Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 170

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Saarbrücken 2006

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 170
submitted: March 20, 2006

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Abstract*

This paper deals with the construction of nonorthogonal and orthogonal polynomial bases for particular subspaces of vector fields defined in the unit ball of $\mathbb{R}^3$. We present two different techniques to obtain such bases. The first approach uses vector spherical harmonics to construct orthogonal sets of solenoidal and potential vector fields by means of ridge functions. The second one applies differential operators of vector analysis to known polynomial basis functions defined in the unit ball. This procedure can be generalized to arbitrary Rimannian metrics without any difficulties. It is shown that both approaches lead to bases according to the subspaces induced by the Helmholtz-Hodge decomposition of square integrable vector fields. In addition the bases relying on the approach by ridge functions are othogonal.

1. INTRODUCTION

Many complex problems of mathematical physics arising in theory of elasticity, gas dynamics, hydrodynamics, geotomography, tensor and vector field tomography, reduce to a mathematical formulation with projections of vector or tensor fields as unknown quantities. Particularly vector fields are the most adequate tools to investigate the distribution of flow velocity, stress in some specimen or electromagnetic fields in plasma physics, see e.g. [21], [22], [23], [28] for details.

The problem of vector tomography consists of reconstructing an unknown vector field in a bounded domain when its ray or normal Radon transform is given. In contrast to scalar computerized tomography the vector tomography problem does not have a unique solution since the ray as well as the normal Radon transform has a non-trivial null space. According to the Helmholtz decomposition every vector field can be represented as the sum of a potential and a solenoidal vector field. Since this decomposition is not unique, different conditions to guarantee uniqueness in certain Hilbert spaces have been discussed in the past, see [29], [26], [6], [1]. The kernel of the ray transform consists precisely of the potential fields with vanishing boundary values. Hence, it turns impossible to recover the potential part of a field from the mere knowledge of its ray transform. Only the solenoidal part can be detected. The null space of the Radon normal transform in turn is equal to the subspace of solenoidal vector fields with vanishing normal component at the boundary, see e.g. [20]. This is why the potential part of a searched field can be recovered from its Radon normal transform. In that sense the ray transform and the Radon normal transform represent a complementary pair of integral transforms: The entire field can be reconstructed if one has both transforms as data available and the null space of one transform consists exactly of those fields which can be recovered by the other one.

About twenty years ago Louis [14] and Maaß [15] published the singular value decompositions (SVD) of the n-dimensional Radon transform and the 3D-ray transform, respectively. To the authors best knowledge corresponding results concerning vector tomography

*The work was partially supported by Deutsche Forschungsgemeinschaft (DFG) (under grant Schu 1978/1-5), RFBR-DFG (grant № 04-01-04003), RFBR (grants № 05-01-00250, № 05-01-00611), and Mathematical Department of RAS (grant 1.3.6)

still are unknown. This preprint is to be understood as the first step towards the determination of SVDs for both, the vectorial ray and Radon normal transform on the unit ball $\mathbb{B}^3$ of $\mathbb{R}^3$. The presented bases of vector field subspaces in $L^2(\mathbb{B}^3, \mathbf{S}_1)$ should be appropriate for a subsequent deduction of SVDs for the two integral operators in the near future.
The construction of the mentioned nonorthogonal and orthogonal bases is accomplished in two different ways. The first approach uses classical spherical harmonics as starting point to generate vector spherical harmonics. Similar results can be found in the work of Michel [17], who constructed a generalization of spherical harmonics for vector fields defined on the unit sphere. With the help of ridge functions we in contrast derive vector valued functions supported in $\mathbb{B}^3$ which build orthogonal sets of potential and solenoidal fields. The second way achieves a set of linearly independent vector fields with certain properties applying differential operators such as ∇, rot and div to polynomials. Polynomial bases of solenoidal and potential vector fields for the unit disk in $\mathbb{R}^2$ were derived in [11], [12].
The paper consists of three parts. Section 2 contains the mathematical setup with all necessary notations and definitions and further provides an introduction to the Helmholtz-Hodge decomposition of vector fields, Zernike polynomials and spherical harmonics. Furthermore some properties and relations between classical orthogonal polynomials are established. In Section 3 we give an outline of the deduction of orthogonal vector spherical harmonics in $\mathbb{B}^3$ from classical spherical harmonics and construct bases of orthogonal ridge functions for subspaces of solenoidal and potential vector fields associated with the Helmholtz-Hodge decomposition. The construction of bases by applying differential operators to polynomials finally is described in Section 4. Some fundamentals of orthogonal polynomials as well as basic properties of Legendre, Gegenbauer, associated Legendre and Jacobi polynomials are listed in the Appendix.

2. PRELIMINARIES

In this first section we collect all necessary tools for our further investigations. We define necessary function spaces, important differential operators from vector analysis, we give a brief outline of the Helmholtz- Hodge decomposition, introduce the Zernike polynomials and some needful properties of orthogonal polynomials.
Let a Cartesian rectangular coordinate system be given in the Euclidean space $\mathbb{R}^3$. For a point $\mathbf{x} \in \mathbb{R}^3$ we use the notation $\mathbf{x} = (x^1, x^2, x^2)$ as well as $\mathbf{x} = (x, y, z)$ for their coordinates. Vectors are written in bold face letters $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)^\top$, or $\mathbf{a} = (a^1, a^2, a^3)^\top$ The scalar product of vectors $\mathbf{x}$, $\mathbf{y}$ is denoted by $\mathbf{x} \cdot \mathbf{y}$, and the Euclidean norm by $|\mathbf{x}| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. By $\mathbb{B}^3 = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| < 1\}$ and $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 \mid |\boldsymbol{\xi}| = 1\} = \partial\mathbb{B}^3$ we define the unit ball and unit sphere in $\mathbb{R}^3$, respectively.
The space of vector fields – which is equal to the space of tensor fields of rank 1 – defined on $\mathbb{B}^3$ and $\mathbb{S}^2$ is written as $\mathbf{S}_1(\mathbb{B}^3)$, $\mathbf{S}_1(\mathbb{S}^2)$, respectively. We omit the domain $\mathbb{B}^3$ or $\mathbb{S}^2$ if a confusion is impossible. Tensor fields of rank 0 are called *scalar fields* or *potentials* if they are used to generate potential fields. For $k \geq 0$ we introduce function spaces $C^k(\mathbb{B}^3)$, $C^k(\mathbb{S}^2)$, $C_0^k(\mathbb{B}^3)$ of k-times continuously differentiable scalar fields and Sobolev spaces of order k which are denoted by $H^k(\mathbb{B}^3)$, $H^k(\mathbb{S}^2)$, $H_0^k(\mathbb{B}^3)$ as usual. We remind that $C_0^k(\mathbb{B}^3)$ consists of those functions from $C^k(\mathbb{B}^3)$ having compact support in $\mathbb{B}^3$ and $H_0^k(\mathbb{B}^3)$ is the H^k-closure of $C_0^\infty(\mathbb{B}^3)$. The spaces $C^k(\mathbb{B}^3, \mathbf{S}_1)$, $C^k(\mathbb{S}^2, \mathbf{S}_1)$, $C_0^k(\mathbb{B}^3, \mathbf{S}_1)$ accordingly contain k-times continously differentiable vector fields and the Sobolev spaces $H^k(\mathbb{B}^3, \mathbf{S}_1)$, $H^k(\mathbb{S}^2, \mathbf{S}_1)$, $H_0^k(\mathbb{B}^3, \mathbf{S}_1)$ consist of vector fields with components being elements of the corresponding

scalar Sobolev spaces.
The space of square integrable vector fields $L_2(\mathbb{B}^3, \mathbf{S}_1) = H^0(\mathbb{B}^3, \mathbf{S}_1)$ is a Hilbert space with inner product $(\cdot, \cdot)$ and norm $||\cdot||$,

$$
\begin{aligned}
(\mathbf{a},\ \mathbf{b}) &= (\mathbf{a},\ \mathbf{b})_{\mathbf{L}_2(\mathbb{B}^3,\mathbf{S}_1)} = \int\limits_{\mathbb{B}^3} (\mathbf{a}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x})) d\mathbf{x}, \\
||\mathbf{a}||^2 &= ||\mathbf{a}||^2_{\mathbf{L}_2(\mathbb{B}^3,\mathbf{S}_1)} = (\mathbf{a},\ \mathbf{a}).
\end{aligned}
$$

The inner product and norm in the space $L_2(\mathbb{S}^2, \mathbf{S}_1) = H^0(\mathbb{S}^2, \mathbf{S}_1)$ are defined accordingly. We continue by recalling the basic differential operators of vector analysis. The operator of gradient $\nabla : H^k(\mathbb{B}^3) \to H^{k-1}(\mathbb{B}^3, \mathbf{S}_1)$, $k \geq 1$, acts on a scalar field $\varphi(\mathbf{x})$ as

$$
\nabla\varphi = \left(\frac{\partial\varphi}{\partial x^1}, \frac{\partial\varphi}{\partial x^2}, \frac{\partial\varphi}{\partial x^3} \right).
$$

A vector field $\mathbf{a} \in H^{k-1}(\mathbb{B}^3, \mathbf{S}_1)$ is called a *potential field*, if $\mathbf{a} = \nabla\varphi$ holds true for some function $\varphi \in H^k(\mathbb{B}^3)$. In that case we call φ the *potential* of $\mathbf{a}$. The operator of divergence $\operatorname{div} : H^k(\mathbb{B}^3, \mathbf{S}_1) \to H^{k-1}(\mathbb{B}^3)$, $k \geq 1$, acts on a vector field $\mathbf{a}$ as

$$
\operatorname{div}\mathbf{a} = \frac{\partial a_1}{\partial x^1} + \frac{\partial a_2}{\partial x^2} + \frac{\partial a_3}{\partial x^3}.
$$

A vector field $\mathbf{a} \in H^k(\mathbb{B}^3, \mathbf{S}_1)$ is called *solenoidal* if its divergence is equal to zero, $\operatorname{div}\mathbf{a} = 0$. The operator of curl $\operatorname{rot} : H^k(\mathbb{B}^3, \mathbf{S}_1) \to H^{k-1}(\mathbb{B}^3, \mathbf{S}_1)$, $k \geq 1$, acts on a vector field $\mathbf{a}(\mathbf{x})$ according to

$$
\operatorname{rot}\mathbf{a} = \left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}, \frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1}, \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right).
$$

Obviously,

$$
\operatorname{rot}(\nabla\varphi) = \mathbf{0}, \quad \operatorname{div}(\operatorname{rot}\mathbf{a}) = 0, \quad \operatorname{div}(\nabla\varphi) = \Delta\varphi
$$

are valid for any scalar field $\varphi \in H^2(\mathbb{B}^3)$ and vector field $\mathbf{a} \in H^2(\mathbb{B}^3, \mathbf{S}_1)$. Moreover, solenoidal and potential fields can be characterized uniquely by means of these three fundamental differential operators.

Proposition 2.1. *a) For every vector field* $\mathbf{a} \in H^1(\mathbb{B}^3, \mathbf{S}_1)$ *satisfying* $\operatorname{rot}\mathbf{a} = \mathbf{0}$*, there exists a potential* $\varphi \in H^2(\mathbb{B}^3)$ *such that* $\mathbf{a} = \nabla\varphi$*.*
b) For every solenoidal field $\mathbf{a} \in H^1(\mathbb{B}^3, \mathbf{S}_1)$*, i.e.* $\operatorname{div}\mathbf{a} = 0$*, there exists a vector field* $\mathbf{v} \in H^2(\mathbb{B}^3, \mathbf{S}_1)$ *such that* $\mathbf{a} = \operatorname{rot}\mathbf{v}$*.*
c) For every solenoidal field $\mathbf{a} \in H^1(\mathbb{B}^3, \mathbf{S}_1)$*,* $\operatorname{div}\mathbf{a} = 0$ *with vanishing normal component at the boundary, i.e.* $\boldsymbol{\nu} \cdot \mathbf{a} = 0$ *on* $\partial\mathbb{B}^3$*, where* $\boldsymbol{\nu}$ *is the outer normal to the unit sphere* $\partial\mathbb{B}^3$*, there exists a vector field* $\mathbf{v} \in H^2(\mathbb{B}^3, \mathbf{S}_1)$ *such that* $\mathbf{a} = \operatorname{rot}\mathbf{v}$ *and* $\mathbf{v}\,|_{\partial\mathbb{B}^3} = \mathbf{0}$*.*

Proof. Statement *a)* and *b)* is a well known fact in vector analysis and can be found e.g. in the book of Kochin [13]. Assertion *c)* is a consequence of Stoke's theorem. □

2.1. The Helmholtz- and Hodge-decomposition of vector fields

In this subsection we summarize the main results about decomposition of vector fields as they have been presented in [24].
The Helmholtz-decomposition for $\mathbf{f} \in H^1(\mathbb{B}^3, \mathbf{S}_1)$ reads

$$\mathbf{f} = \mathbf{f}^s + \nabla v\,, \quad \operatorname{div} \mathbf{f}^s = 0\,, \quad v \in H_0^1(\mathbb{B}^3)\,. \tag{2.1}$$

It says that every vector field can be written as the sum of a solenoidal part $\mathbf{f}^s$ and an irrotational part ∇v. Decomposition (2.1) even holds true for $\mathbf{f} \in L_2(\mathbb{B}^3, \mathbf{S}_1)$ when we define solenoidal fields in the weak sense

$$\int_{\mathbb{B}^3} (\mathbf{f}^s(x) \cdot \nabla p(x))\, dx = 0\,, \quad \text{for all } p \in C_0^\infty(\mathbb{B}^3)\,,$$

which is equivalent to $\operatorname{div} \mathbf{f}^s = 0$ due to the theorem of Gauss-Ostrogradsky. Thus, (2.1) allows for a splitting of $L_2(\mathbb{B}^3, \mathbf{S}_1)$ into orthogonal subspaces

$$L_2(\mathbb{B}^3, \mathbf{S}_1) = H(\mathrm{div}; \mathbb{B}^3) \oplus \nabla H_0^1(\mathbb{B}^3)\,, \tag{2.2}$$

where

$$H(\mathrm{div}; \mathbb{B}^3) := \Big\{ \mathbf{f} \in L_2(\mathbb{B}^3, \mathbf{S}_1) \mid (\mathbf{f}, \nabla p)_{L_2(\mathbb{B}^3, \mathbf{S}_1)} = 0 \quad \text{for all } p \in C_0^\infty(\mathbb{B}^3) \Big\}$$

denotes the space of solenoidal fields in $L_2(\mathbb{B}^3, \mathbf{S}_1)$.
The space $H(\mathrm{div}; \mathbb{B}^3)$ itself again can be split into two orthogonal subspaces. To see this we write every $\mathbf{f}^s \in H_0(\mathrm{div}; \mathbb{B}^3)$ as a sum $\mathbf{f}^s = \mathbf{f}_1^s + \mathbf{f}_2^s$ where $\mathbf{f}_1^s$, $\mathbf{f}_2^s$ solve

$$\left\{ \begin{array}{l} \operatorname{rot} \mathbf{f}_1^s = \operatorname{rot} \mathbf{f}^s \text{ in } \mathbb{B}^3\,, \\ \operatorname{div} \mathbf{f}_1^s = 0 \text{ in } \mathbb{B}^3\,, \\ \boldsymbol{\nu} \cdot \mathbf{f}_1^s = 0 \text{ on } \partial\mathbb{B}^3\,, \end{array} \right\} \quad \left\{ \begin{array}{l} \operatorname{rot} \mathbf{f}_2^s = 0 \text{ in } \mathbb{B}^3\,, \\ \operatorname{div} \mathbf{f}_2^s = 0 \text{ in } \mathbb{B}^3\,, \\ \boldsymbol{\nu} \cdot \mathbf{f}_2^s = \boldsymbol{\nu} \cdot \mathbf{f}^s \text{ on } \partial\mathbb{B}^3\,, \end{array} \right\} \tag{2.3}$$

respectively. Each of the two boundary value problems (BVP) in (2.3) has a unique solution and hence $\mathbf{f}_1^s$, $\mathbf{f}_2^s$ are well-defined. Further we deduce from the second BVP that $\mathbf{f}_2^s = \nabla h$ for a harmonic function h. The vector space of harmonic functions can be described by

$$\mathrm{Harm}(\mathbb{B}^3) := \Big\{ h \in H^1(\mathbb{B}^3) \mid \int_{\mathbb{B}^3} (\nabla h(x) \cdot \nabla p(x)) dx = 0 \text{ for all } p \in C_0^\infty(\mathbb{B}^3) \Big\}\,.$$

Again we used the Gauss-Ostrogradsky theorem to get a weak formulation of harmonic functions which is even valid in $H^1(\mathbb{B}^3)$. Fields $\mathbf{f}_1^s$ fulfilling the first BVP in (2.3) are solenoidal fields with tangential flow at the boundary,

$$H_0(\mathrm{div}; \mathbb{B}^3) := \Big\{ \mathbf{f} \in L_2(\mathbb{B}^3, \mathbf{S}_1) \mid \operatorname{div} \mathbf{f} = 0\,,\ x \in \mathbb{B}^3\,,\ \boldsymbol{\nu} \cdot \mathbf{f}(x) = 0\,, x \in \partial\mathbb{B}^3 \Big\}\,.$$

By means of the Gauss-Ostrogradsky theorem it is easy to prove that $\mathbf{f}_1^s \perp \mathbf{f}_2^s$. This means that the solenoidal fields in $H(\mathrm{div}; \mathbb{B}^3)$ can be decomposed into a sum of solenoidal fields with tangential flow at the boundary and gradients of harmonic functions

$$H(\mathrm{div}; \mathbb{B}^3) = H_0(\mathrm{div}; \mathbb{B}^3) \oplus \nabla \mathrm{Harm}(\mathbb{B}^3)\,. \tag{2.4}$$

The elements of $\nabla\mathrm{Harm}(\mathbb{B}^3)$ are called *harmonic fields*. Putting together (2.2) and (2.4) we get the Hodge-decomposition for square integrable vector fields.

Theorem 2.2. *Every vector field $\mathbf{f} \in L_2(\mathbb{B}^3, \mathbf{S}_1)$ can be decomposed into a solenoidal field $\mathbf{f}_1^s$ with tangential flow at the boundary $\partial\mathbb{B}^3$, a harmonic field $\mathbf{f}_2^s = \nabla h$ with h harmonic and an irrotational part ∇v, $v \in H_0^1(\mathbb{B}^3)$ such that*

$$\mathbf{f} = \mathbf{f}_1^s + \nabla h + \nabla v\,. \tag{2.5}$$

That means, the space of square integrable vector fields $L_2(\mathbb{B}^3, \mathbf{S}_1)$ is an orthogonal sum of three subspaces

$$L_2(\mathbb{B}^3, \mathbf{S}_1) = H_0(\mathrm{div}; \mathbb{B}^3) \oplus \nabla \mathrm{Harm}(\mathbb{B}^3) \oplus \nabla H_0^1(\mathbb{B}^3) .$$

The Hodge decomposition (2.5) can be interpreted as follows: $\mathbf{f}_1^s$ contains all information about the curl of the field $\mathrm{rot}\,\mathbf{f}$, the harmonic field ∇h contains information about the boundary values $\boldsymbol{\nu} \cdot \mathbf{f}$ and the potential part tells everything about the divergence $\mathrm{div}\mathbf{f}$.
The space of harmonic fields $\nabla\mathrm{Harm}(\mathbb{B}^3)$ is closed in $L_2(\mathbb{B}^3, \mathbf{S}_1)$ with respect to the L_2-norm topology. In fact $\nabla\mathrm{Harm}(\mathbb{B}^3)$ is the orthogonal complement of $\nabla H_0^1(\mathbb{B}^3)$ in $\nabla H^1(\mathbb{B}^3)$. A complete system of orthonormal harmonic fields can be generated by means of spherical harmonics, see [24].
So far we have five fundamental subspaces of vector fields $\mathbf{f} \in L_2(\mathbb{B}^3, \mathbf{S}_1)$: The solenoidal fields $H(\mathrm{div}; \mathbb{B}^3)$, the solenoidal fields with tangential flow at the boundary $H_0(\mathrm{div}; \mathbb{B}^3)$, the harmonic fields $\nabla\mathrm{Harm}(\mathbb{B}^3)$, the potential fields $\nabla H^1(\mathbb{B}^3)$ and the potential fields $\nabla H_0^1(\mathbb{B}^3)$ with potentials vanishing on the boundary.
In fluid mechanics the harmonic component of a vector field, which is at the same time solenoidal and irrotational, is also called *laminar component.*
An overview of historical treatises which led to the Helmholtz-Hodge decomposition theorem and the proof for 3D case can be found in [13]. Concerning this problem we refer also to [2], [8], [25], [29].

2.2. Spherical harmonics

Let $d\omega$ be the surface measure on $\mathbb{S}^2$. We have $\int_{\mathbb{S}^2} d\omega(\boldsymbol{\xi}) = 4\pi$. We remind that $L_2(\mathbb{S}^2)$ is a Hilbert space with inner product

$$(\mathbf{a}\,,\, \mathbf{b}) = (\mathbf{a}\,,\, \mathbf{b})_{L_2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} (\mathbf{a}(\boldsymbol{\xi}) \cdot \mathbf{b}(\boldsymbol{\xi})) d\omega(\boldsymbol{\xi})\,,$$

and norm

$$||\mathbf{a}||^2 = ||\mathbf{a}||^2_{L_2(\mathbb{S}^2)} := (\mathbf{a}, \mathbf{a})\,.$$

Any point $\boldsymbol{\xi} \in \mathbb{S}^2$ can be represented in spherical coordinates

$$\boldsymbol{\xi} = \sqrt{1-t^2}(\cos\varphi\, \mathbf{e}_x + \sin\varphi\, \mathbf{e}_y) + t\, \mathbf{e}_z, \quad 0 \le \varphi < 2\pi, \quad -1 \le t \le 1, \quad t = \cos\vartheta, \tag{2.6}$$

where ϑ corresponds to the latitude, φ is the longitude and t the polar distance. Equivalently we have

$$\boldsymbol{\xi} = \sin\vartheta\, \cos\varphi\, \mathbf{e}_x + \sin\vartheta\, \sin\varphi\, \mathbf{e}_y + \cos\vartheta\, \mathbf{e}_z\,,$$

where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are orthogonal cartesian reference vectors. The complex spherical harmonics $\{Y_{n,l}(\boldsymbol{\xi})\}$ are related to the associated Legendre polynomials P_n^m (see Appendix A3 for a definition) by

$$Y_{n,m}(\boldsymbol{\xi}) = N_{n,m}\, e^{\mathrm{i}m\varphi} P_n^m(t), \;\; N_{n,m} = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}, \;\; |m| \le n, \tag{2.7}$$

where $N_{n,m}$ are normalization factors. For negative indices m the associated Legendre polynomials are defined by (5.13), (5.15). The spherical harmonics are orthonormal with respect to the $L_2(\mathbb{S}^2)$-inner product

$$\int_{\mathbb{S}^2} Y_{n,l}(\boldsymbol{\xi})\overline{Y_{m,k}(\boldsymbol{\xi})}\, d\omega(\boldsymbol{\xi}) = \delta_{nm}\delta_{jk}.$$

The set of spherical harmonics of degree n is invariant with respect to rotations in the sense that

$$Y_{n,m}(A_{\alpha,\beta,\gamma}\boldsymbol{\xi}) = \sum_{m'=-n}^{n} D^{(n)}_{m,m'}(\alpha,\beta,\gamma) Y_{n,m'}(\boldsymbol{\xi}) \tag{2.8}$$

for every $A_{\alpha,\beta,\gamma} \in SO(3)$, where α, β and γ denote the corresponding Euler angles. Note that the summation index involves m only and not the degree n of $Y_{n,m}$. If we apply a rotation to a spherical harmonic of degree n as in (2.8), the resulting function is still in $\operatorname{span}\{Y_{n,m} \,|\, m = -n,\dots,n\}$. The matrix $D^{(n)}$ contains the coefficients of that linear combination. Furthermore the matrix $D^{(n)}$ is the $(2n+1)$-dimensional representation of the orthogonal group $SO(3)$ and is known as *matrix of rotation* or *Wigner's matrix.* It satisfies

$$D^{(n)}_{m,m'}(\alpha,\beta,\gamma) = \int_{\mathbb{S}^2} Y_{n,m}(A_{\alpha,\beta,\gamma}\boldsymbol{\xi})\overline{Y_{n,m'}(\boldsymbol{\xi})}\, d\omega(\boldsymbol{\xi})\,.$$

The orthogonality relation for $D^{(n)}$ can be found in standard textbooks on group theory, for instance Inui et al. [Eq. (77.3), 9] or [30]). It is given as

$$\int_{\gamma=0}^{2\pi}\int_{\beta=0}^{2\pi}\int_{\alpha=0}^{\pi} \overline{D^{(n)}_{m,m'}(\alpha,\beta,\gamma)} D^{(N)}_{M,M'}(\alpha,\beta,\gamma) \sin\alpha\, d\alpha\, d\beta\, d\gamma = \frac{8\pi^2}{2n+1}\delta_{nN}\delta_{mM}\delta_{m'M'}. \tag{2.9}$$

Note that $\sin\alpha\, d\alpha\, d\beta\, d\gamma$ is the Haar measure $d\mu$ of the orthogonal group $SO(3)$.
For the special case $\gamma = 0$ which is relevant for this paper we have

$$D^{(n)}_{m,m'}(\alpha,\beta) := D^{(n)}_{m,m'}(\alpha,\beta,0) = N^{(m')}_{n,m}\, e^{\mathrm{i}m\beta}\, P^{m,m'}_{n}(\cos\alpha)\,, \quad N^{(0)}_{n,m} = N_{n,m},$$

where $P^{m,m'}_n$ are the *generalized associated Legendre polynomials* and $N^{(m')}_{n,m}$ are their normalization factors. Explicit representations of $P^{m,m'}_n$ are found in [3] along with more details concerning rotations of spherical harmonics. Putting $\gamma = 0$ weakens the orthogonality relation (2.9) of the Wigner matrix, since the integration with respect to γ is omitted. This yields

$$\int_{\beta=0}^{2\pi}\int_{\alpha=0}^{\pi} \overline{D^{(n)}_{m,m'}(\alpha,\beta,\gamma)} D^{(N)}_{M,m'}(\alpha,\beta,\gamma) \sin\alpha\, d\alpha\, d\beta = \frac{4\pi}{2n+1}\delta_{nN}\delta_{mM}\,. \tag{2.10}$$

Note that we lost the orthogonality with respect to m'. When $\gamma = 0$, equation (2.8) can be rewritten as

$$Y_{n,m}(A_{\boldsymbol{\theta}}\boldsymbol{\xi}) = \sum_{m'=-n}^{n} D^{(n)}_{m,m'}(\boldsymbol{\theta}) Y_{n,m'}(\boldsymbol{\xi})\,,$$

where $\boldsymbol{\theta} = (\cos\beta\sin\alpha, \sin\beta\sin\alpha, \cos\alpha)$.
The *summation formulas for spherical harmonics* read

$$P_n(\boldsymbol{\eta}\cdot\boldsymbol{\xi}) \quad = \frac{4\pi}{2n+1}\sum_{l=-n}^{n} Y_{n,l}(\boldsymbol{\eta})\overline{Y_{n,l}(\boldsymbol{\xi})}, \tag{2.11}$$

$$C^{(3/2)}_n(\boldsymbol{\eta}\cdot\boldsymbol{\xi}) = 4\pi\sum_{k=0}^{[n/2]}\sum_{l=2k-n}^{n-2k} Y_{n-2k,l}(\boldsymbol{\eta})\overline{Y_{n-2k,l}(\boldsymbol{\xi})}. \tag{2.12}$$

Finally we present one of the most important theorems concerning spherical harmonics, the *Funk-Hecke theorem*. The formula was first published by Funk in 1916 and later by Hecke in 1918. A proof of it is found in [p. 247, 5] and [p. 29, 19].

Theorem 2.3. (Funk-Hecke theorem). *Let $F(t)$ be continuous for $-1 \le t \le 1$. Then for every spherical harmonic of degree n we have the identity*

$$\int_{\mathbb{S}^2} F(\boldsymbol{\xi} \cdot \boldsymbol{\eta}) Y_{n,l}(\boldsymbol{\xi}) d\omega(\boldsymbol{\xi}) = 2\pi Y_{n,l}(\boldsymbol{\eta}) \int_{-1}^{1} F(t) P_n(t)\, dt\,,$$

where P_n denote the Legendre polynomials of degree n.

2.3. Zernike polynomials

Zernike polynomials form a complete orthonormal system in $L_2(\mathbb{B}^3)$. They play a key role as singular functions for the Radon transform, see Louis [14] and the X-ray transform, see Maass [15].
First we define radial polynomials $\{R_n^{n-2k},\ k = 0, ..., [n/2],\ n = 0, ...\}$

$$\begin{aligned} R_n^{n-2k}(\rho) &:= \rho^{n-2k} P_k^{(0,n-2k+1/2)}(2\rho^2 - 1) \\ &= \frac{(-1)^k}{2^{2k}} \rho^{n-2k} C_{2k}^k \sum_{p=0}^{k} (-1)^p \frac{C_k^p C_{2n-2k+2p+1}^{2k}}{C_{n-k+p}^k} \rho^{2p},\ \ \rho \in [0,1]\,, \end{aligned} \tag{2.13}$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials of degree n and type (α, β) (Appendix A4.) and $C_n^k = \binom{n}{k}$.

$$P_n^{(\alpha,\beta)}(t) = \frac{(n+\beta)!}{(n+\alpha+\beta)!} \sum_{j=0}^{n} \frac{(-1)^j (2n+\alpha+\beta-j)!)}{j!(n-j)!(n+\beta-j)!} \left(\frac{t+1}{2}\right)^{n-j}, \quad -1 \le t \le 1\,. \tag{2.14}$$

The radial polynomials R_k^{n-2k} coincide with according polynomials in [p. 727,10] up to a normalization factor of $\sqrt{2n+3}$.
The functions R_n^{n-2k} (2.13) are those polynomials of degree n having 0 as $n-2k$-fold zero and satisfying

$$\int_0^1 R_n^{n-2k}(\rho) R_n^{n-2s}(\rho) \rho^2\, d\rho = \frac{1}{2n+3} \delta_{ks}, \quad R_n^{n-2k}(1) = 1\,. \tag{2.15}$$

The Zernike polynomials $\{Z_{n-2k,l}^{(n)}(\mathbf{x})\}$, with $k = 1, ..., [n/2],\ |l| \le n-2k,\ n = 0, 1, 2, \ldots$, are then defined by

$$Z_{n-2k,l}^{(n)}(\mathbf{x}) := 4\pi\, R_n^{n-2k}(\rho)\, Y_{n-2k,l}(\boldsymbol{\phi})\,, \quad \mathbf{x} = \rho\boldsymbol{\phi}, \quad |\boldsymbol{\phi}| = 1\,, \tag{2.16}$$

where $Y_{n-2k,l}$ are spherical harmonics of degree $n-2k$ as introduced in Section 2.2. System $\{Z_{n-2k,l}^{(n)}\}$ (2.16) represents an extension of the *circle polynomials of Zernike* to $\mathbb{R}^3$ and forms an orthogonal basis of $L_2(\mathbb{B}^3)$. The norms compute to

$$||Z_{n-2k,l}^{(n)}||_{L_2(\mathbb{B}^3)} = \frac{4\pi}{\sqrt{2n+3}}\,.$$

Any $f(\mathbf{x}) \in L_2(\mathbb{B}^3)$, hence has an expansion as

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \sum_{|l| \le n-2k} f_{n-2k,l}^{(n)} \, Z_{n-2k,l}^{(n)}(\mathbf{x}). \tag{2.17}$$

Proposition 2.4. *An integral representation of the Zernike polynomials (2.16) in terms of Gegenbauer polynomials is given as*

$$Z_{n-2k,l}^{(n)}(\mathbf{x}) = \int_{\mathbb{S}^2} C_n^{(3/2)}\big(\mathbf{x} \cdot \boldsymbol{\xi}\big) \, Y_{n-2k,l}(\boldsymbol{\xi}) \, d\omega(\boldsymbol{\xi}). \tag{2.18}$$

Representation (2.18) leads to a further useful identity. To distinguish the surface measures on $\mathbb{S}^2$ and $\mathbb{S}^1$ we write $d\omega^2$ and $d\omega^1$, respectively. Let $\boldsymbol{x} = \rho\,\boldsymbol{\phi}$, $\rho = |\mathbf{x}|$ and let $\mathbb{S}^1_{\boldsymbol{\phi}^\perp} = \mathbb{S}^2 \cap \{\boldsymbol{\phi}^\perp\}$ be a great circle, then

$$\begin{aligned}
Z_{n-2k,l}^{(n)}(\mathbf{x}) &= \int_{\mathbb{S}^2} C_n^{(3/2)}\big(\mathbf{x} \cdot \boldsymbol{\xi}\big) Y_{n-2k,l}(\boldsymbol{\xi}) \, d\omega^2(\boldsymbol{\xi}) \\
&= \int_{-1}^{1} \int_{\mathbb{S}^1_{\boldsymbol{\phi}^\perp}} C_n^{(3/2)}\big(\mathbf{x} \cdot (t\boldsymbol{\phi} + \sqrt{1-t^2}\boldsymbol{\eta})\big) Y_{n-2k,l}\big(t\boldsymbol{\phi} + \sqrt{1-t^2}\boldsymbol{\eta}\big) \, d\omega^1(\boldsymbol{\eta}) \, dt \\
&= \int_{-1}^{1} C_n^{(3/2)}(t\rho) \int_{\mathbb{S}^1_{\boldsymbol{\phi}^\perp}} Y_{n-2k,l}\big(t\boldsymbol{\phi} + \sqrt{1-t^2}\boldsymbol{\eta}\big) \sqrt{1-t^2} \, d\omega^1(\boldsymbol{\eta}) \, \frac{dt}{\sqrt{1-t^2}} \\
&= \int_{-1}^{1} C_n^{(3/2)}(t\rho) \int_{\boldsymbol{\xi} \cdot \boldsymbol{\phi} = t} Y_{n-2k,l}(\boldsymbol{\xi}) \, d\omega(\boldsymbol{\xi}) \, \frac{dt}{\sqrt{1-t^2}} \\
&= 2\pi Y_{n-2k,l}(\boldsymbol{\phi}) \int_{-1}^{1} C_n^{(3/2)}(t\rho) P_{n-2k}(t) \, dt \, ,
\end{aligned}$$

where we used

$$\int_{\boldsymbol{\xi} \cdot \boldsymbol{\phi} = t} Y_{n-2k,l}(\boldsymbol{\xi}) d\omega(\boldsymbol{\xi}) = 2\pi \sqrt{1-t^2} P_{n-2k}(t) Y_{n-2k,l}(\boldsymbol{\phi})$$

as well as the replacement $\boldsymbol{\xi} = t\boldsymbol{\phi} + \sqrt{1-t^2}\boldsymbol{\eta}$.
The Gegenbauer polynomials as ridge functions are generating functions for Zernike polynomials. From (2.18) we obtain

$$C_n^{(3/2)}\big(\mathbf{x} \cdot \boldsymbol{\xi}\big) = \sum_{k=0}^{[n/2]} \sum_{|l| \le n-2k} Z_{n-2k,l}^{(n)}(\mathbf{x}) \overline{Y_{n-2k,l}(\boldsymbol{\xi})}.$$

2.4. Additional properties of orthogonal polynomials

The following lemma contains some important integral relations of orthogonal polynomials. Their proofs are omitted here and are mainly based on the *formula of Rodriguez* (5.8), integration by parts and recurrence formulas, see Appendix.

Lemma 2.5. *For $0 \le p \le k \le [n/2]$ we have integral relations*

$$
\begin{aligned}
(a)\quad & \int_{-1}^{1} t^{n-2p} P_{n-2k}(t)\, dt = \frac{\sqrt{\pi}(n-2p)!}{2^{n-2p}(k-p)!\Gamma(n-k-p+3/2)}\,,\\
(b)\quad & \int_{-1}^{1} t C_{n-1}^{(3/2)}(t) P_n(t)\, dt = \frac{2n}{2n+1}\,,\\
(c)\quad & \int_{-1}^{1} t P_{n-1}(t) P_n(t)\, dt = \frac{2n}{(2n-1)(2n+1)}\,,\\
(d)\quad & \int_{-1}^{1} C_n^{3/2}(\rho t) P_{n-2k}(t)\, dt = 2R_k^{n-2k}(\rho)\,,\\
(e)\quad & \int_{-1}^{1} C_n^{(3/2)}(t) P_{n-2k}(t)\, dt = 2\,.
\end{aligned}
$$

From Lemma 2.5(a) we derive for the special cases $p=0$ and $p=k=0$

$$
\begin{aligned}
\int_{-1}^{1} t^n P_{n-2k}(t)\, dt &= \frac{\sqrt{\pi} n!}{2^n k! \Gamma(n-k+3/2)}\,,\\
\int_{-1}^{1} t^n P_n(t)\, dt &= \frac{\sqrt{\pi}(n)!}{2^n \Gamma(n+3/2)} = \frac{(n!)^2 2^{n+1}}{(2n+1)!}\,,
\end{aligned}
$$

respectively.

Proposition 2.6. *For $\mathbf{x} = \rho\boldsymbol{\phi}$, $\boldsymbol{\xi} \in \mathbb{S}^2$, the identities*

$$
\begin{aligned}
(a)\ & C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi}) = \sum_{k=0}^{[n/2]} (2n-4k+1)\rho^{n-2k} P_k^{(0,n-2k+1/2)}(2\rho^2-1) P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi})\,,\\
(b)\ & \int_{\mathbb{S}^2} C_{n-2k}^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi}) C_n^{(3/2)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, d\omega(\boldsymbol{\xi}) = 4\pi C_{n-2k}^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\eta})\,,\ k = 0,\ldots,\left[\frac{n}{2}\right],\\
(c)\ & \int_{\mathbb{B}^3} C_n^{(3/2)}(\boldsymbol{\xi}\cdot\mathbf{x}) C_n^{3/2}(\mathbf{x}\cdot\boldsymbol{\eta})\, d\mathbf{x} = \frac{4\pi}{2n+3} C_n^{3/2}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})
\end{aligned}
$$

are valid.

Proof. a) Suppose that $\mathbf{x} = \rho\boldsymbol{\phi}$, $\rho = |\mathbf{x}|$ and $\boldsymbol{\xi} \in \mathbb{S}^2$. Gegenbauer polynomials have an expansion

$$
C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi}) = \frac{1}{2^n} \sum_{p=0}^{[n/2]} (-1)^p \frac{(2n-2p+1)!}{p!(n-2p)!(n-p)!} (\mathbf{x}\cdot\boldsymbol{\xi})^{n-2p}\,.
$$

The nth power of the inner product $(\boldsymbol{\eta}\cdot\boldsymbol{\xi})^n$ for $\boldsymbol{\eta}$, $\boldsymbol{\xi} \in \mathbb{S}^2$ can be written as

$$
\begin{aligned}
(\boldsymbol{\eta}\cdot\boldsymbol{\xi})^n &= \frac{1}{2^n}\sqrt{\pi} n! \sum_{p=0}^{[n/2]} \frac{n-2p+1/2}{p!\Gamma(n-p+3/2)} P_{n-2p}(\boldsymbol{\eta}\cdot\boldsymbol{\xi})\\
&= \frac{1}{2^{n+1}}\sqrt{\pi} n! \sum_{p=0}^{[n/2]} \frac{n-2p+3/2}{p!\Gamma(n-p+3/2)(n-p+3/2)} C_{n-2p}^{(3/2)}(\boldsymbol{\eta}\cdot\boldsymbol{\xi})\,,
\end{aligned}
$$

which yields

$$
\begin{aligned}
C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi}) &= \frac{1}{2^n}\sum_{p=0}^{[n/2]}(-1)^p\frac{(2n-2p+1)!}{p!(n-2p)!(n-p)!}\frac{1}{2^{n-2p}}\sqrt{\pi}(n-2p)!\rho^{n-2p} \\
&\quad\times\sum_{j=0}^{[n/2]-p}\frac{n-2p-2j+1/2}{j!\Gamma(n-2p-j+3/2)}P_{n-2p-2j}(\boldsymbol{\phi}\cdot\boldsymbol{\xi}).
\end{aligned}
$$

By a change of the summation index $k=j+p$ and a subsequent change of the order of summation we obtain

$$
\begin{aligned}
C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi}) &= \frac{1}{2^n}\sum_{p=0}^{[n/2]}(-1)^p\frac{(2n-2p+1)!}{p!(n-p)!}\frac{1}{2^{n-2p}}\sqrt{\pi}\rho^{n-2p} \\
&\quad\times\sum_{k=p}^{[n/2]}\frac{n-2k+1/2}{(k-p)!\Gamma(n-p-k+3/2)}P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi}) \\
&= \sqrt{\pi}\sum_{k=0}^{[n/2]}(n-2k+1/2)P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi}) \\
&\quad\times\sum_{p=0}^{k}\frac{(-1)^p(2n-2p+1)!\rho^{n-2p}}{2^{2n-2p}p!(n-p)!(k-p)!\Gamma(n-p-k+3/2)} \\
&= \sqrt{\pi}\sum_{k=0}^{[n/2]}(n-2k+1/2)P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi}) \\
&\quad\times\frac{1}{k!}\sum_{p=0}^{k}\frac{(-1)^pC_k^p(2n-2p+1)!\rho^{n-2p}}{2^{2n-2p}(n-p)!\Gamma(n-p-k+3/2)}. \\
&= \sum_{k=0}^{[n/2]}(n-2k+1/2)\rho^{n-2k}P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi}) \\
&\quad\times\frac{1}{k!}\sum_{p=0}^{k}\frac{(-1)^pC_k^p(2n-2p+1)!(n-p-k+1)!}{2^{2(k-1)}(n-p)!(2n-2p-2k+2)!}\rho^{2(k-p)}.
\end{aligned}
$$

Putting $q=k-p$ yields

$$
\begin{aligned}
C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi}) &= \sum_{k=0}^{[n/2]}(n-2k+1/2)\rho^{n-2k}P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi})\frac{(-1)^k}{k!2^{2(k-1)}} \\
&\quad\times\sum_{q=0}^{k}\frac{(-1)^qC_k^q(2n-2k+2q+1)!(n+q-2k+1)!}{(n-k+q)!(2n+2q-4k+2)!}\rho^{2q} \\
&= \sum_{k=0}^{[n/2]}(n-2k+1/2)\rho^{n-2k}P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi})\frac{(-1)^kC_{2k}^k}{2^{2(k-1)}} \\
&\quad\times\sum_{q=0}^{k}\frac{(-1)^qC_k^qk!(2n-2k+2q+1)!(n+q-2k+1)!}{(n-k+q)!(2n+2q-4k+2)!(2k)!}\rho^{2q}
\end{aligned}
$$

$$= \sum_{k=0}^{[n/2]} (n-2k+1/2)\rho^{n-2k} P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi}) \frac{(-1)^k C_{2k}^k}{2^{2k-1}}$$
$$\times \sum_{q=0}^{k} \frac{(-1)^q C_k^q k!(2n-2k+2q+1)!(n+q-2k)!}{(n-k+q)!(2n+2q-4k+1)!(2k)!}\rho^{2q}\,,$$

where we used the notation $C_n^k := \binom{n}{k}$. Hence,

$$C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi}) = \sum_{k=0}^{[n/2]} (2n-4k+1)\rho^{n-2k} P_{n-2k}(\boldsymbol{\phi}\cdot\boldsymbol{\xi}) \frac{(-1)^k C_{2k}^k}{2^{2k}} \sum_{q=0}^{k} \frac{(-1)^q C_k^q C_{2n-2k+2q+1}^{2k}}{C_{n-k+q}^k}\rho^{2q}\,,$$

whence statement a) follows if we take into account that

$$\frac{(-1)^k C_{2k}^k}{2^{2k}} \sum_{q=0}^{k} \frac{(-1)^q C_k^q C_{2n-2k+2q+1}^{2k}}{C_{n-k+q}^k}\rho^{2q} = P_k^{(0,n-2k+1/2)}(2\rho^2-1).$$

b) We only consider the case $k=0$, the proof for $k\neq 0$ follows accordingly. The Gegenbauer polynomials as ridge functions have representations

$$C_n^{(3/2)}(\boldsymbol{\eta}\cdot\boldsymbol{\xi}) = 4\pi \sum_{k=0}^{[n/2]} \sum_{|p|\le n-2k} \overline{Y_{n-2k,p}(\boldsymbol{\eta})} Y_{n-2k,p}(\boldsymbol{\xi})\,, \qquad \boldsymbol{\eta},\ \boldsymbol{\xi}\in\mathbb{S}^2\,, \tag{2.19}$$

$$C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi}) = 4\pi \sum_{k=0}^{[n/2]} R_n^{n-2k}(\rho) \sum_{|p|\le n-2k} Y_{n-2k,p}(\boldsymbol{\phi}) \overline{Y_{n-2k,p}(\boldsymbol{\xi})}\,, \qquad \mathbf{x}=\rho\boldsymbol{\phi}\in\mathbb{B}^3\,,\ \boldsymbol{\xi}\in\mathbb{S}^2\,. \tag{2.20}$$

Using these identities and the orthogonality of spherical harmonics gives

$$\begin{aligned}\int_{\mathbb{S}^2} C_n^{(3/2)}\big(\mathbf{x}\cdot\boldsymbol{\xi}\big) C_n^{(3/2)}\big(\boldsymbol{\xi}\cdot\boldsymbol{\eta}\big)\, d\omega(\boldsymbol{\xi}) &= (4\pi)^2 \sum_{k=0}^{[n/2]} R_n^{n-2k}(\rho) \sum_{|p|\le n-2k} Y_{n-2k,p}(\boldsymbol{\phi}) \overline{Y_{n-2k,p}(\boldsymbol{\eta})} \\ &= 4\pi C_n^{(3/2)}\big(\mathbf{x}\cdot\boldsymbol{\eta}\big)\,,\end{aligned}$$

which is assertion b) for $k=0$.
c) Applying representations (2.19), (2.20) again and taking into account the orthogonality relation (2.15) for the radial polynomials R_n^{n-2k} we derive

$$\begin{aligned}&\int_{\mathbb{B}^3} C_n^{(3/2)}\big(\boldsymbol{\xi}\cdot\mathbf{x}\big) C_n^{(3/2)}\big(\mathbf{x}\cdot\boldsymbol{\eta}\big)\, d\mathbf{x} \\ &\quad= (4\pi)^2 \int_0^1 \int_{\mathbb{S}^2} \sum_{k=0}^{[n/2]} R_n^{n-2k}(\rho) \sum_{|p|\le n-2k} Y_{n-2k,p}(\boldsymbol{\phi}) \overline{Y_{n-2k,p}(\boldsymbol{\xi})} \\ &\quad\times \sum_{k=0}^{[n/2]} R_n^{n-2k}(\rho) \sum_{|p|\le n-2k} \overline{Y_{n-2k,p}(\boldsymbol{\phi})} Y_{n-2k,p}(\boldsymbol{\eta}) \rho^2\, d\rho\, d\omega(\boldsymbol{\phi}) \\ &\quad= (4\pi)^2 \sum_{k=0}^{[n/2]} \int_0^1 \big(R_n^{n-2k}(\rho)\rho\big)^2\, d\rho \sum_{|p|\le n-2k} \overline{Y_{n-2k,p}(\boldsymbol{\xi})} Y_{n-2k,p}(\boldsymbol{\eta}) \\ &\quad= \frac{(4\pi)^2}{2n+3} \sum_{k=0}^{[n/2]} \sum_{|p|\le n-2k} \overline{Y_{n-2k,p}(\boldsymbol{\xi})} Y_{n-2k,p}(\boldsymbol{\eta}) = \frac{4\pi}{2n+3} C_n^{(3/2)}\big(\boldsymbol{\xi}\cdot\boldsymbol{\eta}\big).\end{aligned}$$

□

3. CONSTRUCTION OF A COMPLETE SYSTEM OF ORTHOGONAL VECTOR FIELDS BY MEANS OF RIDGE FUNCTIONS

The construction of complete orthogonal systems in $L_2(\mathbb{B}^3, \mathbf{S}_1)$ corresponding to the subspaces generated by the Hemholtz-Hodge decomposition (2.5) is subject of this section. This construction is derived with the help of *vector spherical harmonics* and *ridge functions.* Ridge functions have the property that they are constant orthogonal to some direction $\omega \in \mathbb{S}^2$. We start by introducing vector spherical harmonics and outline some of their basic properties (Section 3.1). Then we describe the construction of orthogonal bases with the help of ridge functions involving vector spherical harmonics (Section 3.2).

We start by introducing vector spherical harmonics. We denote the unit vectors corresponding to spherical polar coordinates by $\boldsymbol{e}_r,\ \boldsymbol{e}_\varphi$ and $\boldsymbol{e}_\theta$. They form the so called local moving triad

$$\boldsymbol{e}_r = \boldsymbol{\xi}, \quad \mathbf{e}_\varphi = \frac{1}{\sin\,\theta}\frac{\partial \boldsymbol{\xi}}{\partial \varphi}, \quad \mathbf{e}_\theta = \frac{\partial \boldsymbol{\xi}}{\partial \theta}.$$

We will also consider complex reference vectors $\mathbf{e}_\epsilon,\ \epsilon = -1, 0, 1,$

$$\mathbf{e}_{-1} = \frac{\mathbf{e}_x - \mathrm{i}\mathbf{e}_y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \\ 0 \end{pmatrix}, \ \mathbf{e}_0 = \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{e}_{+1} \equiv \mathbf{e}_1 = -\frac{\mathbf{e}_x + \mathrm{i}\mathbf{e}_y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -\mathrm{i} \\ 0 \end{pmatrix},$$

where $\mathbf{e}_x,\ \mathbf{e}_y$ and $\mathbf{e}_z$ are the standard orthogonal cartesian vectors. We have

$$\mathbf{e}_{-1} \times \mathbf{e}_0 = -\mathrm{i}\mathbf{e}_{-1}, \quad \mathbf{e}_1 \times \mathbf{e}_0 = \mathrm{i}\mathbf{e}_1, \quad \mathbf{e}_{-1} \times \mathbf{e}_1 = -\mathrm{i}\mathbf{e}_0. \tag{3.1}$$

3.1. Vector Spherical Harmonics

The concept of spherical harmonics as outlined in Section 2.2 extends to vector fields. We show that we can construct orthogonal sets in $L_2(\mathbb{S}^2, \mathbf{S}_1)$ with spherical harmonics as starting point. The described approach relies on [18], [7], [15], [16].

Let $\{Y_{n,l}(\boldsymbol{\xi}),\ n \in \mathbb{N}_0,\ |l| \le n\}$ be the $L_2(\mathbb{S}^2)$-orthonormal set of scalar spherical harmonics. We define a system of vector spherical harmonics

$$\{\mathbf{y}_{n,l}^{(j)}(\boldsymbol{\xi}),\ n \in \mathbb{N}_0,\ |l| \le n,\ j = 1, 2, 3\}.$$

At first we set

$$\mathbf{y}_{0,l}^{(1)}(\boldsymbol{\xi}) := \boldsymbol{\xi} Y_{0,l}(\boldsymbol{\xi}).$$

For any $n \in \mathbb{N}$ and $|l| \le n$ we define vector fields $\mathbf{y}_{n,l}^{(j)}(\boldsymbol{\xi}),\ j = 1, 2, 3,$ by

$$\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) := \boldsymbol{\xi} Y_{n,l}(\boldsymbol{\xi}), \tag{3.2}$$

$$\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}) := \frac{2n+1}{4\pi} \int_{\mathbb{S}^2} [\boldsymbol{\eta} - (\boldsymbol{\xi} \cdot \boldsymbol{\eta})\boldsymbol{\xi}]\, Y_{n,l}(\boldsymbol{\eta}) C_{n-1}^{(3/2)}(\boldsymbol{\xi} \cdot \boldsymbol{\eta})\, d\omega(\boldsymbol{\eta}), \tag{3.3}$$

$$\mathbf{y}_{n,l}^{(3)}(\boldsymbol{\xi}) := \frac{2n+1}{4\pi} \boldsymbol{\xi} \times \int_{\mathbb{S}^2} \mathbf{y}_{n,l}^{(1)}(\boldsymbol{\eta}) C_{n-1}^{(3/2)}(\boldsymbol{\xi} \cdot \boldsymbol{\eta})\, d\omega(\boldsymbol{\eta}) = \boldsymbol{\xi} \times \mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}), \tag{3.4}$$

which are called *vector spherical harmonics of degree n and type j*.
We continue by summarizing basic properties of the fields $\mathbf{y}_{n,l}^{(j)}$. If we follow Morse and Feshbach [18] and put

$$\begin{aligned}\mathbf{B}_{n,l}(\boldsymbol{\xi}) &= \frac{1}{\sqrt{n(n+1)}}\left[\mathbf{e}_\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin\,\theta}\mathbf{e}_\phi\frac{\partial}{\partial\phi}\right]Y_{n,l}(\boldsymbol{\xi}),\\ \mathbf{C}_{n,l}(\boldsymbol{\xi}) &= -\boldsymbol{\xi}\times\mathbf{B}_{n,l}(\boldsymbol{\xi}),\end{aligned}$$

then Morse and Feshbach's $\mathbf{B}_{n,l}$ and $\mathbf{C}_{n,l}$ are equal to $\mathbf{y}_{n,l}^{(2)}$ and $\mathbf{y}_{n,l}^{(3)}$ up to a normalization factor. Indeed applying the summation fromula (2.11) and $P_n'(t) = C_{n-1}^{3/2}(t)$ we have the identities

$$\begin{aligned}&\frac{1}{\sqrt{n(n+1)}}\left[\mathbf{e}_\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin\,\theta}\mathbf{e}_\varphi\frac{\partial}{\partial\varphi}\right]P_n(\boldsymbol{\eta}\cdot\boldsymbol{\xi}) = \frac{C_{n-1}^{3/2}(\boldsymbol{\eta}\cdot\boldsymbol{\xi})}{\sqrt{n(n+1)}}\left[(\boldsymbol{\eta}\cdot\mathbf{e}_\theta)\mathbf{e}_\theta+(\boldsymbol{\eta}\cdot\mathbf{e}_\varphi)\mathbf{e}_\varphi\right]\\ &= \frac{C_{n-1}^{3/2}(\boldsymbol{\eta}\cdot\boldsymbol{\xi})}{\sqrt{n(n+1)}}\left[\boldsymbol{\eta}-(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\boldsymbol{\xi}\right] = \frac{4\pi}{2n+1}\sum_{|l|\le n}\overline{Y_{n,l}(\boldsymbol{\eta})}\mathbf{B}_{n,l}(\boldsymbol{\xi})\,,\end{aligned}$$

which together with the orthogonality of $Y_{n,l}$ yields

$$\begin{aligned}\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}) &= \sqrt{n(n+1)}\mathbf{B}_{n,l}(\boldsymbol{\xi}),\\ \mathbf{y}_{n,l}^{(3)}(\boldsymbol{\xi}) &= -\sqrt{n(n+1)}\mathbf{C}_{n,l}(\boldsymbol{\xi})\end{aligned}$$

and thus

$$\begin{aligned}\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}) &= \frac{1}{\sin\,\theta}\left[\frac{1}{2n+1}\big[n(n-l+1)Y_{n+1,l}-(n+1)(n+l)Y_{n-1,l}\big]\mathbf{e}_\theta+\mathrm{i}lY_{n,l}\mathbf{e}_\varphi\right],\\ \mathbf{y}_{n,l}^{(3)}(\boldsymbol{\xi}) &= \frac{1}{\sin\,\theta}\left[\frac{1}{2n+1}\big[n(n-l+1)Y_{n+1,l}-(n+1)(n+l)Y_{n-1,l}\big]\mathbf{e}_\theta-\mathrm{i}lY_{n,l}\mathbf{e}_\varphi\right].\end{aligned}$$

The system $\{\mathbf{y}_{n,l}^{(j)}\}$ forms an orthogonal basis in $L_2(\mathbb{S}^2,\mathbf{S}_1)$. From settings (3.2)–(3.4) we immediately read that

$$\boldsymbol{\xi}\times\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) = \mathbf{0},\quad \boldsymbol{\xi}\cdot\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}) = 0,\quad \boldsymbol{\xi}\cdot\mathbf{y}_{n,l}^{(3)}(\boldsymbol{\xi}) = 0. \tag{3.5}$$

The family $\{\mathbf{y}_{n,l}^{(1)}\}$ consists of purely radial vector fields whereas the systems $\{\mathbf{y}_{n,l}^{(2)}\}$, $\{\mathbf{y}_{n,l}^{(3)}\}$ are tangent vector fields.
The fields $\mathbf{y}_{n,l}^{(1)}$ and $\mathbf{y}_{n,l}^{(2)}$ are connected to each other. Using the Funk-Hecke theorem (Theorem 2.3) and formula (b) from Lemma 2.5 we obtain

$$\begin{aligned}\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}) &= \frac{2n+1}{4\pi}\int_{\mathbb{S}^2}\left[\boldsymbol{\eta}-(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\boldsymbol{\xi}\right]Y_{n,l}(\boldsymbol{\eta})C_{n-1}^{(3/2)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})d\omega(\boldsymbol{\eta})\\ &= \frac{2n+1}{4\pi}\int_{\mathbb{S}^2}\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\eta})C_{n-1}^{(3/2)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})d\omega(\boldsymbol{\eta})-\frac{2n+1}{2}\boldsymbol{\xi}Y_{n,l}(\boldsymbol{\xi})\int_{-1}^{1}sC_{n-1}^{(3/2)}(s)P_n(s)ds\\ &= \frac{2n+1}{4\pi}\int_{\mathbb{S}^2}\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\eta})C_{n-1}^{(3/2)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})d\omega(\boldsymbol{\eta})-\frac{2n+1}{2}\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi})\int_{-1}^{1}sC_{n-1}^{(3/2)}(s)P_n(s)ds\\ &= \frac{2n+1}{4\pi}\int_{\mathbb{S}^2}\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\eta})C_{n-1}^{(3/2)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})d\omega(\boldsymbol{\eta})-n\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}).\end{aligned} \tag{3.6}$$

Next we calculate the $L_2(\mathbb{S}^2, \mathbf{S}_1)$-norms of the different systems. From according properties of scalar spherical harmonics we easily get

$$||\mathbf{y}_{n,l}^{(1)}||^2_{L_2(\mathbb{S}^2,\mathbf{S}_1)} = 1.$$

The calculation of $||\mathbf{y}_{n,l}^{(2)}||$ is done with the help of (3.6), Proposition 2.6 (b), the Funk-Hecke theorem and again formula (b) from Lemma 2.5.

$$\begin{aligned}
&||\mathbf{y}_{n,l}^{(2)}||^2_{L_2(\mathbb{S}^2,\mathbf{S}_1)} \\
&= \left(\frac{2n+1}{4\pi}\right)^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} (\boldsymbol{\eta}\cdot\boldsymbol{\eta}')\, Y_{n,l}(\boldsymbol{\eta}) Y_{n,l}(\boldsymbol{\eta}') C_{n-1}^{(3/2)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) C_{n-1}^{(3/2)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}') d\omega(\boldsymbol{\eta}) d\omega(\boldsymbol{\eta}') d\omega(\boldsymbol{\xi}) \\
&\quad - \frac{(2n+1)n}{2\pi}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} (\boldsymbol{\eta}\cdot\boldsymbol{\xi})\, Y_{n,l}(\boldsymbol{\eta}) Y_{n,l}(\boldsymbol{\xi}) C_{n-1}^{(3/2)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) d\omega(\boldsymbol{\eta}) d\omega(\boldsymbol{\xi}) + n^2 ||\mathbf{y}_{n,l}^{(1)}||^2 \\
&= \frac{(2n+1)^2}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} (\boldsymbol{\eta}\cdot\boldsymbol{\eta}')\, Y_{n,l}(\boldsymbol{\eta}) Y_{n,l}(\boldsymbol{\eta}') C_{n-1}^{(3/2)}(\boldsymbol{\eta}\cdot\boldsymbol{\eta}') d\omega(\boldsymbol{\eta}) d\omega(\boldsymbol{\eta}') \\
&\quad - (2n+1)n \int_{\mathbb{S}^2} Y_{n,l}(\boldsymbol{\xi}) Y_{n,l}(\boldsymbol{\xi}) d\omega(\boldsymbol{\xi}) \int_{-1}^{1} s C_{n-1}^{(3/2)}(s) P_n(s) ds + n^2 \\
&= \frac{(2n+1)^2}{2}\int_{\mathbb{S}^2} Y_{n,l}(\boldsymbol{\eta}) Y_{n,l}(\boldsymbol{\eta}) d\omega(\boldsymbol{\eta}) \int_{-1}^{1} s C_{n-1}^{(3/2)}(s) P_n(s) ds \\
&\quad - (2n+1)n \int_{\mathbb{S}^2} Y_{n,l}(\boldsymbol{\xi}) Y_{n,l}(\boldsymbol{\xi}) d\omega(\boldsymbol{\xi}) \int_{-1}^{1} s C_{n-1}^{(3/2)}(s) P_n(s) ds + n^2 \\
&= (2n+1)n - 2n^2 + n^2 = n(n+1)
\end{aligned}$$

The norms of $\mathbf{y}_{n,l}^{(3)}$ coincide with those from $\mathbf{y}_{n,l}^{(2)}$, what follows from (3.4) and(3.5).

$$\begin{aligned}
||\mathbf{y}_{n,l}^{(3)}||^2_{L_2(\mathbb{S}^2,\mathbf{S}_1)} &= \int_{\mathbb{S}^2} (\boldsymbol{\xi}\times\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}))\cdot(\boldsymbol{\xi}\times\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi})) d\omega(\boldsymbol{\xi}) \\
&= \int_{\mathbb{S}^2} \left[(\boldsymbol{\xi}\cdot\boldsymbol{\xi})(\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi})\cdot\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi})) - (\boldsymbol{\xi}\cdot\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}))^2\right] d\omega(\boldsymbol{\xi}) = ||\mathbf{y}_{n,l}^{(2)}||^2
\end{aligned}$$

Proposition 3.1 contains fundamental relations between scalar and vector spherical harmonics. More specifically, it shows that the vector fields $\mathbf{y}_{n,l}^{(j)}$ can be written as linear combinations of only few scalar fields $Y_{n,l}$, which implies that they are orthogonal to almost every $Y_{n,l}$.

Proposition 3.1. *We have the orthogonality relations*

$$\begin{aligned}
&\int_{\mathbb{S}^2} \mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) Y_{m,s}(\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}) = 0, \qquad m \neq n-1,\ n+1 \\
&\int_{\mathbb{S}^2} \mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}) Y_{m,s}(\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}) = 0, \qquad m \neq n-1,\ n+1 \\
&\int_{\mathbb{S}^2} \mathbf{y}_{n,l}^{(3)}(\boldsymbol{\xi}) Y_{m,s}(\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}) = 0, \qquad m \neq n.
\end{aligned}$$

Moreover vector spherical harmonics can be represented as sums of scalar spherical har-

monics as

$$\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) = \sum_{\epsilon=-1}^{1} a_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi})\overline{\mathbf{e}_\epsilon} + \sum_{\epsilon=-1}^{1} b_{n,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\xi})\mathbf{e}_{-\epsilon}, \tag{3.7}$$

$$\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}) = (n+1)\sum_{\epsilon=-1}^{1} a_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi})\overline{\mathbf{e}_\epsilon} - n\sum_{\epsilon=-1}^{1} b_{n,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\xi})\mathbf{e}_{-\epsilon}, \tag{3.8}$$

$$\mathbf{y}_{n,l}^{(3)}(\boldsymbol{\xi}) = \mathrm{i}\sum_{\epsilon=-1}^{1} c_{n,l,\epsilon} Y_{n,l+\epsilon}(\boldsymbol{\xi})\mathbf{e}_{-\epsilon}, \tag{3.9}$$

where the coefficients $a_{n,l,p}$, $b_{n,l,q}$ and $c_{n,l,q}$ are given as

$$a_{n,l,-1} := \sqrt{\frac{(n+l)(n+l-1)}{2(2n-1)(2n+1)}}, \qquad a_{n,l,0} := \sqrt{\frac{(n-l)(n+l)}{(2n-1)(2n+1)}},$$
$$a_{n,l,+1} := \sqrt{\frac{(n-l)(n-l-1)}{2(2n-1)(2n+1)}},$$

$$b_{n,l,-1} := \sqrt{\frac{(n-l+1)(n-l+2)}{2(2n+3)(2n+1)}}, \qquad b_{n,l,0} := \sqrt{\frac{(n+l+1)(n-l+1)}{(2n+3)(2n+1)}},$$
$$b_{n,l,+1} := \sqrt{\frac{(n+l+1)(n+l+2)}{2(2n+3)(2n+1)}},$$

$$c_{n,l,-1} := \sqrt{\frac{(n+l)(n-l+1)}{2}}, \qquad c_{n,l,0} := l,$$
$$c_{n,l,+1} := -\sqrt{\frac{(n+l+1)(n-l)}{2}}.$$

Proof. Since the orthogonality relations are a consequence from the representations (3.7)–(3.9), we restrict the proof to showing the latter ones.
From the recurrence formulas for associated Legendre polynomials, see Appendix A3, and connection (2.7) we get three recurrence relations for spherical harmonic functions.

1) $$e^{\mathrm{i}\varphi}\sqrt{1-t^2}\frac{1}{N_{n,l}}Y_{n,l}(\boldsymbol{\xi}) = e^{\mathrm{i}\varphi}\sqrt{1-t^2}P_n^l(t)e^{\mathrm{i}l\varphi}$$
$$= \frac{1}{2n+1}\left(P_{n+1}^{l+1}(t) - P_{n-1}^{l+1}(t)\right)e^{\mathrm{i}(l+1)\varphi}$$
$$= \frac{1}{2n+1}\left(\frac{1}{N_{n+1,l+1}}Y_{n+1,l+1}(\boldsymbol{\xi}) - \frac{1}{N_{n-1,l+1}}Y_{n-1,l+1}(\boldsymbol{\xi})\right),$$

2) $$e^{-\mathrm{i}\varphi}\sqrt{1-t^2}\frac{1}{N_{n,l}}Y_{n,l}(\boldsymbol{\xi}) = e^{-\mathrm{i}\varphi}\sqrt{1-t^2}P_n^l(t)e^{\mathrm{i}l\varphi}$$
$$= \frac{1}{2n+1}\left((n+l)(n+l-1)P_{n-1}^{l-1}(t) - (n-l+1)(n-l+2)P_{n+1}^{l-1}(t)\right)e^{\mathrm{i}(l-1)\varphi}$$
$$= \frac{1}{2n+1}\left(\frac{(n+l)(n+l-1)}{N_{n-1,l-1}}Y_{n-1,l-1}(\boldsymbol{\xi}) - \frac{(n-l+1)(n-l+2)}{N_{n+1,l-1}}Y_{n+1,l-1}(\boldsymbol{\xi})\right),$$

$$3)\ t\frac{1}{N_{n,l}}Y_{n,l}(\boldsymbol{\xi}) = tP_n^l(t)e^{\mathrm{i}l\varphi}$$
$$= \frac{1}{2n+1}\left((n-l+1)P_{n+1}^l(t) + (n+l)P_{n-1}^l(t)\right)e^{\mathrm{i}l\varphi}$$
$$= \frac{1}{2n+1}\left((n-l+1)\frac{1}{N_{n+1,l}}Y_{n+1,l}(\boldsymbol{\xi}) + (n+l)\frac{1}{N_{n-1,l}}Y_{n-1,l}(\boldsymbol{\xi})\right).$$

We remind that

$$N_{n,m} = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}.$$

From 1) and 2) we deduce

$$1')\ 2\cos\varphi\sqrt{1-t^2}\frac{1}{N_{n,l}}Y_{n,l}(\boldsymbol{\xi}) = (e^{\mathrm{i}\varphi} + e^{-\mathrm{i}\varphi}\sqrt{1-t^2}\frac{1}{N_{n,l}}Y_{n,l}(\boldsymbol{\xi})$$
$$= \frac{1}{2n+1}\left(\frac{1}{N_{n+1,l+1}}Y_{n+1,l+1}(\boldsymbol{\xi}) - (n-l+1)(n-l+2)\frac{1}{N_{n+1,l-1}}Y_{n+1,l-1}(\boldsymbol{\xi})\right.$$
$$\left.-\frac{1}{N_{n-1,l+1}}Y_{n-1,l+1}(\boldsymbol{\xi}) + (n+l)(n+l-1)\frac{1}{N_{n-1,l-1}}Y_{n-1,l-1}(\boldsymbol{\xi})\right),$$

$$2')\ 2\sin\varphi\sqrt{1-t^2}\frac{1}{N_{n,l}}Y_{n,l}(\boldsymbol{\xi}) = \mathrm{i}^{-1}(e^{\mathrm{i}\varphi} - e^{-\mathrm{i}\varphi}\sqrt{1-t^2}\frac{1}{N_{n,l}}Y_{n,l}(\boldsymbol{\xi})$$
$$= \frac{1}{2n+1}\left(\frac{-\mathrm{i}}{N_{n+1,l+1}}Y_{n+1,l+1}(\boldsymbol{\xi}) + (n-l+1)(n-l+2)\frac{-\mathrm{i}}{N_{n+1,l-1}}Y_{n+1,l-1}(\boldsymbol{\xi})\right.$$
$$\left.-\frac{-\mathrm{i}}{N_{n-1,l+1}}Y_{n-1,l+1}(\boldsymbol{\xi}) - (n+l)(n+l-1)\frac{-\mathrm{i}}{N_{n-1,l-1}}Y_{n-1,l-1}(\boldsymbol{\xi})\right).$$

Writing $\boldsymbol{\xi}$ in spherical coordinates as in (2.6) and summarizing 1'), 2') and 3) yields

$$(2n+1)\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) = \frac{N_{n,l}}{2N_{n+1,l+1}}\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix}Y_{n+1,l+1}(\boldsymbol{\xi}) + (n-l+1)\frac{N_{n,l}}{N_{n+1,l}}\begin{pmatrix}0\\0\\1\end{pmatrix}Y_{n+1,l}(\boldsymbol{\xi})$$
$$-(n-l+1)(n-l+2)\frac{N_{n,l}}{2N_{n+1,l-1}}\begin{pmatrix}1\\\mathrm{i}\\0\end{pmatrix}Y_{n+1,l-1}(\boldsymbol{\xi})$$
$$-\frac{N_{n,l}}{2N_{n-1,l+1}}\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix}Y_{n-1,l+1}(\boldsymbol{\xi}) + (n+l)\frac{N_{n,l}}{N_{n-1,l}}\begin{pmatrix}0\\0\\1\end{pmatrix}Y_{n-1,l}(\boldsymbol{\xi})$$
$$+(n+l)(n+l-1)\frac{N_{n,l}}{2N_{n-1,l-1}}\begin{pmatrix}1\\\mathrm{i}\\0\end{pmatrix}Y_{n-1,l-1}(\boldsymbol{\xi}).$$

It remains to evaluate the coefficients. Some straightforward calculations show

$$(n+l)(n+l-1)\frac{N_{n,l}}{\sqrt{2}N_{n-1,l-1}(2n+1)} = \sqrt{\frac{(n-l)!(n+l-2)!(n+l)^2(n+l-1)^2}{2(n+l)!(n-l)!(2n-1)(2n+1)}}$$
$$= \sqrt{\frac{(n+l)(n+l-1)}{2(2n-1)(2n+1)}} = a_{n,l,-1},$$

$$(n+l)\frac{N_{n,l}}{N_{n-1,l}(2n+1)} = \sqrt{\frac{(n-l)!(n+l-1)!(n+l)^2}{(n+l)!(n-l-1)!(2n-3)(2n+1)}}$$
$$= \sqrt{\frac{(n-l)(n+l)}{(2n-1)(2n+1)}} = a_{n,l,0},$$

$$\frac{N_{n,l}}{\sqrt{2}N_{n-1,l+1}(2n+1)} = \sqrt{\frac{(n-l)!(n+l)!}{2(n+l)!(n-l-2)!(2n-1)(2n+1)}}$$
$$= \sqrt{\frac{(n-l)(n-l-1)}{2(2n-1)(2n+1)}} = a_{n,l,1},$$

$$(n-l+1)(n-l+2)\frac{N_{n,l}}{\sqrt{2}N_{n+1,l-1}(2n+1)} = \sqrt{\frac{(n-l)!(n+l)!(n-l+1)^2(n-l+2)^2}{2(n+l)!(n-l+2)!(2n+3)(2n+1)}}$$
$$= \sqrt{\frac{(n-l+1)(n-l+2)}{2(2n+3)(2n+1)}} = b_{n,l,-1},$$

$$(n-l+1)\frac{N_{n,l}}{N_{n+1,l}(2n+1)} = \sqrt{\frac{(n-l)!(n+l+1)!(n-l+1)^2}{(n+l)!(n-l+1)!(2n+3)(2n+1)}}$$
$$= \sqrt{\frac{(n+l+1)(n-l+1)}{(2n+3)(2n+1)}} = b_{n,l,0},$$

$$\frac{N_{n,l}}{\sqrt{2}N_{n+1,l+1}(2n+1)} = \sqrt{\frac{(n-l)!(n+l+2)!}{2(n+l)!(n-l)!(2n+3)(2n+1)}}$$
$$= \sqrt{\frac{(n+l+1)(n+l+2)}{2(2n+3)(2n+1)}} = b_{n,l,1}.$$

That gives the first representation (3.7)

$$\mathbf{y}^{(1)}_{n,l}(\boldsymbol{\xi}) = a_{n,l,-1}Y_{n-1,l-1}(\boldsymbol{\xi})\overline{\mathbf{e}_{-1}} + a_{n,l,0}Y_{n-1,l}(\boldsymbol{\xi})\mathbf{e}_0 + a_{n,l,1}Y_{n-1,l+1}(\boldsymbol{\xi})\overline{\mathbf{e}_{+1}}$$
$$+ b_{n,l,-1}Y_{n+1,l-1}(\boldsymbol{\xi})\mathbf{e}_1 + a_{n,l,0}Y_{n+1,l}(\boldsymbol{\xi})\mathbf{e}_0 + a_{n,l,+1}Y_{n+1,l+1}(\boldsymbol{\xi})\mathbf{e}_{-1}.$$

For sake of simplicity we introduce notations

$$\mathbf{a}_{n,l,\epsilon} = a_{n,l,\epsilon}\overline{\mathbf{e}_\epsilon},\ \mathbf{b}_{n,l,\epsilon} = b_{n,l,\epsilon}\mathbf{e}_{-\epsilon},\ \epsilon = -1, 0, 1,$$

and have

$$\mathbf{y}^{(1)}_{n,l}(\boldsymbol{\xi}) = \sum_{\epsilon=-1}^{1} \mathbf{a}_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi}) + \sum_{\epsilon=-1}^{1} \mathbf{b}_{n,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\xi}).$$

The summation formula (2.12) for spherical harmonics together with according orthogonality properties lead to

$$\begin{aligned}
&\frac{2n+1}{4\pi}\int_{\mathbb{S}^2} \mathbf{y}^{(1)}_{n,l}(\boldsymbol{\eta}) C^{(3/2)}_{n-1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})d\omega(\boldsymbol{\eta}) = \frac{2n+1}{4\pi}\sum_{\epsilon=-1}^{1} \mathbf{a}_{n,l,\epsilon}\int_{\mathbb{S}^2} Y_{n-1,l+\epsilon}(\boldsymbol{\eta}) C^{(3/2)}_{n-1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})d\omega(\boldsymbol{\eta}) \\
&\quad + \frac{2n+1}{4\pi}\sum_{\epsilon=-1}^{1} \mathbf{b}_{n,l,\epsilon} \underbrace{\int_{\mathbb{S}^2} Y_{n+1,l+\epsilon}(\boldsymbol{\eta}) C^{(3/2)}_{n-1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})d\omega(\boldsymbol{\eta})}_{=0} \\
&= (2n+1)\sum_{\epsilon=-1}^{1} \mathbf{a}_{n,l,\epsilon} \sum_{|s|\le n-1}\int_{\mathbb{S}^2} Y_{n-1,l+\epsilon}(\boldsymbol{\eta}) Y_{n-1,s}(\boldsymbol{\xi})\overline{Y_{n-1,s}(\boldsymbol{\eta})}d\omega(\boldsymbol{\eta}) \\
&= (2n+1)\sum_{\epsilon=-1}^{1} \mathbf{a}_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi})
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{y}^{(2)}_{n,l}(\boldsymbol{\xi}) &= \frac{2n+1}{4\pi}\int_{\mathbb{S}^2} \mathbf{y}^{(1)}_{n,l}(\boldsymbol{\eta}) C^{(3/2)}_{n-1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})d\omega(\boldsymbol{\eta}) - n\mathbf{y}^{(1)}_{n,l}(\boldsymbol{\xi}) \\
&= (2n+1)\sum_{\epsilon=-1}^{1} \mathbf{a}_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi}) - n\left(\sum_{\epsilon=-1}^{1} \mathbf{a}_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi}) + \sum_{\epsilon=-1}^{1} \mathbf{b}_{n,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\xi})\right) \\
&= (n+1)\sum_{\epsilon=-1}^{1} \mathbf{a}_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi}) - n\sum_{\epsilon=-1}^{1} \mathbf{b}_{n,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\xi}),
\end{aligned}$$

where we again made use of relation (3.6) and equation (3.7). This is representation (3.8). Recall that $\boldsymbol{\xi} \times \mathbf{y}^{(1)}_{n,l}(\boldsymbol{\xi}) = \mathbf{0}$. Hence, from (3.7) we obtain

$$\sum_{\epsilon=-1}^{1} \mathbf{y}^{(1)}_{n-1,l+\epsilon}(\boldsymbol{\xi}) \times \mathbf{a}_{n,l,\epsilon} + \sum_{\epsilon=-1}^{1} \mathbf{y}^{(1)}_{n+1,l+\epsilon}(\boldsymbol{\xi}) \times \mathbf{b}_{n,l,\epsilon} = \mathbf{0},$$

or equivalently

$$-\sum_{\epsilon=-1}^{1} \mathbf{y}^{(1)}_{n-1,l+\epsilon}(\boldsymbol{\xi}) \times \mathbf{a}_{n,l,\epsilon} = \sum_{\epsilon=-1}^{1} \mathbf{y}^{(1)}_{n+1,l+\epsilon}(\boldsymbol{\xi}) \times \mathbf{b}_{n,l,\epsilon}. \tag{3.10}$$

Applying (3.7) once again gives

$$\mathbf{y}^{(1)}_{n-1,l+\epsilon}(\boldsymbol{\xi}) = \sum_{\alpha=-1}^{1} \mathbf{a}_{n-1,l+\epsilon,\alpha} Y_{n-2,l+\epsilon+\alpha}(\boldsymbol{\xi}) + \sum_{\alpha=-1}^{1} \mathbf{b}_{n-1,l+\epsilon,\alpha} Y_{n,l+\epsilon+\alpha}(\boldsymbol{\xi}),$$

$$\mathbf{y}^{(1)}_{n+1,l+\epsilon}(\boldsymbol{\xi}) = \sum_{\beta=-1}^{1} \mathbf{a}_{n+1,l+\epsilon,\beta} Y_{n,l+\epsilon+\beta}(\boldsymbol{\xi}) + \sum_{\beta=-1}^{1} \mathbf{b}_{n+1,l+\epsilon,\beta} Y_{n+2,l+\epsilon+\beta}(\boldsymbol{\xi}).$$

Inserting these identities into the right- and left-hand side of (3.10), respectively, yields

$$\begin{aligned}
&-\sum_{\epsilon=-1}^{1}\sum_{\alpha=-1}^{1} \mathbf{a}_{n-1,l+\epsilon,\alpha} \times \mathbf{a}_{n,l,\epsilon} Y_{n-2,l+\epsilon+\alpha}(\boldsymbol{\xi}) - \sum_{\epsilon=-1}^{1}\sum_{\alpha=-1}^{1} \mathbf{b}_{n-1,l+\epsilon,\alpha} \times \mathbf{a}_{n,l,\epsilon} Y_{n,l+\epsilon+\alpha}(\boldsymbol{\xi}) \\
&= \sum_{\epsilon=-1}^{1}\sum_{\beta=-1}^{1} \mathbf{a}_{n+1,l+\epsilon,\beta} \times \mathbf{b}_{n,l,\epsilon} Y_{n,l+\epsilon+\beta}(\boldsymbol{\xi}) + \sum_{\epsilon=-1}^{1}\sum_{\beta=-1}^{1} \mathbf{b}_{n+1,l+\epsilon,\beta} \times \mathbf{b}_{n,l,\epsilon} Y_{n+2,l+\epsilon+\beta}(\boldsymbol{\xi}).
\end{aligned}$$

Since the spherical harmonics $Y_{n,l}$ are linearly independent we get as an immediate consequence

$$\sum_{\epsilon=-1}^{1}\sum_{\alpha=-1}^{1} \mathbf{a}_{n-1,l+\epsilon,\alpha} \times \mathbf{a}_{n,l,\epsilon} Y_{n-2,l+\epsilon+\alpha}(\boldsymbol{\xi}) = 0, \tag{3.11}$$

$$\sum_{\epsilon=-1}^{1}\sum_{\alpha=-1}^{1} \mathbf{b}_{n-1,l+\epsilon,\alpha} \times \mathbf{a}_{n,l,\epsilon} Y_{n,l+\epsilon+\alpha}(\boldsymbol{\xi}) + \sum_{\epsilon=-1}^{1}\sum_{\beta=-1}^{1} \mathbf{a}_{n+1,l+\epsilon,\beta} \times \mathbf{b}_{n,l,\epsilon} Y_{n,l+\epsilon+\beta}(\boldsymbol{\xi}) = 0, \tag{3.12}$$

$$\sum_{\epsilon=-1}^{1}\sum_{\beta=-1}^{1} \mathbf{b}_{n+1,l+\epsilon,\beta} \times \mathbf{b}_{n,l,\epsilon} Y_{n+2,l+\epsilon+\beta}(\boldsymbol{\xi}) = 0. \tag{3.13}$$

Since $\mathbf{y}^{(3)}_{n,l}(\boldsymbol{\xi}) = \boldsymbol{\xi} \times \mathbf{y}^{(2)}_{n,l}(\boldsymbol{\xi})$, see (3.4), we may compute

$$\begin{aligned}
\mathbf{y}^{(3)}_{n,l}(\boldsymbol{\xi}) &= (n+1)\sum_{\epsilon=-1}^{1} \boldsymbol{\xi} \times \mathbf{a}_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi}) - n\sum_{\epsilon=-1}^{1} \boldsymbol{\xi} \times \mathbf{b}_{n,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\xi}) \\
&= (n+1)\sum_{\epsilon=-1}^{1} \mathbf{y}^{(1)}_{n-1,l+\epsilon}(\boldsymbol{\xi}) \times \mathbf{a}_{n,l,\epsilon} - n\sum_{\epsilon=-1}^{1} \mathbf{y}^{(1)}_{n+1,l+\epsilon}(\boldsymbol{\xi}) \times \mathbf{b}_{n,l,\epsilon} \\
&= (2n+1)\sum_{\epsilon=-1}^{1} \mathbf{y}^{(1)}_{n-1,l+\epsilon}(\boldsymbol{\xi}) \times \mathbf{a}_{n,l,\epsilon} \\
&= (2n+1)\sum_{\epsilon=-1}^{1}\sum_{\epsilon'=-1}^{1} \mathbf{a}_{n-1,l+\epsilon,\epsilon'} \times \mathbf{a}_{n,l,\epsilon} Y_{n-2,l+\epsilon+\epsilon'}(\boldsymbol{\xi}) \\
&\quad + (2n+1)\sum_{\epsilon=-1}^{1}\sum_{\epsilon'=-1}^{1} \mathbf{b}_{n-1,l+\epsilon,\epsilon'} \times \mathbf{a}_{n,l,\epsilon} Y_{n,l+\epsilon+\epsilon'}(\boldsymbol{\xi}) \\
&= (2n+1)\sum_{\epsilon=-1}^{1}\sum_{\epsilon'=-1}^{1} \mathbf{b}_{n-1,l+\epsilon,\epsilon'} \times \mathbf{a}_{n,l,\epsilon} Y_{n,l+\epsilon+\epsilon'}(\boldsymbol{\xi}) \\
&= (2n+1)\Big((\mathbf{b}_{n-1,l-1,0} \times \mathbf{a}_{n,l,-1} + \mathbf{b}_{n-1,l,-1} \times \mathbf{a}_{n,l,0}) Y_{n,l-1}(\boldsymbol{\xi}) \\
&\qquad + (\mathbf{b}_{n-1,l+1,-1} \times \mathbf{a}_{n,l,1} + \mathbf{b}_{n-1,l-1,1} \times \mathbf{a}_{n,l,-1}) Y_{n,l}(\boldsymbol{\xi}) \\
&\qquad + (\mathbf{b}_{n-1,l+1,0} \times \mathbf{a}_{n,l,1} + \mathbf{b}_{n-1,l,1} \times \mathbf{a}_{n,l,0}) Y_{n,l+1}(\boldsymbol{\xi})\Big)
\end{aligned}$$

$$
\begin{aligned}
&= (2n+1)\Big((b_{n-1,l-1,0}a_{n,l,-1}\mathbf{e}_0 \times \overline{\mathbf{e}_{-1}} + b_{n-1,l,-1}a_{n,l,0}\mathbf{e}_1 \times \mathbf{e}_0)Y_{n,l-1}(\boldsymbol{\xi}) \\
&\qquad + (b_{n-1,l+1,-1}a_{n,l,1}\mathbf{e}_1 \times \overline{\mathbf{e}_1} + b_{n-1,l-1,1}a_{n,l,-1}\mathbf{e}_{-1} \times \overline{\mathbf{e}_{-1}})Y_{n,l}(\boldsymbol{\xi}) \\
&\qquad + (b_{n-1,l+1,0}a_{n,l,1}\mathbf{e}_0 \times \overline{\mathbf{e}_1} + b_{n-1,l,1}a_{n,l,0}\mathbf{e}_{-1} \times \mathbf{e}_0)Y_{n,l+1}(\boldsymbol{\xi})\Big) \\
&= (2n+1)\mathrm{i}\Big((b_{n-1,l-1,0}a_{n,l,-1} + b_{n-1,l,-1}a_{n,l,0})Y_{n,l-1}(\boldsymbol{\xi})\mathbf{e}_1 \\
&\qquad + (-b_{n-1,l+1,-1}a_{n,l,1} + b_{n-1,l-1,1}a_{n,l,-1})Y_{n,l}(\boldsymbol{\xi})\mathbf{e}_0 \\
&\qquad - (b_{n-1,l+1,0}a_{n,l,1} + b_{n-1,l,1}a_{n,l,0})Y_{n,l+1}(\boldsymbol{\xi})\mathbf{e}_{-1})\Big)\,,
\end{aligned}
$$

where we used (3.10) and (3.11) as well as the obvious identities

$$
\begin{aligned}
\mathbf{e}_0 \times \overline{\mathbf{e}_{-1}} &= \mathbf{e}_1 \times \mathbf{e}_0 = \mathrm{i}\mathbf{e}_1, \\
\mathbf{e}_1 \times \overline{\mathbf{e}_1} &= -\mathrm{i}\mathbf{e}_0, \\
\mathbf{e}_{-1} \times \overline{\mathbf{e}_{-1}} &= \mathrm{i}\mathbf{e}_0, \\
\mathbf{e}_0 \times \overline{\mathbf{e}_1} &= \mathbf{e}_{-1} \times \mathbf{e}_0 = -\mathrm{i}\mathbf{e}_{-1} = \mathrm{i}\overline{\mathbf{e}_1}.
\end{aligned}
$$

This proves representation (3.9), since we may evaluate

$$
\begin{aligned}
&b_{n-1,l-1,0}a_{n,l,-1} + b_{n-1,l,-1}a_{n,l,0} \\
&=\sqrt{\frac{(n+l-1)(n-l+1)}{(2n+1)(2n-1)}}\sqrt{\frac{(n+l)(n+l-1)}{2(2n-1)(2n+1)}} + \sqrt{\frac{(n-l)(n-l+1)}{2(2n+1)(2n-1)}}\sqrt{\frac{(n-l)(n+l)}{(2n-1)(2n+1)}} \\
&=\sqrt{\frac{(n+l)(n-l+1)}{2(2n+1)^2(2n-1)^2}}(n+l-1+n-l) = \sqrt{\frac{(n+l)(n-l+1)}{2(2n+1)^2}} = \frac{c_{n,l,-1}}{2n+1},
\end{aligned}
$$

$$
\begin{aligned}
&- b_{n-1,l+1,-1}a_{n,l,1} + b_{n-1,l-1,1}a_{n,l,-1} \\
&= -\sqrt{\frac{(n-l-1)(n-l)}{2(2n+1)(2n-1)}}\sqrt{\frac{(n-l)(n-l-1)}{2(2n-1)(2n+1)}} + \sqrt{\frac{(n+l-1)(n+l)}{2(2n+1)(2n-1)}}\sqrt{\frac{(n+l)(n+l-1)}{2(2n-1)(2n+1)}} \\
&= -\frac{(n-l-1)(n-l)}{2(2n+1)(2n-1)} + \frac{(n+l-1)(n+l)}{2(2n+1)(2n-1)} = \frac{l}{2n+1} = \frac{c_{n,l,0}}{2n+1},
\end{aligned}
$$

$$
\begin{aligned}
&b_{n-1,l+1,0}a_{n,l,1} + b_{n-1,l,1}a_{n,l,0} = \\
&=\sqrt{\frac{(n+l+1)(n-l-1)}{(2n+1)(2n-1)}}\sqrt{\frac{(n-l)(n-l-1)}{2(2n-1)(2n+1)}} + \sqrt{\frac{(n+l)(n+l+1)}{2(2n+1)(2n-1)}}\sqrt{\frac{(n-l)(n+l)}{(2n-1)(2n+1)}} \\
&=\sqrt{\frac{(n+l+1)(n-l)}{2(2n+1)^2(2n-1)^2}}(n-l-1+n+l) = \sqrt{\frac{(n+l+1)(n-l)}{2(2n+1)^2}} = -\frac{c_{n,l,1}}{2n+1}.
\end{aligned}
$$

□

We conclude this subsection by establishing an extension of the Funk-Hecke theorem (Theorem 2.3) to vector spherical harmonics.

Theorem 3.2. *Suppose $F(t)$ to be continuous for $-1 \le t \le 1$. Then for every vector spherical harmonics of degree n and type l*

$$\int_{\mathbb{S}^2} \mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) F(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, d\omega(\boldsymbol{\xi}) = \alpha_{11}\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\eta}) + \alpha_{12}\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\eta}),$$
$$\int_{\mathbb{S}^2} \mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi}) F(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, d\omega(\boldsymbol{\xi}) = \alpha_{21}\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\eta}) + \alpha_{22}\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\eta}),$$
$$\int_{\mathbb{S}^2} \mathbf{y}_{n,l}^{(3)}(\boldsymbol{\xi}) F(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, d\omega(\boldsymbol{\xi}) = \alpha_{33}\mathbf{y}_{n,l}^{(3)}(\boldsymbol{\eta})$$

hold true, where the constants are given as

$$\begin{aligned}
\alpha_{11} &= \frac{2\pi}{2n+1}\left(n\int_{-1}^{1} F(s)P_{n-1}(s)\,ds + (n+1)\int_{-1}^{1} F(s)P_{n+1}(s)\,ds\right),\\
\alpha_{12} &= \frac{2\pi}{2n+1}\left(\int_{-1}^{1} F(s)P_{n-1}(s)\,ds - \int_{-1}^{1} F(s)P_{n+1}(s)\,ds\right),\\
\alpha_{21} &= n(n+1)\alpha_{12},\\
\alpha_{22} &= \frac{2\pi}{2n+1}\left((n+1)\int_{-1}^{1} F(s)P_{n-1}(s)\,ds + n\int_{-1}^{1} F(s)P_{n+1}(s)\,ds\right),\\
\alpha_{33} &= 2\pi\int_{-1}^{1} F(s)P_n(s)\,ds.
\end{aligned}$$

Proof. The identities (3.7) and (3.8) can be reformulated as

$$\sum_{\epsilon=-1}^{1} a_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\xi})\overline{\mathbf{e}_\epsilon} = \frac{1}{2n+1}\left(n\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi})\right),$$
$$\sum_{\epsilon=-1}^{1} b_{n,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\xi})\mathbf{e}_{-\epsilon} = \frac{1}{2n+1}\left((n+1)\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) - \mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi})\right).$$

These equations allow for using the classical Funk-Hecke formula, which together with a further application of (3.7) implies

$$\begin{aligned}
\int_{\mathbb{S}^2} \mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) F(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, d\omega(\boldsymbol{\xi}) &= 2\pi\sum_{\epsilon=-1}^{1} a_{n,l,\epsilon} Y_{n-1,l+\epsilon}(\boldsymbol{\eta})\overline{\mathbf{e}_\epsilon}\int_{-1}^{1} F(s)P_{n-1}(s)\,ds\\
&\quad + 2\pi\sum_{\epsilon=-1}^{1} b_{n,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\eta})\mathbf{e}_{-\epsilon}\int_{-1}^{1} F(s)P_{n+1}(s)\,ds\\
&= \frac{2\pi}{2n+1}\left(n\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi})\right)\int_{-1}^{1} F(s)P_{n-1}(s)\,ds\\
&\quad + \frac{1}{2n+1}\left((n+1)\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\xi}) - \mathbf{y}_{n,l}^{(2)}(\boldsymbol{\xi})\right)\int_{-1}^{1} F(s)P_{n+1}(s)\,ds\\
&= \alpha_{11}\mathbf{y}_{n,l}^{(1)}(\boldsymbol{\eta}) + \alpha_{12}\mathbf{y}_{n,l}^{(2)}(\boldsymbol{\eta}).
\end{aligned}$$

The other identities follow accordingly. □

3.2. Complete orthogonal systems of ridge functions in $L_2(\mathbb{B}^3, \mathbf{S}_1)$

Using vector spherical harmonics as a starting point we are able to present complete orthogonal systems of $L_2(\mathbb{B}^3, \mathbf{S}_1)$-subspaces corresponding to the Helmholtz-Hodge decomposition (2.5).

Proposition 3.3. *The system of vector fields*

$$\begin{aligned}
&\left\{\mathbf{A}^{(n)}_{n-1-2k,l},\ n\in\mathbb{N},\quad k=0,...,\left[\frac{n-1}{2}\right],\ |l|\le n-1-2k\right\}\\
\bigcup&\left\{\mathbf{B}^{(n)}_{n+1-2k,l},\ n\in\mathbb{N}_0,\ k=0,...,\left[\frac{n}{2}\right],\qquad |l|\le n+1-2k\right\}\\
\bigcup&\left\{\mathbf{C}^{(n)}_{n-2k,l},\qquad n\in\mathbb{N},\quad k=0,...,\left[\frac{n-1}{2}\right],\ |l|\le n-2k\right\},
\end{aligned}$$

where for $\mathbf{x}\in\mathbb{B}^3$

$$\mathbf{A}^{(n)}_{n-1-2k,l}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\xi}) C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}), \tag{3.14}$$

$$\mathbf{B}^{(0)}_{1,l}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}^{(1)}_{0,l}(\boldsymbol{\eta})\, d\omega(\boldsymbol{\eta}), \tag{3.15}$$

$$\mathbf{B}^{(n)}_{n+1-2k,l}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\xi}) C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}),\ n\ge 1, \tag{3.16}$$

$$\mathbf{C}^{(n)}_{n-2k,l}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\xi}) C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}), \tag{3.17}$$

forms a complete, orthogonal basis of $\mathbf{L}_2(\mathbb{B}^3, \mathbf{S}_1)$ *consisting of polynomial vector fields. Thus any* $\mathbf{f}\in L_2(\mathbb{B}^3,\mathbf{S}_1)$ *has a unique representation*

$$\begin{aligned}
\mathbf{f}(\mathbf{x}) = \sum_{l=1}^{3} b^{(0)}_{1,l}\mathbf{B}^{(0)}_{1,l}(\mathbf{x}) + \sum_{n=1}^{\infty}\Bigg[&\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\sum_{|l|\le n-1-2k} a^{(n)}_{n-1-2k,l}\mathbf{A}^{(n)}_{n-1-2k,l}(\mathbf{x})\\
&+\sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{l=|1|\le n+1-2k} b^{(n)}_{n+1-2k,l}\mathbf{B}^{(n)}_{n+1-2k,l}(\mathbf{x}) + \sum_{k=1}^{\left[\frac{n-1}{2}\right]}\sum_{|l|\le n-2k} c^{(n)}_{n-2k,l}\mathbf{C}^{(n)}_{n-2k,l}(\mathbf{x})\Bigg].
\end{aligned}$$

Moreover $\mathbf{B}^{(n)}_{n+1-2k,l}$, $\mathbf{C}^{(n)}_{n-2k,l}$ *are solenoidal vector fields, i.e.* $\operatorname{div}\mathbf{B}^{(n)}_{n+1-2k,l}=0$, $\operatorname{div}\mathbf{C}^{(n)}_{n-2k,l}=0$ *and the fields* $\mathbf{A}^{(n)}_{n-1-2k,l}$ *are potential vector fields*

$$\mathbf{A}^{(n)}_{n-1-2k,l}(\mathbf{x}) = \nabla_{\mathbf{x}}\int_{\mathbb{S}^2} Y_{n-1-2k,l}(\boldsymbol{\xi})P_{n+1}(\mathbf{x}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}), \tag{3.18}$$

where the potentials

$$P^{(n)}_{n-1-2k,l}(\mathbf{x}) := \int_{\mathbb{S}^2} Y_{n-1-2k,l}(\boldsymbol{\xi})P_{n+1}(\mathbf{x}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi})$$

have vanishing boundary values for $\boldsymbol{\phi}\in\mathbb{S}^2$

$$\begin{aligned}
P^{(n)}_{n-1-2k,l}(\boldsymbol{\phi}) &= \int_{\mathbb{S}^2} Y_{n-1-2k,l}(\boldsymbol{\xi})P_{n+1}(\boldsymbol{\phi}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi})\\
&= 2\pi\, Y_{n-1-2k,l}(\boldsymbol{\phi})\int_{-1}^{1} P_{n-1-2k}(s)P_{n+1}(s)\, ds = 0.
\end{aligned} \tag{3.19}$$

Proof. The first part of Proposition 3.3 is an immediate consequence from Proposition 3.5. Equation (3.18) follows from (3.2) and $P'_{n+1}(t) = C_n^{(3/2)}(t)$. The orthogonality of the Legendre polynomials P_n leads to (3.19). $\square$

Proposition 3.4 delivers the representation of the systems (3.14)–(3.17) in spherical coordinates. We will see that they are in some sense generalizations of Zernike polynomials (2.16) to vector fields. Restricting $\mathbf{x} = \boldsymbol{\phi} \in \mathbb{S}^2$ we shall realize that the fields $\mathbf{A}^{(n)}_{n-1-2k,l}$, $\mathbf{B}^{(n)}_{n+1-2k,l}$ and $\mathbf{C}^{(n)}_{n-2k,l}$ are identical to the vector spherical harmonics (3.2)–(3.4).

Proposition 3.4. *Let $\mathbf{x} = \rho\boldsymbol{\phi}$, $\rho \geq 0$, $\boldsymbol{\phi} \in \mathbb{S}^2$. The families of vector fields (3.14)–(3.17) are represented in spherical coordinates as*

$$\begin{aligned}\mathbf{A}^{(n)}_{n-1-2k,l}(\mathbf{x}) = {} & \frac{4\pi}{2n-1-4k}\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\phi})\left((n-1-2k)R_n^{n-2-2k}(\rho) + (n-2k)R_n^{n-2k}(\rho)\right)\\ & + \frac{4\pi}{2n-1-4k}\mathbf{y}^{(2)}_{n-1-2k,l}(\boldsymbol{\phi})\left(R_n^{n-2-2k}(\rho) - R_n^{n-2k}(\rho)\right),\end{aligned}$$

$$\begin{aligned}\mathbf{B}^{(n)}_{n+1-2k,l}(\mathbf{x}) = {} & \frac{4\pi(n+1-2k)(n+2-2k)}{2n-4k+3}\mathbf{y}^{(1)}_{n+1-2k,l}(\boldsymbol{\phi})\left(R_n^{n-2k}(\rho) - R_n^{n+2-2k}(\rho)\right)\\ & + \frac{4\pi}{2n-4k+3}\mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\phi})\\ & \times\left((n+2-2k)R_n^{n-2k}(\rho) + (n+1-2k)R_n^{n+2-2k}(\rho)\right),\end{aligned}$$

$$\mathbf{C}^{(n)}_{n-2k,l}(\mathbf{x}) = 4\pi\mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\phi})R_n^{n-2k}(\rho).$$

For $\mathbf{x} = \boldsymbol{\phi} \in \mathbb{S}^2$ we have

$$\begin{aligned}\mathbf{A}^{(n)}_{n-1-2k,l}(\boldsymbol{\phi}) &= 4\pi\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\phi}),\\ \mathbf{B}^{(n)}_{n+1-2k,l}(\boldsymbol{\phi}) &= 4\pi\mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\phi}),\\ \mathbf{C}^{(n)}_{n-2k,l}(\boldsymbol{\phi}) &= 4\pi\mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\phi}).\end{aligned}$$

Proof. Applying the Funk-Hecke formula for vector spherical harmonics (Theorem 3.2) and identity *(d)* from Lemma 2.5 yields

$$\begin{aligned}\mathbf{A}^{(n)}_{n-1-2k,l}(\mathbf{x}) &= \int_{\mathbb{S}^2}\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\xi})C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi})d\omega(\boldsymbol{\xi})\\ &= \frac{4\pi}{2n-1-4k}\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\phi})\left((n-1-2k)R_n^{n-2-2k}(\rho) + (n-2k)R_n^{n-2k}(\rho)\right)\\ &\quad + \frac{4\pi}{2n-1-4k}\mathbf{y}^{(2)}_{n-1-2k,l}(\boldsymbol{\phi})\left(R_n^{n-2-2k}(\rho) - R_n^{n-2k}(\rho)\right),\end{aligned}$$

$$\begin{aligned}\mathbf{B}^{(n)}_{n+1-2k,l}(\mathbf{x}) &= \int_{\mathbb{S}^2}\mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\xi})C_n^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\xi})d\omega(\boldsymbol{\xi})\\ &= \frac{4\pi(n+1-2k)(n+2-2k)}{2n-4k+3}\mathbf{y}^{(1)}_{n+1-2k,l}(\boldsymbol{\phi})\left(R_n^{n-2k}(\rho) - R_n^{n+2-2k}(\rho)\right)\\ &\quad + \frac{4\pi}{2n-4k+3}\mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\phi})\\ &\qquad \times\left((n+2-2k)R_n^{n-2k}(\rho) + (n+1-2k)R_n^{n+2-2k}(\rho)\right),\end{aligned}$$

$$\mathbf{C}^{(n)}_{n-2k,l}(\mathbf{x}) = \int_{\mathbb{S}^2} \mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\xi}) C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi}) d\omega(\boldsymbol{\xi}) = 4\pi \mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\phi}) R^{n-2k}_n(\rho).$$

The boundary values at $\mathbb{S}^2$ then are obvious, since $R^{n-2k}_n(1) = 1$ for all n and $k = 0,\ldots,[n/2]$. □

With the help of the systems (3.14)–(3.17) we can build orthogonal bases of five different subspaces which are inspired by the Helmholtz-Hodge decomposition.

Proposition 3.5. *We have:*

(a) $\left\{\mathbf{A}^{(n)}_{n-1-2k,l}\right\}$ *forms an orthogonal basis for* $\nabla H^1_0(\mathbb{B}^3)$ *with*

$$||\mathbf{A}^{(n)}_{n-1-2k,l}||_{L_2(\mathbb{B}^3,\mathbf{S}_1)} = \sqrt{\frac{16\pi^2}{2n+3}}.$$

(b) $\left\{\mathbf{B}^{(n)}_{n+1,l}\right\}$ *forms an orthogonal basis for* $\nabla \mathrm{Harm}(\mathbb{B}^3)$ *with*

$$||\mathbf{B}^{(n)}_{n+1,l}||_{L_2(\mathbb{B}^3,\mathbf{S}_1)} = \sqrt{\frac{16\pi^2(n+1)(n+2)^2}{(2n+3)^2}}.$$

(c) $\left\{\mathbf{A}^{(n)}_{n-1-2k,l}\right\}\bigcup\left\{\mathbf{B}^{(n)}_{n+1,l}\right\}$ *forms an orthogonal basis for* $\nabla H^1(\mathbb{B}^3)$.

(d) $\left\{\mathbf{B}^{(n)}_{n+1-2k,l},\ k\neq 0\right\}\bigcup\left\{\mathbf{C}^{(n)}_{n-2k,l}\right\}$ *forms an orthogonal basis for* $H_0(\mathrm{div};\mathbb{B}^3)$ *with*

$$\begin{aligned} ||\mathbf{B}^{(n)}_{n+1-2k,l}||_{L_2(\mathbb{B}^3,\mathbf{S}_1)} &= \sqrt{\frac{16\pi^2(n+1-2k)(n+2-2k)}{2n+3}}, \\ ||\mathbf{C}^{(n)}_{n-2k,l}||_{L_2(\mathbb{B}^3,\mathbf{S}_1)} &= \sqrt{\frac{16\pi^2(n-2k)(n-2k+1)}{2n+3}}. \end{aligned}$$

(e) $\{\mathbf{B}^{(n)}_{n+1-2k,l}\}\bigcup\{\mathbf{C}^{(n)}_{n-2k,l}\}$ *forms an orthogonal basis for*

$$H(\mathrm{div};\mathbb{B}^3) = \nabla\mathrm{Harm}(\mathbb{B}^3)\oplus H_0(\mathrm{div};\mathbb{B}^3).$$

Proof. The second part of Proposition 3.3 tells us that $\left\{\mathbf{A}^{(n)}_{n-1-2k,l}\right\}\subset\nabla H^1_0(\mathbb{B}^3)$. Since due to Proposition 3.4

$$\mathbf{A}^{(n)}_{n-1-2k,l}(\boldsymbol{\phi}) = \int_{\mathbb{S}^2}\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\xi})C^{(3/2)}_n(\boldsymbol{\phi}\cdot\boldsymbol{\xi})d\omega(\boldsymbol{\xi}) = 4\pi\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\phi}),$$

we may compute

$$\begin{aligned} ||\mathbf{A}^{(n)}_{n-1-2k,l}||^2 &= \int_{\mathbb{B}^3}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\xi})\cdot\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\xi}')\right) \\ &\qquad\qquad \times C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi})C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi})\,d\mathbf{x} \\ &= \frac{4\pi}{2n+3}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\xi})\cdot\mathbf{y}^{(1)}_{n-1-2k,l}(\boldsymbol{\xi}')\right)C^{(3/2)}_n(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi}) \\ &= \frac{16\pi^2}{2n+3}||\mathbf{y}^{(1)}_{n-1-2k,l}||^2 = \frac{16\pi^2}{2n+3}, \end{aligned}$$

where we applied formula *(c)* from Proposition 2.6, Theorem 3.2 and $||\mathbf{y}^{(1)}_{n-1-2k}||_{L_2} = 1$. Using (3.5) we obtain

$$\operatorname{div} \mathbf{B}^{(n)}_{n+1-2k,l}(\mathbf{x}) = \int_{\mathbb{S}^2} \boldsymbol{\xi} \cdot \mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\xi}) \frac{\mathrm{d}}{\mathrm{d}t} C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}) = 0$$

and in the same way one shows $\operatorname{div} \mathbf{C}^{(n)}_{n-2k,l}(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbb{B}^3$. From the fact that $\mathbf{B}^{(n)}_{n+1-2k,l}(\boldsymbol{\phi}) = 4\pi \mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\phi})$ and $\mathbf{C}^{(n)}_{n-2k,l}(\boldsymbol{\phi}) = 4\pi \mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\phi})$ for $\boldsymbol{\phi} \in \mathbb{S}^2$ (Proposition 3.4) we easily obtain

$$\boldsymbol{\phi} \cdot \mathbf{B}^{(n)}_{n+1-2k,l}(\boldsymbol{\phi}) = 0\,, \quad \boldsymbol{\phi} \cdot \mathbf{C}^{(n)}_{n-2k,l}(\boldsymbol{\phi}) = 0\,, \quad \boldsymbol{\phi} \in \mathbb{S}^2\,,$$

where we have applied (3.5) once more. Hence we have shown that $\left\{\mathbf{B}^{(n)}_{n+1-2k,l},\ k \neq 0\right\} \cup \left\{\mathbf{C}^{(n)}_{n-2k,l}\right\} \subset H_0(\mathrm{div}; \mathbb{B}^3)$.

It remains to show that $\operatorname{rot} \mathbf{B}^{(n)}_{n+1,l}(\mathbf{x}) = 0$ in $\mathbb{B}^3$. Taking into account that $(C^{(\lambda)}_n)' = 2\lambda C^{(\lambda+1)}_{n-1}$ and $\mathbf{y}^{(3)}_{n+1-2k,l}(\boldsymbol{\xi}) = \boldsymbol{\xi} \times \mathbf{y}^{(2)}_{n+1-2k,l}$, we have

$$\begin{aligned} \operatorname{rot} \mathbf{B}^{(n)}_{n+1,l}(\mathbf{x}) &= \int_{\mathbb{S}^2} \boldsymbol{\xi} \times \mathbf{y}^{(2)}_{n+1,l}(\boldsymbol{\xi}) (C^{(3/2)}_{n-1})'(\mathbf{x}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi}) \\ &= 3 \int_{\mathbb{S}^2} \mathbf{y}^{(3)}_{n+1,l}(\boldsymbol{\xi}) C^{(5/2)}_{n-1}(\mathbf{x}\cdot\boldsymbol{\xi})\, d\omega(\boldsymbol{\xi})\,. \end{aligned} \tag{3.20}$$

Since

$$C^{(5/2)}_n(t) = \frac{1}{3} \sum_{s=0}^{[n/2]} (2n+3-4s) C^{(3/2)}_{n-2s}(t)$$

(formula (5.12)) and

$$C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi}) = 4\pi \sum_{k=0}^{[n/2]} (2n-4k+1) R^{n-2k}_n(\rho) \sum_{|p|\le n-2k} Y_{n-2k,p}(\boldsymbol{\phi}) \overline{Y_{n-2k,p}(\boldsymbol{\xi})}\,, \quad \mathbf{x} = \rho\boldsymbol{\phi}\,,$$

which follows from (2.11) and part *(a)* of Proposition 2.6, we may write

$$\begin{aligned} 3C^{(5/2)}_{n-1}(\mathbf{x}\cdot\boldsymbol{\xi}) &= 4\pi \sum_{s=0}^{\left[\frac{n-1}{2}\right]} (2n+1-4s) \sum_{k=0}^{\left[\frac{n-1-2s}{2}\right]} (2n-1-4s-4k) R^{n-1-2s-2k}_{n-1-2s}(\rho) \\ &\qquad \times \sum_{|p|\le n-1-2s-2k} Y_{n-1-2s-2k,p}(\boldsymbol{\phi}) \overline{Y_{n-1-2s-2k,p}(\boldsymbol{\xi})}, \end{aligned} \tag{3.21}$$

where again $\mathbf{x} = \rho\boldsymbol{\phi}$. The last ingredient is the expansion (3.9) with n replaced by $n+1$,

$$\mathbf{y}^{(3)}_{n+1,l}(\boldsymbol{\xi}) = \mathrm{i} \sum_{\epsilon=-1}^{1} c_{n+1,l,\epsilon} Y_{n+1,l+\epsilon}(\boldsymbol{\xi}) \mathbf{e}_{-\epsilon}. \tag{3.22}$$

Putting (3.21) and (3.22) in (3.20) finally shows that $\operatorname{rot} \mathbf{B}^{(n)}_{n+1,l} = 0$ because of $n+1 \neq n-1-2s-2k$ and the orthogonality of the spherical harmonics $Y_{n,l}$.

The norms of $\mathbf{B}^{(n)}_{n+1-2k,l}$ and $\mathbf{C}^{(n)}_{n-2k,l}$ compute to

$$
\begin{aligned}
||\mathbf{B}^{(n)}_{n+1-2k,l}||^2 &= \int_{\mathbb{B}^3}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(\mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\xi})\cdot\mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\xi}')\right)\\
&\qquad\qquad\times C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi})C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi})\,d\mathbf{x}\\
&= \frac{4\pi}{2n+3}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(\mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\xi})\cdot\mathbf{y}^{(2)}_{n+1-2k,l}(\boldsymbol{\xi}')\right)C^{(3/2)}_n(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi})\\
&= \frac{16\pi^2}{2n+3}||\mathbf{y}^{(2)}_{n+1-2k,l}||^2 = \frac{16\pi^2(n+1-2k)(n+2-2k)}{2n+3},\\
||\mathbf{C}^{(n)}_{n-2k,l}||^2 &= \int_{\mathbb{B}^3}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(\mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\xi})\cdot\mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\xi}')\right)\\
&\qquad\qquad\times C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi})C^{(3/2)}_n(\mathbf{x}\cdot\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi})\,d\mathbf{x}\\
&= \frac{4\pi}{2n+3}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(\mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\xi})\cdot\mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\xi}')\right)C^{(3/2)}_n(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi}')\,d\omega(\boldsymbol{\xi})\\
&= \frac{8\pi^2}{2n+3}\left(\int_{-1}^{1}C^{(3/2)}_n(s)P_{n-2k}(s)ds\right)\int_{\mathbb{S}^2}\left(\mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\xi})\cdot\mathbf{y}^{(3)}_{n-2k,l}(\boldsymbol{\xi})\right)d\omega(\boldsymbol{\xi})\\
&= \frac{16\pi^2}{2n+3}||\mathbf{y}^{(3)}_{n-2k,l}||^2 = \frac{16\pi^2(n-2k)(n-2k+1)}{2n+3}.
\end{aligned}
$$

Again we used formula *(c)* from Proposition 2.6, Theorem 3.2 and $||\mathbf{y}^{(2)}_{n,l}||_{L_2} = ||\mathbf{y}^{(3)}_{n,l}||_{L_2} = n(n+1)$.
The density of the systems in the corresponding subspaces follows from the density of polynomials in the according spaces. Note that *(a)* and *(b)* imply *(c)* and *(e)* is a consequence of *(b)* and *(d)*. □

4. CONSTRUCTION OF COMPLETE SYSTEMS OF VECTOR FIELDS BY MEANS OF DIFFERENTIAL OPERATORS

We show an alternative way to construct bases of vector field subspaces. Using bases of polynomial scalar fields defined on the unit ball as starting point complete systems of vector fields are accomplished by the application of differential operators such as gradient or curl. The method described in this section is not restricted to polynomials but does also work for any complete system consisting of smooth functions which are defined on a bounded Riemannian domain with algebraic boundary.
Again we will find bases for spaces of potential fields $\nabla H^1(\mathbb{B}^3)$, $\nabla H^1_0(\mathbb{B}^3)$ (Section 4.1), solenoidal fields $H(\mathrm{div};\mathbb{B}^3)$, $H_0(\mathrm{div};\mathbb{B}^3)$ (Section 4.2) and harmonic fields $\nabla\mathrm{Harm}(\mathbb{B}^3)$ (Section 4.3). In contrast to the systems constructed in Section 3, the bases generated here are non-orthogonal.
Nevertheless polynomial bases are useful not only to develop numerical solvers, since they are simpler to verify on a computer than the ridge functions (3.14)–(3.17), but also to state theoretical results such as convergence and stability estimates or uniqueness theorems. E.g. in [4] the reader finds applications of non-orthogonal polynomial systems in vector and tensor tomography.

4.1. A polynomial basis for $\nabla H^1(\mathbb{B}^3)$ and $\nabla H_0^1(\mathbb{B}^3)$ (potential fields)

Since the multivariate polynomials in (x, y, z) are dense in $H^1(\mathbb{B}^3)$, we find a system of vector fields which is complete in $\nabla H^1(\mathbb{B}^3)$ by simply applying the operator of gradient ∇. The scheme below shows, how the potential fields emerge from polynomials with increasing degree from top to bottom.

$$
\begin{array}{ccccc}
x & \rightarrow & \nabla x & \rightarrow & (1,0,0) \\
y & \rightarrow & \nabla y & \rightarrow & (0,1,0) \\
z & \rightarrow & \nabla z & \rightarrow & (0,0,1) \\
 & & & & \\
x^2 \ , \ xy & \rightarrow & \nabla(x^2) \ , \ \nabla(xy) & \rightarrow & (2x,0,0) \ , \ (y,x,0) \\
xz \ , \ y^2 & \rightarrow & \nabla(xz) \ , \ \nabla(y^2) & \rightarrow & (z,0,x) \ , \ (0,2y,0) \\
yz \ , \ z^2 & \rightarrow & \nabla(yz) \ , \ \nabla(z^2) & \rightarrow & (0,z,y) \ , \ (0,0,2z) \\
 & & & & \\
\cdot \ \cdot \ \cdot \ \cdot & & \cdot \ \cdot & \cdot \ \cdot & \cdot \ \cdot \\
 & & & & \\
x^{s+1} & \rightarrow & \nabla(x^{s+1}) & \rightarrow & ((s+1)x^s,0,0) \\
x^s y & \rightarrow & \nabla(x^s y) & \rightarrow & (sx^{s-1}y, x^s, 0) \\
x^s z & \rightarrow & \nabla(x^s z) & \rightarrow & (sx^{s-1}z, 0, x^s) \\
\cdot \ \cdot \ \cdot \ \cdot & & & \cdot \ \cdot & \cdot \ \cdot \\
x^k y^l z^m & \rightarrow & \nabla(x^k y^l z^m) & \rightarrow & \left(kx^{k-1}y^l z^m, lx^k y^{l-1} z^m, mx^k y^l z^{m-1}\right) \\
\cdot \ \cdot \ \cdot \ \cdot & & \cdot \ \cdot & \cdot \ \cdot & \cdot \ \cdot \\
z^{s+1} & \rightarrow & \nabla(z^{s+1}) & \rightarrow & (0,0,(s+1)z^s)
\end{array}
\tag{4.1}
$$

The following proposition collects basic properties of the space of homogeneous scalar and vectorial polynomials of degree s. The assertions can be proven easily or can be read immediately from scheme (4.1)

Proposition 4.1. *Let n denote the dimension of the Euclidean space.*
(a) The space of homogeneous, multivariate polynomials in $\mathbb{R}^n$ of exact degree s has dimension $N(n,s) = C^s_{s+n-1}$, the dimension of the space of homogeneous, multivariate polynomials of degree less than or equal to s is given as $N(n, \leq s) = C^s_{s+n}$. In particular we have $N(3,s) = (s+1)(s+2)/2$, $N(3, \leq s) = (s+1)(s+2)(s+3)/6$.
(b) The dimension $Nd(n,s)$ of the space of potential vector fields in $\mathbb{R}^n$ with components being homogeneous, multivariate polynomials of exact degree s is equal to $N(n, s+1) = C^{s+1}_{s+n}$, the space of potential vector fields with components being homogeneous, multivariate polynomials of degree less than or equal to s has the dimension $Nd(n, \leq s) = N(n, \leq s+1) - 1 = C^{s+1}_{s+n+1} - 1$. In particular we have $Nd(3,s) = (s+2)(s+3)/2$, $Nd(3, \leq s) = (s+2)(s+3)(s+4)/6 - 1$.
(c) The vector fields generated by method (4.1) are potential fields. They are linearly independent and represent a basis of the subspace which consists of all potential vector fields with components being multivariate polynomials of degree less than or equal to s. Thus, every such field has a unique representation in terms of (4.1).
(d) Letting $s \to \infty$ by (4.1) we obtain a system which is dense in $\nabla H^1(\mathbb{B}^3)$.

We proceed by constructing a polynomial basis for $\nabla H_0^1(\mathbb{B}^3)$ in a similar manner as for $\nabla H^1(\mathbb{B}^3)$. All homogeneous polynomials with vanishing boundary values at $\partial\mathbb{B}^3$ except 0 contain a factor $(1 - r^2)$, where $r^2 = x^2 + y^2 + z^2$. We call the set of polynomials having a representation as $(1 - r^2)p$, where p is homogeneous of degree s in $\mathbb{R}^n$, the set of

$(s, s+2)$-*homogeneous polynomials*. We denote this set by $\Pi^n_{s,s+2}$. We generate a system of nonorthogonal polynomial vector fields in $\nabla H^1_0(\mathbb{B}^3)$ applying the operator of gradient ∇ to $(s, s+2)$-homogeneous polynomials.

$$
\begin{array}{rcccl}
(1-r^2) & \to & \nabla(1-r^2) & \to & (-2x\,,-2y\,,-2z) \\
(1-r^2)x & \to & \nabla[(1-r^2)x] & \to & (1-r^2-2x^2\,,-2xy\,,-2xz) \\
(1-r^2)y & \to & \nabla[(1-r^2)y] & \to & (-2xy\,,1-r^2-2y^2\,,-2yz) \\
(1-r^2)z & \to & \nabla[(1-r^2)z] & \to & (-2xz\,,-2yz\,,1-r^2-2z^2) \\
(1-r^2)x^2 & \to & \nabla[(1-r^2)x^2] & \to & (2x-2xr^2-2x^3\,,-2x^2y\,,-2x^2z) \\
(1-r^2)xy & \to & \nabla[(1-r^2)xy] & \to & (y-yr^2-2x^2y\,,x-xr^2-2xy^2\,, \\
 & & & & -2xyz) \\
\cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
(1-r^2)z^2 & \to & \nabla[(1-r^2)z^2] & \to & (-2xz^2\,,-2yz^2\,,2z-2zr^2-2z^3) \\
(1-r^2)x^{s-1} & \to & \nabla[(1-r^2)x^{s-1}] & \to & ((s-1)x^{s-2}(1-r^2)-2x^s\,, \\
 & & & & -2x^{s-1}y\,,-2x^{s-1}z) \\
(1-r^2)x^{s-2}y & \to & \nabla[(1-r^2)x^{s-2}y & \to & ((s-2)x^{s-3}y(1-r^2)-2x^{s-1}y\,, \\
 & & & & x^{s-2}(1-r^2)-2x^{s-2}y^2\,,-2x^{s-2}yz) \\
\cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
(1-r^2)z^{s-1} & \to & \nabla[(1-r^2)z^{s-1}] & \to & (-2xz^{s-1}\,,-2yz^{s-1}\,, \\
 & & & & (s-1)z^{s-2}(1-r^2)-2z^s)
\end{array}
\tag{4.2}
$$

Proposition 4.2 summarizes the main properties of system (4.2). Note that we consider only polynomials of the type $(1-r^2)x^ky^lz^m$, $k+l+m=s$, $s \geq 0$.

Proposition 4.2. *(a) The space of $(s, s+2)$-homogeneous polynomials in $\mathbb{R}^n$ has dimension $N^0(n, s, s+2) = N(n, s) = C^s_{s+n-1}$, $s \geq 0$. In particular for $n = 3$ we have $N^0(3, s, s+2) = (s+1)(s+2)/2$.*
(b) The dimension of the subspace $\nabla\Pi^n_{s,s+2} \subset \nabla H^1_0(\mathbb{B}^3)$, $s \geq 0$, is $Nd^0(n, s, s+2) = N(n, s) = C^s_{s+n-1}$. The elements of $\nabla\Pi^n_{s,s+2}$ consists of potential vector fields with components being homogeneous, multivariate polynomials of exact degree $s+1$. In particular for $n = 3$ we have $Nd^0(3, s, s+2) = (s+1)(s+2)/2$.
(c) The vector fields generated by method (4.2) are potential vector fields with vanishing potential at the boundary $\partial\mathbb{B}^3$. They are linearly independent and represent a basis for the subspace of all potential vector fields with components being multivariate polynomials of degree less than or equal to s and vanishing potential at $\partial\mathbb{B}^3$. Thus, every such field has a unique representation in terms of $\nabla\Pi^n_{s,s+2}$.
(d) Letting $s \to \infty$ by (4.2) we obtain a system which is dense in $\nabla H^1_0(\mathbb{B}^3)$.

4.2. A complete orthonormal system for the subspace $\nabla\mathrm{Harm}(\mathbb{B}^3)$ (harmonic fields)

To construct an orthonormal basis for $\nabla\mathrm{Harm}(\mathbb{B}^3)$ we use an approach which is different from section 4.1 and revert to spherical harmonics which have been introduced in section 2.2. We remind that there are $2n+1$ linearly independent spherical harmonics of degree n

and they can be represented as

$$\begin{array}{rcl} Y_{n,0}(\boldsymbol{\theta}) & = & P_n(\cos\vartheta)\,, \quad Y_{n,k}(\boldsymbol{\theta}) = P_l^{|k|}(\cos\vartheta)\,\cos(k\varphi)\,, \\ Y_{n,-k}(\boldsymbol{\theta}) & = & Y_{n,-k}(\boldsymbol{\theta}) = P_n^{|k|}(\cos\vartheta)\,\sin(k\varphi)\,, \quad k=1,\ldots,n\,, \end{array}$$

where $\boldsymbol{\theta} = \boldsymbol{\theta}(\vartheta,\varphi) = (\sin\vartheta\,\cos\varphi, \sin\vartheta\,\sin\varphi, \cos\vartheta) \in \mathbb{S}^2$ for $\varphi \in [0,2\pi)$, $\vartheta \in [0,\pi]$ and

$$P_n^k(t) = (-1)^k\,(1-t^2)^{k/2}\,\frac{d^k}{dt^k}P_n(t)\,, \qquad k=0,\ldots,n$$

are the associated Legendre polynomials of degree n. Since spherical harmonics are restrictions of harmonic polynomials to $\mathbb{S}^2$, there is a straightforward manner of extending them to harmonic functions on $\mathbb{R}^3$.

Lemma 4.3. *Let $(r,\boldsymbol{\theta}) \in \mathbb{R}_0^+ \times \mathbb{S}^2$ be spherical coordinates, i.e. $x = r\boldsymbol{\theta}$ for $x \in \mathbb{R}^3$, and $\widetilde{Y}_{n,k}(r\boldsymbol{\theta}) := r^n\,Y_{n,k}(\boldsymbol{\theta})$. Then,*

$$\left\{\widetilde{Y}_{n,k}\right\}, \qquad n \in \mathbb{N}\,,\ -n \le k \le n \tag{4.3}$$

forms a system of harmonic functions in $\mathbb{R}^3$.

Proof. Spherical harmonics are eigenfunctions of the Laplace-Beltrami operator Δ_S. We have

$$\Delta_S Y_{n,k} = -n(n+1)Y_{n,k}.$$

Since

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S$$

we may compute

$$\begin{array}{rcl} \Delta\widetilde{Y}_{n,k} & = & \Delta(r^n Y_{n,k}) \\ & = & n(n-1)r^{n-2}Y_{n,k} + 2nr^{n-2}Y_{n,k} - r^{n-2}n(n+1)Y_{n,k} \\ & = & 0, \qquad n \in \mathbb{N}, \quad -n \le k \le n. \end{array}$$

□

Applying the operator ∇ to the system $\{\widetilde{Y}_{n,k}\}$ yields a complete, orthonormal system of harmonic fields.

Lemma 4.4. *The set*

$$\left\{n^{-1/2}\,\nabla\widetilde{Y}_{n,k}\right\} \subset \nabla\mathrm{Harm}(\mathbb{B}^3)\,, \qquad n \in \mathbb{N}, \quad -n \le k \le n \tag{4.4}$$

with $\widetilde{Y}_{n,k}$ from lemma 4.3 is a complete system of L_2-orthonormal fields in $\nabla\mathrm{Harm}(\mathbb{B}^3)$.

Proof. We obviously have $n^{-1/2}\,\nabla\widetilde{Y}_{n,k} \in \nabla\mathrm{Harm}(\mathbb{B}^3)$ from Lemma 4.3. Assume $n, n' \in \mathbb{N}$,

$-n \leq k \leq n$, $-n' \leq k' \leq n'$. Applying Green's formula shows

$$
\begin{aligned}
& \int_{\mathbb{B}^3} \left(n^{-1/2}\, \nabla \widetilde{Y}_{n,k}(x) \cdot (n')^{-1/2}\, \nabla \widetilde{Y}_{n',k'}(x) \right)\, dx \\
= \;& (nn')^{-1/2} \int_{\partial\mathbb{B}^3} (\partial_{\boldsymbol{\nu}} \widetilde{Y}_{n,k})(x)\, \widetilde{Y}_{n',k'}(x)\, ds_x - (nn')^{-1/2} \int_{\mathbb{B}^3} (\Delta \widetilde{Y}_{n,k})(x)\, Y_{n',k'}(x)\, dx \\
= \;& (nn')^{-1/2} \int_{\mathbb{S}^2} \left\{ \partial_r \widetilde{Y}_{n,k}(r, \boldsymbol{\theta}) \right\}\Big|_{r=1} \widetilde{Y}_{n',k'}(1, \boldsymbol{\theta})\, d\boldsymbol{\theta} \\
= \;& (nn')^{-1/2} \int_{\mathbb{S}^2} n\, Y_{n,k}(\boldsymbol{\theta})\, Y_{n',k'}(\boldsymbol{\theta})\, d\boldsymbol{\theta} \\
= \;& (nn')^{-1/2}\, n\, \delta_{n,n'}\, \delta_{k,k'}
\end{aligned}
$$

since $\{Y_{n,k}\}$ forms a complete orthonormal system in $L_2(\mathbb{S}^2)$. It remains to show the completeness in $\nabla \mathrm{Harm}(\mathbb{B}^3)$. To this end consider an element $\mathbf{f} = \nabla h \in \nabla \mathrm{Harm}(\mathbb{B}^3)$, h harmonic, with

$$
\int_{\mathbb{B}^3} \left(\nabla \widetilde{Y}_{n,k}(x) \cdot \nabla h(x) \right)\, dx = 0 \quad \text{for all } n \in \mathbb{N}, \quad -n \leq k \leq n.
$$

Then, again using Green's formula, we get

$$
\begin{aligned}
0 \;&=\; n^{-1/2} \int_{\mathbb{B}^3} \left(\nabla \widetilde{Y}_{n,k}(x) \cdot \nabla h(x) \right)\, dx \\
&=\; n^{-1/2} \int_{\partial\mathbb{B}^3} (\partial_{\boldsymbol{\nu}} \widetilde{Y}_{n,k})(x)\, h(x)\, ds_x - n^{-1/2} \int_{\mathbb{B}^3} (\Delta \widetilde{Y}_{n,k})(x)\, h(x)\, dx \\
&=\; n^{-1/2} \int_{\mathbb{S}^2} \left\{ \partial_r \widetilde{Y}_{n,k}(r\boldsymbol{\theta}) \right\}\Big|_{r=1} h(\boldsymbol{\theta})\, d\boldsymbol{\theta} = n^{1/2} \int_{\mathbb{S}^2} Y_{n,k}(\boldsymbol{\theta})\, h(\boldsymbol{\theta})\, d\boldsymbol{\theta}
\end{aligned}
$$

which implies that $h = 0$ on $\mathbb{S}^2 = \partial\mathbb{B}^3$ because of the completeness of $\{Y_{n,k}\}$ in $L_2(\mathbb{S}^2)$. The maximum and minimum principle for harmonic functions yield

$$
\begin{aligned}
\sup_{x \in \mathbb{B}^3} h(x) \;&=\; \sup_{\boldsymbol{\theta} \in \mathbb{S}^2} h(\boldsymbol{\theta}) = 0 \\
\inf_{x \in \mathbb{B}^3} h(x) \;&=\; \inf_{\boldsymbol{\theta} \in \mathbb{S}^2} h(\boldsymbol{\theta}) = 0
\end{aligned}
$$

whence $h = 0$ and thus $\nabla h = 0$ in $\overline{\mathbb{B}^3}$ follows. This together with the property $\overline{\nabla \mathrm{Harm}(\mathbb{B}^3)}^{L^2} = \nabla \mathrm{Harm}(\mathbb{B}^3)$ ([24], Lemma 4.14) completes the proof. □

4.3. A polynomial basis for $H(\mathrm{div}; \mathbb{B}^3)$ (solenoidal fields)

We consider again the open unit ball $\mathbb{B}^3$ equipped with the Euclidean metric and use the notations $\mathbf{u} = (u_i) = (u_1, u_2, u_3)$ for a vector field on $\mathbb{B}^3$. We use again homogeneous polynomials to construct a basis of solenoidal vector fields. Let $x^k y^l z^m$, $k, l, m \geq 0$, $k + l + m = s + 2$ be monomials of degree $s + 2$, which we collect in the set

$$
\mathcal{M} \;=\; \{x^{s+2},\, x^{s+1}y,\, x^{s+1}z,\, x^s yz, \ldots, x^k y^l z^m, \ldots, z^{s+2}\}\,.
$$

If we apply the operator ∇ to the set $\mathcal{M}$ we obtain

$$\begin{aligned}\nabla\left(x^k y^l z^m\right) &= \left(\frac{\partial}{\partial x}x^k y^l z^m,\, \frac{\partial}{\partial y}x^k y^l z^m,\, \frac{\partial}{\partial z}x^k y^l z^m\right)\\ &= \left(\frac{\partial}{\partial x}x^k y^l z^m, 0, 0\right) + \left(0, \frac{\partial}{\partial y}x^k y^l z^m, 0\right) + \left(0, 0, \frac{\partial}{\partial z}x^k y^l z^m\right),\\ & k, l, m \geq 0\,, \quad k+l+m = s+2\,.\end{aligned} \tag{4.5}$$

The vector fields (4.5) can be divided in three groups, where each of them corresponds to the number of non-zero components.
1) The first group consists of fields with only one non-zero component. There exists three different types of such fields,

$$\begin{aligned}\nabla(x^{s+2}) &= \left((s+2)x^{s+1}\,,\, 0\,,\, 0\right)\,,\\ \nabla(y^{s+2}) &= \left(0\,,\, (s+2)y^{s+1}\,,\, 0\right)\,,\\ \nabla(z^{s+2}) &= \left(0\,,\, 0\,,\, (s+2)z^{s+1}\right)\,.\end{aligned} \tag{4.6}$$

2) The second group consists of those fields from (4.5) having two non-zero components. Again we have three different types,

$$\begin{aligned}\nabla(x^k y^l) &= \left(kx^{k-1}y^l,\, lx^k y^{l-1},\, 0\right), & k,l>0, && k+l = s+2\\ \nabla(x^k z^m) &= \left(kx^{k-1}x^m,\, 0,\, mx^k z^{m-1}\right), & k,m>0, && k+m = s+2\\ \nabla(y^l z^m) &= \left(0,\, ly^{l-1}z^m,\, my^l z^{m-1}\right), & l,m>0, && l+m = s+2\end{aligned} \tag{4.7}$$

3) The third group finally has no vanishing components,

$$\begin{aligned}\nabla(x^k y^l z^m) = &\left(kx^{k-1}y^l z^m\,,\, lx^k y^{l-1}z^m\,,\, mx^k y^l z^{m-1}\right)\,,\\ & k,l,m>0\,, \quad k+l+m = s+2.\end{aligned} \tag{4.8}$$

An additional application of the operator rot to both sides of (4.5) yields

$$\begin{aligned}\mathbf{0} &= \mathrm{rot}(\nabla(x^k y^l z^m))\\ &= \mathrm{rot}\left(\frac{\partial}{\partial x}x^k y^l z^m, 0, 0\right) + \mathrm{rot}\left(0, \frac{\partial}{\partial y}x^k y^l z^m, 0\right) + \mathrm{rot}\left(0,\, 0,\, \frac{\partial}{\partial z}x^k y^l z^m\right).\end{aligned}$$

The right hand side consists of a sum of solenoidal fields, the left hand side implies that they are linearly dependent. Thus a basis of solenoidal vector fields can be chosen in many ways. We analyze the three groups 1)-3) in order to find the ambuigities and to obtain a linearly independent set of solenoidal fields.
1) The application of rot to elements of the first group delivers no further information, hence they are omitted and will not be included to the searched basis.
2) Applying rot to elements of the second group, we find identities

$$\begin{aligned}k\,\mathrm{rot}(x^{k-1}y^l\,,\, 0\,,\, 0) &= -l\,\mathrm{rot}(0\,,\, x^k y^{l-1}\,,\, 0)\,,\\ 0 < k,l &< s+2\,, \quad k+l = s+2,\\ k\,\mathrm{rot}(x^{k-1}z^m\,,\, 0\,,\, 0) &= -m\,\mathrm{rot}(0\,,\, 0\,,\, x^k z^{m-1})\,,\\ 0 < k,m &< s+2\,, \quad k+m = s+2,\\ l\,\mathrm{rot}(0\,,\, y^{l-1}z^m\,,\, 0) &= -m\,\mathrm{rot}(0\,,\, 0\,,\, y^l z^{m-1})\,,\\ 0 < l,m &< s+2\,, \quad l+m = s+2\,.\end{aligned} \tag{4.9}$$

because of the linearity of rot.
Alltogether we have $3(s+1)$ relations and $6(s+1)$ solenoidal fields in (4.9). Thus we include $3(s+1)$ of them in our searched basis. In particular we take the fields

$$\begin{aligned} \operatorname{rot}\left(kx^{k-1}y^{l}\,,\,0\,,\,0\right) &= \left(0\,,\,0\,,\,-klx^{k-1}y^{l-1}\right)\,, \\ \operatorname{rot}\left(0\,,\,ly^{l-1}z^{m}\,,\,0\right) &= \left(-lmy^{l-1}z^{m-1}\,,\,0\,,\,0\right)\,, \\ \operatorname{rot}\left(0\,,\,0\,,\,mx^{k}z^{m-1}\right) &= \left(0,\,-kmx^{k-1}z^{m-1},\,0\right) \end{aligned} \tag{4.10}$$

as outcome of group 2).
3) Applying rot to the potential fields of the third group, i.e. on the fields of type $\nabla(x^k y^l z^m)$, $0 < k, l, m < s+1$, $k+l+m = s+2$, gives

$$\begin{aligned} \mathbf{0} &= \operatorname{rot}\left(\nabla(x^k y^l z^m)\right) \\ &= \operatorname{rot}\left(kx^{k-1}y^l z^m,\, lx^k y^{l-1} z^m,\, mx^k y^l z^{m-1}\right) \\ &= \operatorname{rot}\left(kx^{k-1}y^l z^m,\, 0,\, 0\right) + \operatorname{rot}\left(0,\, lx^k y^{l-1} z^m,\, 0\right) + \operatorname{rot}\left(0,\, 0,\, mx^k y^l z^{m-1}\right). \\ & k,l,m > 0\,, \quad k+l+m = s+2\,. \end{aligned} \tag{4.11}$$

A little bit of combinatorics shows that these are $s(s+1)/2$ relations and that hence the sum on the right-hand side consists of $3s(s+1)/2$ solenoidal vector fields. Hence we may fix $s(s+1)$ solenoidal fields for the searched basis. The remaining $s(s+1)/2$ fields are omitted since they are linearly dependent. First we rewrite (4.11) as

$$\begin{aligned} \mathbf{0} &= \left(0,\, kmx^{k-1}y^l z^{m-1},\, -klx^{k-1}y^{l-1}z^m\right) \\ &\quad + \left(-lmx^k y^{l-1} z^{m-1},\, 0,\, kmx^{k-1}y^{l-1}z^m\right) \\ &\quad + \left(lmx^k y^{l-1} z^{m-1},\, -kmx^{k-1}y^l z^{m-1},\, 0\right). \end{aligned} \tag{4.12}$$

We omit the last expressions from (4.12) and include

$$\begin{aligned} \operatorname{rot}\left(kx^{k-1}y^l z^m,\, 0,\, 0\right) &= \left(0,\, kmx^{k-1}y^l z^{m-1},\, -klx^{k-1}y^{l-1}z^m\right) \\ \operatorname{rot}\left(0,\, lx^k y^{l-1} z^m,\, 0\right) &= \left(-lmx^k y^{l-1} z^{m-1},\, 0,\, klx^{k-1}y^{l-1}z^m\right) \\ & 0 < k,l,m < s+1, \;\; k+l+m = s+2. \end{aligned} \tag{4.13}$$

into the searched basis.
Thus we have $3(s+1)$ linearly independent solenoidal fields as outcome from the second group and $s(s+1)$ solenoidal fields from group 3). Since obviously the monomials $x^k y^l z^m$, $k, l, m \geq 0$, $k+l+m = s+2$ span the space of homogeneous polynomials of degree $s+2$, the space of solenoidal vector fields with components being homogeneous polynomials of exact degree s has the dimension $3(s+1) + s(s+1) = (s+1)(s+3)$. This result can be generalized to solenoidal, symmetric tensor fields of arbitrary rank q. For $q = 1$ the assertions of proposition 4.5 are consequences from the investigations above.

Proposition 4.5. *(a) The space of solenoidal, symmetric tensor fields of rank q in $\mathbb{R}^n$ with entries being homogeneous, multivariate polynomials of degree s has the dimension $N\delta(n,q,s) = C^q_{q+n-1}C^s_{s+n-1} - C^{q-1}_{q-1+n-1}C^{s-1}_{s-1+n-1}$. In particular for $n = 3$ and $q = 1$ we have $N\delta(3,1,s) = 3C^s_{s+2} - C^{s-1}_{s+1} = (s+1)(s+3)$.*
(b) The elements of the systems (4.10), (4.13) are solenoidal fields, i.e. they are elements of $H(\mathrm{div};\mathbb{B}^3)$. They are linealy independent and represent a basis of the subspace consisting

of solenoidal vector fields with components being homogeneous, multivariate polynomials of degree s. Hence every such field has a unique extension in terms of (4.10), (4.13).
(c) Letting $s \to \infty$ by (4.10), (4.13) we obtain a system which is dense in $H(\mathrm{div};\mathbb{B}^3)$.

We conclude this section giving a brief outline how to perform the described construction of a basis when $\mathbb{B}^3$ is replaced by an arbitrary Riemannian domain $B \subset \mathbb{R}^3$. Let (e^{jkl}) be the *discriminant tensor* with $e^{123} = e^{231} = e^{312} = -e^{213} = e^{132} = e^{321} = 1/\sqrt{g}$ and zero components else. Here g denotes the dterminant of the metric tensor, $g = \det(g_{ij})$.
The operator $\mathfrak{r} : H^k(\mathbf{S}_1(B)) \to H^{k-1}(\mathbf{S}_1(B))$ assigns a vector field (u_j) to a vector field $(\mathfrak{r}u)^k$ by

$$(\mathfrak{r}\, u)^j = e^{jkl} u_{k\,;\,l}\,. \tag{4.14}$$

The operator $\mathfrak{r}$ is a generalization of the operator rot for arbitrary Riemannian metrics as can be seen, if we write (4.14) in an explicit way

$$\begin{aligned} \mathfrak{r}\,\mathbf{u} &= \left((\mathfrak{r}u)^1, (\mathfrak{r}u)^2, (\mathfrak{r}u)^3\right) \\ &= \left(\frac{1}{\sqrt{g}}\left(\frac{\partial u^2}{\partial x^3} - \frac{\partial u^3}{\partial x^2}\right), \frac{1}{\sqrt{g}}\left(\frac{\partial u^3}{\partial x^1} - \frac{\partial u^1}{\partial x^3}\right), \frac{1}{\sqrt{g}}\left(\frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1}\right)\right). \end{aligned}$$

The field $\mathfrak{r}\,\mathbf{u}$ is solenoidal where we have to take into account that the operator of divergence δ is transformed with respect to the given metric tensor (g_{ij}),

$$\delta(\mathfrak{r}u) \;=\; (\mathfrak{r}u)^1_{;1} \;+\; (\mathfrak{r}u)^2_{;2} \;+\; (\mathfrak{r}u)^3_{;3} \;=\; \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}\,(\mathfrak{r}u)^j\right) \;=\; 0\,.$$

Since $\mathfrak{r}$ and δ are analogue expressions for curl and divergence in Riemannian domains, the method to construct a polynomial basis of solenoidal vector fields outlined in this section may be inherited without any difficulties. The resulting basis differs from (4.10), (4.13) by a factor of $(\sqrt{g})^{-1}$ for each term only.

Appendix. **Orthogonal polynomials**

In the appendix we summarize important facts about orthogonal polynomials. We give brief overviews of some well known polynomials and their properties.
Let $w(t)$ be a given real valued function which is non-negative in the interval (a, b) and fulfills $\int_a^b w(t)\,dt > 0$. Assume further that the integrals $\int_a^b t^n w(t)\,dt$ exist for $n = 0, 1, \ldots$. Then there exists a unique sequence of polynomials $p_0(t), p_1(t), \ldots, p_n(t), \ldots$ such that

- the polynomial $p_n(t)$ is of degree n and has a positive leading coefficient of t^n,
- the polynomials $p_0(t), p_1(t), \ldots$ are orthogonal and normalised with respect to the weight function $w(t)$,

  $$\int_a^b p_n(t)p_m(t)w(t)\,dt = \delta_{nm},$$

  where δ_{nm} denotes the Kronecker symbol, $\delta_{nm} = 0$ if $n \neq m$, and $\delta_{nm} = 1$ if $n = m$.

Thus, the family of polynomials $\{p_n(t)\}$, $n \in \mathbb{N}_0$, represent *a system of orthonormal functions in* (a,b) *with respect to the weight function* $w(t)$, which is complete in $L_2(a,b)$.
Let a_n be the leading coefficient of $p_n(t)$. Then we have the summation formula

$$\sum_{k=0}^{n} p_k(u)p_k(v) = \frac{a_n}{a_{n+1}} \frac{p_{n+1}(u)p_n(v) - p_n(u)p_{n+1}(v)}{u-v} \qquad u,v \in (a,b). \tag{5.1}$$

Orthogonal polynomials satisfy further a three term recurrence formula

$$p_n(t) = (A_n t + B_n)p_{n-1}(t) - C_n p_{n-2}(t), \quad n = 2,3,\dots \tag{5.2}$$

where A_n, B_n, C_n are constants. In particular, $A_n = a_n/a_{n-1}$, $C_n = a_n a_{n-2}/a_{n-1}$.
The most important examples of normalized orthogonal polynomials are listed in Table 1.

Table 1. Orthogonal polynomials

Polynomials $p_n(t)$	(a,b)	$w(t)$
Legendre: $\left(n+\frac{1}{2}\right)^{\frac{1}{2}} P_n(t)$	$(-1,\ 1)$	1
Gegenbauer: $2^p\Gamma(p)\left(\frac{(n+p)n!}{2\pi\Gamma(2p+n)}\right)^{\frac{1}{2}} C_n^{(p)}(t)$	$(-1,\ 1)$	$(1-t^2)^{p-\frac{1}{2}}$
Hermite: $2^{-\frac{n}{2}}\pi^{-\frac{1}{4}}(n!)^{-\frac{1}{2}} H_n(t)$	$(-\infty,\ \infty)$	$\exp^{-t^2}$
Jacobi: $\left(\frac{q(q+1)\cdots(q+n-1)(p+2n)\Gamma(p+n)}{n!\Gamma(q)\Gamma(p-q+n+1)}\right)^{\frac{1}{2}} G(p,q,t)$	$(0,\ 1)$	$t^{q-1}(1-t)^{p-q}$
Laguerre: $(n!\Gamma(p+n+1))^{-\frac{1}{2}} L_n^p(t)$	$(0,\ \infty)$	$t^p \exp^{-t}$

A1. Legendre polynomials.
The *Legendre polynomials* P_n of degree n have the representation

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2-1\right)^n = \frac{1}{2^n}\sum_{p=0}^{[n/2]}(-1)^p \frac{(2n-2p)!}{p!(n-2p)!(n-p)!}t^{n-2p}. \tag{5.3}$$

They are orthogonal in $L_2(-1,1)$ and satisfy

$$\int_{-1}^{1} P_n(t)P_m(t)dt = \frac{2}{2n+1}\delta_{nm}. \tag{5.4}$$

The integral representation is given as

$$\begin{aligned} P_n(t) &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(t+(t^2-1)^{\frac{1}{2}}\cos\psi\right)^n d\psi \\ &= \frac{1}{\pi}\int\limits_{0}^{\pi}\left(t+(t^2-1)^{\frac{1}{2}}\cos\psi\right)^n d\psi. \end{aligned}$$

The generating function for the Legendre polynomials is

$$\frac{1}{\sqrt{1-2tz+z^2}} = \sum_{n=0}^{\infty} P_n(t)z^n.$$

Putting $z=r$, $t=\cos\varphi$ we get

$$\frac{1}{\sqrt{1-2r\cos\varphi+r^2}} = \sum_{n=0}^{\infty} P_n(\cos\varphi)r^n.$$

The three term recurrence reads

$$(n+1)P_{n+1}(t)-(2n+1)tP_n(t)+nP_{n-1}(t)=0, \quad n=1,2,\dots, \tag{5.5}$$
$$P_1(t)-tP_0(t)=0.$$

The following relations involving derivatives can be proved with the help of (5.3) and the recurrence (5.5).

$$(t^2-1)\frac{dP_n(t)}{dt} = n\left(tP_n(t)-P_{n-1}(t)\right) = \frac{n(n+1)}{2n+1}\left(P_{n+1}(t)-P_{n-1}(t)\right), \tag{5.6}$$

$$P_n(t) = \frac{dP_{n+1}(t)}{dt}+\frac{dP_{n-1}(t)}{dt}-2t\,\frac{dP_n(t)}{dt},$$

$$nP_n(t) = t\,\frac{dP_n(t)}{dt}-\frac{dP_{n-1}(t)}{dt},$$

$$(2n+1)P_n(t) = \frac{dP_{n+1}(t)}{dt}-\frac{dP_{n-1}(t)}{dt}.$$

From (5.6) we deduce

$$\int_{-1}^{1} t\, f(t)P_n(t)dt = \frac{1}{2n+1}\left((n+1)\int_{-1}^{1} f(t)P_{n+1}(t)dt + n\int_{-1}^{1} f(t)P_{n-1}(t)dt\right). \tag{5.7}$$

Applying integration by parts to (5.7) we obtain the *Rodrigues rule* for Legendre polynomials (see[19], p. 23): For any $f\in C^n([-1,1])$

$$\int_{-1}^{1} f(t)P_n(t)dt = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dt^n}f(t)(1-t^2)^n dt\,. \tag{5.8}$$

The Legendre polynomials up to degree 5 are explicitly given as

$$P_0(t)=1, \qquad P_1(t)=t=\cos\varphi,$$

$$P_2(t) = \frac{1}{2}\left(3t^2 - 1\right) = \frac{1}{4}\left(3\cos 2\varphi + 1\right),$$
$$P_3(t) = \frac{1}{2}\left(5t^3 - 3t\right) = \frac{1}{8}\left(5\cos 3\varphi + 3\cos\varphi\right),$$
$$P_4(t) = \frac{1}{8}\left(35t^4 - 30t^2 + 3\right) = \frac{1}{64}\left(35\cos 4\varphi + 20\cos 2\varphi + 9\right),$$
$$P_5(t) = \frac{1}{8}\left(63t^5 - 70t^3 + 15t\right) = \frac{1}{128}\left(63\cos 5\varphi + 35\cos 3\varphi + 30\cos\varphi\right).$$

A2. Gegenbauer polynomials.
The *Gegenbauer* (*generalized Legendre*) *polynomials* $C_n^{(\mu)}$ of degree n are defined by

$$\begin{aligned} C_n^{(\mu)}(t) &= \frac{(-1)^n}{2^n\, n!} \frac{\Gamma(2\mu+n)\Gamma\left(\frac{2\mu+1}{2}\right)}{\Gamma(2\mu)\Gamma\left(\frac{2\mu+1}{2}+n\right)} \left(1-t^2\right)^{\frac{1}{2}-\mu} \frac{d^n}{dt^n}\left((1-t^2)^{\mu+n-\frac{1}{2}}\right) \\ &= \frac{1}{\Gamma(\mu)} \sum_{p=0}^{[n/2]} (-1)^p \frac{\Gamma(\mu+n-p)}{p!(n-2p)!}(2t)^{n-2p}, \end{aligned} \tag{5.9}$$

For $\mu = 1/2$ the Gegenbauer polynomials and the Legendre polynomials coincide,

$$C_n^{(1/2)}(t) = P_n(t).$$

In case $\mu = 3/2$, which plays an important role in this paper, the Gegenbauer polynomials are explicitly given as

$$\begin{aligned} C_n^{(3/2)}(t) &= \frac{1}{\Gamma(3/2)} \sum_{p=0}^{[n/2]} (-1)^p \frac{\Gamma(n-p+3/2)}{p!(n-2p)!}(2t)^{n-2p} \\ &= \frac{1}{2^n} \sum_{p=0}^{[n/2]} (-1)^p \frac{(2n-2p+1)!}{p!(n-2p)!(n-p)!} t^{n-2p}. \end{aligned} \tag{5.10}$$

Gegenbauer polynomials fulfill the orthogonality relation

$$\int_{-1}^{1} C_n^{(\mu)}(t) C_m^{(\mu)}(t) \left(1-t^2\right)^{\mu-\frac{1}{2}}\, dt = \frac{\pi 2^{1-2\mu}\Gamma(n+2\mu)}{n!(n+\mu)\Gamma(\mu)\Gamma(\mu)}\delta_{nm} \tag{5.11}$$

and have the integral representation

$$C_n^{(\mu)}(t) = \frac{1}{\sqrt{\pi} n!} \frac{\Gamma(2\mu+n)\Gamma\left(\frac{2\mu+1}{2}\right)}{\Gamma(2\mu)\Gamma(\mu)} \int_0^{\pi} \left(t + (t^2-1)^{\frac{1}{2}}\cos\psi\right)^n \sin^{2\mu-1}\psi\, d\psi.$$

The generating function of $C_n^{(\mu)}(t)$ is given by

$$(1 - 2tz + z^2)^{-\mu} = \sum_{n=0}^{\infty} C_n^{(\mu)}(t) z^n.$$

We have the three term recurrence formulas

$$(n+1)C_{n+1}^{(\mu)}(t) - 2(n+\mu)tC_n^{(\mu)}(t) + (n+2\mu-1)C_{n-1}^{(\mu)}(t) = 0, \quad n = 1, 2, \dots,$$

$$(n+1)C_{n+1}^{(\mu)}(t) - 2\mu\left(tC_n^{(\mu+1)}(t) - C_{n-1}^{(\mu+1)}(t)\right) = 0,$$

$$(n+2\mu)C_n^{(\mu)}(t) - 2\mu\left(C_n^{(\mu+1)}(t) - tC_{n-1}^{(\mu+1)}(t)\right) = 0,$$

$$nC_n^{(\mu)}(t) - (n+2\mu-1)tC_{n-1}^{(\mu)}(t) + 2\mu(1-t^2)C_{n-2}^{(\mu-1)}(t) = 0,$$

and for k-th order derivatives formula

$$\frac{d^k}{dt^k}C_n^{(\mu)}(t) = 2^k\,\frac{\Gamma(k+\mu)}{\Gamma(\mu)}\,C_{n-k}^{(\mu+k)}(t)$$

is valid. The polynomials $C_n^{(\mu)}(t)$ have further an expansion in terms of Gegenbauer polynomials of different type

$$C_n^{(\mu)}(t) = \sum_{k=0}^{[n/2]} \frac{n-2k+\mu-1}{\mu-1} C_{n-2k}^{(\mu-1)}(t). \tag{5.12}$$

The relations above lead to numerous other identities such as

$$\frac{d}{dt}C_n^{(\mu)}(t) = 2\mu\, C_{n-1}^{(\mu+1)}(t),$$

$$\begin{aligned}(1-t^2)\frac{dC_n^{(\mu)}(t)}{dt} &= (n+2\mu-1)C_{n-1}^{(\mu)}(t) - ntC_n^{(\mu)}(t)\\ &= (n+2\mu)tC_n^{(\mu)}(t) - (n+1)C_{n+1}^{(\mu)}(t),\end{aligned}$$

$$(1-t^2)\frac{d^2}{dt^2}C_n^{(\mu)}(t) - (2\mu+1)t\frac{d}{dt}C_n^{(\mu)}(t) + n(n+2\mu)C_n^{(\mu)}(t) = 0.$$

In section 3 we often use relations between $C_n^{(3/2)}(t)$ and Legendre polynomials $P_n(t)$. In particular we have

$$C^{(3/2)}{}_n(t) = \frac{dP_{n+1}}{dt}(t),$$

$$C_n^{(3/2)}(t) = \sum_{k=0}^{[n/2]} \frac{n-2k+1/2}{1/2} C_{n-2k}^{(1/2)}(t) = \sum_{k=0}^{[n/2]} (2n-4k+1)P_{n-2k}(t).$$

The Gegenbauer polynomials up to degree 5 are explicitly given as

$$C_0^{(\mu)}(t) = 1, \qquad C_1^{(\mu)}(t) = 2\mu t,$$

$$C_2^{(\mu)}(t) = 2\mu\,(\mu+1)\,t^2 - \mu,$$

$$C_3^{(\mu)}(t) = \frac{4}{3}\mu\,(\mu+1)\,(\mu+2)\,t^3 - 2\mu\,(\mu+1)\,t,$$

$$C_4^{(\mu)}(t) = \frac{2}{3}\mu\,(\mu+1)\,(\mu+2)\,(\mu+3)\,t^4 - 2\mu\,(\mu+1)\,(\mu+2)\,t^2 + \frac{1}{2}\mu\,(\mu+1)\,,$$

$$C_5^{(\mu)}(t) = \frac{4}{15}\mu\,(\mu+1)\,(\mu+2)\,(\mu+3)\,(\mu+4)\,t^5 - \frac{4}{3}\mu\,(\mu+1)\,(\mu+2)\,(\mu+3)\,t^3$$

$+\mu\,(\mu+1)\,(\mu+2)\,t.$

A3. Associated Legendre polynomials (Legendre functions of the first kind).
The *associated Legendre polynomials* P_n^m of degree n and order m are defined by means of the Rodrigues formula

$$\begin{aligned} P_n^m(t) &= (1-t^2)^{m/2}\frac{d^m}{dt^m}P_n(t) = \frac{1}{2^n n!}(1-t^2)^{m/2}\frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n \\ &= \frac{(2m)!}{2^m m!}(1-t^2)^{m/2}C_{n-m}^{(m+1/2)}(t), \qquad t\in[-1,1]. \end{aligned} \tag{5.13}$$

They obey the orthogonality relation

$$\int_0^1 P_n^m(t)P_l^k(t)\,dt = \frac{2(n+m)!}{(2n+1)(n-m)!}\delta_{km}\delta_{ln} \tag{5.14}$$

and their values at 0 are

$$P_{m+2l+1}^m(0) = 0\,, \quad P_{m+2l}^m(0) = \frac{(-1)^l(2m+2l)!}{2^{m+2l}l!(m+l)!}.$$

Three term recurrences are given in different ways

$$\begin{aligned} (2n+1)\sqrt{1-t^2}P_n^m(t) &= P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t), \\ (2n+1)\sqrt{1-t^2}P_n^m(t) &= (n+m)(n+m-1)P_{n-1}^{m-1}(t) - (n-m+1)(n-m+2)P_{n+1}^{m-1}(t), \\ (2n+1)tP_n^m(t) &= (n-m+1)P_{n+1}^m(t) + (n+m)P_{n-1}^m(t). \end{aligned}$$

These formulas extend to negative m, if we define

$$P_n^{-m}(t) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(t), \qquad 0\le m\le n. \tag{5.15}$$

For $|m|>n$ we set $P_n^m = 0$.

Associated Legendre polynomials P_n^m are related to Gegenbauer polynomials $C_n^{(\mu)}$ and derivatives of Legendre polynomials P_n by

$$C_n^{(\mu)}(t) = \frac{\Gamma(2\mu+n)\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(2\mu)\Gamma(n+1)}\left(\frac{1}{4}(t^2-1)\right)^{\frac{1}{4}-\frac{\mu}{2}}P_{n+\mu-\frac{1}{2}}^{\frac{1}{2}-\mu}(t),$$

$$C_{n-m}^{(m+\frac{1}{2})}(t) = \frac{1}{(2m-1)!!}\frac{d^m P_n(t)}{dt^m} = (-1)^m\frac{(1-t^2)^{-\frac{m}{2}}m!2^m}{(2m)!}P_n^m(t).$$

A4. Jacobi polynomials.
The *Jacobi polynomials* $P_n^{(p,q)}$ of degree n are defined by the Rodrigues formula

$$P_n^{(p,q)}(t) = \frac{t^{1-q}(1-t)^{q-p}}{q(q+1)\dots(q+n-1)}\frac{d^n}{dt^n}\left(t^{q+n-1}(1-t)^{p+n-q}\right). \tag{5.16}$$

They have the orthogonality property

$$\int_0^1 t^{q-1}(1-t)^{p-q} P_n^{(p,q)}(t) P_m^{(p,q)}(t)\, dt = \frac{n!\Gamma(q)\Gamma(p-q+n+1)}{q(q+1)\dots(q+n-1)(p+2n)\Gamma(p+n)}\, \delta_{nm}. \tag{5.17}$$

The explicit expansion in terms of monomials t^k is given as

$$P_n^{(p,q)}(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(p+n)(p+n+1)\dots(p+n+k-1)}{q(q+1)\dots(q+k-1)} t^k.$$

Legendre and Gegenbauer polynomials can be generated from Jacobi polynomials. More explicitly we have

$$P_n(t) = P_n^{(1,1)}\left(\frac{1-t}{2}\right),$$

$$C_n^{(\mu)} = (-1)^n \frac{(2\mu+n-1)!}{n!(2\mu-1)!} P_n^{(2\mu,\mu+\frac{1}{2})}\left(\frac{1+t}{2}\right).$$

Jacobi polynomials have the generating function

$$\frac{(1-t)^{1-q}(1+t)^{q-p}\left(z-1+\sqrt{1-2zt+z^2}\right)^{q-1}\left(z+1-\sqrt{1-2zt+z^2}\right)^{p-q}}{z^{p-1}\sqrt{1-2zt+z^2}}$$

$$= \sum_{k=0}^{\infty} \frac{(q+k-1)!}{k!(q-1)!} P_k^{(p,q)}\left(\frac{1-t}{2}\right) z^k.$$

The Jacobi polynomials up to degree 5 are explicitly given as

$$P_0^{(p,q)}(t) = 1, \qquad P_1^{(p,q)}(t) = 1 - \frac{p+1}{q} t,$$

$$P_2^{(p,q)}(t) = 1 - 2\frac{p+2}{q} t + \frac{(p+2)(p+3}{q(q+1)} t^2,$$

$$P_3^{(p,q)}(t) = 1 - 3\frac{p+3}{q} t + 3\frac{(p+3)(p+4)}{q(q+1)} t^2 - \frac{(p+3)(p+4)(p+5)}{q(q+1)(q+2)} t^3,$$

$$\begin{aligned} P_4^{(p,q)}(t) &= 1 - 4\frac{p+4}{q} t + 6\frac{(p+4)(p+5)}{q(q+1)} t^2 - 4\frac{(p+4)(p+5)(p+6)}{q(q+1)(q+2)} t^3 \\ &\quad + \frac{(p+4)(p+5)(p+6)(p+7)}{q(q+1)(q+2)(q+3)} t^4, \end{aligned}$$

$$\begin{aligned} P_5^{(p,q)}(t) &= 1 - 5\frac{p+5}{q} t + 10\frac{(p+5)(p+6)}{q(q+1)} t^2 - 10\frac{(p+5)(p+6)(p+7)}{q(q+1)(q+2)} t^3 \\ &\quad + 5\frac{(p+5)(p+6)(p+7)(p+8)}{q(q+1)(q+2)(q+3)} t^4 - \frac{(p+5)(p+6)(p+7)(p+8)(p+9)}{q(q+1)(q+2)(q+3)(q+4)} t^5. \end{aligned}$$

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