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On the Hilbert-Samuel multiplicity of Fredholm tuples

Jörg Eschmeier[1]

Abstract. For commuting tuples $R \in L(Z)^n$ of Banach-space operators that arise as quotients of lower semi-Fredholm systems $T \in L(X)^n$ with constant cohomology dimension $\dim H^n(z - T, X)$ near the origin $0 \in \mathbb{C}^n$, we show that the Hilbert-Samuel multiplicity of R calculates the rank of the cohomology sheaf $\mathcal{H}^n(z - R, \mathcal{O}^Z_{\mathbb{C}^n})$ at $z = 0$.

0. **Introduction**

Let $T \in L(X)^n$ be a commuting tuple of bounded linear operators on a complex Banach space X, and let $K^\bullet(T, X)$ be the Koszul complex of T. The tuple T is said to be *lower semi-Fredholm* if the last cohomology group $H^n(T, X) = X/\sum_{i=1}^n T_i X$ of its Koszul complex is finite dimensional. In this case all the spaces $M_k(T) = \sum_{|\alpha|=k} T^\alpha X \quad (k \in \mathbb{N})$ are finite codimensional, and the direct sum $\oplus_{k\geq 0} M_k(T)/M_{k+1}(T)$ can be turned into a graded finitely generated $\mathbb{C}[z]$-module. It is a fundamental result of commutative algebra that to any such module there is a polynomial $p \in \mathbb{Q}[x]$ of degree $\leq n$, the *Hilbert-Samuel polynomial*, with $\dim X/M_k(T) = p(k)$ for large k and such that the leading coefficient multiplied with $n!$ is a natural number, the so-called *Hilbert-Samuel multiplicity*.

On the other hand, for a given lower semi-Fredholm tuple $T \in L(X)^n$, there is an open neighbourhood U of $0 \in \mathbb{C}^n$ such that $\dim H^n(z - T, X) < \infty$ for all $z \in U$ and such that the last cohomology sheaf $\mathcal{H} = \mathcal{H}^n(z - T, \mathcal{O}^X_U)$ of the induced complex $K^\bullet(z - T, \mathcal{O}^X_U)$ of $\mathcal{O}_U$-modules is isomorphic to a quotient of a free module $\mathcal{O}^N_U$ on U. In particular, the stalk $\mathcal{H}_0$ is a noetherian module over the local ring $\mathcal{O}_0$ of all convergent power series at $z = 0$ and hence possesses a Hilbert-Samuel polynomial $p_{an} \in \mathbb{Q}[x]$ such that $\dim \mathcal{H}_0/\mathfrak{m}^k\mathcal{H}_0 = p_{an}(k)$ for large k. Here $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_0$.

Both versions of the Hilbert-Samuel polynomial and their leading coefficients were introduced by Douglas and Yan in [3]. In a series of papers Xiang Fang studied the properties of the Hilbert-Samuel multiplicity

$$c(T) = n! \lim_{k\to\infty} \frac{\dim X/M_k(T)}{k^n}$$

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and its relations to other invariants in operator theory. In [6] a complete description of the relation between Fredholm index and the Hilbert-Samuel multiplicity for single operators is given, while [8] contains a variety of results showing the close connection between Hilbert-Samuel multiplicity and the Fredholm index in multivariable operator theory, including the observation that, for row contractions with finite defect, the Hilbert-Samuel multiplicity coincides with Arveson's curvature invariant [1].

In this note we extend a method originating in [7] to show that, for a suitable class of lower semi-Fredholm tuples $R \in L(Z)^n$ on a Banach space Z, the Hilbert-Samuel multiplicity coincides with the rank of the cohomology sheaf $\mathcal{H} = \mathcal{H}^n(z - R, \mathcal{O}_U^Z)$ at $z = 0$, or equivalently, that there are an open neighbourhood U of 0 and a proper analytic subset $S \subset U$ such that

$$c(R) = \min_{z \in U} \dim H^n(z - R, Z) = \dim H^n(w - R, Z)$$

for $w \in U \setminus S$. Furthermore, the restriction of $\mathcal{H}$ to $U \setminus S$ is a locally free analytic sheaf of rank $c(R)$. These results are shown to hold for tuples $R \in L(Z)^n$ arising as quotients of lower semi-Fredholm tuples $T \in L(X)^n$ which are regular in the sense that the cohomology groups $H^n(z - T, X)$ have constant dimension near 0. Hence, in this setting, we prove that the Hilbert-Samuel multiplicity calculates the stabilized dimension of $H^n(z - R, Z)$ at $z = 0$, and give a natural explanation of results previously known in more particular situations.

1. **Main results**

Let $T = (T_1, \ldots, T_n) \in L(X)^n$ be a commuting tuple of continuous linear operators on a complex Banach space X. The Koszul complex of T

$$K^\bullet(T, X): \ 0 \to \Lambda^0 X \xrightarrow{T} \Lambda^1 X \xrightarrow{T} \ldots \xrightarrow{T} \Lambda^n X \to 0$$

is a finite complex of bounded operators between Banach spaces (see §2.2 in [4]). The vector spaces

$$H^p(T, X) = \mathrm{Ker}(\Lambda^p X \xrightarrow{T} \Lambda^{p+1} X)/\mathrm{Im}(\Lambda^{p-1} X \to \Lambda^p X) \quad (p = 0, \ldots, n)$$

are called the cohomology groups of $K^\bullet(T, X)$.

Let us suppose that the space

$$H^n(T, X) \cong X / \sum_{i=1}^n T_i X$$

is finite dimensional. Fix a basis $([x_1],\dots,[x_N])$ of this space. Then the spaces $M_k = \sum_{|\alpha|=k} T^\alpha X$ $(k \in \mathbb{N})$ form a decreasing sequence of finite-codimensional subspaces of X such that

$$M_{k+1} = \sum_{j=1}^{n} T_j M_k \quad (k \in \mathbb{N}).$$

On the algebraic direct sum $\overline{X} = \bigoplus_{k=0}^{\infty} (M_k/M_{k+1})$ we define a commuting tuple $\overline{T} = (\overline{T}_1,\dots,\overline{T}_n)$ of linear maps $\overline{T}_j : \overline{X} \to \overline{X}$ by

$$\overline{T}_j\big((x_k + M_{k+1})_{k\in\mathbb{N}}\big) = (T_j x_{k-1} + M_{k+1})_{k\in\mathbb{N}}.$$

It is elementary to check that the induced module structure

$$\mathbb{C}[z] \times \overline{X} \to \overline{X}, \quad (p,x) \mapsto p(\overline{T})x$$

turns $\overline{X}$ into a finitely generated graded $\mathbb{C}[z]$-module. More precisely, let $\mathcal{V}_k = \{p \in \mathbb{C}[z];\ \deg(p) \le k\}$. Then one can show that

$$\bigoplus_{j=0}^{k-1} M_j/M_{j+1} = \mathrm{LH}\big(\{p(\overline{T})(x_i + M_1);\ p \in \mathcal{V}_{k-1} \text{ and } i = 1,\dots,N\}\big)$$

for all $k \ge 1$. In particular, one obtains the estimates

$$\dim(X/M_k) \le N \ \dim\, \mathcal{V}_{k-1} = N\ \frac{k(k+1)\cdot\ldots\cdot\big(k+n-1\big)}{n!}$$

for all $k \ge 0$. Using a theorem going back to Hilbert (Theorem 1.11 in [5]) we conclude that there is a polynomial $p \in \mathbb{Q}[x]$ with $\deg(p) \le n$ such that

$$\dim(X/M_k) = p(k)$$

for all sufficiently large natural numbers k. Furthermore, in this case the limit

$$c(T) = n! \lim_{k\to\infty} \dim(X/M_k)/k^n$$

exists and defines a natural number $c(T) \in \{0,1,2,\dots,N\}$. We call p the *Hilbert-Samuel polynomial* and $c(T)$ the *Hilbert-Samuel multiplicity* of T.

The condition that $H^n(T,X)$ is finite dimensional implies that the spaces $H^n(z-T,X) \cong X/\sum_{i=1}^n (z_i - T_i)X$ are finite dimensional for all points z in a suitable open neighbourhood U of 0 in $\mathbb{C}^n$ (Section 2.6 in [4]). For a Banach space E, we denote by $\mathcal{O}_U^E$ the analytic sheaf of all germs of analytic E-valued functions on U. The boundary maps in the Koszul complexes $K^\bullet(z-T,X)$ depend analytically on z and induce a corresponding sequence

$$K^\bullet(z-T,\mathcal{O}_U^X) : 0 \longrightarrow \mathcal{O}_U^{\Lambda^1 X} \xrightarrow{z-T} \dots \xrightarrow{z-T} O_U^{\Lambda^n X} \longrightarrow 0$$

of analytic sheaves on U. Let us denote by

$$\mathcal{F} = \mathcal{F}_T = \mathcal{H}^n(z - T, \mathcal{O}_U^X) \cong \mathcal{O}_U^X/(z-T)\mathcal{O}_U^{X^n}$$

its last cohomology sheaf. Let $([x_1], \ldots, [x_r])$ be a basis of $H^n(T, X)$. After shrinking U one may suppose that there is an epimorphism

$$\mathcal{O}_U^r \xrightarrow{h} \mathcal{F}$$

of analytic sheaves (see the proof of Proposition 9.4.5 in [4]). It follows that the stalk $\mathcal{F}_0$ of $\mathcal{F}$ at $z = 0$ is a noetherian module over the local ring $\mathcal{O}_0$ of all convergent power series at $z = 0$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_0$. It is well known that there is a polynomial $p_{an} \in \mathbb{Q}[x]$ with $\deg(p) \leq n$ such that

$$\dim_{\mathbb{C}}(\mathcal{F}_0/\mathfrak{m}^k\mathcal{F}_0) = p_{an}(k)$$

for all sufficiently large natural numbers k. Again the limit

$$c_{an}(T) = n! \lim_{k\to\infty} \dim(\mathcal{F}_0/\mathfrak{m}^k\mathcal{F}_0)/k^n$$

exists and defines a natural number $c_{an}(T) \in \{0, 1, 2, \ldots\}$. We call p_{an} the *analytic Hilbert-Samuel polynomial* and $c_{an}(T)$ the *analytic Hilbert-Samuel multiplicity* of T.

The relationship between the invariants $p, c(T)$ and their analytic counterparts p_{an}, $c_{an}(T)$ was studied before by Douglas and Yan [3], although in a slightly different language. We recall the main results.

Define $M = \{p \in \mathbb{C}[z];\ p(0) = 0\}$. Regard X as a $\mathbb{C}[z]$-module via

$$\mathbb{C}[z] \times X \to X, \quad (p, x) \mapsto p(T)x.$$

It is elementary to check that $M^kX = \sum_{|\alpha|=k} T^\alpha X$ and that the map

$$\varphi : X \to \mathcal{F}_0, \quad x \mapsto x + (z-T)\mathcal{O}_0^{X^n}$$

is a morphism of $\mathbb{C}[z]$-modules such that $\varphi(M^kX) \subset \mathfrak{m}^k\mathcal{F}_0$ for all $k \geq 0$. One can show (cf. Proposition 5 in [3]) that the induced maps

$$\varphi_k : X/M^kX \to \mathcal{F}_0/\mathfrak{m}^k\mathcal{F}_0, \quad x + M^kX \mapsto \varphi(x) + \mathfrak{m}^k\mathcal{F}_0$$

are onto for all $k \geq 0$.

Proposition 1 *(Douglas-Yan)* *Let $T \in L(X)^n$ be a commuting tuple such that $\dim H^n(T, X) < \infty$. Let p and p_{an} be the Hilbert-Samuel polynomial and analytic Hilbert-Samuel polynomial of T, respectively. Then*

$$p_{an}(k) \leq p(k)$$

for all sufficiently large natural numbers k. In particular, the inequality $c_{an}(T) \leq c(T)$ holds.

Let $Z = X/Y$ be the quotient of X modulo a closed invariant subspace Y of T. We denote by $S = T|Y$ and $R = T/Y$ the restriction of T to Y and the quotient tuple of T modulo Y, respectively. The short exact sequences

$$0 \to K^\bullet(z - S, Y) \to K^\bullet(z - T, X) \to K^\bullet(z - R, Z) \to 0$$

($z \in \mathbb{C}^n$) of Koszul complexes induce long exact sequences of cohomology

$$\begin{array}{cccccc}
0 & \longrightarrow & H^0(z-S,Y) & \xrightarrow{j} & H^0(z-T,X) & \xrightarrow{q} & H^0(z-R,Z) \\
 & \xrightarrow{d_z} & H^1(z-S,Y) & \xrightarrow{j} & \ldots & & \ldots \\
 & & \ldots & & \ldots & & \ldots \\
 & \xrightarrow{d_z} & H^n(z-S,Y) & \xrightarrow{j} & H^n(z-T,X) & \xrightarrow{q} & H^n(z-R,Z) \longrightarrow 0,
\end{array}$$

where j and q are the linear maps induced by the inclusion $Y \hookrightarrow X$ and the quotient map $X \to Z$ and the maps d_z are the connecting homomorphisms (cf. [5]).

Our aim is to study the relation between the Hilbert-Samuel multiplicities of T and R. We shall clarify this relation under the additional hypothesis that there is an open neighbourhood U of $0 \in \mathbb{C}^n$ such that all the spaces

$$H^n(z - T, X) \cong X / \sum_{i=1}^{n} (z_i - T_i) X$$

have the same finite dimension $N \geq 1$. It is well known that the set S of all discontinuity points of the function

$$U \to \mathbb{N}, \quad z \mapsto \dim H^n(z - R, Z)$$

is a nowhere dense analytic subset of U (Satz 1.5 in [10]) and that there is a natural number $r \in \{0, \ldots, N\}$ with

$$\dim H^n(z - R, Z) = r < \dim H^n(w - R, Z)$$

for $z \in U \setminus S$ and $w \in S$.

It is the main aim of this note to prove the following result.

Theorem 2 *Let $R \in L(Z)^n$ be the quotient of a commuting tuple $T \in L(X)^n$ such that*

$$\dim H^n(z - T, X) \equiv \text{const.} < \infty$$

near zero. Then there is nowhere dense analytic subset $S \subset U$ of an open neighbourhood U of $0 \in \mathbb{C}^n$ with the property that

$$c(R) = \dim H^n(z - R, Z) \quad \textit{for } z \in U \setminus S.$$

To prove this result we need some preparations. Let U be a connected open neighbourhood of $0 \in \mathbb{C}^n$ such that

$$\dim H^n(z - T, X) = N \quad (z \in U).$$

Choose a direct complement D of $\sum_{i=1}^n T_i X$ in X. Since the analytically parametrized complex

$$T(z) : X^n \oplus D \to X, \quad \big((x_i), y\big) \mapsto \sum_{i=1}^n (z_i - T_i)x_i + y$$

is onto at $z = 0$, we can achieve (by shrinking U) that the induced map

$$\mathcal{O}(U, X^n \oplus D) \to \mathcal{O}(U, X)$$

is onto again (Lemma 2.1.5 in [4]). By comparing dimensions we see that the surjective linear maps

$$D \to H^n(z - T, X), \quad y \mapsto [y]$$

are vector-space isomorphisms for all $z \in U$. Hence, for each $x \in X$ and each $z \in U$, there is a unique vector $x(z) \in D$ with

$$x - x(z) \in \sum_{i=1}^n (z_i - T_i)X.$$

Since $x(z)$ depends analytically on z, we obtain a linear map

$$\rho : X \to \mathcal{O}(U, D), \quad (\rho x)(z) = x(z).$$

The relation

$$T_j x - z_j x(z) \in (T_j - z_j)x(z) + \sum_{i=1}^n (z_i - T_i)X,$$

valid for $x \in X$, $z \in U$ and $j = 1, \ldots, n$, shows that

$$\rho\big(p(T)x\big) = p\rho(x) \quad (x \in X, p \in \mathbb{C}[z]).$$

By Lemma 2.1.5 in [4], the morphism

$$\mathcal{O}_U^{X^n \oplus D} \xrightarrow{T(\cdot)} \mathcal{O}_U^X$$

of analytic sheaves remains onto. Therefore the map

$$\mathcal{O}_U^D \to \mathcal{F}, \quad f_\lambda \mapsto f_\lambda + (z - T)\mathcal{O}_\lambda^{X^n}$$

defines an isomorphism of analytic sheaves.

Let as before $S \subset U$ be a nowhere dense analytic subset such that

$$\dim H^n(z - R, Z) = r$$

for all $z \in U \setminus S$. The above arguments applied to R instead of T show that the sheaf $\mathcal{F}_R$ is locally free of rank r on $U \setminus S$. We call r the *rank* of the sheaf $\mathcal{F}_R$ on U and write $\operatorname{rank}(\mathcal{F}_R) = r$.

Lemma 3 *Let $T \in L(X)^n$ be a commuting tuple such that there is an open neighbourhood U of 0 in $\mathbb{C}^n$ with $\dim H^n(z - T, X) = N$ for $z \in U$. Then we have $c(T) = c_{an}(T) = N$.*

Proof. Denote by $\mathcal{F} = \mathcal{H}^n(z - T, \mathcal{O}_U^X)$ the n-th cohomology sheaf of the complex $K^\bullet(z - T, \mathcal{O}_U^X)$. As seen above, there is an isomorphism $\psi : \mathcal{O}_0^N \longrightarrow \mathcal{F}_0$ of $\mathcal{O}_0$-modules. This map induces isomorphisms of $\mathcal{O}_0$-modules

$$\psi_k : \mathcal{O}_0^N / \mathfrak{m}^k \mathcal{O}_0^N \longrightarrow \mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0 \quad (k \geq 0).$$

Because of $\mathfrak{m}^k \mathcal{O}_0^N = (\mathfrak{m}^k \mathcal{O}_0)^N = (\mathfrak{m}^k)^N$ we obtain induced isomorphisms of $\mathcal{O}_0$-modules

$$\mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0 \cong \mathcal{O}_0^N / (\mathfrak{m}^k)^N \cong (\mathcal{O}_0 / \mathfrak{m}^k)^N.$$

Using the canonical vector-space isomorphisms $\mathcal{O}_0 / \mathfrak{m}^k \cong \mathcal{V}_{k-1}$, we find that

$$\dim_{\mathbb{C}}(\mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0) = \dim(\mathcal{V}_{k-1}^N) = N \, \frac{k(k+1) \cdot \ldots \cdot (k+n-1)}{n!}$$

for all $k \geq 0$. But then $N = c_{an}(T) \leq c(T) \leq N$, and the assertion follows.

$\square$

For $z \in U$, define

$$\delta_z : X \to D, \quad x \mapsto x(z)$$

and

$$Y_z = \{y(z);\ y \in Y\}.$$

Using the explicit definition of the connecting homomorphisms d_z occurring in the long exact cohomology sequences explained before Theorem 2 one easily checks that the sequences

$$H^{n-1}(z-R,Z) \xrightarrow{d_z} H^n(z-S,Y) \xrightarrow{\widehat{\delta_z}} Y_z \longrightarrow 0$$

are exact for $z \in U$. Using this sequence together with the above long exact cohomology sequence we find that

$$\dim H^n(z-R,Z) = N - \dim Y_z$$

for $z \in U$.

Hence the minimal dimension of $H^n(z-R,Z)$ on U corresponds to the maximal dimension of Y_z on U. The last number admits an algebraic representation (cf. Lemma 4 in [7]). To formulate the relevant result, let us denote by

$$T_k : \mathcal{O}(U,D) \to \mathcal{O}(U,D), \quad f \mapsto \sum_{|\alpha| \le k} \frac{f^{(\alpha)}(0)}{\alpha!}\, z^\alpha$$

the linear maps that associate with each analytic function on U its k-th Taylor polynomial.

Lemma 4 *Let $U \subset \mathbb{C}^n$ be an open neighbourhood of 0 and let D be a finite-dimensional vector space. Given a $\mathbb{C}[z]$-submodule $M \subset \mathcal{O}(U,D)$, define $M_z = \{f(z);\ f \in M\}$ for $z \in U$. Then there is a nowhere dense analytic subset $S \subset U$ such that*

$$\dim M_z = \max_{w \in U} \dim M_w = n! \lim_{k \to \infty} \frac{\dim T_k(M)}{k^n}$$

for all $z \in U \setminus S$.

Proof. It is elementary to check that the proof given in [7] (see Lemma 4) for submodules of Hilbert modules remains valid. For completeness sake, we indicate the main steps.

Fix a basis $(e_1, \dots, e_N)$ of D. Define $m = \max_{w \in U} \dim M_w$ and choose $h_1, \dots, h_m \in M$ such that $h_1(z_0), \dots, h_m(z_0)$ are linearly independent vectors in D for some point $z_0 \in U$. Each function h_j has a representation of the form

$$h_j(z) = \sum_{i=1}^{N} h_j^i(z) e_i \qquad (z \in U)$$

with uniquely determined analytic functions $h^i_j \in \mathcal{O}(U)$. After permuting the given basis of D we may suppose that the analytic matrix-valued function

$$\Theta : U \to \mathbb{C}^{m,m}, \quad \Theta(\lambda) = \big(h^i_j(\lambda)\big)_{1 \leq i,j \leq m}$$

is invertible at z_0. Define $c = \mathrm{ord}_0\big(\det(\Theta)\big)$. Basic linear algebra allows us to choose an analytic function $A = (A_{ij}) : U \to \mathbb{C}^{m,m}$ with

$$\Theta(z)A(z) = A(z)\Theta(z) = \det\big(\Theta(z)\big)\, I_m \quad (z \in U).$$

For each $k \in \mathbb{N}$, the analytic functions

$$\Theta^k_{ij} = \big[\Theta\big(A - (T_k A_{ij})\big)\big]_{i,j} = \sum_{\mu=1}^{m} h^i_\mu (A_{\mu j} - T_k A_{\mu j})$$

have order at least $k+1$ at $z = 0$.

Set $D_0 = \mathrm{LH}\{e_1, \ldots, e_m\}$. Let $Q_0 : D \to D_0$ be the projection onto D_0 with $Q_0 e_i = 0$ for $i = m+1, \ldots, N$. Denote by

$$Q = 1 \otimes Q_0 : \mathcal{O}(U, D) \to \mathcal{O}(U, D_0)$$

the induced projection on $\mathcal{O}(U, D)$. For each function $P = (P_{ij}) : U \to \mathbb{C}^{m,m}$ with polynomial entries $P_{ij} \in \mathbb{C}[z]$, we have

$$\sum_{i=1}^{m} [\Theta P]_{ij} e_i = Q\Big(\sum_{\mu=1}^{m} P_{\mu j} h_\mu\Big) \in QM$$

for $j = 1, \ldots, m$.

Fix $k \in \mathbb{N}$ and a polynomial $p \in \mathbb{C}[z]$. For $j = 1, \ldots, m$, the function $T_k\big(p \det(\Theta) e_j\big)$ is obtained by applying the linear map T_k coefficientwise to the matrix

$$p \det(\Theta)\, I_m - p(\Theta^k_{\mu\nu})_{\mu,\nu} = \Theta p (T_k A_{\mu\nu})_{\mu,\nu},$$

then multiplying the i-th coefficient of the j-th column of this matrix with e_i and adding up over all $i = 1, \ldots, m$. It follows that

$$T_k\big(p \det(\Theta) e_j\big) \in T_k(Q\, M) \quad (p \in \mathbb{C}[z],\ j = 1, \ldots, m).$$

Let $\mathcal{V}_i = \{p \in \mathbb{C}[z];\ \deg(p) \leq i\}$ for $i \in \mathbb{N}$. For $k \geq c$ and $j = 1, \ldots, m$, the linear maps

$$\mathcal{V}_{k-c} \to T_k(QM), \quad p \mapsto T_k(p \det(\Theta) e_j)$$

are obviously injective and therefore

$$\begin{aligned} \dim T_k(M) \ &\geq \dim Q\, T_k(M) = \dim T_k(Q\, M) \\ &\geq m \dim \mathcal{V}_{k-c} = \frac{m(k-c+1) \cdot \ldots \cdot (k-c+n)}{n!} \end{aligned}$$

We conclude that

$$m \;\leq\; n! \liminf_{n\to\infty} \frac{\dim T_k(M)}{k^n}\;.$$

Let us turn to the proof of the opposite inequality. Since the right-hand side of the last inequality can be estimated from above against $n!\ \lim_{k\to\infty}(N \dim \mathcal{V}_k)/k^n = N$, we may suppose that $m < N$. Let $f = \sum_{i=1}^N f^i e_i \in M$ be arbitrary. The maximality of m implies that the vectors $h_1(z), \ldots, h_m(z)$, $f(z) \in D$ are linearly dependent for each $z \in U$. Hence the determinant of the matrix

$$\begin{pmatrix} h_1^1 & \ldots & h_m^1 & f^1 \\ \ldots & \ldots & \ldots & \ldots \\ h_1^m & \ldots & h_m^m & f^m \\ h_1^i & \ldots & h_m^i & f^i \end{pmatrix}$$

is identically zero for every fixed $i = m+1, \ldots, N$. Expanding this determinant according to the last column, we find that

$$g_1 f^1 + \ldots + g_m f^m + g_i f^i \equiv 0$$

with suitable functions $g_1, \ldots, g_m \in \mathcal{O}(U)$ and $g_i = \det(\Theta)$.

Fix $k \in \mathbb{N}$. Let $g = T_k(f)$ with $f = \sum\limits_{i=1}^{N} f^i e_i$ be as above, but assume in addition that $Q\,g = 0$. Then

$$g = \sum_{i=1}^{N} T_k(f^i) e_i = \sum_{i=m+1}^{N} T_k(f^i) e_i,$$

and for $i = m+1, \ldots, N$, we obtain the relations

$$T_k(g_i f^i) = -T_k(g_1 f^1 + \ldots + g_m f^m) = 0.$$

It follows that $\mathrm{ord}(f^i) \geq k - c + 1$ for $i = m+1, \ldots, N$ and that

$$g = (T_k - T_{k-c}) \sum_{i=m+1}^{N} f^i e_i \in (I - Q)(T_k - T_{k-c})\mathcal{O}(U, D).$$

The above arguments show that

$$\dim\big(T_k(I-Q)M\big) \leq \mathrm{rank}(I-Q)(T_k - T_{k-c}) = (N-m)\left[\binom{n+k}{n} - \binom{n+k-c}{n}\right].$$

Since the right-hand side is a polynomial of degree at most $(n-1)$ in k, we conclude that

$$
\begin{aligned}
n! \quad \limsup_{k\to\infty} \frac{\dim T_k(M)}{k^n} &\le n! \ \limsup_{k\to\infty} \frac{\dim Q\, T_k(M)}{k^n} \\
&\le n! \lim_{k\to\infty} \frac{T_k \mathcal{O}(U, D_0)}{k^n} = m.
\end{aligned}
$$

□

The significance of the maximal dimension of the spaces M_z, usually referred to as the fibre dimension of M, was recognized before in the context of analytic functional Hilbert spaces by Gleason, Richter and Sundberg [9].

Let us return to the operator-theoretic situation described before Lemma 3. With the notation fixed there, the subspace

$$M = \rho Y \subset \mathcal{O}(U, D)$$

is a $\mathbb{C}[z]$-submodule. Applying Lemma 4 to this submodule we obtain the next result.

Corollary 5 *With the notation explained above, there is a nowhere dense analytic subset $S \subset U$ of a connected open neighbourhood U of $0 \in \mathbb{C}^n$ such that*

$$\dim H^n(z - R, Z) = \min_{w \in U} H^n(w - R, Z) = N - n! \lim_{k\to\infty} \frac{\dim T_k(\rho Y)}{k^n}$$

for all $z \in U \setminus S$.

Our next aim is to relate the limit occurring in Corollary 5 to the Hilbert-Samuel multiplicity of R. For a commuting tuple $T \in L(X)^n$, we use the notation $M_k(T) = \sum_{|\alpha|=k} T^\alpha X$ $(k \in \mathbb{N})$.

Lemma 6 *Let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace of a commuting tuple $T \in L(X)^n$, let $Z = X/Y$ and let $R = T/Y \in L(Z)^n$ be the induced quotient tuple. Suppose that*

$$\dim H^n(T, X) < \infty.$$

Then the Hilbert-Samuel multiplicities of T and R satisfy

$$c(R) = c(T) - n! \lim_{k\to\infty} \frac{\dim \left[\big(Y + M_k(T)\big)/M_k(T)\right]}{k^n} .$$

Proof. It suffices to observe that the inclusion map $j : Y \to X$ and the quotient map $q : X \to Z$ induce short exact sequences

$$0 \longrightarrow \frac{Y + M_k(T)}{M_k(T)} \xrightarrow{\ j\ } X/M_k(T) \xrightarrow{\ q\ } Z/M_k(R) \longrightarrow 0.$$

Using the fact that the alternating sum of the dimensions of the three spaces forming this sequence is zero, one deduces the assertion.

□

Let us return to the case where $\dim H^n(z-T,X) = N$ near $z = 0 \in \mathbb{C}^n$. By Lemma 3 we know that $c(T) = N$. With the notation fixed before Lemma 4 we obtain that

$$\rho\big(M_k(T)\big) \subset \{f \in \mathcal{O}(U,D);\ \mathrm{ord}_0(f) \geq k\}.$$

Hence the maps $T_{k-1} \circ \rho$ induce surjective linear maps

$$\tau_k : \big(Y + M_k(T)\big)/M_k(T) \to T^{k-1}(\rho Y)\,.$$

It follows that the limit occurring in Lemma 6 is at most larger than the corresponding limit in Corollary 5. We complete the proof of Theorem 2 by showing that both limits actually coincide.

Corollary 7 *Let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace of a commuting tuple $T \in L(X)^n$, let $Z = X/Y$ and let $R = T/Y \in L(Z)^n$ be the induced quotient tuple. Suppose that in some connected open neighbourhood U of $0 \in \mathbb{C}^n$*

$$\dim H^n(z-T,X) = N \quad (z \in U).$$

Then there is a nowhere dense analytic subset $S \subset U$ such that

$$c(R) = \min_{w \in U} \dim H^n(w-R,Z) = \dim H^n(z-R,Z)$$

for $z \in U \setminus S$.

Proof. Let $\mathcal{F} = \mathcal{H}^n(z-T,\mathcal{O}_U^X)$ be the n-th cohomology sheaf of the Koszul complex $K^\bullet(z-T,\mathcal{O}_U^X)$. Fix a direct complement D of $\sum_{i=1}^n T_iX$ in X. As seen before, after shrinking U, we may suppose that the map

$$\mathcal{O}_U^D \to \mathcal{F}, \quad f_\lambda \mapsto f_\lambda + (z-T)\mathcal{O}_\lambda^{X^n}$$

is an isomorphism of analytic sheaves. The composition

$$X \xrightarrow{\rho} \mathcal{O}(U,D) \cong \mathcal{F}(U) \xrightarrow{\delta_0} \mathcal{F}_0,$$

where the last map is the point evaluation $\delta_0(\gamma) = \gamma(0)$, is precisely the map

$$\varphi : X \to \mathcal{F}_0, \quad \varphi(x) = x + (z-T)\mathcal{O}_0^{X^n}$$

that we defined in the section leading to Proposition 1. The cited result from [3] implies that the compositions

$$\varphi_k : X/M^kX \xrightarrow{[\rho]} \mathcal{O}(U,D)/M^k\mathcal{O}(U,D) \longrightarrow \mathcal{F}(U)/M^k\mathcal{F}(U) \longrightarrow \mathcal{F}_0/\mathfrak{m}^k\mathcal{F}_0$$

are onto for all $k \geq 0$. Let p and p_{an} be the Hilbert-Samuel polynomial and analytic Hilbert-Samuel polynomial of T, respectively. As an application of Lemma 3 we obtain that $q = p - p_{an}$ is a polynomial with $\deg(q) \leq n-1$. By construction

$$\dim \ \mathrm{Ker}\ \varphi_k = \dim(X/M^kX) - \dim(\mathcal{F}_0/\mathfrak{m}^k\mathcal{F}_0) = q(k)$$

for sufficiently large k. Since τ_k acts as the composition

$$\big(Y + M_k(T)\big)/M_k(T) \xrightarrow{[\rho]} \mathcal{O}(U,D)/M^k\mathcal{O}(U,D) \xrightarrow{\hat{T}_{k-1}} \mathcal{O}(U,D),$$

where the map $\hat{T}_{k-1}$ defined by $\hat{T}_{k-1}([f]) = T_{k-1}(f)$ is injective, we conclude that

$$\dim \ \mathrm{Ker}\ \tau_k \leq \dim \ \mathrm{Ker}\ \varphi_k \qquad (k \in \mathbb{N}).$$

The observation that

$$\dim\big[\big(Y + M_k(T)\big)/M_k(T)\big] - \dim T_{k-1}(\rho Y) = \dim(\mathrm{Ker}\ \tau_k) \leq q(k)$$

for all sufficiently large k, completes the proof.

$\square$

The question whether $c(R) = c_{an}(R)$ in the setting of Corollary 7 remains open here, at least in the Banach-space case. In the case of Hilbert spaces the cohomology sheaf $\mathcal{H} = \mathcal{H}^n(z - R, \mathcal{O}_U^Z)$ is known to be coherent [11]. Then standard results from analytic geometry (Theorem 7.4 in [2]) imply that the map

$$U \to \mathbb{N}, \quad z \mapsto c_{an}(z - R) = n! \lim_{k\to\infty} \frac{\dim(\mathcal{H}_z/\mathfrak{m}_z^k\mathcal{H}_z)}{k^n}$$

is upper semicontinuous. Since

$$c_{an}(z - R) = \min_{w\in U} \dim H^n(w - R, Z) = c(R) \quad (z \in U \setminus S)$$

for a proper analytic subset $S \subset U$, it follows that $c_{an}(R) \geq c(R)$. Since the reverse inequality always holds, we have equality. The sheaf $\mathcal{H}$ is also known to be coherent, when the tuple R is Fredholm, that is, when $\dim H^p(R,Z) < \infty$ for $p = 0,\ldots,n$. Hence also in this case we obtain equality $c(R) = c_{an}(R)$ in the setting of Corollary 7.

A second natural question is whether the assertion of Theorem 2 remains true, when we replace the hypothesis that R is a quotient of a tuple $T \in L(X)^n$ for which $\dim H^n(z - T, X)$ is constant for z near 0 simply by the condition that $\dim H^n(R,Z) < \infty$.

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