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Abstract

We prove interior $C^{1,\alpha}$-regularity of minimizing displacement fields for a class of nonlinear Hencky materials in the $2D$-case.

Let $\Omega \subset \mathbb{R}^2$ denote a bounded open set on which the displacements u of an elastic body are defined. If the case of linear elasticity is considered, then the elastic energy of the deformation is given by

$$J_0[u] = \int_\Omega \left[\frac{1}{2}\lambda(\operatorname{div} u)^2 + \kappa|\varepsilon(u)|^2\right] \mathrm{d}x, \tag{1}$$

where λ, $\kappa > 0$ denote physical constants and $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric gradient of u. In order to model a nonlinear material behaviour, in particular the nonlinear Hencky material, see [Ze], (1) is replaced by the energy

$$J[u] = \int_\Omega \left[\frac{1}{2}\lambda(\operatorname{div} u)^2 + \varphi\big(|\varepsilon^D(u)|^2\big)\right] \mathrm{d}x \tag{2}$$

for some nonlinear function φ. Here $\varepsilon^D(u)$ is the deviatoric part of $\varepsilon(u)$, i.e. $\varepsilon^D(u) = \varepsilon(u) - \frac{1}{2}(\operatorname{div} u)\mathbf{1}$. The purpose of our short note is to investigate the regularity properties of local minimizers of the functional J under suitable assumptions on the function φ. To be precise and to have more flexibility, we replace the quantity $\varphi(|\varepsilon^D(u)|^2)$ in the expression (2) for the energy by $F(\varepsilon^D(u))$, where F: $\mathbb{S}^2 \to [0, \infty)$ is a function of class C^2 defined on the space $\mathbb{S}^2$ of all symmetric (2×2)-matrices satisfying for some exponent $s \in (1, \infty)$ and with positive constants a, A the ellipticity estimate

$$a\big(1 + |\varepsilon|^2\big)^{\frac{s-2}{2}} |\sigma|^2 \le D^2F(\varepsilon)(\sigma, \sigma) \le A\big(1 + |\varepsilon|^2\big)^{\frac{s-2}{2}} |\sigma|^2 \tag{3}$$

for all ε, $\sigma \in \mathbb{S}^2$.

DEFINITION 1 *A function u from the Sobolev class $W^1_{1,loc}(\Omega; \mathbb{R}^2)$ is called a local minimizer of the functional*

$$I[v, \Omega] := \int_\Omega \left[\frac{\lambda}{2}(\operatorname{div} v)^2 + F(\varepsilon^D(v))\right] \mathrm{d}x \tag{4}$$

iff $I[u, \Omega'] < \infty$ and $I[u, \Omega'] \le I[v, \Omega']$ for all $v \in W^1_{1,loc}(\Omega; \mathbb{R}^2)$ such that $\operatorname{spt}(u - v) \Subset \Omega'$, Ω' being an arbitrary subdomain with compact closure in Ω.

Then we have

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THEOREM 1 *Let u denote a local minimizer of the functional $I[\cdot,\Omega]$ defined in (4) with F satisfying (3). Then u is in the local Hölder space $C^{1,\alpha}(\Omega;\mathbb{R}^2)$ for any $0<\alpha<1$ provided that $s\in(1,4)$.*

REMARK 1 *In the case that $s\leq 2$ the result from Theorem 1 in principle is a consequence of the work of Frehse and Seregin [FrS] on plastic materials with logarithmic hardening. They consider the function $F(\varepsilon)=|\varepsilon|\ln(1+|\varepsilon|)$ but it is not hard to show that their arguments actually cover the case of exponents $s\leq 2$.*

REMARK 2 *For $s\in(1,2]$ the functional $I[\cdot,\Omega]$ also serves as a model for plasticity with power hardening, we refer to [Ka], [Kl] and [NH]. It is worth remarking that Seregin proved partial regularity in the 3D-case for the above mentioned range of exponents, see e.g. [Se1], [Se2].*

Proof of Theorem 1. Let $f(\varepsilon):=\frac{\lambda}{2}(\operatorname{tr}\varepsilon)^2+F(\varepsilon^D), \varepsilon\in\mathbb{S}^2$. Here $\operatorname{tr}\varepsilon$ is the trace of the matrix ε and $\varepsilon^D=\varepsilon-\frac{1}{2}\operatorname{tr}\varepsilon\mathbf{1}$. Clearly $f\colon\ \mathbb{S}^2\to[0,\infty)$ is of class C^2 and satisfies for all $\varepsilon,\sigma\in\mathbb{S}^2$

$$D^2f(\varepsilon)(\sigma,\sigma)=\lambda(\operatorname{tr}\sigma)^2+D^2F(\varepsilon^D)(\sigma^D,\sigma^D). \tag{5}$$

If $s\geq 2$, then (3) and (5) imply with positive constants ν and μ

$$\nu|\sigma|^2\leq D^2f(\varepsilon)(\sigma,\sigma)\leq\mu\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}|\sigma|^2 \tag{6}$$

for arbitrary matrices ε, $\sigma\in\mathbb{S}^2$. If $1<s<2$, then we observe (see (3) and (5))

$$D^2f(\varepsilon)(\sigma,\sigma)\geq\lambda\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}(\operatorname{tr}\sigma)^2+a\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}|\sigma^D|^2$$

which follows from $(1+|\varepsilon|^2)^{(s-2)/2}\leq 1$ and $(1+|\varepsilon|^2)^{(s-2)/2}\leq(1+|\varepsilon^D|^2)^{(s-2)/2}$. Thus, for a suitable constant $\overline{\nu}>0$ we find that

$$D^2f(\varepsilon)(\sigma,\sigma)\geq\overline{\nu}\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}\Big[|\sigma^D|^2+(\operatorname{tr}\sigma)^2\Big]\geq\overline{\nu}\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}|\sigma|^2,$$

and (see again (3))

$$D^2f(\varepsilon)(\sigma,\sigma)\leq\lambda(\operatorname{tr}\sigma)^2+A|\sigma^D|^2\leq\overline{\mu}|\sigma|^2$$

for some $\overline{\mu}>0$. Putting together both cases by letting $q:=\max\{2,s\}$, $p:=\min\{2,s\}$ we deduce from (6) and the calculations following (6) that

$$\overline{\alpha}\big(1+|\varepsilon|^2\big)^{\frac{p-2}{2}}|\sigma|^2\leq D^2f(\varepsilon)(\sigma,\sigma)\leq\overline{\beta}\big(1+|\varepsilon|^2\big)^{\frac{q-2}{2}}|\sigma|^2 \tag{7}$$

is true for all ε, $\sigma\in\mathbb{S}^2$ with constants $\overline{\alpha}$, $\overline{\beta}>0$. But in [BFZ] we showed that any local minimizer $u\in W^1_{1,loc}(\Omega;\mathbb{R}^2)$ of the energy $\int_\Omega f(\varepsilon(v))\,\mathrm{d}x$ with f satisfying (7) is of class $C^{1,\alpha}$ in the interior of Ω provided that the exponents p and q are related through the condition

$$q<\min(2p,p+2). \tag{8}$$

Recalling the definitions of p and q it is immediate that the latter condition on p and q holds for $s\in(1,4)$. The reader should note that in [BFZ] all comparison functions have to satisfy the incompressibility condition $\operatorname{div}v=0$ but of course the situation now simplifies in comparison to [BFZ] and all results remain valid if this condition is dropped. $\square$

REMARK 3 *If Ω is a domain in $\mathbb{R}^3$ and if F satisfies (3), then local minimizers u of the functional $I[\cdot, \Omega]$ are of class $C^{1,\alpha}$ on an open subset Ω_0 of Ω such that $\Omega - \Omega_0$ is of Lebesgue measure zero, provided $s < 10/3$. This follows from the results in [BF] if again the incompressibility condition is neglected.*

REMARK 4 *If we replace the term $\frac{\lambda}{2}(\mathrm{div}\, u)^2$ in the functional (4) by an expression like $g(\mathrm{div}\, u)$ with function g of growth rate $r \in (1, \infty)$, then a regularity result like Theorem 1 follows along the lines of [BFZ] if we require (8) to hold with the choices $q := \max\{s, r\}$, $p := \min\{s, r\}$.*

Let us now partially remove the restriction that $s < 4$.

THEOREM 2 *Suppose that $u \in W^1_{1,loc}(\Omega; \mathbb{R}^2)$ is a local minimizer of the energy $I[\cdot, \Omega]$ defined in (4), where F satisfies (3) for some exponent $s \geq 4$. Then the first derivatives of u are α-continuous functions in the interior of Ω for any $\alpha \in (0, 1)$, provided we assume that $\mathrm{div}\, u \in L^s_{loc}(\Omega)$.*

Proof. We define the class

$$V := \left\{ v \in L^2(\Omega; \mathbb{R}^2) : \mathrm{div}\, v \in L^2(\Omega),\, \varepsilon^D(v) \in L^s(\Omega; \mathbb{S}^2) \right\}$$

being the subspace of $W^1_{1,loc}(\Omega; \mathbb{R}^2)$ on which the functional I is well defined. If $v \in V$ is compactly supported in Ω, then it follows from

$$I[u, \mathrm{spt} v] \leq I[u + tv, \mathrm{spt} v], \quad t \in \mathbb{R},$$

that

$$\int_\Omega \Big[\lambda \mathrm{div}\, u\, \mathrm{div}\, v + DF(\varepsilon^D(u)) : \varepsilon^D(v) \Big] \,\mathrm{d}x = 0. \tag{9}$$

For $h \in \mathbb{R} - \{0\}$ and $k \in \{1, 2\}$ let

$$\Delta_h w(x) := \frac{1}{h}[w(x + he_k) - w(x)]$$

denote the difference quotient of the function w. For $\varphi \in C^\infty_0(\Omega)$ it is easy to check that $v := \Delta_{-h}(\varphi^2 \Delta_h(u - Px))$ is admissible in (9) for any constant matrix P. Following for example the calculations carried out in [BF], it is not hard to show that after passing to the limit $h \to 0$ the next inequality can be deduced (from now on summation w.r.t. k)

$$\begin{aligned} &\int_\Omega D^2W(\varepsilon(u))(\partial_k \varepsilon(u), \partial_k \varepsilon(u)) \varphi^2 \,\mathrm{d}x \\ &\quad \leq \; - \int_\Omega D^2W(\varepsilon(u))(\partial_k \varepsilon(u), \nabla\varphi^2 \odot \partial_k[u - Px]) \,\mathrm{d}x, \end{aligned} \tag{10}$$

where

$$W(\sigma) := \frac{\lambda}{2}(\mathrm{tr}\, \sigma)^2 + F(\sigma^D), \quad \sigma \in \mathbb{S}^2,$$

$\odot$ being the symmetric product of tensors. In particular, all the derivatives of u occuring in (10) exist in the weak sense. Let

$$H := \big(D^2W(\varepsilon(u))(\partial_k \varepsilon(u), \partial_k \varepsilon(u)) \big)^{\frac{1}{2}}$$

and observe that $H^2 \in L^1_{loc}(\Omega)$ which can be justified in strength via difference quotient technique but can be made plausible as follows: since we assume $\operatorname{div} u \in L^s_{loc}(\Omega)$, we have

$$\varepsilon(u) = \varepsilon^D(u) + \frac{1}{2}(\operatorname{div} u)\mathbf{1} \in L^s_{loc}(\Omega; \mathbb{S}^2)\,,$$

hence $\nabla u \in L^s_{loc}(\Omega; \mathbb{R}^{2\times 2})$ by Korn's inequality. If we apply the Cauchy-Schwarz inequality to the bilinear form $D^2W(\varepsilon(u))$, we get

$$\begin{aligned} &\Big| D^2W(\varepsilon(u))(\partial_k \varepsilon(u), \nabla\varphi^2 \odot \partial_k u)\Big| \\ &\quad \le\; 2\big[D^2W(\varepsilon(u))(\partial_k\varepsilon(u), \partial_k\varepsilon(u))\varphi^2\big]^{\frac{1}{2}}\big[D^2W(\varepsilon(u))(\nabla\varphi \odot \partial_k u, \nabla\varphi\odot\partial_k u)\big]^{\frac{1}{2}}. \end{aligned}$$

Applying this estimate on the r.h.s. of (10) (choosing $P = 0$ for the moment) and using Young's inequality we obtain

$$\begin{aligned} \int_\Omega \varphi^2 H^2 \,\mathrm{d}x &= \int_\Omega \varphi^2 D^2W(\varepsilon(u))(\partial_k\varepsilon(u), \partial_k\varepsilon(u))\,\mathrm{d}x \\ &\le c\int_\Omega |D^2W(\varepsilon(u))||\nabla\varphi|^2|\nabla u|^2\,\mathrm{d}x \\ &\le c\int_\Omega |\nabla\varphi|^2(1+|\nabla u|^2)^{\frac{s}{2}}\,\mathrm{d}x < \infty \end{aligned}$$

on account of our assumption. This “shows” the local integrability of the function H^2, and a similar argument gives that the integral on the r.h.s. of (10) is well defined. Note that $H^2 \in L^1_{loc}(\Omega)$ is equivalent to

$$\operatorname{div}\partial_k u \in L^2_{loc}(\Omega), \qquad \big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{4}}|\partial_k\varepsilon^D(u)| \in L^2_{loc}(\Omega). \tag{11}$$

In fact, the definition of W implies for any ε, τ, $\sigma \in \mathbb{S}^2$

$$D^2W(\varepsilon)(\tau,\sigma) = \lambda(\operatorname{tr}\tau)(\operatorname{tr}\sigma) + D^2F(\varepsilon^D)(\tau^D, \sigma^D),$$

thus

$$H^2 = \lambda|\nabla\operatorname{div} u|^2 + D^2F(\varepsilon^D(u))(\partial_k\varepsilon^D(u), \partial_k\varepsilon^D(u))$$

and therefore (11) is immediate. Moreover, we have

$$\begin{aligned} &\int_\Omega \big|D^2W(\varepsilon(u))(\partial_k\varepsilon(u), \nabla\varphi^2\odot\partial_k[u-Px])\big|\,\mathrm{d}x \\ &\quad\le\; c\int_\Omega \Big[|\nabla\operatorname{div} u||\nabla\varphi||\nabla u - P| + \big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{2}}|\nabla\varepsilon^D(u)||\nabla\varphi||\nabla u - P\Big]\,\mathrm{d}x \\ &\quad\le\; c\int_\Omega \big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{4}}|\nabla u - P||\nabla\varphi|\big[|\nabla\operatorname{div} u| + \big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{4}}|\nabla\varepsilon^D(u)|\big]\,\mathrm{d}x \\ &\quad\le\; c\int_\Omega \big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{4}}|\nabla u - P|H|\nabla\varphi|\,\mathrm{d}x. \end{aligned}$$

Letting $h := (1+|\varepsilon^D(u)|^2)^{(s-2)/4}$ and returning to (10) it is shown that

$$\int_{B_R} H^2\,\mathrm{d}x \le \frac{c}{R}\int_{B_{2R}} hH|\nabla u - P|\,\mathrm{d}x, \tag{12}$$

provided we choose $\varphi \in C_0^\infty(B_{2R})$, $\varphi \equiv 1$ on B_R, $|\nabla\varphi| \leq c/R$, where B_R is any open disc with compact closure in Ω. Now (12) exactly corresponds to (2.4) in [BFZ], and with $\gamma := 4/3$ we end up with (2.5) of [BFZ], i.e. we deduce from (12) (choosing $P = \fint_{B_{2R}} \nabla u \, \mathrm{d}x$)

$$\fint_{B_R} H^2 \, \mathrm{d}x \leq c \Bigg[\fint_{B_{2R}} (Hh)^\gamma \, \mathrm{d}x \Bigg]^{\frac{1}{\gamma}} \Bigg[\fint_{B_{2R}} |\nabla^2 u|^\gamma \, \mathrm{d}x \Bigg]^{\frac{1}{\gamma}}. \tag{13}$$

Noting that

$$|\nabla^2 u| \leq c|\nabla\varepsilon(u)| \leq c(|\nabla \mathrm{div}\, u| + |\nabla\varepsilon^D(u)|) \leq cHh,$$

(13) turns into

$$\Bigg[\fint_{B_R} H^2 \, \mathrm{d}x \Bigg]^{\frac{\gamma}{2}} \leq c \fint_{B_R} (hH)^\gamma \, \mathrm{d}x. \tag{14}$$

But (14) is the starting inequality for applying Lemma 1.2 of [BFZ], provided we let $d := 2/\gamma$, $f := H^\gamma$, $g := h^\gamma$ in the lemma. Note that as in Section 2 of [BFZ] all the assumptions of the lemma are satisfied since by (11)

$$\Phi := \big(1 + |\varepsilon^D(u)|^2\big)^{\frac{s}{4}} \in W^1_{2,loc}(\Omega)$$

and therefore

$$\int_{B_\rho} \exp(\beta h^2) \, \mathrm{d}x \leq c(\rho, \beta) < \infty$$

follows as outlined after (2.7) in [BFZ]. We conclude from the lemma that (with a suitable constant c_0)

$$\int_{B_\rho} H^2 \log^{c_0\beta}(e + H) \, \mathrm{d}x \leq c(\beta, \rho) < \infty \tag{15}$$

is valid for any $\beta > 0$ and all $\rho < 2R$. Finally, let $\sigma := DW(\varepsilon(u))$. Then

$$|\nabla\sigma| \leq c\big[|\nabla \mathrm{div}\, u| + \big(1 + |\varepsilon^D(u)|^2\big)^{\frac{s-2}{2}} |\nabla\varepsilon^D(u)|\big] \leq chH,$$

and we may proceed as in [BFZ] (see the calculations after (2.11)) to combine the latter inequality with (15) in order to get

$$\int_{B_R} |\nabla\sigma|^2 \log^\alpha(e + |\nabla\sigma|) \, \mathrm{d}x \leq c(R, \alpha) \tag{16}$$

for any $\alpha > 0$ (compare (2.11) of [BFZ]). By results outlined in [KKM] (in particular Example 5.3) (16) shows the continuity of σ. Since $\varepsilon(u) = (DW)^{-1}(\sigma)$, the continuity of $\varepsilon(u)$ follows as well. If $v := \partial_k u$, $k = 1, 2$, then

$$0 = \int_\Omega D^2W(\varepsilon(u))(\varepsilon(v), \varepsilon(\varphi)) \, \mathrm{d}x \tag{17}$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^2)$, and (17) can be seen as an elliptic system with continuous coefficients. Then we use Campanato-type estimates for systems with constant coefficients (see, e.g. [GM]) combined with a freezing argument to deduce $v \in C^{0,\alpha}(\Omega; \mathbb{R}^2)$, $0 < \alpha < 1$. A detailed proof can be found for example in [ABF], Corollary 5.1, where of course the incompressibility condition can be dropped. $\square$

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