}_{\mathfrak{m}}\}$ to define distributions. But these distributions are special case of those on $\{T_{\mathfrak{m}}\}$. The family $\{f_{\mathfrak{m}}\}$ is called a punctured distribution if for each $\mathfrak{m}$ the function $f_{\mathfrak{m}}$ is not defined at the integral ideals divisible by $\mathfrak{m}$ and they satisfy Eqs.(1.1) for all $\mathfrak{a}$ with $\mathfrak{n} \nmid \mathfrak{a}$. Notice that, by Lem.1.1(1), if $\mathfrak{n} \nmid \mathfrak{a}$ and $\mathfrak{b} \sim_{\mathfrak{n}} \mathfrak{a}$ then $\mathfrak{n} \nmid \mathfrak{b}$ and thus $\mathfrak{m} \nmid \mathfrak{b}$ for $\mathfrak{n} \mid \mathfrak{m}$. Let $G = \varprojlim G_{\mathfrak{m}}$. For $\sigma \in G$, let $\sigma_{\mathfrak{b}} \in G_{\mathfrak{m}}$ be the image of σ under the natural map $G \to G_{\mathfrak{m}}$, where $\sigma_{\mathfrak{b}}$ is the Artin morphism associated to the integral ideal $\mathfrak{b}$. We define the action of G on the distribution $\{f_{\mathfrak{m}}\}$ by $\sigma f_{\mathfrak{m}}(\mathfrak{a}) = f_{\mathfrak{m}}(\mathfrak{b}\mathfrak{a})$. From Lem.1.1(2) we see that the family $\{\sigma f_{\mathfrak{m}}\}$ is also a distribution on the system.

In $\bar{T}_0$, we define an equivalence relation $\sim$ as follows: $\mathfrak{u} \sim \mathfrak{v}$ if and only if $\mathfrak{v} = x\mathfrak{u}$ for some totally positive $x \in 1 + \mathfrak{u}^{-1}$. We mention that if $\mathfrak{u} = \mathfrak{a}\mathfrak{m}^{-1}$ and $\mathfrak{v} = \mathfrak{b}\mathfrak{n}^{-1}$ are the coprime fractional expressions, then $\mathfrak{u} \sim \mathfrak{v}$ if and only if $\mathfrak{m} = \mathfrak{n}$ and $\mathfrak{a} \sim_{\mathfrak{m}} \mathfrak{b}$.

The distribution $\{f_{\mathfrak{m}}\}$ is called ordinary if whenever $\mathfrak{a}\mathfrak{m}^{-1} = \mathfrak{b}\mathfrak{n}^{-1}$ we have $f_{\mathfrak{m}}(\mathfrak{a}) = f_{\mathfrak{n}}(\mathfrak{b})$. In this case, the map $f_{\mathfrak{m}}$ for each $\mathfrak{m}$ can be obtained by composing the map $T_0 \longrightarrow \bar{T}_0$ multiplying by $\mathfrak{m}^{-1}$ with a map $f : \bar{T}_0 \longrightarrow V$ which factors through $\bar{T}_0 \to \bar{T}_0/\sim$. Thus an ordinary distribution of k can be replaced by a single function f on $\bar{T}_0$ modulo $\sim$, which satisfies the relations, from (1.1),

$$f(\mathfrak{u}) = \sum_{\substack{\mathfrak{v}\sim\mathfrak{u} \\ \mathfrak{v}\mathfrak{n}^{-1} \bmod \sim}} f(\mathfrak{v}\mathfrak{n}^{-1}) = \sum_{x\in w(\mathfrak{u}^{-1}/\mathfrak{u}^{-1}\mathfrak{n})} f(x\mathfrak{u}\mathfrak{n}^{-1}) \tag{1.2}$$

for all $\mathfrak{u} \in \bar{T}_0$ and $\mathfrak{n} \in T_0$. Sometimes we call f a distribution of the field k for simplicity.

The action of G on $\{f_{\mathfrak{m}}\}$ induces an action of G on f. Clearly σf for $\sigma \in G$ is also an ordinary distribution. If $\{f_{\mathfrak{m}}\}$ is punctured, then f is only defined on $\bar{T}_0 \setminus T_0$. In this case, we call f punctured.

Let $J_{\mathfrak{m}}$ be the subgroup of $G_{\mathfrak{m}}$ consisting of the $\sim_{\mathfrak{m}}$-equivalence classes of ideals of the form (a) for some $a \in \mathbb{A}$ such that $a \equiv 1 (\mod \mathfrak{m})$. The sign-subgroup is defined to be $J = \varprojlim J_{\mathfrak{m}} \subset G$. We now describe it more clearly, cf. [2.24, DR]. Let v be one of r_1 real places of k. Take $\alpha \in 1 + \mathfrak{m}$ such that it is negative at v and positive at each real places of k different from v. The class $\sigma^{\mathfrak{m}}_v$ of (α) in $G_{\mathfrak{m}}$ is independent of α and has order 1 or 2. It is trivial if and only if α may be chosen to be a unit. In fact, $\sigma^{\mathfrak{m}}_v$ is the complex conjugation after we choose an embedding $K_{\mathfrak{m}} \hookrightarrow \mathbb{C}$ which induces v on k. The subgroup $J_{\mathfrak{m}}$ of $G_{\mathfrak{m}}$ is generated by all $\sigma^{\mathfrak{m}}_{v_i}$. Let $\sigma_v = (\sigma^{\mathfrak{m}}_v)_{\mathfrak{m}} \in G$. Then each σ_v has order 2. The sign-subgroup J of G is generated by all σ_v. Thus J is an elementary 2-group of order 2^{r_1}.

Let $\epsilon = (\epsilon_1, \cdots, \epsilon_{r_1})$, where $\epsilon_i = 0$ or 1. An ordinary distribution f is called an ϵ-distribution if for all $1 \le i \le r_1$ and all $\mathfrak{u} \in \bar{T}_0$ ($\mathfrak{u} \notin T_0$ if f is punctured) one has

$$\sigma_{v_i} f(\mathfrak{u}) = (-1)^{\epsilon_i} f(\mathfrak{u}). \tag{1.3}$$

We can construct ϵ-distributions as follows. Let f be an ordinary distribution. Set

$$\alpha^{\epsilon} = \prod_{\epsilon_i=0}(1+\sigma_{v_i}) \prod_{\epsilon_i=1}(1-\sigma_{v_i}). \tag{1.4}$$

Then $f^{\epsilon} = \alpha^{\epsilon} f$ is an ϵ-distribution. We mention that the concept of ϵ-distributions can be defined for general distributions on $\{T_{\mathfrak{m}}\}$, not only the ordinary ones. In the general case it would be more natural to introduce some concepts such as G-action if we used another definition of a distribution as a locally constant function on the compact space $\varprojlim T_{\mathfrak{m}}$.

For an integral ideal $\mathfrak{m}$, we set $\epsilon^{\mathfrak{m}} = (\epsilon^{\mathfrak{m}}_1, \cdots, \epsilon^{\mathfrak{m}}_{r_1})$, where $\epsilon^{\mathfrak{m}}_i = 0$ if $\epsilon_i = 0$ or if the image $\sigma^{\mathfrak{m}}_{v_i}$ of σ_{v_i} in $J_{\mathfrak{m}}$ is trivial and $\epsilon^{\mathfrak{m}}_i = 1$ otherwise. We call it $\mathfrak{m}$-reduction of ϵ. If f is an ϵ-distribution and if $\epsilon^{\mathfrak{m}} \neq \epsilon$, then $f(\mathfrak{a}\mathfrak{m}^{-1})$ must be 2-torsion for all $\mathfrak{a} \in T_0$ by Eq.(1.3).

Lemma 1.2. *The equivalence relations $\sim_{\mathfrak{m}}^{\epsilon}$ and $\sim_{\mathfrak{m}}^{\epsilon^{\mathfrak{m}}}$ are the same. Further we have $G_{\mathfrak{m}}^{\epsilon} \simeq G_{\mathfrak{m}}/ < \sigma_{v_i}^{\mathfrak{m}} \mid \epsilon_i = 0 >$.*

Proof. Clearly if $\mathfrak{a} \sim_{\mathfrak{m}}^{\epsilon} \mathfrak{b}$ for integral ideals $\mathfrak{a}$ and $\mathfrak{b}$, then $\mathfrak{a} \sim_{\mathfrak{m}}^{\epsilon^{\mathfrak{m}}} \mathfrak{b}$. Conversely, we assume $\epsilon' = (0, \epsilon_2, \cdots)$ and $\sigma_{v_1}^{\mathfrak{m}} = 1$. If $\mathfrak{a} \sim_{\mathfrak{m}}^{\epsilon'} \mathfrak{b}$, then $\mathfrak{b} = x\mathfrak{a}$ for some $x \in 1 + \mathfrak{a}^{-1}\mathfrak{m}$ such that $v_i(x) > 0$ when $\epsilon_i = 1$. Let $\epsilon = (1, \epsilon_2, \cdots)$. If $v_1(x) > 0$, then $\mathfrak{a} \sim_{\mathfrak{m}}^{\epsilon} \mathfrak{b}$. If $v_1(x) < 0$, take a unit $\alpha \in 1 + \mathfrak{m}$ such that $v_1(\alpha) < 0$ and $v_i(\alpha) > 0$ for $i > 1$. Since $\sigma_{v_1}^{\mathfrak{m}} = 1$, there does exist such α. We have $\mathfrak{b} = \alpha x \mathfrak{a}$, $\alpha x \in 1 + \mathfrak{a}^{-1}\mathfrak{m}$, $v_1(\alpha x) > 0$ and $v_i(\alpha x)/v_i(x) > 0$ for $i > 1$. Thus also $\mathfrak{a} \sim_{\mathfrak{m}}^{\epsilon} \mathfrak{b}$. Continuing this process, one shows the first claim in the lemma. The homomorphism $G_{\mathfrak{m}} \to G_{\mathfrak{m}}^{\epsilon}$ defined by $\mathfrak{a}\mathrm{mod} \sim \mapsto \mathfrak{a}\mathrm{mod} \sim^{\epsilon}$ is surjective and has kernel $< \sigma_{v_i}^{\mathfrak{m}} \mid \epsilon_i = 0 >$. This shows the second claim.

Recall that the distribution f is called even if J acts on it trivially and odd if $s(J)f$ is the trivial distribution, where $s(J) = \sum_{\tau \in J} \tau$. An ϵ-distribution is even if $\epsilon = 0$ and is odd otherwise. If f takes values in a ring on which 2 is invertible, then f can be uniquely expressed as a sum of even and odd distributions, and as a sum of ϵ-distributions. In fact, we have $|J|f = s(J)f + (|J| - s(J))f$ and $|J|f = \sum_{\epsilon} f^{\epsilon}$, where f^{ϵ} is defined as above and the sum is over all vectors in $\mathbb{F}_2^{r_1}$. The latter equality is induced from $\sum_{\epsilon} \alpha^{\epsilon} = |J|$

The level $\mathfrak{m}$ group of an V-valued distribution f is defined to be the subgroup of V generated by $f(\mathfrak{a}\mathfrak{m}^{-1})$ with $\mathfrak{a} \in T_0$ (and $\mathfrak{m} \nmid \mathfrak{a}$ if f is punctured). Since σf for $\sigma \in G$ is also a distribution, the level $\mathfrak{m}$ group is a $G_{\mathfrak{m}}$-module via the action of G on f. If $f = f^+ + f^-$ is the even-odd decomposition of f, then the rank of level $\mathfrak{m}$ group of f is the sum of those of f^+ and f^-. The same is true for the ϵ-decomposition.

The V-valued distribution f is called universal if for any distribution $g : \bar{T}_0 \to W$ there exists a unique homomorphism $h : V \to W$ such that $g = h \circ f$. In this case we must have $V = \mathrm{Im} f$. We often ignore this easily satisfied condition. If the universal condition holds for all g with torsion free values, we call f universal with values in torsion free abelian groups or universal up to torsion. We denote the level $\mathfrak{m}$ group of a universal ordinary (resp. ϵ-, even(odd), punctured, punctured even(odd)) distribution by $A_{\mathfrak{m}}$ (resp. $A_{\mathfrak{m}}^{\epsilon}, A_{\mathfrak{m}}^{\pm}, A_{\mathfrak{m}}^{0}, A_{\mathfrak{m}}^{0\pm}$), and call them the universal level $\mathfrak{m}$ groups of k. They are $G_{\mathfrak{m}}$-modules. A distribution f is universal up to torsion if and only if the rank of level $\mathfrak{m}$ group of f is equal to that of $A_{\mathfrak{m}}$ for all integral ideals $\mathfrak{m}$. We have showed $\mathrm{rank} A_{\mathfrak{m}} = |G_{\mathfrak{m}}|$, see [Th.3.4, Yi1]. In this paper, we will determine all other ranks. The following lemma is useful, whose proof is easy.

Lemma 1.3. *Assume that $f = \sum_{\epsilon} f^{\epsilon}$ is the ϵ-decomposition of the (punctured) distribution f. Then f is a universal ordinary (punctured) distribution up to torsion if and only if f^{ϵ} is a universal (punctured) ϵ-distribution up to torsion.*

Let $\Omega = \varinjlim \mathbb{C}[G_{\mathfrak{m}}]$. Let f be a distribution with complex values. The Stickelberger distribution associated to f is defined to be $\mathrm{St}(f) : \bar{T}_0 \longrightarrow \Omega$, and for $\mathfrak{u} = \mathfrak{a}\mathfrak{m}^{-1} \in \bar{T}_0$,

$$\mathrm{St}(f)(\mathfrak{u}) = \sum_{\mathfrak{b} \in G_{\mathfrak{m}}} f(\mathfrak{u}\mathfrak{b}) \sigma_{\mathfrak{b}}^{-1},$$

where the sum over $\mathfrak{b} \in G_{\mathfrak{m}}$ means that $\mathfrak{b}$ runs over a complete set of representatives of the classes in $G_{\mathfrak{m}}$. We will often use this notation in the paper for simplicity.

2. ϵ-partial zeta function.

In this section, we introduce ϵ-partial zeta functions to give examples of ϵ-distributions. We determine the ranks of the universal level $\mathfrak{m}$ groups $A_{\mathfrak{m}}^{\epsilon}$ and $A_{\mathfrak{m}}^{\pm}$. We also prove the universality of some distributions derived from the ϵ-partial zeta functions.

The notations are as those in last section. Let $[k : \mathbb{Q}] = r_1 + 2r_2$ be the degree. Let $v_1, \cdots, v_{r_1}$ be all the real places of k. Fix an r_1-vector $\epsilon = (\epsilon_1, \cdots, \epsilon_{r_1})$, where $\epsilon_i = 0$ or 1. Let $N : \bar{T}_0 \to \mathbb{Q}$ be the absolute norm. For $x \in k^*$, write $\| x \| = Nx\mathbb{A}$. Let $\mathfrak{a}, \mathfrak{m} \in T_0$. Set $\mathfrak{m}_1 = \mathfrak{m}(\mathfrak{a}, \mathfrak{m})^{-1}$. Let $U_{\mathfrak{m}_1}^{+}$ be the totally positive unit group in $1 + \mathfrak{m}_1$ and let $U_{\mathfrak{m}_1}^{\epsilon}$ be the unit group in $1 + \mathfrak{m}_1$ whose elements are positive at v_i when $\epsilon_i = 1$. For $\mathrm{Re}(s) > 1$ we define

$$Z_{\mathfrak{m}}^{\epsilon}(s, \mathfrak{a}) = N\mathfrak{a}^{-s} \sum_{\substack{0 \neq x \in (1 + \mathfrak{a}^{-1}\mathfrak{m})/U^{+} \\ v_i(x) > 0 \text{ when } \epsilon_i = 1}} \| x \|^{-s} = [U_{\mathfrak{m}_1}^{\epsilon} : U_{\mathfrak{m}_1}^{+}] \sum_{\mathfrak{b} \sim_{\mathfrak{m}}^{\epsilon} \mathfrak{a}} N\mathfrak{b}^{-s},$$

and call it ϵ-partial zeta function of the class of $\mathfrak{a}$ modulo $\sim^\epsilon_\mathfrak{m}$. It can be extended to the whole complex plane and is holomorphic except for a simple pole at $s = 1$. Furthermore, the following equality, for $\mathfrak{n} \mid \mathfrak{m}$ and all $s \neq 1$,

$$\sum_{\substack{0\neq x\in(1+\mathfrak{a}^{-1}\mathfrak{n})/U^+ \\ v_i(x)>0 \text{ when } \epsilon_i=1}} \| x \|^{-s} = \sum_{y\in w} \sum_{\substack{0\neq z\in(y+\mathfrak{a}^{-1}\mathfrak{m})/U^+ \\ v_i(z)>0 \text{ when } \epsilon_i=1}} \| z \|^{-s},$$

where $w = w(\mathfrak{a}^{-1}\mathfrak{n}/\mathfrak{a}^{-1}\mathfrak{m})$, implies the following

Proposition 2.1. *For each $s \neq 1$, the family of functions $\{Z^\epsilon_\mathfrak{m}(s, *) \mid \mathfrak{m} \in T_0\}$ on T_0 is a distribution on the system $\{T_\mathfrak{m}\}$.*

We now prepare to give some information about the order of vanishing of $Z^\epsilon_\mathfrak{m}(s, *)$ at $s = 0$. Write $|\epsilon| = \sum_i \epsilon_i$. Let $\epsilon^\mathfrak{m}$ be the $\mathfrak{m}$-reduction of ϵ.

Lemma 2.2. *Let $\mathfrak{a}, \mathfrak{m} \in T_0$. We have $Z^\epsilon_\mathfrak{m}(s, \mathfrak{a}) = 2^{|\epsilon^\mathfrak{m}|-|\epsilon|} Z^{\epsilon^\mathfrak{m}}_\mathfrak{m}(s, \mathfrak{a})$. Thus $Z^\epsilon_\mathfrak{m}(s, \mathfrak{a})$ and $Z^{\epsilon^\mathfrak{m}}_\mathfrak{m}(s, \mathfrak{a})$ have the same order of vanishing at $s = 0$.*

Proof. Without loss of generality, we assume $\epsilon = (1, \epsilon_2, \cdots)$ and $\sigma^\mathfrak{m}_{v_1} = 1$. Write $\epsilon' = (0, \epsilon_2, \cdots)$. Take a unit $\alpha \in 1 + \mathfrak{m}$ such that $v_1(\alpha) < 0$ and $v_i(\alpha) > 0$ when $i > 1$. Then

$$2Z^\epsilon_\mathfrak{m}(s, \mathfrak{a}) = Z^\epsilon_\mathfrak{m}(s, \mathfrak{a}) + Z^\epsilon_\mathfrak{m}(s, \alpha\mathfrak{a}) = Z^{\epsilon'}_\mathfrak{m}(s, \mathfrak{a}).$$

Continuing this process, we get the lemma.

An idele class character χ of conductor $\mathfrak{f}$ is called an ϵ-character if for all $1 \le i \le r_1$ we have $\chi(\sigma_{v_i}) := \chi(\sigma^\mathfrak{f}_{v_i}) = (-1)^{\epsilon_i}$. In this case one must have $\epsilon = \epsilon^\mathfrak{f}$. Such a character induces a primitive character of $G^\epsilon_\mathfrak{f}$ by Lem.1.2. We denote it also by χ. For $\mathfrak{a} \in T_0$, we define $\chi(\mathfrak{a}) = 0$ if $(\mathfrak{f}, \mathfrak{a}) \neq \mathfrak{e}$ and $\chi(\mathfrak{a}) = \chi(\sigma_\mathfrak{a})$ otherwise, where $\sigma_\mathfrak{a} \in G^\epsilon_\mathfrak{f}$ is the class of $\mathfrak{a}$ modulo $\sim^\epsilon_\mathfrak{f}$. Let $\hat{G}_\mathfrak{m} = \mathrm{Hom}(G_\mathfrak{m}, \mathbb{C})$ be the complex character group of $G_\mathfrak{m}$. Let $u : \bar{T}_0 \to \mathbb{C}$ be a function defined modulo $\sim$. For a character χ of conductor dividing $\mathfrak{m}$, we set

$$u_\mathfrak{m}(\chi) = \sum_{\mathfrak{a}\in G_\mathfrak{m}} \chi(\mathfrak{a}) u(\mathfrak{a}\mathfrak{m}^{-1}). \tag{2.1}$$

In [Prop.3.1, Yi1], we showed the following result.

Lemma 2.3. *Assume $u : \bar{T}_0 \to \mathbb{C}$ is defined modulo $\sim$. Then u is an ordinary distribution if and only if for all $\mathfrak{m} \in T_0$ and all $\chi \in \hat{G}_\mathfrak{m}$ of conductor $\mathfrak{f}_\chi$ one has*

$$u_\mathfrak{m}(\chi) = \prod_{\mathfrak{p}\mid\mathfrak{m}} (1 - \chi(\mathfrak{p})) \cdot u_{\mathfrak{f}_\chi}(\chi),$$

where the product is over all prime divisors of $\mathfrak{m}$.

Write $\bar{\epsilon} = r_1 + r_2 - |\epsilon|$. We have

Proposition 2.4. *Assume $|\epsilon| \neq 0$. The order of vanishing of $Z^\epsilon_\mathfrak{m}(s, \mathfrak{a})$ at $s = 0$ is greater than or equal to $\bar{\epsilon}$. Assume $|\epsilon| = 0$. This order is equal to $r_1 + r_2$ if $\mathfrak{m} \nmid \mathfrak{a}$ and to $r_1 + r_2 - 1$ if $\mathfrak{m} \mid \mathfrak{a}$.*

Proof. Before the proof, we first mention a basic fact. Let A be a finite abelian group and let $\hat{A} = \mathrm{Hom}(A, \mathbb{C}^*)$. Then the determinant of the matrix $(\chi(a))_{\chi\in\hat{A}, a\in A}$ is not zero.

The order in the case $|\epsilon| = 0$ has been determined in [Prop.2.2, Yi1]. We mainly consider the case $|\epsilon| \neq 0$ here. Let $l \geq 0$ be the minimal order of vanishing of $Z^\epsilon_\mathfrak{m}(s, \mathfrak{a})$ at $s = 0$ for all integral ideals $\mathfrak{m}$ and $\mathfrak{a}$. Let $f^\epsilon(\mathfrak{a}\mathfrak{m}^{-1})$ be the coefficient of s^l in the Taylor expansion of $Z^\epsilon_\mathfrak{m}(s, \mathfrak{a})$ at $s = 0$. By Prop.2.1, the function $f^\epsilon : \bar{T} \to \mathbb{C}$ is a distribution. Suppose $\mathrm{ord}_{s=0} Z^\epsilon_{\mathfrak{m}_0}(s, \mathfrak{a}_0) = l$. We can assume that $\mathfrak{a}_0$ and $\mathfrak{m}_0$ are coprime. Let $\chi \in \hat{G}^\epsilon_{\mathfrak{m}_0}$ be of conductor $\mathfrak{f}$. Then χ induces a character of $G^\epsilon_\mathfrak{f}$. The L-function $L(s, \chi)$ can be expressed as

$$L(s, \chi) = \sum_{\mathfrak{a}\in G^\epsilon_\mathfrak{f}} \chi(\mathfrak{a}) \sum_{\mathfrak{b}\sim^\epsilon_\mathfrak{f}\mathfrak{a}} N\mathfrak{b}^{-s} = \frac{1}{[U^\epsilon_\mathfrak{f} : U^+_\mathfrak{f}]} \sum_{\mathfrak{a}\in G^\epsilon_\mathfrak{f}} \chi(\mathfrak{a}) Z^\epsilon_\mathfrak{f}(s, \mathfrak{a}). \tag{2.2}$$

Determine $\epsilon_i^\chi = 0$ or 1 by the condition $\chi(\sigma_{v_i}) = (-1)^{\epsilon_i^\chi}$. Set $\epsilon_\chi = (\epsilon_1^\chi, \cdots, \epsilon_{r_1}^\chi)$. By the well-known function equation of L-functions, see for example [Chap.8, CF], we know that the order of vanishing of $L(s,\chi)$ at $s=0$ is equal to $\bar{\epsilon_\chi}$. Assume $|\epsilon| \neq 0$ and assume $l < \bar{\epsilon}$. Since $\bar{\epsilon_\chi} \geq \bar{\epsilon} > l$, we have $\sum_{\mathfrak{a}\in G_\mathfrak{f}^\epsilon} \chi(\mathfrak{a}) f^\epsilon(\mathfrak{a}\mathfrak{f}^{-1}) = 0$ by Eq.2.2. Because f^ϵ is a distribution, by Lem.2.3 we see that for all $\chi \in \hat{G}_{\mathfrak{m}_0}^\epsilon$ the following equality holds

$$\sum_{\mathfrak{a}\in G_{\mathfrak{m}_0}^\epsilon} \chi(\mathfrak{a}) f^\epsilon(\mathfrak{a}\mathfrak{m}_0^{-1}) = 0.$$

Thus $f^\epsilon(\mathfrak{a}\mathfrak{m}_0^{-1}) = 0$ for all $\mathfrak{a}$ coprime to $\mathfrak{m}_0$. This is a contradiction.

Since the order of vanishing of $L(s,\chi_0)$ at $s=0$, where χ_0 is the trivial character, is equal to r_1+r_2-1, not r_1+r_2, one must modify the discussion above in the case $|\epsilon| = 0$.

We guess that the order of $Z_\mathfrak{m}^\epsilon(s,\mathfrak{a})$ at $s=0$ is equal to $\epsilon^{\bar{\mathfrak{m}}}$ if $\mathfrak{a}, \mathfrak{m}$ are coprime. Let $f^\epsilon(\mathfrak{a}\mathfrak{m}^{-1})$ be the coefficient of $s^{\bar{\epsilon}}$ in the Taylor expansion of $Z_\mathfrak{m}^\epsilon(s,\mathfrak{a})$ at $s=0$. Here we assume $\mathfrak{m} \nmid \mathfrak{a}$ if $\epsilon = 0$. By Props.2.1 and 2.4, the function f^ϵ is an ordinary distribution, but it is punctured in the case $\epsilon = 0$. Let χ be an idele class character of conductor $\mathfrak{f} = \mathfrak{f}_\chi$. Let ϵ_χ be defined as above. Then χ is an ϵ_χ-character of $G_\mathfrak{f}$. The character χ is called real if $\epsilon_\chi = 0$ and non-real otherwise. In the real case, we assume χ is ramified. The $\bar{\epsilon}_\chi$-th derivative of $L(s,\chi)$ at $s=0$ can be expressed as, by Eq.(2.2),

$$L^{(\bar{\epsilon}_\chi)}(0,\chi) = \frac{1}{[U_\mathfrak{f}^{\epsilon_\chi} : U_\mathfrak{f}^+]} \sum_{\mathfrak{a}\in G_\mathfrak{f}^{\epsilon_\chi}} \chi(\mathfrak{a}) f^{\epsilon_\chi}(\mathfrak{a}\mathfrak{f}^{-1}) = \frac{|G_\mathfrak{f}^{\epsilon_\chi}|}{[U_\mathfrak{f}^{\epsilon_\chi} : U_\mathfrak{f}^+]|G_\mathfrak{f}|} \sum_{\mathfrak{a}\in G_\mathfrak{f}} \chi(\mathfrak{a}) f^{\epsilon_\chi}(\mathfrak{a}\mathfrak{f}^{-1}). \tag{2.3}$$

Let $f_\mathfrak{e}^+(\mathfrak{a})$ be the coefficient of $s^{r_1+r_2}$ in the expansion of $Z_\mathfrak{e}^+(s,\mathfrak{a})$ at $s=0$. Here Z^+ is the ϵ-partial zeta function in the case $\epsilon = 0$ and $\mathfrak{e}$ is the unit ideal of $\mathbb{A}$. When $\chi \neq 1$ is real and unramified, we have

$$L^{(r_1+r_2)}(0,\chi) = \frac{1}{[U:U^+]} \sum_{\mathfrak{a}\in G_\mathfrak{e}^+} \chi(\mathfrak{a}) f_\mathfrak{e}^+(\mathfrak{a}) = \frac{|G_\mathfrak{e}^+|}{[U:U^+]|G_\mathfrak{e}|} \sum_{\mathfrak{a}\in G_\mathfrak{e}} \chi(\mathfrak{a}) f_\mathfrak{e}^+(\mathfrak{a}). \tag{2.4}$$

By the function equation of L-functions, the two equalities above give an expression of the value $L(1,\chi)$ as a finite summation for each non-trivial idele class character χ. Since for each $\epsilon \in \mathbb{F}_2^{r_1}$ there exist idele class characters χ such that $\epsilon_\chi = \epsilon$, the distribution f^ϵ is non-trivial. We denote f^ϵ by f^+ in the case $\epsilon = 0$. It is punctured and is even. We now consider the case $\epsilon \neq 0$.

Let $1 \leq j \leq r_1$. Let $\alpha_j \in 1+\mathfrak{m}$ be a number such that α_j is negative at v_j and positive at each real places of k different from v_j. Assume $\mathfrak{u} = \mathfrak{a}\mathfrak{m}^{-1}$. If $\epsilon_j = 0$ we have

$$\sigma_{v_j} Z_\mathfrak{m}^\epsilon(s,\mathfrak{a}) = Z_\mathfrak{m}^\epsilon(s,\alpha_j\mathfrak{a}) = Z_\mathfrak{m}^\epsilon(s,\mathfrak{a}),$$

and thus $\sigma_{v_j} f^\epsilon(\mathfrak{u}) = f^\epsilon(\mathfrak{u})$. If $\epsilon_j = 1$, we denote $\epsilon(j)$ the r_1-vector by replacing the j-component of ϵ by 0. We have

$$(1+\sigma_{v_j}) Z_\mathfrak{m}^\epsilon(s,\mathfrak{a}) = Z_\mathfrak{m}^\epsilon(s,\mathfrak{a}) + Z_\mathfrak{m}^\epsilon(s,\alpha_j\mathfrak{a}) = Z_\mathfrak{m}^{\epsilon(j)}(s,\mathfrak{a}). \tag{2.5}$$

¿From Prop.2.4, we see $\sigma_{v_j} f^\epsilon(\mathfrak{u}) = -f^\epsilon(\mathfrak{u})$ except for the case when $\mathfrak{m} = \mathfrak{e}$ and $|\epsilon(j)| = 0$. Thus f^ϵ is an ϵ-distribution when $|\epsilon| \geq 2$ and a punctured ϵ-distribution when $|\epsilon| = 1$. In the latter case, we can make f^ϵ become a non-punctured ϵ-distribution by redefining its values at integral ideals. In fact, for any distribution g we can redefine its value at $\mathfrak{e}$ freely. For any α define $\bar{g}(\mathfrak{a}) = g(\mathfrak{a}) - g(\mathfrak{e}) + \alpha$ if $\mathfrak{a} \in T_0$ and $\bar{g}(\mathfrak{u}) = g(\mathfrak{u})$ if $\mathfrak{u} \in \bar{T}_0 \setminus T_0$. Then $\bar{g}$ is also a distribution. Assume $|\epsilon| = 1$. We define $\bar{f}^\epsilon(\mathfrak{a}) = f^\epsilon(\mathfrak{a}) - f^\epsilon(\mathfrak{e})$ if $\mathfrak{a} \in T_0$ and $\bar{f}^\epsilon(\mathfrak{u}) = f^\epsilon(\mathfrak{u})$ if $\mathfrak{u} \notin T_0$. Then $\bar{f}^\epsilon$ is a non-punctured ϵ-distribution. Finally, we get

Theorem 2.5. *If $|\epsilon| \geq 2$, f^ϵ is an ϵ-distribution. If $|\epsilon| = 1$, f^ϵ is a punctured ϵ-distribution, but we can make it become a non-punctured ϵ-distribution $\bar{f}^\epsilon$ by redefining its values at integral ideals. If $|\epsilon| = 0$, f^+ is punctured and even.*

In the next section, we will see that in any case f^+ can not be made to be non-punctured.

Let $F^\epsilon = \mathrm{St}(f^\epsilon)$ be the Stickelberger distribution associated to f^ϵ. Then F^ϵ is punctured when $|\epsilon| \leq 1$. Set $\Omega' = \varinjlim \mathbb{C}[G_\mathfrak{m}]/(s(G_\mathfrak{m}))$. Let F^ϵ take values in Ω' via the natural map $\Omega \to \Omega'$. The first non-zero coefficient in the expansion of $Z_\mathfrak{e}^+(s,\mathfrak{a})$ at $s=0$ is a constant. In fact, it is equal to $-R/w$, where R and w are the regulator and the number of roots of unity of k, respectively. When $|\epsilon|=1$, we see from Eq.(2.5) that F^ϵ becomes a non-punctured ϵ-distribution. Actually F^ϵ is the composition of $\mathrm{St}(\bar f^\epsilon)$ with $\Omega\to\Omega'$. When $|\epsilon|=0$, we write $F^+=F^\epsilon$ and define for $\mathfrak{a}\in T_0$

$$F^+(\mathfrak{a})=\sum_{\mathfrak{b}\in G_\mathfrak{m}}(f^+_\mathfrak{e}(\mathfrak{ab})--f^+_\mathfrak{e}(\mathfrak{b}))\sigma_\mathfrak{b}^{-1},$$

which is independent of $\mathfrak{m}$ as an element in Ω'. Then F^+ becomes a non-punctured even distribution [Prop.2.4, Yi1]. Note that $F^+(\mathfrak{e})=0$.

Our next purpose is to determine the ranks of $A^\epsilon_\mathfrak{m}$ and $A^\pm_\mathfrak{m}$ and to prove the universility of F^ϵ. We need some preparations. We first recall the definition of Iwasawa distribution introduced in [Sect.3, Yi1].

There exists a unique function $u_0:\bar T_0\to\mathbb{Q}$ defined modulo $\sim$ such that for all $\mathfrak{m}\in T_0$ and all $\chi\in\hat G_\mathfrak{m}$ one has

$$\sum_{\mathfrak{a}\in G_\mathfrak{m}}\chi(\mathfrak{a})u_0(\mathfrak{am}^{-1})=\prod_{\mathfrak{p}|\mathfrak{m}}(1-\chi(\mathfrak{p})),\tag{2.6}$$

where the product is over all prime divisors of $\mathfrak{m}$. By Lem.2.3 u_0 is a distribution. The Stickelberger distribution $I=\mathrm{St}(u_0)$ of u_0 is called the Iwasawa distribution of the field k. It is universal up to torsion [Th.3.4, Yi1]. Set $I^\epsilon=\alpha^\epsilon I/|J|$, where α^ϵ is defined in Eq.(1.4). We have ϵ-decomposition $I=\sum_\epsilon I^\epsilon$. By Lemma 1.3, I^ϵ is a universal ϵ-distribution up to torsion. We call it Iwasawa ϵ-distribution of k. For $\chi\in\hat G_\mathfrak{m}$, We extend the definition of χ linearly to $\mathbb{C}[G_\mathfrak{m}]$. Let $r^\epsilon_\mathfrak{m}$ be the number of ϵ-characters of $G_\mathfrak{m}$.

Theorem 2.6. $\mathrm{rank}A^\epsilon_\mathfrak{m}=r^\epsilon_\mathfrak{m}$. *Especially,* $\mathrm{rank}A^+_\mathfrak{m}=|G^+_\mathfrak{m}|$ *and* $\mathrm{rank}A^-_\mathfrak{m}=|G_\mathfrak{m}|-|G^+_\mathfrak{m}|$.

Proof. Let $I^\epsilon_\mathfrak{m}$ be the level $\mathfrak{m}$ group of I^ϵ. By Thm.3.1 in Chap.1 in [KL], we have

$$\mathrm{rank}A^\epsilon_\mathfrak{m}=\mathrm{rank}I^\epsilon_\mathfrak{m}=\dim_\mathbb{C}\mathbb{C}\otimes_\mathbb{Z}I^\epsilon_\mathfrak{m}=\#\{\chi\in\hat G_\mathfrak{m}\mid\chi(I^\epsilon_\mathfrak{m})\neq0\}.$$

Let $\chi\in\hat G_\mathfrak{m}$ be of conductor $\mathfrak{f}$. If χ is an ϵ-character, we have $\chi(I^\epsilon(\mathfrak{f}))=\chi(I(\mathfrak{f}))\neq0$. If χ is not an ϵ-character, we claim $\chi(I^\epsilon_\mathfrak{m})=0$. Since $I^\epsilon_\mathfrak{m}$ is generated as $G_\mathfrak{m}$-modules by $I^\epsilon(\mathfrak{n}^{-1})$ with $\mathfrak{n}\mid\mathfrak{m}$, we only need to show $\chi(I^\epsilon(\mathfrak{n}^{-1}))=0$ for all $\mathfrak{n}\mid\mathfrak{m}$. We first assume $\mathfrak{f}\nmid\mathfrak{n}$. Since $\mathrm{Gal}(K_\mathfrak{n}/k)\simeq G_\mathfrak{m}/\mathrm{Gal}(K_\mathfrak{m}/K_\mathfrak{n})$, there exists $\sigma\in\mathrm{Gal}(K_\mathfrak{m}/K_\mathfrak{n})$ such that $\chi(\sigma)\neq1$. Thus $\chi(I^\epsilon(\mathfrak{n}^{-1}))=0$ as $\sigma I^\epsilon(\mathfrak{n}^{-1})=I^\epsilon(\mathfrak{n}^{-1})$. Now we assume $\mathfrak{f}\mid\mathfrak{n}$. By Lem.2.3 it is enough to show $\chi(I^\epsilon(\mathfrak{f}^{-1}))=0$. Since χ is not an ϵ-character, there exists i such that $\chi(\sigma_{v_i})\neq(-1)^{\epsilon_i}$. But we have $\sigma_{v_i}I^\epsilon(\mathfrak{f}^{-1})=(-1)^{\epsilon_i}I^\epsilon(\mathfrak{f}^{-1})$. Thus $\chi(I^\epsilon(\mathfrak{f}^{-1}))=0$. This shows the claim. Therefore, $\chi(I^\epsilon_\mathfrak{m})\neq0$ if and only if χ is an ϵ-character of $G_\mathfrak{m}$. We complete the proof.

Let u be a complex valued distribution and let $\chi\in\hat G_\mathfrak{m}$ be of conductor $\mathfrak{f}_\chi$, we set

$$\omega_\mathfrak{m}=\omega_\mathfrak{m}(u)=\sum_{\chi\in\hat G_\mathfrak{m}}u(\bar\chi)e_\chi\in\mathbb{C}[G_\mathfrak{m}],$$

where $\bar\chi$ is the inverse of χ, $u(\bar\chi)=u_{\mathfrak{f}_\chi}(\bar\chi)$ is defined in Eq.(2.1) and e_χ is the idempotent element of χ in $\mathbb{C}[G_\mathfrak{m}]$. Let $U=\mathrm{St}(u)$. Let $U_\mathfrak{m}$ and $I_\mathfrak{m}$ be the level $\mathfrak{m}$ groups of U and I respectively. The following lemma will be useful in next section.

Lemma 2.7. *Assume that u is an ϵ-distribution. Then* $U_\mathfrak{m}=\omega_\mathfrak{m}I_\mathfrak{m}=\omega_\mathfrak{m}I^\epsilon_\mathfrak{m}$.

Proof. Since u is an ϵ-distribution, we have $\alpha^\epsilon U=|J|U$. Thus the first equality implies the second. To show the first, we will prove $U(\mathfrak{n}^{-1})=\omega_\mathfrak{m}I(\mathfrak{n}^{-1})$ for all $\mathfrak{n}\mid\mathfrak{m}$. It is enough to prove

$$\chi(U(\mathfrak{n}^{-1}))=\chi(\omega_\mathfrak{m}I(\mathfrak{n}^{-1}))\tag{2.7}$$

for all $\chi\in\hat G_\mathfrak{m}$. Let $\mathfrak{f}=\mathfrak{f}_\chi$ be the conductor of χ. If $\mathfrak{f}\nmid\mathfrak{n}$, we see that both sides of Eq.(2.7) are zero as in the proof of Th.2.6. If $\mathfrak{f}\mid\mathfrak{n}$, using Lem.2.3, one can show Eq.(2.7) easily.

Now we show the following

Theorem 2.8. *F^+ is a universal even distribution subject to the condition $F^+(\mathfrak{e}) = 0$ and F^ϵ (where $\epsilon \neq 0$) is a universal ϵ-distribution, up to torsion.*

Proof. We denote $F^\epsilon_{\mathfrak{m}}$ the level $\mathfrak{m}$ group of F^ϵ. We need to show $\mathrm{rank} F^+_{\mathfrak{m}} = |G^+_{\mathfrak{m}}| - 1$ and $\mathrm{rank} F^\epsilon_{\mathfrak{m}} = r^\epsilon_{\mathfrak{m}}$. Let $S^\epsilon_{\mathfrak{m}}$ be the level $\mathfrak{m}$ group of F^ϵ with values in Ω. For $M \subseteq G_{\mathfrak{m}}$, write $s(M) = \sum_{\sigma\in M}\sigma$. Then $F^\epsilon_{\mathfrak{m}} = S^\epsilon_{\mathfrak{m}}/s(G_{\mathfrak{m}})S^\epsilon_{\mathfrak{m}}$. We have

$$\mathrm{rank} F^\epsilon_{\mathfrak{m}} = \#\{\chi \in \hat{G}_{\mathfrak{m}} \mid \chi(F^\epsilon_{\mathfrak{m}}) \neq 0\} = \#\{1 \neq \chi \in \hat{G}_{\mathfrak{m}} \mid \chi(S^\epsilon_{\mathfrak{m}}) \neq 0\}.$$

Let χ be a non-trivial ϵ-character of $G_{\mathfrak{m}}$ of conductor $\mathfrak{f}$. When χ is real, we assume $\mathfrak{f} \neq \mathfrak{e}$. We have, by Eq.(2.3),

$$\chi(F^\epsilon(\mathfrak{f}^{-1})) = \sum_{\mathfrak{a}\in G_{\mathfrak{m}}} f^\epsilon(\mathfrak{a}\mathfrak{f}^{-1})\bar{\chi}(\mathfrak{a}) = \frac{[U^\epsilon_{\mathfrak{f}} : U^+_{\mathfrak{f}}]|G_{\mathfrak{m}}|}{|G^\epsilon_{\mathfrak{f}}|} L^{(\bar{\epsilon})}(0, \bar{\chi}) \neq 0.$$

Composing with Th.2.6, we get $\mathrm{rank} F^\epsilon_{\mathfrak{m}} = r^\epsilon_{\mathfrak{m}}$ when $\epsilon \neq 0$.

When $\chi \neq 1$ is real and $\mathfrak{f} = \mathfrak{e}$, let $\mathfrak{a} \in T_0$ be such that $\chi(\mathfrak{a}) \neq 1$. We have, by Eq.(2.4),

$$\chi(F^+(\mathfrak{a})) = \frac{[U : U^+]|G_{\mathfrak{m}}|}{|G^+_{\mathfrak{e}}|}(\chi(\mathfrak{a}) - 1) \cdot L^{(r_1+r_2)}(0, \bar{\chi}) \neq 0.$$

We showed $\mathrm{rank} F^+_{\mathfrak{m}} = |G^+_{\mathfrak{m}}| - 1$.

It is easy to supplement F^+ to obtain a universal even distribution $\bar{F}^\epsilon$ with no restriction on the value at $\mathfrak{e}$, using the method ahead Th.2.5. Let $F^- = \sum_{\epsilon\neq 0} F^\epsilon$ and let $F = \bar{F}^+ + F^-$. Then F is a universal ordinary distribution and F^- is a universal odd distribution with values in torsion free abelian groups.

Here we give no information about the torsion of $A^\pm_{\mathfrak{m}}$ and $A^\epsilon_{\mathfrak{m}}$. In [Th.3.4, Yi1] we showed that $A_{\mathfrak{m}}$ has no p-torsion if the prime p does not divide $|G_{\mathfrak{m}}|$. Recently, Belliard and Oukhaba [BO] gave another sufficient condition when $A_{\mathfrak{m}}$ is p-torsion free (better than ours). Their results [Prop.3, BO] imply that if prime p does not divide the order of $\mathrm{Gal}(K_{\mathfrak{m}}/K_{\mathfrak{e}})$ then $A_{\mathfrak{m}}$ has no p-torsion. Especially when k is imaginary quadratic, they gave more detailed information about the torsion. In the case $k = \mathbb{Q}$, the group $A = \varinjlim A_{\mathfrak{m}}$ is free [Ku1], and the torsion of $A^\pm_{\mathfrak{m}}$ was first determined by Yamamoto [Ya] and later by Sinnott [Si] using the cohomology method. In the function field case, these results were given by Anderson [An] by double complex method. For a general number field k, the difficulty to determine the torsion of $A_{\mathfrak{m}}$, compared to cases $k = \mathbb{Q}$ and a function field, arises mainly from the totally positive units of k. In the following, we explain some arithmetic meaning of the torsion.

Let $A = \varinjlim A_{\mathfrak{m}}$ and $A^\epsilon = \varinjlim A^\epsilon_{\mathfrak{m}}$. We first give a clear description of A and A^ϵ. Let $\mathcal{A} = \mathcal{A}(k)$ be the free abelian group generated by all the classes $[\mathfrak{u}]$ in $\bar{T}_0/\sim$. We define the action of G on $\mathcal{A}$ as before. We identify A with the quotient of $\mathcal{A}$ by the subgroup generated by all elements in $\mathcal{A}$ of the form

$$[\mathfrak{u}] - \sum_{x\in w(\mathfrak{u}^{-1}/\mathfrak{u}^{-1}\mathfrak{n})} [(1+x)\mathfrak{u}\mathfrak{n}^{-1}] \tag{2.8}$$

for $\mathfrak{n} \in T_0$ and $\mathfrak{u} \in \bar{T}_0$. We identify A^ϵ with the quotient of $\mathcal{A}$ by the subgroups generated by all elements of $\mathcal{A}$ of the form (2.8), along with those of the form $[\mathfrak{u}] - \sigma_{v_i}[\mathfrak{u}]$ when $\epsilon_i = 0$ and $[\mathfrak{u}] + \sigma_{v_i}[\mathfrak{u}]$ when $\epsilon_i = 1$. Let $\mathbf{a}=\sum_i m_i[\mathfrak{u}_i]$ be an element in $\mathcal{A}$ and let $\mathfrak{m}$ be the lcm of the fractional parts of the $\mathfrak{u}_i$. If $\mathfrak{a} \in T_0$, we set $f^+(\mathfrak{a}) = f^+_{\mathfrak{e}}(\mathfrak{a})$. We have the following criteria.

Corollary 2.9. *$\mathbf{a}=\sum_i m_i[\mathfrak{u}_i] \in \mathrm{tor}(A^\epsilon)$ if and only if $\sum_i m_i f^\epsilon(\mathfrak{b}\mathfrak{u}_i)$ is independent of integral ideals $\mathfrak{b}$ coprime to $\mathfrak{m}$. In the case $\epsilon = 0$ we must have $\sum_j m_j = 0$, where the summation is over all j such that $\mathfrak{u}_j$ is integral.*

Proof. When $\epsilon \neq 0$ we have the exact sequence, by Th.2.8,

$$0 \longrightarrow \mathrm{tor}(A^\epsilon) \longrightarrow A^\epsilon \longrightarrow \varinjlim F^\epsilon_{\mathfrak{m}} \longrightarrow 0.$$

Thus $\mathbf{a}=\sum_i m_i[\mathfrak{u}_i] \in \mathrm{tor}(A^\epsilon)$ if and only if $\sum_i m_i F^\epsilon(\mathfrak{u}_i) = 0$ in Ω', i.e., $\sum_i m_i f^\epsilon(\mathfrak{b}\mathfrak{u}_i)$ is independent of integral ideals $\mathfrak{b}$, where $\mathfrak{b}$ runs over the representatives of $G_{\mathfrak{m}}$. In the case $\epsilon = 0$, replaceing F^+ by $\bar{F}^+$, we still have the exact sequence. Similarly we get the criterion when $\mathbf{a}\in \mathrm{tor}(A^+)$. In the even case, since $\bar{F}^+(\mathfrak{e})=\alpha$ is $\mathbb{Z}$-linear independent with all the values of F^+, we get $\sum_j m_j=0$, the sum over all j such that $\mathfrak{u}_j$ is integral.

We know that f^ϵ itself is an ϵ-distribution when $|\epsilon|\geq 2$. The proof of Th.2.8 implies that the ϵ-distribution $F^\epsilon=\mathrm{St}(f^\epsilon)$ with values in Ω, not in Ω', is universal up to torsion. As the proof of Cor.2.9, we have

Corollary 2.10. *Assume $|\epsilon|\geq 2$. Then $\mathbf{a}=\sum_i m_i[\mathfrak{u}_i]\in \mathrm{tor}(A^\epsilon)$ if and only if for all $\mathfrak{b}$ coprime to $\mathfrak{m}$ one has $\sum_i m_i f^\epsilon(\mathfrak{b}\mathfrak{u}_i)=0$.*

A natural question is whether these f^ϵ themselves are universal up to torsion. Let $f^\epsilon_{\mathfrak{m}}$ be the level $\mathfrak{m}$ group of f^ϵ. If f^ϵ has the property that $\sum_i m_i f^\epsilon(\mathfrak{a}_i\mathfrak{b}\mathfrak{m}^{-1})=0$ for all integral ideals $\mathfrak{b}$ coprime to $\mathfrak{m}$ whenever $\sum_i m_i f^\epsilon(\mathfrak{a}_i\mathfrak{m}^{-1})=0$, where $m_i\in\mathbb{Z}$, then $\mathrm{rank} f^\epsilon_{\mathfrak{m}}=\mathrm{rank} F^\epsilon_{\mathfrak{m}}$ and thus f^ϵ is universal up to torsion. Conversely, if f^ϵ is universal up to torsion, we have the exact sequence, for all integral ideals $\mathfrak{m}$,

$$0\longrightarrow \mathrm{tor}(A^\epsilon_{\mathfrak{m}})\longrightarrow A^\epsilon_{\mathfrak{m}}\longrightarrow f^\epsilon_{\mathfrak{m}}\longrightarrow 0.$$

If $\sum_i m_i f^\epsilon(\mathfrak{a}_i\mathfrak{m}^{-1})=0$, where $m_i\in\mathbb{Z}$, then $\sum_i m_i[\mathfrak{a}_i\mathfrak{m}^{-1}]\in \mathrm{tor}(A^\epsilon_{\mathfrak{m}})$. By Cor.2.10 (or since $\mathrm{tor}(A^\epsilon_{\mathfrak{m}})$ is an $G_{\mathfrak{m}}$-submodule of $A^\epsilon_{\mathfrak{m}}$), we have $\sum_i m_i f^\epsilon(\mathfrak{a}_i\mathfrak{b}\mathfrak{m}^{-1})=0$ for all integral ideals $\mathfrak{b}$ coprime to $\mathfrak{m}$. In the case $|\epsilon|=1$, if we replace f^ϵ by $\bar{f}^\epsilon$ defined ahead Th.2.5, the discussions above are also valid.

It is impossible that f^ϵ are universal up to torsion for all number field k and all ϵ with $|\epsilon|\geq 2$. For example, when k is totally real, by Siegel's results, see [Si] or [DR], we know that f^ϵ takes rational values for $\epsilon=1$ and thus it is not universal. We do not know whether there exist some ϵ such that f^ϵ is universal. So we propose the following question. Write $c=\sum_j m_j f^\epsilon(\mathfrak{e})$, the sum over j such that $\mathfrak{m}\mid\mathfrak{a}_j$.

Question 2.11. *For which number field k and for which ϵ with $|\epsilon|\geq 2$ (resp. $|\epsilon|=1$), f^ϵ (resp. $\bar{f}^\epsilon$) is a universal ϵ-distribution up to torsion? Or equivalently, when does f^ϵ have the property that $\sum_i m_i f^\epsilon(\mathfrak{a}_i\mathfrak{m}^{-1})=0$ (resp. $=c$) implies $\sum_i m_i f^\epsilon(\mathfrak{a}_i\mathfrak{b}\mathfrak{m}^{-1})=0$ (resp. $=c$) for all integral ideals $\mathfrak{b}$ coprime to $\mathfrak{m}$? where $m_i\in\mathbb{Z}$.*

3. Punctured distributions.

In this section, we first give a sufficient and necenssary condition when a punctured distribution can be completed to a non-punctured distribution, which implies an upper bound for the rank of $A^0_{\mathfrak{m}}$. We then construct punctured even and odd distributions whose sum of ranks of level $\mathfrak{m}$ groups is equal to the upper bound. To obtain this result, we need to study $\mathbb{Z}$-linear independence of some values of f^+. We conjectured that f^+ is universal up to torsion in [Conj.3.1, BGY]. This motivates a conjecture on the Galois action of some special values of f^+. In the cases when $k=\mathbb{Q}$ and is imaginary quadratic, the conjecture can be proved easily by the explicit reciprocity law via the obvious expression of f^+.

Let $g:\bar{T}_0\longrightarrow V$ be a distribution. If we only consider g on $\bar{T}_0\setminus T_0$, we get a punctured distribution and call it the punctured reduction of g. When can a punctured distribution be got in this way? Or equivalently, when can it be completed to a non-punctured distribution? The answer is the follows.

Theorem 3.1. *Punctured distribution g can be completed to a non-punctured distribution if and only if $\sum_{0\neq x\in w(\mathbb{A}/x_0\mathbb{A})} g(xx_0^{-1}\mathbb{A})=0$ for all totally positive $x_0\in\mathbb{A}\setminus U^+$. Furthermore, this non-punctured distribution is unique up to its value at the unit ideal $\mathfrak{e}$.*

Proof. Assume that g can be completed to a non-punctured distribution, denoted also by g. Let $x_0\in\mathbb{A}\setminus U^+$ be totally positive. We have

$$g(\mathfrak{e})-g(x_0\mathfrak{e})=\sum_{0\neq x\in w(\mathbb{A}/x_0\mathbb{A})} g(xx_0^{-1}\mathbb{A})=0.$$

Conversely, we define $g(\mathfrak{e})=0$. Let $\mathfrak{a}\in T_0$. Take $\mathfrak{m}\in T_0$ such that $\mathfrak{a}\mathfrak{m}=(x_0)$ for some totally positive x_0. We define

$$g(\mathfrak{a})=-\sum_{0\neq x\in w(\mathbb{A}/\mathfrak{m})} g(x\mathfrak{m}^{-1})=-\sum_{0\neq x\in w(\mathbb{A}/x_0\,\mathfrak{a}^{-1})} g(xx_0^{-1}\mathfrak{a}).$$

Using the condition in the theorem and the distribution relations of g at fractional ideals, we have

$$
\begin{aligned}
g(\mathfrak{a}) &= -\sum_{0\neq x\in w(\mathbb{A}/x_0\,\mathfrak{a}^{-1})}\ \sum_{y\in w(x^{-1}x_0\mathfrak{a}^{-1}/x^{-1}x_0\mathbb{A})} g(xyx_0^{-1}\mathbb{A}) \\
&= -\sum_{0\neq x\in(\mathbb{A}/x_0\,\mathfrak{a}^{-1})/U^+}\ \sum_{z\in(x+(x_0\mathfrak{a}^{-1}/x_0\mathbb{A}))/U^+} g(zx_0^{-1}\mathbb{A}) \\
&= -\sum_{0\neq x\in w(\mathbb{A}/x_0\,\mathbb{A})} g(xx_0^{-1}\mathbb{A}) + \sum_{0\neq x\in(x_0\mathfrak{a}^{-1}/x_0\mathbb{A})/U^+} g(xx_0^{-1}\mathbb{A}) \\
&= \sum_{0\neq y\in w(\mathfrak{a}^{-1}/\mathbb{A})} g(y\mathbb{A}).
\end{aligned}
\tag{3.1}
$$

Thus $g(\mathfrak{a})$ is independent of the choice of $\mathfrak{m}$ or x_0 and g factors through $T_0\to T_0/\sim$.

Now we show that g satisfies the distribution relations (1.2) at integral ideals. Let $\mathfrak{a}=x_0\mathfrak{m}^{-1}$ be as above and let $\mathfrak{e}\neq\mathfrak{n}\in T_0$. Let $y_0\in\mathfrak{n}$ be totally positive. By the definition of g at integral ideals and the distribution relations of g at fractional ideals, we have

$$
\begin{aligned}
g(\mathfrak{a})-g(y_0\mathfrak{a}\mathfrak{n}^{-1}) &= \sum_{0\neq y\in w(\mathbb{A}/\mathfrak{m}\mathfrak{n})} g(y\mathfrak{m}^{-1}\mathfrak{n}^{-1}) - -\sum_{0\neq x\in w(\mathbb{A}/\mathfrak{m})} g(x\mathfrak{m}^{-1}) \\
= \sum_{0\neq y\in w(\mathbb{A}/\mathfrak{m}\mathfrak{n})} g(y\mathfrak{m}^{-1}\mathfrak{n}^{-1}) &- -\sum_{0\neq x\in(\mathbb{A}/\mathfrak{m})/U^+}\ \sum_{z\in(x+(\mathfrak{m}/\mathfrak{m}\mathfrak{n}))/U^+} g(z\mathfrak{m}^{-1}\mathfrak{n}^{-1}) \\
= \sum_{0\neq z\in(\mathfrak{m}/\mathfrak{m}\mathfrak{n})/U^+} g(z\mathfrak{m}^{-1}\mathfrak{n}^{-1}) &= \sum_{0\neq y\in w(\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{n})} g(y\mathfrak{a}\mathfrak{n}^{-1}).
\end{aligned}
\tag{3.2}
$$

This is the distribution relation we required. From the distribution relations (1.2) we can see that the completed non-punctured distribution is unique up to its values at $\mathfrak{e}$.

The condition in the theorem is hard to deal with. We wish to replace it in some cases by a weaker but simpler condition. We mention that the condition in the theorem implies $\sum_{\mathfrak{a}\in T_\mathfrak{m},\mathfrak{m}\nmid\mathfrak{a}} g(\mathfrak{a}\mathfrak{m}^{-1})=0$ for all $\mathfrak{m}\in T_0$ and $\mathfrak{m}\neq\mathfrak{e}$ since the latter holds if g is a non-punctured distribution, where the sum over $\mathfrak{a}\in T_\mathfrak{m},\mathfrak{m}\nmid\mathfrak{a}$ means that $\mathfrak{a}$ runs over all representatives of the classes in $T_\mathfrak{m}$ such that $\mathfrak{m}\nmid\mathfrak{a}$. Note that the condition $\mathfrak{m}\nmid\mathfrak{a}$ is independent of the choice of the representative in the class by Lem.1.1(1).

Proposition 3.2. *Let g be a punctured distribution. Assume $\mathfrak{m}\neq\mathfrak{e}$. We have*
(1) Both $\sum_{\mathfrak{a}\in T_\mathfrak{m},\mathfrak{m}\nmid\mathfrak{a}} g(\mathfrak{a}\mathfrak{m}^{-1})$ and $\sum_{\mathfrak{a}\in G_\mathfrak{m}} g(\mathfrak{a}\mathfrak{m}^{-1})$ are $\mathbb{Z}$-linear combinations of the elements $\sum_{\mathfrak{a}\in G_\mathfrak{p}} g(\mathfrak{a}\mathfrak{p}^{-1})$, where $\mathfrak{p}$ are prime divisors of $\mathfrak{m}$.
(2) Assume $\mathfrak{m}=\alpha\mathbb{A}$, where α is totally positive. Write $T_{\alpha\mathbb{A}}=T_\alpha$. Then

$$
\sum_{\mathfrak{a}\in T_\alpha,\alpha\nmid\mathfrak{a}} g(\mathfrak{a}\alpha^{-1}) = |G_\mathfrak{e}| \sum_{0\neq x\in w(\mathbb{A}/\alpha\mathbb{A})} g(x\alpha^{-1}\mathbb{A}).
$$

Proof. (1) Assume $\mathfrak{m}=\mathfrak{p}\mathfrak{n}$, where $\mathfrak{p}$ is prime and $\mathfrak{n}\neq\mathfrak{e}$. Since $\mathfrak{a}\mathfrak{p}\sim_\mathfrak{m}\mathfrak{b}\mathfrak{p}$ if and only if $\mathfrak{a}\sim_\mathfrak{n}\mathfrak{b}$, we have

$$
\begin{aligned}
\sum_{\substack{\mathfrak{a}\in T_\mathfrak{m}\\ \mathfrak{m}\nmid\mathfrak{a}}} g(\mathfrak{a}\mathfrak{m}^{-1}) &= \sum_{\substack{\mathfrak{a}\in T_\mathfrak{m}\\ \mathfrak{p}|\mathfrak{a},\mathfrak{m}\nmid\mathfrak{a}}} g(\mathfrak{a}\mathfrak{m}^{-1}) + \sum_{\substack{\mathfrak{a}\in T_\mathfrak{m}\\ \mathfrak{p}\nmid\mathfrak{a}}} g(\mathfrak{a}\mathfrak{m}^{-1}) \\
= \sum_{\substack{\mathfrak{a}\in T_\mathfrak{n}\\ \mathfrak{n}\nmid\mathfrak{a}}} g(\mathfrak{a}\mathfrak{n}^{-1}) + \sum_{\substack{\mathfrak{b}\in T_\mathfrak{p}\\ \mathfrak{p}\nmid\mathfrak{b}}}\ \sum_{\substack{\mathfrak{a}\sim_\mathfrak{p}\mathfrak{b}\\ \mathfrak{a}\bmod\sim_\mathfrak{m}}} g(\mathfrak{a}\mathfrak{m}^{-1}) &= \sum_{\substack{\mathfrak{a}\in T_\mathfrak{n}\\ \mathfrak{n}\nmid\mathfrak{a}}} g(\mathfrak{a}\mathfrak{n}^{-1}) + \sum_{\mathfrak{a}\in G_\mathfrak{p}} g(\mathfrak{a}\mathfrak{p}^{-1}).
\end{aligned}
$$

Continuing this process, we get the result on $\sum_{\mathfrak{a}\in T_\mathfrak{m},\mathfrak{m}\nmid\mathfrak{a}} g(\mathfrak{a}\mathfrak{m}^{-1})$. Furthermore, using the Mobius inversion formula to the following equality

$$
\sum_{\substack{\mathfrak{a}\in T_\mathfrak{m}\\ \mathfrak{m}\nmid\mathfrak{a}}} g(\mathfrak{a}\mathfrak{m}^{-1}) = \sum_{\substack{\mathfrak{d}|\mathfrak{m}\\ \mathfrak{d}\neq\mathfrak{m}}}\ \sum_{\substack{\mathfrak{a}\in T_\mathfrak{m}\\ (\mathfrak{a},\mathfrak{m})=\mathfrak{d}}} g(\mathfrak{a}\mathfrak{m}^{-1}) = \sum_{\substack{\mathfrak{n}|\mathfrak{m}\\ \mathfrak{n}\neq\mathfrak{e}}}\ \sum_{\mathfrak{b}\in G_\mathfrak{n}} g(\mathfrak{b}\mathfrak{n}^{-1}),
\tag{3.3}
$$

we get the result on $\sum_{\mathfrak{a}\in G_{\mathfrak{m}}} g(\mathfrak{a}\mathfrak{m}^{-1})$.

(2) Let $\mathfrak{a}\in T_0$. For totally positive $x_0\in\mathfrak{a}$, set $g_{x_0}(\mathfrak{a}) = -\sum_{0\neq x\in w(\mathbb{A}/x_0\mathfrak{a}^{-1})} g(xx_0^{-1}\mathfrak{a})$. On one hand, we have by taking $\mathfrak{n}=\alpha\mathbb{A}$ in Eqs.(3.2)

$$g_{x_0}(\mathfrak{a}) - g_{x_0\alpha}(\mathfrak{a}) = \sum_{0\neq y\in w(\mathfrak{a}^{-1}/\alpha\mathfrak{a}^{-1})} g(y\mathfrak{a}\alpha^{-1}).$$

On the other hand, from the third equality of Eqs.(3.1) we see

$$\begin{aligned} g_{x_0}(\mathfrak{a}) - g_{x_0\alpha}(\mathfrak{a}) &= -\sum_{0\neq x\in w(\mathbb{A}/x_0\mathbb{A})} g(xx_0^{-1}\mathbb{A}) + \sum_{0\neq x\in w(\mathbb{A}/x_0\alpha\mathbb{A})} g(xx_0^{-1}\alpha^{-1}\mathbb{A}) \\ &= \sum_{0\neq x\in w(\mathbb{A}/\alpha\mathbb{A})} g(x\alpha^{-1}\mathbb{A}). \end{aligned}$$

Here the second equality was got by the method in the proof of Eqs.(3.1). Thus we have

$$\sum_{\mathfrak{a}\in T_\alpha,\alpha\nmid\mathfrak{a}} g(\alpha^{-1}\mathfrak{a}) = \sum_{\mathfrak{a}\in G_{\mathfrak{e}}}\ \sum_{0\neq x\in w(\mathfrak{a}^{-1}/\alpha\mathfrak{a}^{-1})} g(x\alpha^{-1}\mathfrak{a}) = |G_{\mathfrak{e}}| \sum_{0\neq x\in w(\mathbb{A}/\alpha\mathbb{A})} g(x\alpha^{-1}\mathbb{A}).$$

We complete the proof.

Corollary 3.3. *Assume $\epsilon\neq 0$. Any punctured ϵ-distribution with values in a group on which 2 and $|G_{\mathfrak{e}}|$ are invertible can be completed to a unique non-punctured ϵ-distribution with value 0 at the unit ideal $\mathfrak{e}$.*

Proof. Let g be such a punctured distribution. By the two results above, it is enough to show $\sum_{\mathfrak{a}\in G_{\mathfrak{p}}} g(\mathfrak{a}\mathfrak{p}^{-1}) = 0$ for all primes $\mathfrak{p}$. Assume $\epsilon_i = 1$. Take $\alpha_i\in 1+\mathfrak{p}$ such that $v_i(\alpha_i)<0$ and $v_j(\alpha_i)>0$ for $j\neq i$. We have

$$-\sum_{\mathfrak{a}\in G_{\mathfrak{p}}} g(\mathfrak{a}\mathfrak{p}^{-1}) = \sum_{\mathfrak{a}\in G_{\mathfrak{p}}} \sigma_{v_i} g(\mathfrak{a}\mathfrak{p}^{-1}) = \sum_{\mathfrak{a}\in G_{\mathfrak{p}}} g(\alpha_i\mathfrak{a}\mathfrak{p}^{-1}) = \sum_{\mathfrak{a}\in G_{\mathfrak{p}}} g(\mathfrak{a}\mathfrak{p}^{-1}).$$

Since 2 is invertible, we get the result.

Applying Th.3.1 and Prop.3.2, we can give an upper bound of the rank of $A^0_{\mathfrak{m}}$. Let $N_{\mathfrak{m}}$ (resp. $N^+_{\mathfrak{m}}$) be the subgroup of $G_{\mathfrak{e}}$ (resp. $G^+_{\mathfrak{e}}$) generated by the classes including $\mathfrak{p}$ in $G_{\mathfrak{e}}$ (resp. $G^+_{\mathfrak{e}}$) with $\mathfrak{p}\mid\mathfrak{m}$. Let $t=[G_{\mathfrak{e}}:N_{\mathfrak{m}}]$ and $t^+=[G^+_{\mathfrak{e}}:N^+_{\mathfrak{m}}]$ be the indices. Let s be the number of distinct prime divisors of $\mathfrak{m}$.

Let $g:\bar{T}_0\setminus T_0\to V$ be a punctured distribution. Let W and R be the subgroups of V generated by $\sum_{0\neq x\in w(\mathbb{A}/\alpha\mathbb{A})} g(x\alpha^{-1}\mathbb{A})$ with all totally positive $\alpha\in\mathbb{A}\setminus U^+$ and by $\sum_{\mathfrak{a}\in G_{\mathfrak{p}}} g(\mathfrak{a}\mathfrak{p}^{-1})$ with all primes $\mathfrak{p}$, respectively. We have $|G_{\mathfrak{e}}|W\subseteq R$ by Prop.3.2. Let $g_{\mathfrak{m}}$ be the level $\mathfrak{m}$ group of g. Let $W_{\mathfrak{m}} = W\cap g_{\mathfrak{m}}$ and let $R_{\mathfrak{m}} = R\cap g_{\mathfrak{m}}$. The latter is generated by $\sum_{\mathfrak{a}\in G_{\mathfrak{p}}} g(\mathfrak{a}\mathfrak{p}^{-1})$ with primes $\mathfrak{p}\mid\mathfrak{m}$. We have $R_{\mathfrak{m}}\supseteq |G_{\mathfrak{e}}|W\cap g_{\mathfrak{m}}\supseteq |G_{\mathfrak{e}}|W_{\mathfrak{m}}$.

Let g take values in V/W via the natural map $V\to V/W$. Then g can be completed to a non-punctured distribution, say $\bar{g}$, by Th.3.1. It takes values in the group generated by V/W and by $\bar{g}(\mathfrak{a})$ with $\mathfrak{a}\in T_0$. Let $\bar{g}_{\mathfrak{m}}$ be the level $\mathfrak{m}$ group of $\bar{g}$. In the level case, the values of $\bar{g}$ at some integral ideals can be determined by the values of $\bar{g}$ at the other integral ideals and the values of $\bar{g}$ at fractional ideals (i.e. the elements in $g_{\mathfrak{m}}/W_{\mathfrak{m}}$). Let $\bar{\mathfrak{p}} = \alpha_0\mathfrak{p}^{-1}$, where $\alpha_0\in\mathfrak{p}$ is totally positive. If we gave the value of $\bar{g}(\mathfrak{e})$, then $\bar{g}(\bar{\mathfrak{p}}^i)$ would be determined for all prime divisors $\mathfrak{p}$ of $\mathfrak{m}$ and all $i\geq 1$ by applying the distribution relations (1.2) at the unit ideal $\mathfrak{e}$ and the integral ideal $\bar{\mathfrak{p}}$ repeatedly. Let $N'_{\mathfrak{m}}$ be the subgroup of $G_{\mathfrak{e}}$ generated by the classes including $\bar{\mathfrak{p}}$ in $G_{\mathfrak{e}}$ with $\mathfrak{p}\mid\mathfrak{m}$. Then $N'_{\mathfrak{m}}$ and $N_{\mathfrak{m}}$ have the same order. Let integral ideals $\mathfrak{a}_1=\mathfrak{e},\mathfrak{a}_2,\cdots,\mathfrak{a}_t$ represent all the classes in $G_{\mathfrak{e}}/N'_{\mathfrak{m}}$. To determine the values of $\bar{g}$ at all integral ideals, it is enough to add generaters $\bar{g}(\mathfrak{a}_i)$ to $g_{\mathfrak{m}}/W_{\mathfrak{m}}$ with $1\leq i\leq t$. We can assume that the $\bar{g}(\mathfrak{a}_i)$ themselves and they with the elements in $g_{\mathfrak{m}}$ are $\mathbb{Z}$-linear independent. We obtain an exact sequence

$$0\longrightarrow W_{\mathfrak{m}}\longrightarrow g_{\mathfrak{m}}\oplus(\oplus_{i=1}^t\mathbb{Z}\bar{g}(\mathfrak{a}_i))\longrightarrow \bar{g}_{\mathfrak{m}}\longrightarrow 0. \tag{3.4}$$

Since $\operatorname{rank} W_{\mathfrak{m}} = \operatorname{rank}|G_{\mathfrak{e}}|W_{\mathfrak{m}}\leq \operatorname{rank} R_{\mathfrak{m}}\leq s$ and $\operatorname{rank}\bar{g}_{\mathfrak{m}}\leq |G_{\mathfrak{m}}|$, we get

Proposition 3.4. *For any punctured distribution g we have* $\operatorname{rank} g_{\mathfrak{m}} \le |G_{\mathfrak{m}}| + s - t$.

We will construct a punctured distribution whose rank of level $\mathfrak{m}$ group is equal to the upper bound. We need a long preparation.

Let I be the Iwasawa distribution of k and let I^0 be the punctured reduction of I. Let $I^0 = I^{0+} + I^{0-}$ be the even-odd decomposition of I^0. Let $I^0_{\mathfrak{m}}$, $I^{0+}_{\mathfrak{m}}$ and $I^{0-}_{\mathfrak{m}}$ be their respective level $\mathfrak{m}$ group.

Lemma 3.5. $\operatorname{rank} I^{0+}_{\mathfrak{m}} = |G^+_{\mathfrak{m}}| - t^+$ *and* $\operatorname{rank} I^{0-}_{\mathfrak{m}} = (|G_{\mathfrak{m}}| - t) - (|G^+_{\mathfrak{m}}| - t^+)$.

Proof. As in the proof of theorem 2.6 we have

$$\operatorname{rank} I^0_{\mathfrak{m}} = \#\{\chi \in \hat{G}_{\mathfrak{m}} \mid \chi(I^0_{\mathfrak{m}}) \ne 0\}.$$

If χ is ramified, clearly $\chi(I^0_{\mathfrak{m}}) \ne 0$. If χ is unramified, we have, for $\mathfrak{n} \mid \mathfrak{m}$ and $\mathfrak{n} \ne \mathfrak{e}$,

$$\chi(I(\mathfrak{n}^{-1})) = \frac{|G_{\mathfrak{m}}|}{|G_{\mathfrak{n}}|} \cdot \prod_{\mathfrak{p}\mid\mathfrak{n}} (1 - \chi(\mathfrak{p})).$$

Here we regard $I(\mathfrak{n}^{-1})$ as an element in $\mathbb{C}[G_{\mathfrak{m}}]$. Thus $\chi(I(\mathfrak{n}^{-1})) \ne 0$ if and only if $\chi(\mathfrak{p}) \ne 1$ for all $\mathfrak{p} \mid \mathfrak{n}$. We get $\{\chi \mid \chi(I^0_{\mathfrak{m}}) \ne 0\} = \hat{G}_{\mathfrak{m}} - \widehat{G_{\mathfrak{e}}/N_{\mathfrak{m}}}$, which implies $\operatorname{rank} I^0_{\mathfrak{m}} = |G_{\mathfrak{m}}| - t$. In the same way as above, we can show $\operatorname{rank} I^{0+}_{\mathfrak{m}} = |G^+_{\mathfrak{m}}| - t^+$. This completes the proof.

Let f^+ and f^ϵ (where $\epsilon \ne 0$) be the distributions defined in last section. We know f^+ is punctured. Let $f^{0\epsilon}$ be the punctured reduction of f^ϵ. Let $f^- = \sum_{\epsilon\ne 0} f^{0\epsilon}$. Let $F^{0+} = \mathrm{St}(f^+)$ and $F^{0\epsilon} = \mathrm{St}(f^{0\epsilon})$ be their respective Stickelberger distribution. Now they take values in Ω, not in Ω'. Let $F^{0-} = \sum_{\epsilon\ne 0} F^{0\epsilon}$ and let $F^0 = F^{0+} + F^{0-}$. Let $F^0_{\mathfrak{m}}, F^+_{\mathfrak{m}}$ and $F^{0-}_{\mathfrak{m}}$ be their respective level $\mathfrak{m}$ group. The $G_{\mathfrak{m}}$-module structures of these level groups defined before are the same as those multiplying by the elements in $G_{\mathfrak{m}}$. Let $e_{\mathfrak{m}} = s(G_{\mathfrak{m}})/|G_{\mathfrak{m}}|$.

Lemma 3.6. $\operatorname{rank} F^{0-}_{\mathfrak{m}} = (|G_{\mathfrak{m}}| - t) - (|G^+_{\mathfrak{m}}| - t^+)$ *and* $\operatorname{rank}(1 - e_{\mathfrak{m}}) F^{0+}_{\mathfrak{m}} = |G^+_{\mathfrak{m}}| - t^+$.

Proof. From f^+ we construct a new distribution $\bar{f}^+$ by $\bar{f}^+(\mathfrak{a}\mathfrak{m}^{-1}) = (1 - e_{\mathfrak{m}}) f^+(\mathfrak{a}\mathfrak{m}^{-1})$, which is independent of the fractional part $\mathfrak{m}$. Since $\sum_{\mathfrak{a}\in G_{\mathfrak{p}}} (1 - e_{\mathfrak{p}}) f^+(\mathfrak{a}\mathfrak{p}^{-1}) = 0$ for all primes $\mathfrak{p}$ and $|G_{\mathfrak{e}}|$ is invertible in Ω, $\bar{f}^+$ can be completed to a unique non-punctured even distribution with value 0 at $\mathfrak{e}$ by Th.3.1 and Prop.3.2. We denote it also by $\bar{f}^+$. By Th.2.5 or Cor.3.3, f^- can also be completed to a unique non-punctured odd distribution, say $\bar{f}^-$, with value 0 at $\mathfrak{e}$. Let $\bar{F}^\pm = \mathrm{St}(\bar{f}^\pm)$. Then $F^{0\pm}$ above are the punctured reductions of $\bar{F}^\pm$, respectively. Let $\chi \in \hat{G}_{\mathfrak{m}}$ be of conductor $\mathfrak{f}_\chi$. We define $\bar{f}^\pm(\chi) = \sum_{\mathfrak{a}\in G_{\mathfrak{f}_\chi}} \chi(\mathfrak{a}) \bar{f}^\pm(\mathfrak{a}\mathfrak{f}_\chi^{-1})$ as in Eq.(2.1). When $\chi = 1$ we have $\bar{f}^+(\chi) = \sum_{\mathfrak{a}\in G_{\mathfrak{e}}} (1 - e_{\mathfrak{e}}) f^+(\mathfrak{a}) = 0$. Set

$$\omega^\pm_{\mathfrak{m}} = \sum_{\chi\in\hat{G}_{\mathfrak{m}}} \bar{f}^\pm(\bar{\chi}) e_\chi \in \mathbb{C}[G_{\mathfrak{m}}],$$

respectively. By Lem.2.7, we have, for $\mathfrak{n} \mid \mathfrak{m}$,

$$\bar{F}^\pm(\mathfrak{n}^{-1}) = \omega^\pm_{\mathfrak{m}} I(\mathfrak{n}^{-1}) = \omega^\pm_{\mathfrak{m}} I^\pm(\mathfrak{n}^{-1}),$$

where the second equality comes from the fact that $s(J)\bar{F}^+ = |J|\bar{F}^+$ and $s(J)\bar{F}^- = 0$. Thus $F^{0-}_{\mathfrak{m}} = \omega^-_{\mathfrak{m}} I^{0-}_{\mathfrak{m}}$ and $(1 - e_{\mathfrak{m}}) F^{0+}_{\mathfrak{m}} = \omega^+_{\mathfrak{m}} I^{0+}_{\mathfrak{m}}$. Since $\chi(\omega^+_{\mathfrak{m}}) \ne 0$ if $\chi \ne 1$ is real (note $\chi(I^{0+}_{\mathfrak{m}}) = 0$ when $\chi = 1$) and $\chi(\omega^-_{\mathfrak{m}}) \ne 0$ if χ is non-real by the proof of Th.2.8, we get the ranks of $(1 - e_{\mathfrak{m}}) F^{0+}_{\mathfrak{m}}$ and $F^{0-}_{\mathfrak{m}}$ from last lemma.

We now consider the rank of the whole $F^{0+}_{\mathfrak{m}}$. We have by the lemma above

$$\operatorname{rank} F^{0+}_{\mathfrak{m}} = \operatorname{rank}(1 - e_{\mathfrak{m}}) F^{0+}_{\mathfrak{m}} + \operatorname{rank} e_{\mathfrak{m}} F^{0+}_{\mathfrak{m}} = \operatorname{rank} e_{\mathfrak{m}} F^{0+}_{\mathfrak{m}} + |G^+_{\mathfrak{m}}| - t^+. \tag{3.5}$$

Here $e_{\mathfrak{m}} F^{0+}_{\mathfrak{m}}$ is generated by $s(G_{\mathfrak{m}}) \sum_{\mathfrak{b}\in G_{\mathfrak{m}}} f^+(\mathfrak{a}\mathfrak{b}\mathfrak{m}^{-1})/|G_{\mathfrak{m}}|$ with $\mathfrak{a} \in T_0$ and $\mathfrak{m} \nmid \mathfrak{a}$. Let $R_{\mathfrak{m}}$ is the $\mathbb{Z}$-submodule of $\mathbb{C}$ generated by $\sum_{\mathfrak{a}\in G_{\mathfrak{p}}} f^+(\mathfrak{a}\mathfrak{p}^{-1})$ with $\mathfrak{p} \mid \mathfrak{m}$. By Prop.3.2(1), the rank of $e_{\mathfrak{m}} F^{0+}_{\mathfrak{m}}$ is equal to that of $R_{\mathfrak{m}}$. To determine this rank, we need to study $\mathbb{Q}$-linear independence of some special values of f^+. We will do more.

Let $Z(s) = \sum_{\mathfrak{a}\in T_0} (N\mathfrak{a})^{-s}$ be the Dedekind zeta function of k. It is well-known that the order of vanishing of $Z(s)$ at $s = 0$ is equal to $r_1 + r_2 - 1$. Let c be the coefficient of $s^{r_1+r_2-1}$ in the Taylor expansion of $Z(s)$ at $s = 0$. Let $T^+_{\mathfrak{m}} = T^\epsilon_{\mathfrak{m}}$ be the set of equivalence classes of T_0 under $\sim^\epsilon_{\mathfrak{m}}$ in the case $\epsilon = 0$ (see Sect.1). Define $\Lambda : T_0 \to \mathbb{C}$ by $\Lambda(\mathfrak{m}) = \log N\mathfrak{p}$ if $\mathfrak{m}$ is a power of the prime $\mathfrak{p}$ and $\Lambda(\mathfrak{m}) = 0$ otherwise.

Theorem 3.7. *Let $\mathfrak{m} \in T_0$ and assume $\mathfrak{m} \neq \mathfrak{e}$. We have*

$$\sum_{\mathfrak{a} \in T^+_{\mathfrak{m}}, \mathfrak{m} \nmid \mathfrak{a}} f^+(\mathfrak{a}\mathfrak{m}^{-1}) = c \cdot \log N\mathfrak{m} \quad \text{and} \quad \sum_{\mathfrak{a} \in G^+_{\mathfrak{m}}} f^+(\mathfrak{a}\mathfrak{m}^{-1}) = c \cdot \Lambda(\mathfrak{m}).$$

Thus $\sum_{\mathfrak{a} \in G_{\mathfrak{p}}} f^+(\mathfrak{a}\mathfrak{p}^{-1})$ are $\mathbb{Q}$-linear independent for all primes $\mathfrak{p}$ of k. Furthermore, if $\epsilon \neq 0$ we have

$$\sum_{\mathfrak{a} \in T^\epsilon_{\mathfrak{m}}, \mathfrak{m} \nmid \mathfrak{a}} f^\epsilon(\mathfrak{a}\mathfrak{m}^{-1}) = \sum_{\mathfrak{a} \in G^\epsilon_{\mathfrak{m}}} f^\epsilon(\mathfrak{a}\mathfrak{m}^{-1}) = 0.$$

Proof. Write $r = r_1 + r_2$. We have

$$\sum_{\mathfrak{a} \in T^+_{\mathfrak{m}}, \mathfrak{m} \nmid \mathfrak{a}} f^+(\mathfrak{a}\mathfrak{m}^{-1}) = \sum_{\mathfrak{a} \in T^+_{\mathfrak{m}}, \mathfrak{m} \nmid \mathfrak{a}} \frac{\mathrm{d}^r}{\mathrm{d}s^r} \left(\sum_{\mathfrak{b} \sim^+_{\mathfrak{m}} \mathfrak{a}} N\mathfrak{b}^{-s} \right)_{s=0} = \frac{\mathrm{d}^r}{\mathrm{d}s^r} \left(\sum_{\mathfrak{b} \in T_0, \mathfrak{m} \nmid \mathfrak{b}} N\mathfrak{b}^{-s} \right)_{s=0}$$
$$= \frac{\mathrm{d}^r}{\mathrm{d}s^r} (Z(s) - (N\mathfrak{m})^{-s} Z(s))|_{s=0} = c \cdot \log N\mathfrak{m}.$$

Let $\mu(\mathfrak{m})$ be the Mobius function defined on T_0, that is, $\mu(\mathfrak{m}) = (-1)^r$ if $\mathfrak{m}$ is a product of r distinct primes and $\mu(\mathfrak{m}) = 0$ otherwise. By the equality above and by Mobius inversion formula (cf. Eq.(3.3) and [Sect.6.4, Hua]), we have

$$\sum_{\mathfrak{a} \in G^+_{\mathfrak{m}}} f^+(\mathfrak{a}\mathfrak{m}^{-1}) = c \cdot \sum_{\mathfrak{d} | \mathfrak{m}} \mu(\mathfrak{m}\mathfrak{d}^{-1}) \log N\mathfrak{d} = c \cdot \Lambda(\mathfrak{m}).$$

The last equality comes from the following fact, write $\mathfrak{m} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}$,

$$\sum_{\mathfrak{d} | \mathfrak{m}} \Lambda(\mathfrak{d}) = \sum_{l_1=1}^{e_1} \cdots \sum_{l_s=1}^{e_s} \Lambda(\mathfrak{p}_1^{l_1} \cdots \mathfrak{p}_s^{l_s}) = \sum_{l_1=1}^{e_1} \Lambda(\mathfrak{p}_1^{l_1}) + \cdots + \sum_{l_s=1}^{e_s} \Lambda(\mathfrak{p}_s^{l_s}) = \log N\mathfrak{m}.$$

We get the two equalities on f^+. Notice that $\frac{\mathrm{d}^{\bar\epsilon}}{\mathrm{d}s^{\bar\epsilon}}(Z(s) - (N\mathfrak{m})^{-s}Z(s))|_{s=0} = 0$ since $\bar\epsilon < r$. We get the result on f^ϵ in the same way.

For a prime ideal $\mathfrak{p}$ of k, let $f_{\mathfrak{p}}$ be the residue class degree of $\mathfrak{p}$ in the extension $k/\mathbb{Q}$. Let p be the rational prime lying down $\mathfrak{p}$. Then $N\mathfrak{p} = p^{f_{\mathfrak{p}}}$. The second equality in the theorem on f^+ implies $\sum_{\mathfrak{a} \in G_{\mathfrak{p}}} f^+(\mathfrak{a}\mathfrak{p}^{-1}) = c|J_{\mathfrak{p}}| f_{\mathfrak{p}} \cdot \log p$ and thus they are $\mathbb{Q}$-linear independent for all prime ideals $\mathfrak{p}$.

We remark that in the case $k = \mathbb{Q}$, the function $\Lambda(m)$ on positive integers is the well-known Mangoldt's function, see [Sect.6.1, Hua], which is useful in the study of primes. Here we gave another description of the function using the cyclotomic units. We mention that Ths.3.1 and 3.7 and Prop.3.2 imply that f^+ can not be completed to a non-punctured distribution.

We have rank $F^{0+}_{\mathfrak{m}} = |G^+_{\mathfrak{m}}| + s - t^+$ for all $\mathfrak{m} \neq \mathfrak{e}$ by Eq.(3.5) and Th.3.7. Furthermore, by Prop.3.4, Lem.3.6 and this equality, we get

Theorem 3.8. rank $A^{0+}_{\mathfrak{m}} = |G^+_{\mathfrak{m}}| + s - t^+$ *and* rank $A^{0-}_{\mathfrak{m}} = (|G_{\mathfrak{m}}| - t) - (|G^+_{\mathfrak{m}}| - t^+)$. *The distributions F^{0+} and F^{0-} are universal punctured even and odd distributions with values in torsion free abelian groups, respectively.*

We also have rank $A^0_{\mathfrak{m}} = |G_{\mathfrak{m}}| + s - t$ and that $F^0 = F^{0+} + F^{0-}$ is a universal punctured distribution up to torsion. By Lemma 1.3 $I^{0\epsilon}$ and $F^{0\epsilon}$ for $\epsilon \neq 0$ are universal ϵ-distributions up to torsion. The following result is a direct corollary of the theorem.

Corollary 3.9. *Let* $\mathbf{a}= \sum_i n_i[\mathfrak{u}_i]$ *be an element in the free abelian group generated by $(\bar T_0 \setminus T_0)/\sim$. Let $\mathfrak{m}$ be the lcm of the fractional parts of the $\mathfrak{u}_i$. Then* $\mathbf{a} \in$ *tor*$(A^{0\pm})$ *if and only if $\sum_i n_i f^\pm(\mathfrak{b}\mathfrak{u}_i) = 0$ for all integral ideals $\mathfrak{b}$ coprime to $\mathfrak{m}$, respectively.*

We have conjectured that f^+ is a universal punctured even distribution up to torsion [Conj.3.1, BGY]. It requires the equality rank $f^+_{\mathfrak{m}} =$ rank $F^{0+}_{\mathfrak{m}}$, where $f^+_{\mathfrak{m}}$ is the level $\mathfrak{m}$ group of f^+. As the discussion at the end of last section, the conjecture is equivalent to the following conjecture on Galois action of some values of f^+.

Conjecture 3.10. *Assume k is a number field. Let $\mathfrak{e} \neq \mathfrak{m} \in T_0$. If $\sum_i n_i f^+(\mathfrak{a}_i\mathfrak{m}^{-1}) = 0$, where $n_i \in \mathbb{Z}$ and $\mathfrak{m} \nmid \mathfrak{a}_i$, then $\sum_i n_i f^+(\mathfrak{a}_i\mathfrak{b}\mathfrak{m}^{-1}) = 0$ for all integral ideals $\mathfrak{b}$ coprime to $\mathfrak{m}$.*

This conjecture would imply that the complex numbers $f^+(\mathfrak{a}\mathfrak{p}^{-1})$ for all primes $\mathfrak{p}$ and all integral ideals $\mathfrak{a}$ with $\mathfrak{p} \nmid \mathfrak{a}$ are $\mathbb{Q}$-linear independent. In fact, if $\sum_i n_i f^+(\mathfrak{a}_i\mathfrak{p}_i^{-1}) = 0$, where $n_i \in \mathbb{Z}$, $\mathfrak{p}_i \nmid \mathfrak{a}_i$ and $\mathfrak{p}_i$ are diferent primes, we set $\mathfrak{m} = \prod_i \mathfrak{p}_i$. By the conjecture, we have $\sum_i n_i f^+(\mathfrak{b}\mathfrak{a}_i\mathfrak{p}_i^{-1}) = 0$ for all $\mathfrak{b}$ coprime to $\mathfrak{m}$, which shows

$$\sum_{\mathfrak{b}\in G_{\mathfrak{m}}} \sum_i n_i f^+(\mathfrak{b}\mathfrak{a}_i\mathfrak{p}_i^{-1}) = \sum_i n_i \frac{|G_{\mathfrak{m}}|}{|G_{\mathfrak{p}_i}|} \sum_{\mathfrak{b}\in G_{\mathfrak{p}_i}} f^+(\mathfrak{b}\mathfrak{p}_i^{-1}) = 0,$$

and thus $n_i = 0$ by Th.3.7.

In some sense, Stark's conjecture [St] (or its generalization by Rubin [Ru]) predicts the existence of global units over k encoded in the special values of L-functions of k. Theorem 3.7 implies evidence for this conjecture. If conjecture 3.10 is right, then the level $\mathfrak{m}$ group $f_{\mathfrak{m}}^+$ of f^+ has rank $|G_{\mathfrak{m}}^+| + s - t^+$. Here s comes from the part $s(G_{\mathfrak{m}})f_{\mathfrak{m}}^+$ and $|G_{\mathfrak{m}}^+| - t^+$ from the part $(|G_{\mathfrak{m}}| - s(G_{\mathfrak{m}}))f_{\mathfrak{m}}^+$. The latter encodes global units of $K_{\mathfrak{m}}$. It is reasonable to conjecture that $f_{\mathfrak{m}}^+$ encodes a unit subgroup of $K_{\mathfrak{m}}$ of rank $|G_{\mathfrak{m}}^+| - t^+$. In the cases $k = \mathbb{Q}$, imaginary quadratic and a function field, we will see this fact later.

Are there some $\epsilon \neq 0$ such that $f^{0\epsilon}$ are universal punctured ϵ-distributions up to torsion? The following result transfers this question to Question 2.11 in last section.

Proposition 3.11. *Assume $\epsilon \neq 0$ and u is a universal ϵ-distribution up to torsion. Then the punctured reduction u^0 of u is a universal punctured ϵ-distribution up to torsion.*

Proof. Assume that u takes values in the group V. Composing u with the homomorphisms

$$V \longrightarrow V/\mathrm{tor}(V) \longrightarrow (V/\mathrm{tor}(V)) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{3.6}$$

we get a distribution $\bar{u}$. Let $\bar{u}^0$ be the punctured reduction of $\bar{u}$. The ranks of level $\mathfrak{m}$ groups of u and $\bar{u}$ are equal since the second map in (3.6) is injective. The same is true for u^0 and $\bar{u}^0$. Thus we can assume $V \subset \mathbb{C}$. Let $U = \mathrm{St}(u)$ and $U^0 = \mathrm{St}(u^0)$. The latter is the punctured reduction of the former. Let $u_{\mathfrak{m}}^0$ and $U_{\mathfrak{m}}^0$ be their respective level $\mathfrak{m}$ group. By the proof of Lem.2.7 we have $U_{\mathfrak{m}}^0 = \omega_{\mathfrak{m}} I_{\mathfrak{m}}^{0\epsilon}$. The universality of u implies that of U. Thus $\chi(\omega_{\mathfrak{m}}) \neq 0$ for all ϵ-characters χ of $G_{\mathfrak{m}}$. This fact implies $\mathrm{rank} U_{\mathfrak{m}}^0 = \mathrm{rank} I_{\mathfrak{m}}^{0\epsilon}$ and thus U^0 is universal up to torsion.

If $\sum_i n_i u^0(\mathfrak{a}_i\mathfrak{m}^{-1}) (= \sum_i n_i u(\mathfrak{a}_i\mathfrak{m}^{-1})) = 0$, where $n_i \in \mathbb{Z}$ and $\mathfrak{m} \nmid \mathfrak{a}_i$, then $\sum_i n_i[\mathfrak{a}_i\mathfrak{m}^{-1}] \in \mathrm{tor}(A_{\mathfrak{m}}^{\epsilon})$ since u is universal up to torsion. Thus $\sum_i n_i u(\mathfrak{a}_i\mathfrak{b}\mathfrak{m}^{-1}) = \sum_i n_i u^0(\mathfrak{a}_i\mathfrak{b}\mathfrak{m}^{-1}) = 0$ for all $\mathfrak{b}$ coprime to $\mathfrak{m}$ as $\mathrm{tor}(A_{\mathfrak{m}}^{\epsilon})$ is an $G_{\mathfrak{m}}$-submodule of $A_{\mathfrak{m}}^{\epsilon}$. We get $\mathrm{rank} u_{\mathfrak{m}}^0 = \mathrm{rank} U_{\mathfrak{m}}^0$. This completes the proof.

Finally, we consider the conjecture in the cases when $k = \mathbb{Q}$ and is imaginary quadratic. In these cases, f^+ has an obvious expression. In fact, we have $f^+(\mathfrak{a}\mathfrak{m}^{-1}) = \log|\Phi_{\mathfrak{m}}(\mathfrak{a})|$ up to a constant factor, where $\Phi_{\mathfrak{m}}(\mathfrak{a}) = 1 - \exp(2\pi a i/m)$ if $\mathfrak{a}\mathfrak{m}^{-1} = \frac{a}{m}\mathbb{Z}$, when $k = \mathbb{Q}$, and $\Phi_{\mathfrak{m}}(\mathfrak{a})$ is the Remachandra invariant of the ideal class $\mathfrak{a} \bmod \mathfrak{m}$, see [Sects.4-6, Ra] or [Sect.7, GR], when k is imaginary quadratic. Notice that $\Phi_{\mathfrak{m}}(\mathfrak{a})$ is an algebraic integer in the ray class field $K_{\mathfrak{m}}$. The following well-known result is a direct conclusion of theorem 3.7.

Corollary 3.12. *Assume $\mathfrak{a}, \mathfrak{m}$ are coprime. Then $\Phi_{\mathfrak{m}}(\mathfrak{a})$ is a unit in $K_{\mathfrak{m}}$ if $\mathfrak{m}$ has two or more prime factors and is a $\mathfrak{p}$-unit if $\mathfrak{m}$ is a power of the prime $\mathfrak{p}$.*

The conjecture in these two cases can be proved easily by the explicit reciprocity law. Recall that $\Phi_{\mathfrak{m}}(\mathfrak{a})^{\sigma_{\mathfrak{b}}} = \Phi_{\mathfrak{m}}(\mathfrak{a}\mathfrak{b})$, where $\sigma_{\mathfrak{b}} \in G_{\mathfrak{m}}^+$ is the Artin symbol of the integral ideal $\mathfrak{b}$. If $\sum_i n_i f^+(\mathfrak{a}_i\mathfrak{m}^{-1}) = 0$, where $n_i \in \mathbb{Z}, \mathfrak{m} \nmid \mathfrak{a}_i$, we have $\prod_i |\Phi_{\mathfrak{m}}(\mathfrak{a}_i)|^{n_i} = 1$. The explicit reciprocity law implies that $\sum_i n_i f^+(\mathfrak{a}_i\mathfrak{b}\mathfrak{m}^{-1}) = 0$ for all integral ideals $\mathfrak{b}$ coprime to $\mathfrak{m}$. Thus our conjecture could be regarded as a weak analytic version of the explicit reciprocity law. The subgroup of $K_{\mathfrak{m}}^*$ generated by $\Phi_{\mathfrak{m}}(\mathfrak{a})$ with all $\mathfrak{a} \in T_0$ has rank $|G_{\mathfrak{m}}^+| + s - t^+$ by Th.3.7 and the unit subgroup in it has rank $|G_{\mathfrak{m}}^+| - t^+$ by Lem.3.6. We showed

Theorem 3.13. *Assume $k = \mathbb{Q}$ or imaginary quadratic. Then conjecture 3.10 is valid, and thus f^+ is a universal punctured even distribution up to torsion. Furthermore, the subgroup of $K_{\mathfrak{m}}^*$ generated by $\Phi_{\mathfrak{m}}(\mathfrak{a})$ with $\mathfrak{a} \in T_0$ contains a unit subgroup of $K_{\mathfrak{m}}$ of rank $|G_{\mathfrak{m}}^+| - t^+$.*

When $k = \mathbb{Q}$, this is the Bass' theorem [Ba] (conjectured by Milnor) in the classical cyclotomic theory. But Bass did not consider the torsion. The fact that 2-torsion must be considered in Bass' theorem was first recognized by Ennola [En]. The torsion of $A^{0\pm}_{\mathfrak{m}}$ in the rational case has been determined by Schmidt [Sc]. When k is imaginary quadratic, the result here is new. For the torsion of $A^{0+}_{\mathfrak{m}}(=A^0_{\mathfrak{m}})$ in this case, we refer the reader to [BO]. We will give the analogue result in the function field case in next section.

4. Division points of rank one Drinfeld modules.

In this last section, we consider the function field case. We show that the level $\mathfrak{m}$ group of the universal punctured even distribution also has rank $|G^+_{\mathfrak{m}}|+s-t^+$ by the division points of sign-normalized rank one Drinfeld modules.

Assume that k is the function field of a projective smooth irreducible curve $\mathcal{C}$ over the finite field $\mathbb{F}_q$ of q elements. Fix a closed point $\infty\in\mathcal{C}$. Let $\mathbb{A}$ be the ring of entire functions on $\mathcal{C}\setminus\{\infty\}$. Let k_∞ be the completion of k at ∞. Fix a sign-function sgn: $k^*_\infty\to\mathbb{F}^*_\infty$, where $\mathbb{F}_\infty$ is the residue class field of k at ∞. For the definition of sign-functions, we refer the reader to [Def.2.1 in Ch.4, Ge]. An element $x\in k$ is called (totally) positive if $\operatorname{sgn}(x)=1$. The sign-subgroup J of k is isomorphic to $\mathbb{F}^*_\infty$. All the concepts about distributions in sections 1-2 can be made for the triple $(k,\infty,\operatorname{sgn})$. But in this case, we only classify the distributions into even and odd parts since k has only one "real" place ∞. We remark that we can also fix a finite set $\{\infty_1,\cdots,\infty_r\}$ of places of k and fix sign-functions $\{\operatorname{sgn}_1,\cdots,\operatorname{sgn}_r\}$ at each place ∞_i. Let $\mathbb{A}$ be the Dedekind subring of k of functions regular outside of all ∞_i. Then all the concepts before about distributions can be extended to the general case. Now the ϵ should be a vector in the space $\mathbb{F}^*_{\infty_1}\times\cdots\times\mathbb{F}^*_{\infty_r}$. In this paper we only consider the simple case of a single place.

By Weil's theorem or the Riemann-Roch theorem, the function f^+, which is defined in the same way as that in the number field case, takes rational values up to the factor $\log q$. Thus the values $\sum_{a\in G_{\mathfrak{p}}}f^+(\mathfrak{a}\mathfrak{p}^{-1})$ for primes $\mathfrak{p}$ are not $\mathbb{Q}$-independent. Also Conjecture 3.9 is not meaningful in the function field case. However, we will see that the function f^+ encodes a (mutiplicative) even distribution we expect. We first recall the division points of sgn-normalized rank one Drinfeld modules.

Let C_k be the completion of an algebraic closure of k_∞. A rank 1 Drinfeld $\mathbb{A}$-module ρ of generic characteristic is called sgn-normalized, if for all $x\in\mathbb{A}$ the coefficient $\mu_\rho(x)$ of the highest term of ρ_x is a twisting of the sign function sgn, i.e., there exists $\sigma\in\operatorname{Gal}(\mathbb{F}_\infty/\mathbb{F}_q)$ such that $\mu_\rho=\sigma\circ\operatorname{sgn}$. For $\mathfrak{u}\in\bar{T}_0$, let $\xi(\mathfrak{u})\in C_k$ be the ξ-invariant of $\mathfrak{u}$, which is characterized by the condition that the rank one $\mathbb{A}$-lattice $\xi(\mathfrak{u})\mathfrak{u}$ corresponds to some sgn-nomalized rank one Drinfeld $\mathbb{A}$-module. By a general result of Yu, we know that $\xi(\mathfrak{u})$ is transcendence over k [Yu]. Let $e_{\mathfrak{u}}(z)=z\prod_{0\neq x\in\mathfrak{u}}(1-z/x)$ be the exponential function associated to $\mathfrak{u}$. Hayes [Sect.5, Ha2] showed that the set of $\mathfrak{m}$-division points of ρ, where ρ corresponds to the lattice $\xi(\mathfrak{m})\mathfrak{m}$, is equal to $\xi(\mathfrak{m})e_{\mathfrak{m}}(\mathbb{A})$ and $\lambda_{\mathfrak{m}}=\xi(\mathfrak{m})e_{\mathfrak{m}}(1)$ is a generator. The Galois action is $\lambda^{\sigma_{\mathfrak{a}}}_{\mathfrak{m}}=\xi(\mathfrak{a}^{-1}\mathfrak{m})e_{\mathfrak{a}^{-1}\mathfrak{m}}(1)$, where $(\mathfrak{a},\mathfrak{m})=\mathfrak{e}$. Hayes [Ha2, Thm 6.1] also showed that, for $\mathfrak{u}\in\bar{T}_0\setminus T_0$,

$$f^+(\mathfrak{u})=\deg\infty\cdot\log q\cdot v_\infty(\xi(\mathfrak{u}^{-1})e_{\mathfrak{u}^{-1}}(1)),$$

where v_∞ is the extension to C_k of the normalized valuation of k_∞ at ∞. Thus the valuations of these division points at ∞ satisfy the distribution relations. In fact, the division points themselves satisfy the relations. Define $g^+:\bar{T}_0\setminus T_0\to C^*_k/\operatorname{tor}(C^*_k)$ by $g^+(\mathfrak{u})=\xi(\mathfrak{u}^{-1})e_{\mathfrak{u}^{-1}}(1)$. Then g^+ is a (multiplicative) punctured even distribution, see [Prop.7.1, BGY].

We next consider the rank of the level $\mathfrak{m}$ group $g^+_{\mathfrak{m}}$ of g^+. By the definition, it is generated by $\xi(\mathfrak{a}^{-1}\mathfrak{m})e_{\mathfrak{a}^{-1}\mathfrak{m}}(1)$ with $\mathfrak{a}\in T_0,\mathfrak{m}\nmid\mathfrak{a}$. The group $g^+_{\mathfrak{m}}$ is nothing but the group of cyclotomic numbers of the cyclotomic extension $K_{\mathfrak{m}}$ of k of conductor $\mathfrak{m}$ defined in [Def.1.1, Yi2]. Let $C_{\mathfrak{m}}$ be the unit subgroup in $g^+_{\mathfrak{m}}$. We have the following exact sequence, see the proof of [Th.A, Yi2],

$$0\longrightarrow C_{\mathfrak{m}}\longrightarrow g^+_{\mathfrak{m}}\longrightarrow\mathbb{Z}^s\longrightarrow 0, \tag{4.1}$$

where s is the number of distinct prime divisors of $\mathfrak{m}$. In [Th.A, Yi2] we have showed $\operatorname{rank}C_{\mathfrak{m}}=|G^+_{\mathfrak{m}}|-t^+$. Thus $\operatorname{rank}g^+_{\mathfrak{m}}=|G^+_{\mathfrak{m}}|+s-t^+$. The sequence (4.1) comes from the fact that $\lambda_{\mathfrak{p}}^{s(G_{\mathfrak{p}})}$ are $\mathbb{Z}$-linear independent (in multiplicative) for all primes $\mathfrak{p}$. This is a multiplicative version of Th.3.7.

Let f^- be the function defined in section 2 by taking $\epsilon=1$. We know that it is a punctured odd distribution. Let F^{0-} be the corresponding Stickelberger distribution with values in Ω. The level $\mathfrak{m}$ group $F^{0-}_{\mathfrak{m}}$ is the Stickelberger ideal defined in [Yi3] and has rank $(|G_{\mathfrak{m}}|-t)-(|G^+_{\mathfrak{m}}|-t^+)$ by Lem.3.6. We have by Prop.3.4

Theorem 4.1. $\mathrm{rank} A_{\mathfrak{m}}^{0+} = |G_{\mathfrak{m}}^+| + s - t^+$ *and* $\mathrm{rank} A_{\mathfrak{m}}^{0-} = (|G_{\mathfrak{m}}| - t) - (|G_{\mathfrak{m}}^+| - t^+)$.

We mention that g^+ satisfies the multiplicative version of the conjecture 3.9. In fact, if $\prod_i g^+(\mathfrak{a}_i\mathfrak{m}^{-1})^{n_i} = 1$, the explicit reciprocity law implies $\prod_i g^+(\mathfrak{a}_i\mathfrak{b}\mathfrak{m}^{-1})^{n_i} = 1$ for any integral ideal $\mathfrak{b}$ coprime to $\mathfrak{m}$.

Let $F^0 = g^+ \oplus F^{0-}$. Let $F_{\mathfrak{m}}^0$ be the level $\mathfrak{m}$ group. It is a free abelian group of rank $|G_{\mathfrak{m}}| + s - t$. We claim that F^0 is universal with values in abelian groups on which $|G_{\mathfrak{e}}|$ is invertible. Let $g : \bar{T}_0 \setminus T_0 \to V$ be a punctured distribution. Assume $|G_{\mathfrak{e}}|$ is invertible on V. By Prop. 3.2 the condition $\sum_{\mathfrak{a}\in G_{\mathfrak{p}}} g(\mathfrak{a}\mathfrak{p}^{-1}) = 0$ for all primes $\mathfrak{p}$ implies the condition $\sum_{0\neq x\in w(\mathbb{A}/\alpha\mathbb{A})} g(x\alpha^{-1}\mathbb{A}) = 0$ for all totally positive $\alpha \in \mathbb{A} \setminus U^+$. Thus the exact sequence (3.4) is also valid if we replace $W_{\mathfrak{m}}$ by $R_{\mathfrak{m}}$. Since there exist $|G_{\mathfrak{m}}|$ elements which generate $A_{\mathfrak{m}}$ [Prop. 4.1, Yi1], $g_{\mathfrak{m}}$ requires at most $|G_{\mathfrak{m}}| + s - t$ generators and thus is a quotient of $F_{\mathfrak{m}}^0$. This shows the claim. Thus we have

Proposition 4.2. $A_{\mathfrak{m}}^0$ *has rank* $|G_{\mathfrak{m}}| + s - t$. *If prime* p *does not divide* $|G_{\mathfrak{e}}|$, *there is no* p*-torsion in* $A_{\mathfrak{m}}^0$.

In the theory of global distributions we developed in this paper, a basic question unsolved is how to determine the torsion of various kinds of universal level groups of a global field k. The results will describe the arithmetic, especially the unit groups, of the ray class fields of k. Up to now, we have truly satisfactory answer on this question only in the case $k = \mathbb{Q}$, see [Ya], [Sin], [Ku2], [An], [Sc] and [Th.12.18, Wa], and we have partial results in the two cases when k is imaginary quadratic and when k is a fuction field, see [BO] and [An] respectively. For a general number field, the author thinks that it is a hard question.

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