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Beurling-type representation of invariant subspaces in reproducing kernel Hilbert spaces

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Abstract

By Beurling's theorem, the orthogonal projection onto an invariant subspace M of the Hardy space $H^2(\mathbb{D})$ on the complex unit disk can be represented as $P_M = M_\phi M_\phi^*$ where ϕ is a suitable multiplier of $H^2(\mathbb{D})$. This concept can be carried over to arbitrary Nevanlinna-Pick spaces but fails in more general settings. This paper introduces the notion of Beurling decomposability of subspaces. An invariant subspace M of a reproducing kernel space will be called Beurling decomposable if there exist (operator-valued) multipliers ϕ_1, ϕ_2 such that $P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*$ and $M = \mathrm{ran} M_{\phi_1}$. We characterize the finite-codimensional and the finite-rank Beurling-decomposable subspaces by means of the core function and the core operator. As an application, we show that in many analytic Hilbert modules $\mathcal{H}$, every finite-codimensional submodule M can be written as $M = \sum_{i=1}^r p_i \mathcal{H}$ with suitable polynomials p_i.

1 Introduction

In many areas of analysis, reproducing kernel spaces and their multipliers play an important role. Probably the best understood reproducing kernel spaces are the Hardy space $H^2(\mathbb{D})$ and the Bergman space $L_a^2(\mathbb{D})$ on the open unit disk in $\mathbb{C}$. The unilateral shift on $H^2(\mathbb{D})$, that is, the multiplication by the independent variable z, is one of the few operators whose lattice of invariant subspaces is completely known. By Beurling's theorem, a subspace M of $H^2(\mathbb{D})$ is invariant under M_z exactly if it is of the form $\phi \cdot H^2(\mathbb{D})$ for some inner function ϕ, or equivalently, if the orthogonal projection on M can be represented as $P_M = M_\phi M_\phi^*$ with some function $\phi \in H^\infty(\mathbb{D})$. When passing to the Bergman space, the situation becomes more complicated, and only weaker formulations of Beurling's theorem remain valid([1]). As it turned out in recent years, the reason for the failure of Beurling's theorem in the Bergman space is that, contrary to the Hardy space, the Bergman space is not a Nevanlinna-Pick space. Recall that a reproducing kernel space $\mathcal{H}$ with reproducing kernel K is said to be a Nevanlinna-Pick space if $1 - \frac{1}{K}$ is a positive definite function. It is well known that Nevanlinna-Pick spaces are essentially the only spaces for which the Nevanlinna-Pick interpolation problem can be solved ([19]). A possible formulation of Beurling's theorem for Nevanlinna-Pick spaces, as stated in [11] and [15], reads as follows:

Theorem. *Suppose that $\mathcal{H}$ is a Nevanlinna-Pick space over an arbitrary set D and that M is an invariant subspace of $\mathcal{H}$ (that is, M is closed and $\gamma \cdot M \subset M$ holds for all multipliers γ). Then there exist a Hilbert space $\mathcal{D}$ and a multiplier $\phi : D \to L(\mathcal{D}, \mathbb{C})$ such that $P_M = M_\phi M_\phi^*$.*

One easily checks that the existence of such a multiplier ϕ implies and, in fact, is equivalent to the positive definiteness of the so called core function $G_M = \frac{K_M}{K}$, where K_M is the reproducing kernel of the reproducing kernel space M. The core function appeared in [16], [17] as a function-theoretic tool in the study of invariant subspaces. With these notations, the above theorem can be restated in the following way:

Theorem. *Suppose that $\mathcal{H}$ is a Nevanlinna-Pick space over an arbitrary set D. Then, for every invariant subspace M of $\mathcal{H}$, the core function $G_M = \frac{K_M}{K}$ is positive definite.*

Suppose that $\mathcal{H}$ is a reproducing kernel space with kernel K such that there exists a distinguished point $z_0 \in D$ with $K(\cdot, z_0) = \mathbf{1}$ and such that $\|\mathbf{1}\| = 1$. Then the core function of the invariant subspace $M = \{f \in \mathcal{H} \; ; \; f(z_0) = 0\}$ is $1 - \frac{1}{K}$. Thus Nevanlinna-Pick spaces are basically the only reproducing kernel spaces admitting a Beurling-type theorem of the above form. Motivated by this observation, we introduce the notion of Beurling-decomposable subspaces. To be able to use the concept of the core function, we require that the kernel of the underlying reproducing kernel space $\mathcal{H} \subset \mathbb{C}^D$ has no zeroes. Furthermore we shall always assume that $\mathcal{H}$ contains the constant functions and that the functions $K(\cdot, w)$ are multipliers of $\mathcal{H}$ for all $w \in D$. Finally, we suppose that the inverse kernel admits a representation of the form

$$\frac{1}{K(z,w)} = \beta(z)\beta(w)^*(1) - \gamma(z)\gamma(w)^*(1)$$

with suitable multipliers $\beta \in \mathcal{M}(\mathcal{H} \otimes \mathcal{B}, \mathcal{H})$ and $\gamma \in \mathcal{M}(\mathcal{H} \otimes \mathcal{C}, \mathcal{H})$. We shall see that Nevanlinna-Pick spaces as well as the standard reproducing kernel spaces on bounded symmetric domains fulfill these conditions. A closed subspace M of a reproducing kernel space $\mathcal{H}$ will be called Beurling decomposable if the orthogonal projection on M admits a representation $P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*$ with multipliers $\phi_i : D \to L(\mathcal{D}_i, \mathbb{C})$ such that $M = \text{ran } M_{\phi_1}$. Obviously, any such subspace is invariant. The first main result of this paper (Theorem 3.3) gives a characterization of the Beurling-decomposable subspaces by means of the core function.

Theorem. *A closed subspace M of $\mathcal{H}$ is Beurling decomposable if and only if its core function can be written as*

$$G_M(z,w) = \phi_1(z)\phi_1(w)^*(1) - \phi_2(z)\phi_2(w)^*(1)$$

with multipliers $\phi_i \in \mathcal{M}(\mathcal{H} \otimes \mathcal{D}_i, \mathcal{H})$.

Since multipliers of $\mathcal{H}$ are necessarily bounded functions, the core function of a Beurling-decomposable subspace must be bounded as well. Furthermore, we shall see in Section 3 that every Beurling-decomposable subspace contains non-trivial multipliers. Examples in [17] and [20] show that even in very familiar spaces not all invariant subspaces are Beurling decomposable. The concept of subordinate kernels, as introduced in [8], turns out to be a powerful tool in the study of Beurling decomposability. In particular, we shall see that there always exists a unique operator $\Delta_M \in L(\mathcal{H})$ such that

$$G_M(z,w) = \langle \Delta_M K(\cdot,w), K(\cdot,z)\rangle$$

holds for all $z, w \in D$. Following [16], this operator will be called the core operator of M. The core operator allows us to use more operator-theoretic methods in the study of Beurling-decomposable subspaces. At the end of Section 3 (Propositions 3.5 and 3.6), we solve the problem of Beurling decomposability for finite-codimensional spaces and spaces whose core operator has finite rank.

In Section 4, we turn our attention to the class of analytic Hilbert modules as introduced in [10]. Under suitable conditions which are satisfied, for instance, by the standard reproducing kernel spaces on bounded symmetric domains, we shall prove that all finite-codimensional invariant subspaces are Beurling decomposable. As an application we compute the right essential spectrum of the commuting tuple $M_{\mathbf{z}} = (M_{\mathbf{z}_1}, \ldots, M_{\mathbf{z}_d})$ consisting of the multiplication operators with the coordinate functions on analytic Hilbert modules of this type. In these spaces, the finite-codimensional invariant spaces turn out to be exactly the subspaces M of the form $M = \sum_{i=1}^r p_i \cdot \mathcal{H}$, where $p_1, \ldots, p_r$ are polynomials with common zero set contained in D. In particular, we obtain a solution of Gleason's problem for a large class of spaces.

2 Preliminaries

A Hilbert space $\mathcal{H}$ of complex-valued functions on an arbitrary set D is called a reproducing kernel space if all evaluation functionals

$$\delta_w : \mathcal{H} \to \mathbb{C}\ ,\ f \mapsto f(w) \quad (w \in D)$$

are continuous. In this case there exists a unique function (the reproducing kernel of $\mathcal{H}$) $K : D \times D \to \mathbb{C}$ such that $K(\cdot, w)$ belongs to $\mathcal{H}$ for all $w \in D$ and satisfies

$$\langle f, K(\cdot,w)\rangle = f(w) \quad (f \in \mathcal{H}).$$

It is easy to see that K is a positive definite function in the sense that, for all finite sequences $z_1, \dots, z_n$ in D, the matrices $(K(z_i, z_j))_{i,j}$ are positive semidefinite.
It is a well-known fact (see [5] for more information) that, for every positive definite function F, one can construct a unique reproducing kernel space $\mathcal{F} \subset \mathbb{C}^D$ whose reproducing kernel is given by F. We call $\mathcal{F}$ the reproducing kernel space associated to F.
We shall write $F \leq G$ to indicate that $G - F$ is positive definite. In this way we obtain a partial ordering on the set of all positive definite functions on D.
Suppose that $F_1, F_2 : D \times D \to \mathbb{C}$ are positive definite functions. Then F_1 and F_2 are said to be disjoint if the only positive definite function F which satisfies $F \leq F_1$ and $F \leq F_2$ is $F = 0$. It can be shown (see [21] for details) that F_1 and F_2 are disjoint if and only if the associated reproducing kernel spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ have trivial intersection, that is, $\mathcal{F}_1 \cap \mathcal{F}_2 = \{0\}$.
The following lemma provides a useful tool to decide whether or not a given function $f : D \to \mathbb{C}$ belongs to a given reproducing kernel space.

Lemma 2.1. *Let $\mathcal{H} \subset \mathbb{C}^D$ denote a reproducing kernel space with reproducing kernel K. For a function $f : D \to \mathbb{C}$, the following assertions are equivalent:*

(i) f belongs to $\mathcal{H}$.

(ii) There exists a real number $c \geq 0$ such that the function

$$D \times D \to \mathbb{C} \ , \ (z, w) \mapsto c^2 K(z, w) - f(z)\overline{f(w)}$$

is positive definite.

In this case, $\|f\|$ is the minimum of all constants c satisfying (ii).

A proof of this well-known result can be found in [9].
A Kolmogorov factorization of a positive definite function F is a pair $(\mathcal{D}, d)$ consisting of a Hilbert space $\mathcal{D}$ and a function $d : D \to L(\mathcal{D}, \mathbb{C})$ such that

$$\mathcal{D} = \bigvee \{d(w)^*(1) \ ; \ w \in D\}$$

and $F(z, w) = d(z)d(w)^*(1)$ holds for all $z, w \in D$. Obviously, the reproducing kernel space $\mathcal{F}$ associated to F and the mapping $d : D \to L(\mathcal{F}, \mathbb{C})$, $z \mapsto \delta_z$, define a possible Kolmogorov factorization of F.
If $\mathcal{E}$ is a Hilbert space and $\mathcal{H}$ is a reproducing kernel space with kernel K, then $\mathcal{H}_{\mathcal{E}}$ will denote the Hilbert space of all functions $f : D \to \mathcal{E}$ such that for every $x \in \mathcal{E}$ the function

$$f_x : D \to \mathbb{C} \ , \ f_x(z) = \langle f(z), x \rangle$$

belongs to $\mathcal{H}$ and such that

$$\|f\|^2 = \sum_i \|f_{e_i}\|^2 < \infty$$

for some (equivalently every) orthonormal basis $(e_i)_i$ of $\mathcal{E}$. One easily verifies that the above norm $\|\cdot\|$ on $\mathcal{H}_{\mathcal{E}}$ does not depend on the choice of the orthonormal basis. The space $\mathcal{H}_{\mathcal{E}}$ can also be thought of as the reproducing kernel space with operator-valued kernel $K \cdot 1_{\mathcal{E}}$. We refer to [9] for further treatment of vector-valued reproducing kernel spaces. It is quite standard to show that there exists a unique isometric isomorphism

$$U : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H}_{\mathcal{E}} \quad \text{with} \quad U(f \otimes x) = f \cdot x \quad (f \in \mathcal{H},\ x \in \mathcal{E})$$

between the Hilbertian tensor product $\mathcal{H} \otimes \mathcal{E}$ and $\mathcal{H}_{\mathcal{E}}$. In the sequel, we will use this identification without further mentioning.
Assume now that $\mathcal{H}$ is a reproducing kernel space with kernel K and that $\mathcal{E}, \mathcal{E}_*$ are arbitrary Hilbert spaces. In this setting, a function $\phi : D \to L(\mathcal{E}, \mathcal{E}_*)$ is called an $L(\mathcal{E}, \mathcal{E}_*)$-valued multiplier of $\mathcal{H}$ if, for every function $f \in \mathcal{H} \otimes \mathcal{E}$, the pointwise product $\phi \cdot f$ belongs to $\mathcal{H} \otimes \mathcal{E}_*$. The collection of all such multipliers will be denoted by $\mathcal{M}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}_*)$. A standard application of the closed graph theorem shows that each $\phi \in \mathcal{M}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}_*)$ defines a bounded linear operator

$$M_\phi : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H} \otimes \mathcal{E}_*\ ,\ f \mapsto \phi \cdot f.$$

Obviously, the operator norm of $L(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}_*)$ induces a norm on the space $\mathcal{M}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}_*)$ which is called the multiplier norm and turns $M(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}_*)$ into a Banach space. It is a well-known fact that the functions $K(\cdot, w)$ $(w \in D)$ are eigenfunctions for the adjoints of multiplication operators. More generally, if ϕ belongs to $\mathcal{M}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}_*)$, then the equality

$$M_\phi^*(K(\cdot, w)x) = K(\cdot, w)(\phi(w)^* x)$$

holds for all $x \in \mathcal{E}_*$ and $w \in D$. For a multiplier $\phi \in \mathcal{M}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H})$, we obtain the formula

$$\left(M_\phi M_\phi^* K(\cdot, w)\right)(z) = \phi(z)\phi(w)^*(1)\ K(z, w) \quad (z, w \in D)$$

which will be intensively used in this paper.

Lemma 2.2. *Let $\mathcal{H}$ be a reproducing kernel space with kernel K and let $\mathcal{E}, \mathcal{E}_*$ be arbitrary Hilbert spaces. For a function $\phi : D \to L(\mathcal{E}, \mathcal{E}_*)$, the following are equivalent:*

(i) *ϕ belongs to $\mathcal{M}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}_*)$.*

(ii) *There exists a real number $c \geq 0$ such that*

$$D \times D \to L(\mathcal{E}_*) \ , \ (z, w) \mapsto K(z, w)(c^2 - \phi(z)\phi(w)^*)$$

is an operator-valued positive definite function.

In this case $\|M_\phi\|$ is the minimum of all constants c satisfying (ii).

Analogously to the scalar definition, a function $F : X \times X \to L(\mathcal{D})$ is called positive definite if, for all finite sequences $z_1, \ldots, z_n$, the matrix $(F(z_i, z_j))_{i,j}$ is a positive operator on $\mathcal{D}^n$. A more general form of this result treating the case of arbitrary vector-valued reproducing kernel spaces and their multipliers can be found in [9].
Next we recall the concept of subordinate kernels which was introduced in [5] and refined in [8]. In this context, a kernel simply is a complex-valued function on $D \times D$. A kernel is called positive, if it is a positive definite function. A kernel L is said to be hermitian if $L(z, w) = \overline{L(w, z)}$ holds for all $z, w \in D$.

Definition 2.3. Let $K : D \times D \to \mathbb{C}$ denote a positive kernel and let $\mathcal{H}$ be the associated reproducing kernel space. A kernel $L : D \times D \to \mathbb{C}$ is said to be subordinate to K ($L \prec K$) if there exists a (necessarily unique) operator $T \in L(\mathcal{H})$ such that

$$L(z, w) = \langle TK(\cdot, w), K(\cdot, z)\rangle \quad (z, w \in D).$$

In this case, T is called the representing operator for L. We write $S(K)$ for the set of all kernels that are subordinate to K.

Note that a subordinate kernel is hermitian (positive) if and only if its representing operator is selfadjoint (positive). Furthermore, every hermitian kernel in $S(K)$ can be written as a difference of two positive kernels in $S(K)$, and $S(K)$ is the linear span of its positive kernels. To prove this, observe that the analogous statements are true in $L(\mathcal{H})$.
If $L \prec K$ is a positive kernel, one may ask for the relation between the associated reproducing kernel spaces. The following lemma answers this question.

Lemma 2.4. *Let $K, L : D \times D \to \mathbb{C}$ denote positive kernels and let $\mathcal{H}, \mathcal{L}$ be the associated reproducing kernel spaces. Then the following are equivalent:*

(i) *L is subordinate to K.*

(ii) *There exists a real number $c \geq 0$ such that $cK - L$ is a positive kernel.*

(iii) $\mathcal{L}$ *is continuously embedded in* $\mathcal{H}$*.*

(iv) $\mathcal{L}$ *is a linear subspace of* $\mathcal{H}$*.*

If in this case, $T \in L(\mathcal{H})$ *is the the (positive) representing operator of* L*, then* $\mathcal{L} = ran\, T^{\frac{1}{2}}$*.*

Proof. For the sake of completeness, we include a proof of this well-known fact. Suppose that L is subordinate to K with representing operator T. Then we can choose $c \geq 0$ such that $c1_{\mathcal{H}} - T$ is a positive operator. Consequently, $cK - L$ is a positive kernel. Now fix a function $f \in \mathcal{L}$ with $\|f\|_{\mathcal{L}} = 1$. By Lemma 2.1, the kernel

$$cK(z,w) - f(z)\overline{f(w)} = (cK(z,w) - L(z,w)) + (L(z,w) - f(z)\overline{f(w)})$$

is positive, and another application of Lemma 2.1 yields that f belongs to $\mathcal{H}$ with $\|f\|_{\mathcal{H}} \leq \sqrt{c}$. Therefore, $\mathcal{L}$ is contained in $\mathcal{H}$ and the inclusion mapping has norm at most $\sqrt{c}$. If $\mathcal{L}$ is contained in $\mathcal{H}$ and the inclusion mapping $i : \mathcal{L} \to \mathcal{H}$ is bounded, then it is easy to verify that

$$i^* K(\cdot, w) = L(\cdot, w)$$

holds for all $w \in D$ and therefore L is subordinate to K and is represented by the operator $ii^* \in L(\mathcal{H})$. This settles the equivalence of $(i) - (iii)$. A simple application of the closed graph theorem furnishes the equivalence of (iii) and (iv).
Now let $T \in L(\mathcal{H})$ denote the (positive) representing operator for L. The identity

$$\langle L(\cdot, w), L(\cdot, z)\rangle_{\mathcal{L}} = L(z,w) = \langle T^{\frac{1}{2}} K(\cdot, w), T^{\frac{1}{2}} K(\cdot, z)\rangle_{\mathcal{H}}$$

valid for all $z, w \in D$ implies that there exists a unitary operator

$$\alpha : \mathcal{L} \to \overline{\text{ran } T^{\frac{1}{2}}} \quad \text{with} \quad \alpha L(\cdot, w) = T^{\frac{1}{2}} K(\cdot, w).$$

The calculation

$$\begin{aligned} \langle T^{\frac{1}{2}} \alpha L(\cdot, w), K(\cdot, z)\rangle &= \langle TK(\cdot, w), K(\cdot, z)\rangle \\ &= L(z,w) \\ &= \langle iL(\cdot, w), K(\cdot, z)\rangle \quad (z, w \in D) \end{aligned}$$

proves that $i = T^{\frac{1}{2}}\alpha$. Finally, the observation

$$i(\mathcal{L}) = T^{\frac{1}{2}}\alpha(\mathcal{L}) = T^{\frac{1}{2}}(\overline{\text{ran } T^{\frac{1}{2}}}) = \text{ran } T^{\frac{1}{2}},$$

completes the proof. □

Throughout the rest of this section, we will examine those positive kernels which can be factorized by multipliers.

Lemma 2.5. *Let $K : D \times D \to \mathbb{C}$ be a positive kernel and let $\mathcal{H}$ be the associated reproducing kernel space. For a positive kernel $G : X \times X \to \mathbb{C}$, the following assertions are equivalent:*

(i) $G \cdot K \in S(K)$.

(ii) $G \cdot L \in S(K)$ *for all* $L \in S(K)$.

(iii) There exists a Hilbert space $\mathcal{D}$ and a multiplier $\phi \in \mathcal{M}(\mathcal{H} \otimes \mathcal{D}, \mathcal{H})$ such that $G(z, w) = \phi(z)\phi(w)^*(1)$ holds for all $z, w \in D$.

If in this case, $\mathcal{G}$ denotes the reproducing kernel space associated to G, then $\mathcal{G}$ is contained in $\mathcal{M}(\mathcal{H})$. Furthermore, the set of all positive kernels G satisfying the equivalent conditions above, is closed under pointwise addition and multiplication.

Proof. By choosing a Kolmogorov decomposition $(\mathcal{D}, \phi)$ of G and using Lemma 2.4, the equivalence of (i) and (iii) becomes a reformulation of Lemma 2.2. Now suppose that (i) holds. Since every kernel $S(K)$ can be written as a linear combination of positive kernels in $S(K)$, it suffices to show that $G \cdot L \in S(K)$ holds for all positive $L \in S(K)$. To this end, let c, c' be positive constants such that $cK - G \cdot K$ and $c'K - L$ are positive. Then

$$cc'K - G \cdot L = c'(cK - G \cdot K) + G \cdot (c'K - L)$$

is positive definite as sum and product of positive definite functions. Hence $G \cdot L$ belongs to $S(K)$. The implication (ii) to (i) is obvious.
We are now going to prove the inclusion $\mathcal{G} \subset \mathcal{M}(\mathcal{H})$. Choose a positive number c such that $cK - G \cdot K$ is positive and let ϕ be a function in $\mathcal{G}$ with $\|\phi\|_{\mathcal{G}} = 1$. Since by Lemma 2.1, the kernel

$$\begin{aligned} & K(z, w)(c - \phi(z)\overline{\phi(w)}) \\ = \quad & (cK(z, w) - K(z, w)G(z, w)) + K(z, w)(G(z, w) - \phi(z)\overline{\phi(w)}) \end{aligned}$$

is positive, Lemma 2.2 ensures that ϕ is a multiplier of $\mathcal{H}$.
To prove the final assertion, fix two positive kernels G_1, G_2 satisfying (i). Obviously $(G_1 + G_2) \cdot K = G_1 \cdot K + G_2 \cdot K$ belongs to $S(K)$, since $S(K)$ is a linear space. Now choose positive constants c_i such that $c_i K - G_i \cdot K$ are positive. Then

$$c_1 c_2 K - G_1 \cdot G_2 \cdot K = c_1(c_2 K - G_2 \cdot K) + G_2 \cdot (c_1 K - G_1 \cdot K)$$

is positive as well. Hence $(G_1 \cdot G_2) \cdot K \in S(K)$. □

3 Beurling decomposition of subspaces

Throughout this section, let $\mathcal{H} \subset \mathbb{C}^D$ be a reproducing kernel space with reproducing kernel K such that K has no zeroes and such that $\mathcal{H}$ contains the constant functions. Furthermore, we suppose that the inverse kernel admits a representation of the form

$$\frac{1}{K(z,w)} = \beta(z)\beta(w)^*(1) - \gamma(z)\gamma(w)^*(1) \quad (z, w \in D) \tag{3.1}$$

with multipliers $\beta \in \mathcal{M}(\mathcal{H} \otimes \mathcal{B}, \mathcal{H})$ and $\gamma \in \mathcal{M}(\mathcal{H} \otimes \mathcal{C}, \mathcal{H})$, where $\mathcal{B}, \mathcal{C}$ are appropriate Hilbert spaces. Since the functions $\beta(\cdot)\beta(w)^*(1)$ and $\gamma(\cdot)\gamma(w)^*(1)$ are complex-valued multipliers, the functions $\frac{1}{K(\cdot,w)}$ belong to $\mathcal{M}(\mathcal{H})$ for all $w \in D$. In addition, we require that also the functions $K(\cdot, w)$ are multipliers. We will now discuss three classes of spaces which fulfill these requirements.

Example 1.

(a) Suppose that K is a Nevanlinna-Pick kernel. This means by definition that K has no zeroes and that the kernel $1 - \frac{1}{K}$ is positive definite. Therefore the kernel $K - \mathbf{1} = K \cdot (1 - \frac{1}{K})$ is positive as well and, by Lemma 2.1, $\mathcal{H}$ contains the constant function $\mathbf{1}$. Choose a Kolmogorov decomposition $(\mathcal{C}, \gamma)$ of $1 - \frac{1}{K}$. Since the kernel

$$X \times X \to L(\mathbb{C}) \ , \ (z, w) \mapsto K(z,w)(1 - \gamma(z)\gamma(w)^*(1)) = 1$$

is positive, Lemma 2.2 implies that γ is a multiplier with multiplier norm less or equal to 1. Since $\|\gamma(w)\|^2 = 1 - \frac{1}{K(w,w)} < 1$ holds for all $w \in D$, we conclude that for $w \in D$, the function

$$\phi_w : D \to \mathbb{C} \ , \ \phi_w(z) = \gamma(z)\gamma(w)^*(1)$$

belongs to $\mathcal{M}(\mathcal{H})$ with multiplier norm strictly less than 1. Therefore the series $\sum_{n=0}^{\infty} \phi_w^n$ converges in $\mathcal{M}(\mathcal{H})$. On the other hand, the series converges pointwise to $K(\cdot, w)$. Consequently, the functions $K(\cdot, w)$ are multipliers for all w.

A simple argument shows that the class of kernels we consider is closed under pointwise multiplication. Hence products of Nevanlinna-Pick kernels belong to this class as well.

(b) Assume that D is a bounded domain in $\mathbb{C}^d$ and that K is sesquianalytic on $D \times D$, or equivalently, that $\mathcal{H}$ consists of holomorphic functions on D. Let us suppose further that the coordinate functions $\mathbf{z}_i$ $(1 \leq i \leq d)$) are multipliers on $\mathcal{H}$ such that the Taylor spectrum of the commuting tuple $M_{\mathbf{z}} = (M_{\mathbf{z}_1}, \ldots, M_{\mathbf{z}_d}) \in L(\mathcal{H})^d$ is contained in $\overline{D}$. Finally, we suppose that $\frac{1}{K}$ is defined and sesquianalytic on an open neighbourhood of $\overline{D} \times \overline{D}$. In [8] (proof of Theorem 3.3) it is shown that every sesquianalytic kernel on a domain is subordinate to the reproducing kernel of some weighted Bergman space. Since we can find a domain $U \supset \overline{D}$ such that $\frac{1}{K}$ is sesquianalytic on $U \times U$, the hermitian kernel $\frac{1}{K}$ can be written as a difference of two positive definite sesquianalytic kernels defined on $U \times U$. To prove this, choose an appropriate decomposition of the representing operator of $\frac{1}{K}$. Taking Kolmogorov decompositions of these positive kernels, we obtain functions β and γ which satisfy the identity (3.1) and, in addition, are analytic on U. The assumption on the spectrum of $M_{\mathbf{z}}$ guarantees that every operator-valued function which is analytic on a neighbourhood of $\overline{D}$, belongs to $\mathcal{M}(\mathcal{H})$ (see for example [3] for a proof). Thus, the functions β, γ are in fact multipliers of $\mathcal{H}$. Therefore a decomposition of the form (3.1) automatically exists in this situation.

(c) We now focus on reproducing kernel spaces over bounded symmetric domains in $\mathbb{C}^d$. To this end, we fix a Cartain domain in $\mathbb{C}^d$ of rank r and characteristic multiplicities a, b. Let us denote by h the Jordan triple determinant of D and let $\mathcal{H} = \mathcal{H}_\nu$ be the reproducing kernel space associated to the kernel

$$K(z, w) = K_\nu(z, w) = h(z, \overline{w})^{-\nu},$$

where ν is in the Wallach set of D. It is well known that K has no zeroes and $\mathcal{H}$ contains the constant functions. It is shown in [13] that, under the additional hypothesis that $\nu \geq \frac{r-1}{2}a + 1$, the inverse kernel admits a representation of the form (3.1). For ν in the continuous Wallach set (this means $\nu > \frac{r-1}{2}a$), the functions $K(\cdot, w)$ are multipliers for all $w \in D$. In fact, it is proved in [4] that the Taylor spectrum of the tuple $M_{\mathbf{z}}$ is $\overline{D}$. Therefore, by the same argument as in the previous example, it suffices to show that $K(\cdot, w)$ is analytic on an open neighbourhood of $\overline{D}$. To see this, fix $w \in D$ and choose a real number $0 < \rho < 1$ such that $\frac{w}{\rho} \in D$. By homogeneous expansion, it can easily be checked that K satisfies the equation $K(z, w) = K(\rho z, \frac{w}{\rho})$ for all $z \in D$. Obviously the right-hand side defines an analytic extension of $K(\cdot, w)$ on the set $\frac{1}{\rho}D$ which is an open neighbourhood of $\overline{D}$.

Following [16] we define the core function and the core operator of a closed subspace of $\mathcal{H}$. But first, we indicate that, by (3.1) and Lemma 2.5, the space $S(K)$ is closed under pointwise multiplication by the inverse kernel $\frac{1}{K}$. Hence, for any $L \in S(K)$, the kernel $\frac{L}{K}$ has a (necessarily unique) representing operator in $L(\mathcal{H})$.

Definition 3.1. Let M be a closed subspace of $\mathcal{H}$ and let K_M denote the kernel

$$K_M : D \times D \to \mathbb{C}\ ,\ \ K_M(z,w) = \langle P_M K(\cdot, w), K(\cdot, z)\rangle.$$

Then $G_M = \frac{K_M}{K} \in S(K)$ is called the core function of M. The core operator $\Delta_M \in L(\mathcal{H})$ of M is by definition the representing operator of G_M. The rank of M is defined to be the rank of Δ_M, that is,

$$\text{rank } M = \text{rank } \Delta_M = \dim \text{ ran } \Delta_M.$$

Note that the kernel K_M is in fact the reproducing kernel of M considered as a reproducing kernel space. Obviously G_M is a hermitian kernel and therefore Δ_M is a selfadjoint operator. It can easily be verified that the diagonal evaluation $G_M(z,z)$ coincides with the Berezin transform of P_M as defined in [6], [7].
In many cases, the core operator can be expressed in a very concrete form.

Example 2.

(a) Suppose that D is an open set in $\mathbb{C}^d$ and that $\frac{1}{K}$ is a polynomial in z and $\overline{w}$,

$$\frac{1}{K}(z,w) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \overline{w}^\beta.$$

Assume further that the coordinate functions $\mathbf{z}_i$ $(1 \le i \le d)$ are multipliers of $\mathcal{H}$. Let $M_{\mathbf{z}}$ denote the commuting tuple $(M_{\mathbf{z}_1}, \ldots, M_{\mathbf{z}_d})$. Then

$$\Delta_M = \sum_{\alpha,\beta} c_{\alpha,\beta} M_{\mathbf{z}}^\alpha P_M M_{\mathbf{z}}^{*\beta}$$

is the core operator of a given subspace M of $\mathcal{H}$.

It is clear that $G_M + G_{M^\perp} = \mathbf{1}$ holds for every closed subspace M of $\mathcal{H}$. Let $P_{\mathbb{C}}$ denote the orthogonal projection onto the one-dimensional subspace of all constant functions in $\mathcal{H}$. Then the constant kernel $\mathbf{1}$ is represented by $\|\mathbf{1}\|^2 P_{\mathbb{C}}$. Hence $\Delta_M + \Delta_{M^\perp} = \|\mathbf{1}\|^2 P_{\mathbb{C}}$.

This observation and the above formula for Δ_M show that the finite dimension of M or $M^\perp$ implies that both Δ_M and $\Delta_{M^\perp}$ have finite rank.

(b) Suppose that D is a bounded symmetric domain in $\mathbb{C}^d$ and adopt the notations of Example 1. In view of the Faraut-Koranyi formula

$$\frac{1}{K(z,w)} = \sum_{\mathbf{m}} (-\nu)_{\mathbf{m}} K_{\mathbf{m}}(z,w) \quad (z, w \in D)$$

(see [14] for details), we show that

$$\Delta_M = \sum_{\mathbf{m}} (-\nu)_{\mathbf{m}} K_{\mathbf{m}}(L_{M_{\mathbf{z}}}, R_{M_{\mathbf{z}}^*})(P_M)$$

(at least if $\nu \geq \frac{r-1}{2}a + 1$). In the above expression, $L_{M_{\mathbf{z}}}$ and $R_{M_{\mathbf{z}}^*}$ denote the tuples of left and right multiplications with the operators $M_{\mathbf{z}_i}$ and $M_{\mathbf{z}_i}^*$, respectively. Since the kernels $K_{\mathbf{m}}$ are polynomials in z and $\overline{w}$, the terms of the series are well defined. Moreover, $K_{\mathbf{m}}$ is positive definite and hence

$$0 \leq K_{\mathbf{m}}(L_{M_{\mathbf{z}}}, R_{M_{\mathbf{z}}^*})(P_M) \leq K_{\mathbf{m}}(L_{M_{\mathbf{z}}}, R_{M_{\mathbf{z}}^*})(1_{\mathcal{H}}).$$

The convergence of the series above now follows directly by a result in [13], where it is shown that the series

$$\sum_{\mathbf{m}} |(-\nu)_{\mathbf{m}}| \| K_{\mathbf{m}}(L_{M_{\mathbf{z}}}, R_{M_{\mathbf{z}}^*})(1_{\mathcal{H}}) \|$$

converges (for $\nu \geq \frac{r-1}{2}a + 1$).

We now turn to the study of invariant subspaces. A closed subspace M of $\mathcal{H}$ will be called K-invariant ($\frac{1}{K}$-invariant) if it is invariant under multiplication by all functions $K(\cdot, w)$ ($\frac{1}{K(\cdot,w)}$, respectively). As usual, M is said to be invariant if $\phi \cdot M \subset M$ for all $\phi \in \mathcal{M}(\mathcal{H})$.

Definition 3.2. A closed subspace M of $\mathcal{H}$ is called Beurling decomposable if there exist Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$ and multipliers $\phi_1 \in \mathcal{M}(\mathcal{H} \otimes \mathcal{E}_1, \mathcal{H})$, $\phi_2 \in \mathcal{M}(\mathcal{H} \otimes \mathcal{E}_2, \mathcal{H})$ such that

$$P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^* \quad \text{and} \quad \text{ran } M_{\phi_1} = M.$$

In this case, the pair (ϕ_1, ϕ_2) is called a Beurling decomposition of M.

Let M be a Beurling-decomposable subspace of $\mathcal{H}$. It is obvious that M is invariant. A simple calculation shows that the equality $P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*$ holds if and only if

$$G_M(z, w) = \phi_1(z)\phi_1(w)^*(1) - \phi_2(z)\phi_2(w)^*(1)$$

for all $z, w \in D$. Thus G_M can be written as the difference of two positive kernels G_1, G_2 which satisfy $K \cdot G_i \prec K$ for $i = 1, 2$. As we shall see in the following theorem, the existence of such a decomposition is basically sufficient for the Beurling decomposability of M.

But first let us observe that unfortunately not all invariant subspaces are Beurling decomposable. Since the reproducing kernel K_M of a Beurling-- decomposable subspace M can be expressed as

$$\begin{aligned} K_M(z,w) &= \langle P_M K(\cdot,w), K(\cdot,z)\rangle \\ &= (\phi_1(z)\phi_1(w)^*(1) - \phi_2(z)\phi_2(w)^*(1))K(z,w) \quad (z,w \in D) \end{aligned}$$

and all functions $K(\cdot,w)$ are supposed to be multipliers, the functions $K_M(\cdot,w)$ define multipliers as well. Hence the set $M \cap \mathcal{M}(\mathcal{H})$ is dense in M.
An example given by Rudin ([20], Theorem 4.1.1) shows that there exists an invariant subspace of the Hardy space $H^2(\mathbb{D}^2)$ over the bidisk which does not contain any nonzero multiplier $\phi \in \mathcal{M}(H^2(\mathbb{D}^2)) = H^\infty(\mathbb{D}^2)$. Therefore we cannot expect all invariant subspaces to be Beurling decomposable.
However, all invariant subspaces M of the Hardy space $H^2(\mathbb{D})$ on the open unit disk are Beurling decomposable. By Beurling's theorem there exists an inner function ϕ on $\mathbb{D}$ such that $P_M = M_\phi M_\phi^*$. This result can be generalized (in a weaker form) to arbitrary Nevanlinna-Pick spaces. It was shown by several authors ([11] or [15]) that in Nevanlinna-Pick spaces the projection onto an invariant subspace M can always be represented as $P_M = M_\phi M_\phi^*$ with a multiplier $\phi \in \mathcal{M}(\mathcal{H}\otimes\mathcal{E},\mathcal{H})$, where $\mathcal{E}$ is a suitable Hilbert space. In particular, M_ϕ is a partial isometry and ran $M_\phi = M$ holds. Consequently, in Nevanlinna-Pick spaces, all invariant subspaces are Beurling decomposable.

Theorem 3.3. *Let M be a closed subspace of $\mathcal{H}$ which is K-invariant and $\frac{1}{K}$-invariant. Then M is Beurling decomposable if and only if there exist positive kernels G_1, G_2 on D such that*

(i) $G_M = G_1 - G_2$

(ii) $K \cdot G_i \prec K$ *for* $i = 1, 2$.

Furthermore, G_1 and G_2 can always be chosen disjoint. If G_1, G_2 are disjoint, then any pair of Kolmogorov factorizations

$$\phi_1 : D \to L(\mathcal{E}_1, \mathbb{C}) \quad , \quad \phi_2 : D \to L(\mathcal{E}_2, \mathbb{C})$$

of G_1 and G_2 defines a Beurling decomposition of M.

Proof. Suppose that M is Beurling decomposable. Then the above discussion proves the existence of positive kernels G_1, G_2 satisfying conditions (i) and (ii).
In order to prove the opposite direction, let us first point out that we may assume G_1, G_2 to be disjoint. In fact, one can show that the set

$$\{G : D \times D \to \mathbb{C} \; ; \; 0 \le G \le G_1, G_2\}$$

is inductively ordered (see [2] or [21] for details). Let G_{max} be a maximal element in this set and write

$$G_1' = G_1 - G_{max} \quad \text{and} \quad G_2' = G_2 - G_{max}.$$

By construction, G_1', G_2' are disjoint positive kernels which satisfy condition (i). As

$$K \cdot G_i' \prec K \cdot G_i \prec K \quad (i = 1, 2),$$

condition (ii) holds as well.
Thus let us suppose that G_1 and G_2 are disjoint. Choose functions

$$\phi_1 : D \to L(\mathcal{E}_1, \mathbb{C}) \quad , \quad \phi_2 : D \to L(\mathcal{E}_2, \mathbb{C})$$

such that

$$G_1(z, w) = \phi_1(z)\phi_1(w)^*(1) \quad \text{and} \quad G_2(z, w) = \phi_2(z)\phi_2(w)^*(1)$$

holds for all $z, w \in D$. Condition (ii) guarantees that ϕ_1, ϕ_2 are in fact multipliers. It follows that

$$\begin{aligned} \langle \left(M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^* \right) K(\cdot, w), K(\cdot, z) \rangle &= (G_1(z, w) - G_2(z, w)) K(z, w) \\ &= K_M(z, w) \\ &= \langle P_M K(\cdot, w), K(\cdot, z) \rangle \quad (z, w \in D), \end{aligned}$$

and therefore

$$M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^* = P_M.$$

It remains to show that ran $M_{\phi_1} = M$. To this end, we note that G_1, G_2 belong to $S(K)$ by Lemma 2.5 since the constant kernel $\mathbf{1}$ belongs to $S(K)$. Let $\Delta_1, \Delta_2 \in L(\mathcal{H})$ denote the (positive) representing operators for G_1, G_2. Since G_1, G_2 are disjoint, the associated reproducing kernel spaces $\mathcal{G}_1$ and $\mathcal{G}_2$ have trivial intersection. By Lemma 2.4 we obtain that

$$\text{ran } \Delta_1^{\frac{1}{2}} \cap \text{ran } \Delta_2^{\frac{1}{2}} = \{0\}$$

and hence that

$$\text{ran } \Delta_1 \cap \text{ran } \Delta_2 = \{0\}.$$

Now it is an elementary exercise to verify that the ranges of Δ_1, Δ_2 must necessarily be contained in the closure of the range of $\Delta_M = \Delta_1 - \Delta_2$.
Since all the functions

$$\Delta_M K(\cdot, w) = G_M(\cdot, w) = \frac{1}{K(\cdot, w)} \cdot K_M(\cdot, w) \quad (w \in D)$$

are contained in M, it follows that ran $\Delta_M \subset M$ and hence that

$$\text{ran } \Delta_1 \subset \overline{\text{ran } \Delta_M} \subset M.$$

Therefore the functions $G_1(\cdot, w) = \Delta_1 K(\cdot, w)$ are contained in M as well for all $w \in D$. Using the K-invariance of M, we see that

$$M_{\phi_1} M_{\phi_1}^* K(\cdot, w) = G_1(\cdot, w) K(\cdot, w) \in M$$

for every $w \in D$. Thus ran $M_{\phi_1} \subset M$.
The opposite inclusion is easier to prove. First, it is elementary to show and well known that for Hilbert spaces H_1, H_2, H and operators $A_1 \in L(H_1, H)$, $A_2 \in L(H_2, H)$ with $A_1 A_1^* \geq A_2 A_2^*$, there exists a contraction $C \in L(H_1, H_2)$ with $CA_1^* = A_2^*$. In view of

$$A_1 A_1^* - A_2 A_2^* = A_1(1_{G_1} - C^* C) A_1^*,$$

it is obvious that ran $A_1 A_1^* - A_2 A_2^* \subset$ ran A_1. To prove that $M \subset$ ran M_{ϕ_1}, it suffices to apply this remark with $A_1 = M_{\phi_1}$ and $A_2 = M_{\phi_2}$. □

Corollary 3.4. *For every $\lambda \in D$, the invariant subspace*

$$M_\lambda = \{f \in \mathcal{H} \ ; \ f(\lambda) = 0\} = \{K(\cdot, \lambda)\}^\perp$$

is Beurling decomposable.

Proof. An easy calculation shows that

$$\begin{aligned} G_{M_\lambda}(z, w) &= 1 - \frac{K(z,\lambda)\overline{K(w,\lambda)}}{K(\lambda,\lambda)K(z,w)} \\ &= \left(1 + \frac{K(z,\lambda)\overline{K(w,\lambda)}}{K(\lambda,\lambda)} \gamma(z)\gamma(w)^*(1)\right) - \left(\frac{K(z,\lambda)\overline{K(w,\lambda)}}{K(\lambda,\lambda)} \beta(z)\beta(w)^*(1)\right) \end{aligned}$$

holds for all $z, w \in D$. Since the function $K(\cdot, \lambda)$ is a multiplier of $\mathcal{H}$, this furnishes the desired decomposition of G_{M_λ}. □

The spaces M_λ considered above have codimension one and form, in some sense, the simplest type of invariant subspaces of $\mathcal{H}$. Now is natural to examine arbitrary subspaces of finite codimension.

Proposition 3.5. *If $M \subset \mathcal{H}$ is a finite-codimensional subspace of $\mathcal{H}$ which is K-invariant and $\frac{1}{K}$-invariant, then the following assertions are equivalent:*

(i) $M^\perp \subset \mathcal{M}(\mathcal{H})$.

(ii) M is Beurling decomposable.

Proof. Let M be Beurling decomposable. By the remarks following Definition 3.2, $K_M(\cdot, w)$ is a multiplier for every $w \in D$. As the functions $K(\cdot, w)$ are supposed to belong to $\mathcal{M}(\mathcal{H})$, the functions

$$K_{M^\perp}(\cdot, w) = K(\cdot, w) - K_M(\cdot, w) \quad (w \in D)$$

define multipliers as well. Thus $M^\perp$, being the linear span of the $K_{M^\perp}(\cdot, w)$, is a subset of $\mathcal{M}(\mathcal{H})$.
Suppose conversely that $M^\perp \subset \mathcal{M}(\mathcal{H})$. Choose an orthonormal basis $(u_i)_{i=1}^m$ of $M^\perp$, and note that

$$K_{M^\perp}(z, w) = \langle P_{M^\perp} K(\cdot, w), K(\cdot, z)\rangle = \sum_{i=1}^m u_i(z)\overline{u_i(w)} \quad (z, w \in D).$$

As the functions u_i are all multipliers, Lemma 2.5 yields $K \cdot K_{M^\perp} \in S(K)$. We define $B(z, w) = \beta(z)\beta(w)^*(1)$ and $C(z, w) = \gamma(z)\gamma(w)^*(1)$. As B and C are positive kernels with $K \cdot B, K \cdot C \in S(K)$, an application of Lemma 2.5 proves that the decomposition

$$G_M = 1 - \frac{K_{M^\perp}}{K} = (1 + K_{M^\perp} \cdot C) - (K_{M^\perp} \cdot B).$$

fulfills the hypotheses of Theorem 3.3. □

Later we will see that in many cases of practical interest, condition (i) of the above proposition is automatically fulfilled for all finite-codimensional invariant subspaces.
We conclude this section by giving a characterization of Beurling decomposability of finite-rank subspaces. Let M be a Beurling-decomposable subspace. From Definition 3.2, it is immediately clear that all functions $G_M(\cdot, w) = \Delta_M K(\cdot, w)$ $(w \in D)$ belong to $\mathcal{M}(\mathcal{H})$. Moreover, the range of the core operator Δ_M consists of multipliers. In order to prove this, we choose G_1, G_2 as in Theorem 3.3 and operators $\Delta_1, \Delta_2 \in L(\mathcal{H})$ representing G_1, G_2. Let $\mathcal{G}_1, \mathcal{G}_2$ denote the associated kernel spaces and note that, by Lemma 2.5 and 2.4,

$$\text{ran } \Delta_i \subset \text{ran } \Delta_i^{\frac{1}{2}} = \mathcal{G}_i \subset \mathcal{M}(\mathcal{H}) \quad (i = 1, 2).$$

Hence

$$\text{ran } \Delta_M \subset \text{ran } \Delta_1 + \text{ran } \Delta_2 \subset \mathcal{M}(\mathcal{H}).$$

For finite-rank invariant subspaces M, the condition ran $\Delta_M \subset \mathcal{M}(\mathcal{H})$ is also sufficient for the Beurling decomposability of M.

Proposition 3.6. *Let M be a closed subspace of $\mathcal{H}$ which is K-invariant and $\frac{1}{K}$-invariant. Suppose that M has finite rank. Then M is Beurling decomposable if and only if ran $\Delta_M \subset \mathcal{M}(\mathcal{H})$. In this case, for every decomposition $G_M = G_1 - G_2$ with disjoint positive kernels $G_1, G_2 \in S(K)$, it follows that $K \cdot G_i \prec K$ for $i = 1, 2$. In particular, there exist multipliers $\phi_1, \ldots, \phi_s, \psi_1, \ldots, \psi_t \in$ ran Δ_M $(s + t =$ rank $M)$ such that*

$$P_M = \sum_{i=1}^{s} M_{\phi_i} M_{\phi_i}^* - \sum_{j=1}^{t} M_{\psi_j} M_{\psi_j}^*$$

and

$$M = \sum_{i=1}^{s} \phi_i \ \mathcal{H}.$$

Proof. Suppose that the inclusion ran $\Delta_M \subset \mathcal{M}(\mathcal{H})$ holds. Fix an arbitrary decomposition $G_M = G_1 - G_2$ with disjoint positive kernels $G_1, G_2 \in S(K)$. Let $\Delta_M = \Delta_1 - \Delta_2$ denote the corresponding decomposition of Δ_M. As seen in the proof of Theorem 3.3, the disjointness of G_1, G_2 and the finite rank of Δ_M imply that ran $\Delta_1 \cap$ ran $\Delta_2 = \{0\}$ and ran $\Delta_M =$ ran $\Delta_1 +$ ran Δ_2. Since in particular ran $\Delta_i \subset \mathcal{M}(\mathcal{H})$, there exist multipliers $\phi_1, \ldots, \phi_s$ and $\psi_1, \ldots, \psi_t$ $(s + t = \text{rank } M)$ with

$$\Delta_1 = \sum_{i=1}^{s} \phi_i \otimes \phi_i \quad \text{ and } \quad \Delta_2 = \sum_{j=1}^{t} \psi_j \otimes \psi_j.$$

Since

$$G_1(z, w) = \langle \Delta_1 K(\cdot, w), K(\cdot, z) \rangle = \sum_{i=1}^{s} \phi_i(z) \overline{\phi_i(w)},$$

and analogously $G_2(z, w) = \sum_{j=1}^{t} \psi_j(z) \overline{\psi_j(w)}$, an application of Lemma 2.5 shows that $K \cdot G_i \in S(K)$ for $i = 1, 2$. Hence G_1 and G_2 are disjoint kernels satisfying the hypotheses of Theorem 3.3. But then the Beurling decomposability of M and all remaining assertions follow directly from Theorem 3.3. □

4 Application to analytic Hilbert modules

Throughout this section, we fix a bounded open set $D \subset \mathbb{C}^d$ and suppose that $\mathcal{H} \subset \mathcal{O}(D)$ is an analytic Hilbert module in the sense of [10] having some additional properties which allow us to apply the results of the preceding section. To be more precise, we shall suppose that

(A) $\mathcal{H}$ contains the constant functions;

(B) $\mathcal{H}$ is a $\mathbb{C}[z]$-module, or equivalently, the coordinate functions $\mathbf{z}_i$ $(1 \leq i \leq d)$ are multipliers of $\mathcal{H}$;

(C) the polynomials are dense in $\mathcal{H}$;

(D) there are no points $z \in \mathbb{C}\backslash D$ for which the mapping

$$\mathbb{C}[z] \to \mathbb{C}\ ,\ p \mapsto p(z)$$

extends to a continuous linear form on all of $\mathcal{H}$. In the language of [10] this means that the set of virtual points of $\mathcal{H}$ coincides with D.

In [10] a reproducing kernel space $\mathcal{H} \subset \mathcal{O}(D)$ satisfying the above conditions is called an ananalytic Hilbert module. To be able to apply the results of Section 3 we require in addition that:

(E) the reproducing kernel K of $\mathcal{H}$ has no zeroes and the inverse kernel $\frac{1}{K}$ admits a representation of the form

$$\frac{1}{K(z,w)} = \beta(z)\beta(w)^*(1) - \gamma(z)\gamma(w)^*(1) \quad (z, w \in D),$$

with multipliers

$$\beta \in \mathcal{M}(\mathcal{H} \otimes \mathcal{B}, \mathcal{H}) \quad \text{and} \quad \gamma \in \mathcal{M}(\mathcal{H} \otimes \mathcal{C}, \mathcal{H})$$

such that the functions

$$\beta(\cdot)\beta(w)^*(1) \quad \text{and} \quad \gamma(\cdot)\gamma(w)^*(1)$$

belong to $\mathcal{O}(\overline{D})$ for every $w \in D$;

(F) the Taylor spectrum $\sigma(M_{\mathbf{z}})$ of the tuple $M_{\mathbf{z}} = (M_{\mathbf{z}_1}, \ldots, M_{\mathbf{z}_d}) \in L(\mathcal{H})^d$ is contained in $\overline{D}$;

(G) for all $z \in D$, there exist open neighbourhoods $U \subset D$ of z and V of $\overline{D}$ such that $K_{|U \times D}$ admits a sesquianalytic extension on $U \times V$.

Although these conditions seem to be rather technical, they are general enough to cover in particular the standard reproducing kernel spaces on bounded symmetric domains.

Example 3.

(a) Suppose that D is a bounded symmetric domain with rank r and characteristic multiplicities a, b and that ν is in the continuous Wallach set of D, that is, $\nu > \frac{r-1}{2}a$. It is well known that the reproducing spaces $\mathcal{H}_\nu$ contain the polynomials as a dense subset. By a recent result of Arazy and Zhang ([4]) the coordinate functions are multipliers of $\mathcal{H}_\nu$. For the special case that $\mathcal{H}_\nu$ is the Bergman space on D, it is shown in [18] that there are no virtual points outside D. But it is easy to see that the given proof remains valid for all $\nu > \frac{r-1}{2}a$. According to [4], the Taylor spectrum of $M_{\mathbf{z}}$ is $\overline{D}$. To show that condition (G) is fulfilled, we fix $z \in D$ and a positive number $0 < \rho < 1$ such that $\frac{z}{\rho} \in D$. If $K_\nu : D \times D \to \mathbb{C}$ denotes the reproducing kernel of $\mathcal{H}_\nu$, then the function

$$\rho D \times \frac{1}{\rho} D \to \mathbb{C} \ , \ (\zeta, \omega) \mapsto K_\nu(\frac{\zeta}{\rho}, \rho\omega)$$

is a sesquianalytic extension of $K_{\nu|\rho D \times D}$. This can be seen by use of the Faraut-Koranyi expansion

$$K_\nu(z, w) = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(z, w) \quad (z, w \in D),$$

where the sum ranges over all signatures $\mathbf{m}$ of length r, the numbers $(\nu)_{\mathbf{m}}$ are the generalized Pochhammer symbols and the functions $K_{\mathbf{m}}$ are the reproducing kernels of the homogeneous spaces $\mathcal{P}_{\mathbf{m}}$ of the Peter-Weyl decomposition $\mathcal{H}_\nu = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}$. Turning towards condition (E), we have to require that $\nu \geq \frac{r-1}{2}a + 1$. For these parameters ν, it was shown in [13] that the decomposition

$$\begin{aligned} \frac{1}{K_\nu} = & \sum_{(-\nu)_{\mathbf{m}} < 0} |(-\nu)_{\mathbf{m}}| \left(K_{\mathbf{m}}(e, e) - K_{\mathbf{m}} \right) \\ & - \sum_{(-\nu)_{\mathbf{m}} > 0} |(-\nu)_{\mathbf{m}}| \left(K_{\mathbf{m}}(e, e) - K_{\mathbf{m}} \right), \end{aligned}$$

yields the existence of multipliers β, γ satisfying

$$\frac{1}{K_\nu(z, w)} = \beta(z)\beta(w)^*(1) - \gamma(z)\gamma(w)^*(1) \quad (z, w \in D).$$

Using the defining homogeneous expansions for β and γ, we obtain by similar arguments that $\beta(\cdot)\beta(w)^*(1)$ and $\gamma(\cdot)\gamma(w)^*(1)$ belong to $\mathcal{O}(\overline{D})$.

(b) If the inverse kernel $\frac{1}{K}$ happens to be a polynomial in z and $\overline{w}$, then condition (E) is automatically satisfied. It is an easy exercise to show that in this case there exist polynomials $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ such that

$$\frac{1}{K(z,w)} = \sum_{i=1}^{m} p_i(z)\overline{p_i(w)} - \sum_{j=1}^{n} q_j(z)\overline{q_j(w)} = B(z,w) - C(z,w)$$

for all $z, w \in \mathbb{C}^d$. Since polynomials are supposed to be multipliers of $\mathcal{H}$, this decomposition has all required properties.

We collect some consequences of our hypotheses. As mentioned before, every function $\phi \in \mathcal{O}(\overline{D})$ automatically is a multiplier of $\mathcal{H}$ and the equality $M_\phi = \phi(M_{\mathbf{z}})$ holds, where the right-hand side is formed with the help of Taylor's functional calculus. Since this fact is of central importance for the following, we indicate a proof (see [3] for details). First note that because of condition (F), the commuting tuple $M_{\mathbf{z}}$ admits an $\mathcal{O}(U)$-calculus for every open neighbourhood U of $\overline{D}$. Since for every $z \in D$, the function $K(\cdot, z)$ is an eigenvector of the operators $M_{\mathbf{z}_i}^*$ with eigenvalue $\overline{z_i}$, it follows by basic properties of the analytic functional calculus that $K(\cdot, z)$ also is an eigenvector of $\phi(M_{\mathbf{z}})^*$ to the eigenvalue $\overline{\phi(z)}$. Now for every $f \in \mathcal{H}$, we obtain

$$\phi(M_{\mathbf{z}})f(z) = \langle f, \phi(M_{\mathbf{z}})^* K(\cdot, z)\rangle = \phi(z)\langle f, K(\cdot, z)\rangle = \phi(z)f(z) \quad (z \in D).$$

Hence ϕ is a multiplier and $\phi(M_{\mathbf{z}}) = M_\phi$.

When dealing with analytic Hilbert modules, there is a natural notion of submodules. A linear subspace M of $\mathcal{H}$ is called a submodule of $\mathcal{H}$ if it is closed in $\mathcal{H}$ and a submodule of $\mathcal{H}$ as a $\mathbb{C}[z]$-module (in other words, a common invariant subspace of the tuple $M_{\mathbf{z}}$). Of course, this concept differs from the definition of invariant subspaces as given before. Obviously, every invariant subspace is a submodule, but the converse is not true.

However, because of condition (F) every finite-codimensional submodule M of $\mathcal{H}$ automatically is an $\mathcal{O}(\overline{D})$-submodule of $\mathcal{H}$ and hence K-invariant by condition (G) and $\frac{1}{K}$-invariant by condition (E). To see this, first note that by Theorem 2.2.5 in [10], the canonical mapping

$$\mathbb{C}[z]/(M \cap \mathbb{C}[z]) \to \mathcal{H}/M \ , \ [p] \mapsto [p]$$

is an isomorphism of (finite dimensional) linear spaces and the inclusion

$$\sigma_p(M_{\mathbf{z}}, \mathbb{C}[z]/(M \cap \mathbb{C}[z])) \subset D$$

holds. Therefore we have

$$\sigma(M_{\mathbf{z}}, \mathcal{H}/M) = \sigma(M_{\mathbf{z}}, \mathbb{C}[z]/(M \cap \mathbb{C}[z])) = \sigma_p(M_{\mathbf{z}}, \mathbb{C}[z]/(M \cap \mathbb{C}[z])) \subset D$$

and, by Lemma 2.2.3 in [12], we obtain

$$\sigma(M_{\mathbf{z}|M}) \subset \sigma(M_{\mathbf{z}}) \cup \sigma(M_{\mathbf{z}}, \mathcal{H}/M) = \overline{D} = \sigma(M_{\mathbf{z}}).$$

It is a well-known property of the analytic functional calculus (see Lemma 2.5.8 in [12]) that in this case M is invariant for $\phi(M_{\mathbf{z}})$, whenever ϕ is analytic on an open neighbourhood of $\sigma(M_{\mathbf{z}})$.
Finally we point out that in many cases all submodules of $\mathcal{H}$ are $\mathcal{O}(\overline{D})$-submodules. For example, this follows by the continuity of the functional calculus and the Oka-Weil Theorem whenever $\overline{D}$ is polynomially convex.
Before we proceed, we need to formulate the concept of "higher order kernels".

Lemma 4.1. *For every multiindex $\alpha \in \mathbb{N}_0^d$ and every $w \in D$, there exists a unique function $K_w^{(\alpha)} \in \mathcal{O}(\overline{D})$ satisfying*

$$D^\alpha f(w) = \langle f, K_w^{(\alpha)} \rangle$$

for all $f \in \mathcal{H}$. If $((w_1, \alpha_1), \ldots, (w_m, \alpha_m))$ are pairwise different, then the functions $K_{w_1}^{(\alpha_1)}, \ldots, K_{w_m}^{(\alpha_m)}$ are linearly independent in $\mathcal{H}$.

Proof. Since the inclusion mapping $\mathcal{H} \hookrightarrow \mathcal{O}(D)$ is continuous, the higher order point evaluation

$$\delta_w^{(\alpha)} : \mathcal{H} \to \mathbb{C} \ , \ f \mapsto D^\alpha f(w)$$

defines a continuous linear functional for every $\alpha \in \mathbb{N}_0^d$ and $w \in D$. Hence $K_w^{(\alpha)} = \delta_w^{(\alpha)^*}(1)$ is the unique function in $\mathcal{H}$ with

$$D^\alpha f(w) = \langle f, K_w^{(\alpha)} \rangle$$

for all functions $f \in \mathcal{H}$. Let us observe that

$$K_w^{(\alpha)}(z) = \langle \delta_w^{(\alpha)^*}(1), K(\cdot, z) \rangle = \langle 1, \delta_w^{(\alpha)} K(\cdot, z) \rangle = \overline{(D^\alpha K(\cdot, z))(w)}$$

for all $z, w \in D$ and $\alpha \in \mathbb{N}_0^d$.
It remains to show that the functions $K_w^{(\alpha)}$ belong to $\mathcal{O}(\overline{D})$. By assumption (G), there exist open neighbourhoods V of $\overline{D}$ and $U \subset D$ of w such that $K_{|U \times D}$ extends to a sesquianalytic function $H : U \times V \to \mathbb{C}$. But then

$$h : \tilde{V} \to \mathcal{O}(U) \ , \ z \mapsto H(\cdot, \overline{z}),$$

defined on the set $\tilde{V} = \{\overline{z} \ ; \ z \in V\}$, is analytic as a function with values in the Fréchet space $\mathcal{O}(U)$. Since continuous linear maps preserve analyticity, it follows that the function

$$V \to \mathbb{C} \ , \ z \mapsto \overline{(D^\alpha H(\cdot, z))(w)}$$

is analytic again and, as seen above, extends the function $K_w^{(\alpha)}$.
To see that the functions $K_{w_i}^{(\alpha_i)}$ $(1 \leq i \leq m)$ are linearly independent, choose polynomials $p_1, \ldots, p_m$ such that

$$D^{\alpha_i} p_j(w_i) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{else} \end{cases} .$$

The observation that

$$\overline{c_j} = \sum_{i=1}^{m} \overline{c_i} D^{\alpha_i} p_j(w_i) = \langle p_j, \sum_{i=1}^{m} c_i K_{w_i}^{(\alpha_i)} \rangle \quad (1 \leq j \leq m)$$

holds for any choice of complex numbers $c_1, \ldots, c_m$, completes the proof.

$\square$

The following definitions are, up to a slight reformulation, taken from [10]. Let $w \in D$ be arbitrary. For a polynomial $p = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C}[z]$ set

$$K_w^{(p)} = \sum_\alpha \overline{c_\alpha} K_w^{(\alpha)}.$$

Then

$$\langle f, K_w^{(p)} \rangle = \sum_\alpha c_\alpha D^\alpha f(w)$$

for $f \in \mathcal{H}$, and the mapping

$$\gamma_w : \mathbb{C}[z] \to \mathcal{H} \ , \ p \mapsto K_w^{(p)}$$

is antilinear and one-to-one by the preceding lemma.
Let M be a submodule of $\mathcal{H}$. Then

$$M_w = \gamma_w^{-1}(M^\perp) \subset \mathbb{C}[z]$$

is a linear subspace and the enveloping space of M defined by

$$M_w^e = (\gamma_w(M_w))^\perp \subset \mathcal{H}$$

is a submodule containing M. We refer to [10] for more details.
For an arbitrary subspace N of $\mathcal{H}$, we denote by $Z(N)$ the zero variety of N, that is,

$$Z(N) = \{z \in D \ ; \ f(z) = 0 \text{ for all } f \in N\}.$$

Now consider a finite-codimensional submodule M of $\mathcal{H}$. Then the zero sets of the enveloping spaces M_w^e have a very simple structure. More precisely, we observe that

$$Z(M_w^e) = \begin{cases} \{w\} & \text{if} \quad w \in Z(M) \\ \emptyset & \text{else} \end{cases}$$

holds for all $w \in D$. To prove this, we suppose first that $z \in Z(M_w^e)$. Then the function $K(\cdot, z)$ is contained in $\overline{\gamma_w(M_w)} = \gamma_w(M_w)$ since M_w has finite dimension by hypothesis. Therefore $K(\cdot, z)$ is a linear combination of the elements $K_w^{(\alpha)}$ and hence $z = w$. This proves the inclusion $Z(M_w^e) \subset \{w\}$. For obvious reasons, we have $Z(M_w^e) \subset Z(M)$. So it remains to show that $w \in Z(M_w^e)$ whenever $w \in Z(M)$. But $w \in Z(M)$ is equivalent to $\mathbf{1} \in M_w$ which implies $K(\cdot, w) \in \gamma_w(M_w)$. Hence $w \in Z(M_w^e)$.
The following result completely describes the finite-codimensional submodules of $\mathcal{H}$ by means of the enveloping spaces M_w^e and appears as Corollary 2.2.6 in [10] .

Lemma 4.2. *Suppose M is a finite-codimensional submodule of $\mathcal{H}$. Then we have*

1. *$Z(M)$ is a finite subset of D.*
2. *$M = \bigcap_{w \in Z(M)} M_w^e$.*
3. *$\dim M^\perp = \sum_{w \in Z(M)} \dim M_w$.*

We are now ready to conclude that, for every finite-codimensional submodule of M, the orthogonal complement of M consists of multipliers.

Proposition 4.3. *Assume that M is a finite-codimensional submodule of $\mathcal{H}$. Then the inclusions $M^\perp \subset \mathcal{O}(\overline{D}) \subset \mathcal{M}(\mathcal{H})$ hold.*

Proof. Assume first that $Z(M) = \{w\}$ for some $w \in D$. By Lemma 4.2, we obtain $M = M_w^e = (\gamma_w(M_w))^\perp$, and therefore $M^\perp = \gamma_w(M_w)$. Since every $K_w^{(p)}$ belongs to $\mathcal{O}(\overline{D})$ by Lemma 4.1, it follows that ran $\gamma_w \subset \mathcal{O}(\overline{D})$, . If $Z(M)$ is arbitrary, then for every $w \in Z(M)$, the subspace M_w^e is a finite-codimensional submodule with $Z(M_w^e) = \{w\}$, and thus $(M_w^e)^\perp \subset \mathcal{O}(\overline{D})$. Another application of Lemma 4.2 yields

$$M^\perp = \sum_{w \in Z(M)} (M_w^e)^\perp \subset \mathcal{O}(\overline{D}).$$

□

The main result of this section can now be stated.

Theorem 4.4. *Suppose that M is a finite-codimensional submodule of $\mathcal{H}$. Then M is Beurling decomposable. If in addition M has finite rank, then there exist multipliers $\phi_1, \ldots, \phi_s$ and $\psi_1, \ldots, \psi_t$ ($s + t =$ rank M) such that*

$$P_M = \sum_{i=1}^{s} M_{\phi_i} M_{\phi_i}^* - \sum_{j=1}^{t} M_{\psi_j} M_{\psi_j}^*$$

and

$$M = \sum_{i=1}^{s} \phi_i \cdot \mathcal{H}.$$

The functions $\phi_1, \ldots, \phi_s$ and $\psi_1, \ldots, \psi_t$ can be chosen in $\mathcal{O}(\overline{D})$.

Proof. By Propositions 4.3 and 3.5, the space M is Beurling decomposable. Suppose, in addition, that M has finite rank. Since, by condition (E), the functions

$$\beta(\cdot)\beta(w)^*(1) \quad \text{and} \quad \gamma(\cdot)\gamma(w)^*(1)$$

belong to $\mathcal{O}(\overline{D})$, it follows that $G_M(\cdot, w) \in \mathcal{O}(\overline{D})$ for $w \in D$ as well. To see this, recall that, by the proof of Proposition 3.5, the core function can be written as

$$G_M(z, w) = (1 + \gamma(z)\gamma(w)^*(1)K_{M^\perp}(z, w)) - (\beta(z)\beta(w)^*(1)K_{M^\perp}(z, w)).$$

Therefore ran Δ_M, being the linear span of the functions $G_M(\cdot, w)$, is contained in $\mathcal{O}(\overline{D})$. By Proposition 3.6, there are multipliers $\phi_1, \ldots, \phi_s$ and $\psi_1, \ldots, \psi_t$ in ran Δ_M allowing the claimed representations of P_M and M. □

As an application, we compute the right essential spectrum $\sigma_{re}(M_{\mathbf{z}})$ of the commuting tuple $M_{\mathbf{z}}$. Recall that the right essential spectrum of a commuting tuple $T \in L(H)^d$ is the set of all $\lambda \in \mathbb{C}^d$ for which the last cohomology group in the Koszul complex of $\lambda - T$ has infinite dimension. Equivalently, $\lambda \in \mathbb{C}^d$ is not in the right essential spectrum of T exactly if the row operator $(T_1, \ldots, T_d) \in L(H^d, H)$ has finite-codimensional range.

Proposition 4.5. *Suppose that the inverse kernel is a polynomial in $z, \overline{w}$. Then $\sigma_{re}(M_{\mathbf{z}}) = \partial D$.*

Proof. First of all, observe that $\sigma_{re}(M_{\mathbf{z}}) \subset \sigma(M_{\mathbf{z}}) \subset \overline{D}$. We are now going to prove that

$$\sigma_{re}(M_{\mathbf{z}}) \cap D = \emptyset.$$

To this end, fix $\lambda \in D$ and let M_λ be the finite-codimensional submodule

$$M_\lambda = \{f \in \mathcal{H} \; ; \; f(\lambda) = 0\} = \{K(\cdot, \lambda)\}^\perp.$$

By Example 2, the submodule M_λ has finite rank, and Theorem 4.4 shows that there exist multipliers $\phi_1, \ldots, \phi_s \in \mathcal{O}(\overline{D})$, such that

$$M_\lambda = \sum_{i=1}^{s} \phi_i \cdot \mathcal{H}.$$

The row operator $(M_{\phi_1}, \ldots, M_{\phi_s}) \in L(\mathcal{H}^s, \mathcal{H})$ consequently has finite-codimensional range. This means that 0 is not in the right essential spectrum of the commuting tuple

$$M_\phi = (M_{\phi_1}, \ldots, M_{\phi_s}) \in L(\mathcal{H})^s.$$

By the spectral mapping theorem for the right essential spectrum (Corollary 2.6.9 in [12]), we have

$$\sigma_{re}(M_\phi) = \phi(\sigma_{re}(M_{\mathbf{z}})).$$

Since $\phi(\lambda) = 0$, it follows that $\lambda \notin \sigma_{re}(M_{\mathbf{z}})$. This proves that $\sigma_{re}(M_{\mathbf{z}}) \subset \partial D$. Suppose conversely that λ is in the boundary of D. Then λ is not a virtual point of $\mathcal{H}$. As observed in [10], this is equivalent to the fact that the maximal ideal of $\mathbb{C}[z]$ at λ is dense in $\mathcal{H}$, in other words

$$\overline{\sum_{i=1}^{d} (\lambda_i - M_{\mathbf{z}_i})\mathcal{H}} = \overline{\sum_{i=1}^{d} (\lambda_i - M_{\mathbf{z}_i})\mathbb{C}[z]} = \mathcal{H}.$$

Assume now that $\lambda \notin \sigma_{re}(M_{\mathbf{z}})$. Then the space

$$\sum_{i=1}^{d} (\lambda_i - M_{\mathbf{z}_i})\mathcal{H} \subset \mathcal{H}$$

is closed and therefore equals $\mathcal{H}$. Since the surjectivity spectrum is closed, there exists some $r > 0$ such that

$$\sum_{i=1}^{d} (\mu_i - M_{\mathbf{z}_i})\mathcal{H} = \mathcal{H}$$

holds for all $\mu \in \mathbb{C}^d$ with $|\mu - \lambda| < r$. Hence there would have to be a point $\mu \in D$ with $\mathbf{1} \in \sum_{i=1}^{d} (\mu_i - M_{\mathbf{z}_i})\mathcal{H}$. This contradiction completes the proof. $\square$

We are now able to give the following supplement to the Ahern-Clark type result stated in [10] as Theorem 2.2.3.

Corollary 4.6. *Suppose that $\frac{1}{K}$ is a polynomial in z and $\overline{w}$. Then the finite-codimensional submodules of $\mathcal{H}$ are exactly the closed subspaces M of the form $M = \sum_{i=1}^{r} p_i \cdot \mathcal{H}$ where $r \in \mathbb{N}$ and $p = (p_1, \ldots, p_r)$ is a tuple of polynomials with $Z(p) \subset D$.*

Proof. Suppose that M is a finite-codimensional submodule of $\mathcal{H}$. By Theorem 2.2.3 in [10], the intersection $M \cap \mathbb{C}[z]$ is a finite-codimensional ideal in $\mathbb{C}[z]$ with $Z(I) \subset D$ and $M = \overline{I}$. Now we choose a generating set $p = (p_1, \ldots, p_r)$ of I and claim that $M = \sum_{i=1}^r p_i \cdot \mathcal{H}$. Since

$$M = \overline{I} = \overline{\sum_{i=1}^r p_i \cdot \mathbb{C}[z]} = \overline{\sum_{i=1}^r p_i \cdot \mathcal{H}},$$

it suffices to show that the row operator $(M_{p_1}, \ldots, M_{p_r}) \in L(\mathcal{H}^r, \mathcal{H})$ has closed range. But this is obvious, because $Z(p) = Z(I) \subset D$ and $\sigma_{re}(M_{\mathbf{z}}) = \partial D$, and hence

$$0 \notin \sigma_{re}(M_{p_1}, \ldots, M_{p_r}) = p(\sigma_{re}(M_{\mathbf{z}})).$$

□

The proof shows that the polynomials $p_1, \ldots, p_r$ can be chosen as a generating set of the Ideal $M \cap \mathbb{C}[z]$. If in particular $d = 1$, then we can achieve that $r = 1$.

Note also that, under the same hypotheses, Gleason's problem can be solved in $\mathcal{H}$. Recall that Gleason's problem is, for a given function $f \in \mathcal{H}$ and $\lambda \in D$, to find functions $g_1, \ldots, g_d \in \mathcal{H}$ satisfying

$$f(z) - f(\lambda) = \sum_{i=1}^d (z_i - \lambda_i) g_i(z) \quad (z \in D).$$

To solve Gleason's problem, it is therefore sufficient to apply Corollary 4.6 to the submodule $M_\lambda = \{h \in \mathcal{H} \ ; \ h(\lambda) = 0\}$.

References

[1] A. Aleman, S. Richter and C. Sundberg, *Beurling's theorem for the Bergman space*, Acta Math. **177** (1996), 275-310

[2] D. Alpay, *Some remarks on reproducing kernel Krein spaces*, Rocky Mt. J. Math. **21** (1991), 1189-1205

[3] C. Ambrozie and J.Eschmeier, *A commutant lifting theorem on analytic polyhedra*, Warsaw: Polish Academy of Sciences, Institute of Mathematics. Banach Center Publications **67** (2005), 83-108

[4] J. Arazy and G. Zhang, *Homogeneous multiplication operators on bounded symmetric domains*, J.Funct. Anal. **202** (2003), 44-66

[5] N. Aronszajn, *Theory of reproducing kernels*, Trans. Am. Math. Soc. **68** (1950), 337-404

[6] F.A. Berezin, *Covariant and contravariant symbols of operators*, Math. USSR Izv. **6** (1972), 1117-1151

[7] F.A. Berezin, *Quantization*, Math. USSR Izv. **8** (1974), 1109-1163

[8] F. Beatrous Jr. and J. Burbea, *Positive Definiteness and its Applications to Interpolation Problems for Holomorphic Functions*, Trans. Am. Math. Soc. **284** (1984), 247-270

[9] J. Burbea and P. Masani, *Banach and Hilbert spaces of vector-valued functions. Their general theory and applications to holomorphy*, Research Notes in Mathematics **90**, Boston-London-Melbourne, Pitman Advanced Publishing Program (1984)

[10] X. Chen and K. Guo, *Analytic Hilbert Modules*, Chapman & Hall/CRC (2003)

[11] S. McCullough and T.T. Trent, *Invariant Subspaces and Nevanlinna-Pick Kernels*, J. Funct. Anal. **178** (2000), 226-249

[12] J. Eschmeier and M. Putinar, *Spectral decompositions and analytic sheaves*, London Mathematical Society Monographs, Clarendon Press, Oxford (1996)

[13] M. Englis, *Some Problems in Operator Theory on Bounded Symmetric Domains*, Act. Appl. Math. **81** (2004), 51-71

[14] J. Faraut and A. Koranyi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), 64-89

[15] D. Greene, S. Richter and C. Sundberg, *The structure of inner multipiers on spaces with complete Nevanlinna Pick kernels*, J. Funct. Anal. **194** (2002), 311-331

[16] K. Guo, *Defect operators, defect functions and defect indices for analytic submodules*, J. Funct. Anal. **213** (2000), 380-411

[17] K. Guo and R. Yang, *The core function of submodules over the bidisk*, Indiana Univ. Math. J. **53** (2004), 205-222

[18] K. Guo and D. Zheng, *Invariant subspaces, quasi-invariant subspaces, and Hankel operators*, J. Funct. Anal. **187** (2001), 308-342

[19] P. Quiggin, *For which reproducing kernel Hilbert spaces is Pick's theorem true?*, Integral Equations Oper. Theory **16** (1993), 244-266

[20] W. Rudin, *Function theory in polydiscs*, Mathematics Lecture Notes Series. New York Amsterdam, W.A. Benjamin, Inc.(1969)

[21] L. Schwartz, *Sous-espaces d'espaces vectoriels topologiques et noyaux associés. (Noyaux reproduisants.)*, J. Anal. Math **13** (1964), 115-256