Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 173

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Saarbrücken 2006

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 173
submitted: May 24, 2006

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Samuel multiplicity and Fredholm theory

Jörg Eschmeier

Abstract. In this note we prove that, for a given Fredholm tuple $T = (T_1, \dots, T_n)$ of commuting bounded operators on a complex Banach space X, the limits $c_p(T) = \lim_{k\to\infty} \dim H^p(T^k, X)/k^n$ exist and calculate the generic dimension of the cohomology groups $H^p(z - T, X)$ of the Koszul complex of T near $z = 0$. To deduce this result we show that the above limits coincide with the Samuel multiplicities of the stalks of the cohomology sheaves $H^p(z-T, \mathcal{O}^X_{\mathbb{C}^n})$ of the associated complex of analytic sheaves at $z = 0$.

0 Introduction

Let $T = (T_1, \dots, T_n) \in L(X)^n$ be a commuting tuple of bounded linear operators on a complex Banach space X. A fundamental principle of multivariable operator theory is that all basic spectral properties of T should be understood as properties of its Koszul complex. The Koszul complex $K^\bullet(z - T, X)$ is a finite complex of Banach spaces with coboundary maps

$$K^p(z - T, X) \to K^{p+1}(z - T, X), \quad x s_I \mapsto \sum_{j=1}^{n} (z_j - T_j)x \; s_j \wedge s_I$$

that depend analytically on the parameter $z \in \mathbb{C}^n$. The commuting tuple T is said to be invertible if the Koszul complex $K^\bullet(T, X)$ is exact. The joint spectrum $\sigma(T)$ of T consists of all points $z \in \mathbb{C}^n$ for which the tuple $z - T = (z_1 - T_1, \dots, z_n - T_n)$ is not invertible. It was a breakthrough [12] when J.L.Taylor introduced this notion of joint spectrum and showed that it carries an analytic functional calculus, that is, there exists a continuous algebra homomorphism

$$\mathcal{O}(\sigma(T)) \to L(X), \; f \mapsto f(T)$$

extending the natural $\mathcal{O}(\mathbb{C}^n)$-module structure of X given by T.

2000 Mathematics Subject Classification. Primary 47A13; Secondary 47A53, 13D40, 32C35.

The commuting tuple T is said to be Fredholm if all cohomology groups $H^p(T,X)$ $(p = 0,\dots,n)$ of its Koszul complex $K^\bullet(T,X)$ are finite dimensional. The Fredholm index of T is defined as the Euler characteristic

$$\operatorname{ind}(T) = \sum_{p=0}^{n}(-1)^p \dim H^p(T,X)$$

of its Koszul complex. The essential spectrum $\sigma_e(T)$ of T consists of all points $z \in \mathbb{C}^n$ for which $z - T$ is not Fredholm. The observation that T is Fredholm if and only if all cohomology sheaves of the associated complex $K^\bullet(z-T, \mathcal{O}^X_{\mathbb{C}^n})$ of Banach-free analytic sheaves are coherent near $0 \in \mathbb{C}^n$ allows the application of methods from complex analytic geometry to multivariable Fredholm theory. For instance, the Fredholm spectrum $\sigma(T) \cap \rho_e(T)$ is an analytic subset of the essential resolvent set $\rho_e(T) = \mathbb{C}^n \setminus \sigma_e(T)$, since it is the support of the coherent sheaf $\oplus_{p=0}^n H^p(z-T, \mathcal{O}^X_{\rho_e(T)})$. The discontinuity points of the functions

$$\rho_e(T) \to \mathbb{C},\ z \mapsto \dim H^p(z-T,X) \quad (p = 0,\dots,n)$$

form proper analytic subsets of $\rho_e(T)$. Suppose that T is Fredholm. Then the stalks of the cohomology sheaves $\mathcal{H}^p = H^p(z-T, \mathcal{O}^X_{\rho_e(T)})$ at $0 \in \mathbb{C}^n$ are finitely generated modules over the Noetherian local ring $\mathcal{O}_0$ of all convergent power series at 0. Hence there are polynomials $q_{an,p} \in \mathbb{Q}[x]$, the Hilbert-Samuel polynomials of $\mathcal{H}^p_0$, with $\deg(q_{an,p}) \le n$ such that

$$\dim(\mathcal{H}^p_0/\mathfrak{m}^k\mathcal{H}^p_0) = q_{an,p}(k)$$

for sufficiently large natural numbers k and such that the limits

$$c_{an,p}(T) = n! \lim_{k\to\infty} \dim(\mathcal{H}^p_0/\mathfrak{m}^k\mathcal{H}^p_0)/k^n \qquad (p = 0,\dots,n)$$

exist and define natural numbers, the Samuel multiplicities of $\mathcal{H}^p_0$. Here $\mathfrak{m}$ denotes the maximal ideal of the local ring $\mathcal{O}_0$.

On the other hand, if T is Fredholm, then the spaces $M_k(T) = \sum_{|\alpha|=k} T^\alpha X$ are finite codimensional in X and the direct sum $\oplus_{k\ge 0} M_k(T)/M_{k+1}(T)$ becomes in a natural way a finitely generated graded $\mathbb{C}[z]$-module. By a classical result of Hilbert, there is a polynomial $q \in \mathbb{Q}[x]$ of degree at most n such that

$$\dim\big(X/M_k(T)\big) = q(k)$$

for large values of k and such that the limit

$$c(T) = n! \lim_{k\to\infty} \dim\big(X/M_k(T)\big)/k^n$$

exists and defines a natural number, the algebraic Samuel multiplicity of T.

In a paper [1] of Douglas and Yan from 1993 the algebraic Hilbert-Samuel polynomial q and its analytic counterpart $q_{an,n}$ were studied and it was suggested that their nth order coefficients and degrees should have a natural meaning in operator theory.

In recent papers of Xiang Fang [6] and the author [3] it was shown that $c(T) = c_{an,n}(T)$ and that this number calculates the generic dimension of the last cohomology groups $H^n(z-T, X)$ of the Koszul complex $K^\bullet(z-T, X)$ near $z = 0$. More precisely, for every connected open neighbourhood U of 0, the number $c(T)$ coincides with the constant value of the function $\dim H^n(z-T, X)$ outside of its discontinuity set. Moreover, it was suggested that the functions

$$h_p(k) = \dim H^p(T^k, X) \qquad (k \in \mathbb{N})$$

with $T^k = (T_1^k, \ldots, T_n^k)$ should be the algebraic analogues of the pth order analytic Hilbert-Samuel polynomials $q_{an,p}$.

In this paper we show that indeed, for $p = 0, \ldots, n$, the limit formula

$$c_{an,p}(T) = \lim_{k\to\infty} \dim H^p(T^k, X)/k^n$$

holds and that $c_{an,p}(T)$ is the generic dimension of $H^p(z-T, X)$ near $z = 0$. It follows that

$$\mathrm{ind}(T) = \sum_{p=0}^{n} (-1)^p c_{an,p}(T).$$

As a first step, we show in Section 1 that, for every natural number k, there are canonical vector space isomorphisms

$$H^p(T^k, X) \cong H^p\big(z-T, \mathcal{O}_0^X/(z^k)\mathcal{O}_0^X\big)$$

for $p = 0, \ldots, n$. In Section 2 we use results on analytically parametrized complexes of Banach spaces and methods from commutative algebra to deduce that

$$c_{an,p}(T) = \lim_{k\to\infty} \dim H^p\big(z-T, \mathcal{O}_0^X/(z^k)\mathcal{O}_0^X\big)/k^n$$

for $p = 0, \ldots, n$. Using the fact (cf. [6]) that the leading coefficient of the Samuel multiplicity of the stalk of a coherent sheaf at a given point z calculates its rank near z, we find that the above limits represent the generic dimension of $H^p(z-T, X)$ for z near 0.

1 Analytic functional calculus

To compute the cohomology groups $H^p(T^k, X)$ of the powers T^k of a Fredholm tuple $T \in L(X)^n$ we apply a construction which was used in [4] (Theorem 10.3.13) to prove an analytic index formula for the Fredholm index.

Theorem 1.1 *Let $T \in L(X)^n$ be a commuting tuple of bounded linear operators on a complex Banach space X. Suppose that $0 \in \sigma(T) \setminus \sigma_e(T)$. Let $f \in \mathcal{O}(U)^n$ be an n-tuple of analytic functions defined on an open neighbourhood U of $\sigma(T)$ such that $f^{-1}(\{0\}) = \{0\}$ and such that*

$$H^p\big(f, \mathcal{O}(V)\big) = \{0\} \qquad (p = 0, \ldots, n-1)$$

for some Stein open neighbourhood $V \subset U$ of $0 \in \mathbb{C}^n$. Then there are vector space isomorphisms

$$H^p\big(f(T), X\big) \cong H^p\big(z - T, H^n(f, \mathcal{O}(V, X))\big)$$

for $p = 0, \ldots, n$.

Proof. We follow closely the lines of the proof of Theorem 10.3.13 in [4]. In particular, we use the notations established there.

Choose a Stein open cover $\mathfrak{A} = (U_i)_{i \in \mathbb{N}}$ of U with $U_0 = V$ and $0 \notin U_i$ for $i > 0$. We denote by $\mathcal{C}^\bullet(\mathfrak{A})$ the alternating Čech complex with coefficients in $\mathcal{O}_U$ relative to the open cover $\mathfrak{A}$. Let us regard $\mathcal{O}(V)$ as the trivial complex with $\mathcal{O}(V)$ as the space in degree 0 and zero elsewhere. The kernel $K^\bullet$ of the canonical epimorphism $r : \mathcal{C}^\bullet(\mathfrak{A}) \to \mathcal{O}(V)$ becomes a complex of Fréchet $\mathcal{O}(U)$-modules. We denote by

$$0 \to K_1 \to K_2 \to K_3 \to 0$$

the induced short exact sequence of double complexes

$$\begin{aligned} K_1 &= K^\bullet \hat{\otimes}_{\mathcal{O}(U)} K^\bullet\big(z - T, \mathcal{O}(U, X)\big), \\ K_2 &= \mathcal{C}^\bullet(\mathfrak{A}) \hat{\otimes}_{\mathcal{O}(U)} K^\bullet\big(z - T, \mathcal{O}(U, X)\big), \\ K_3 &= \mathcal{O}(V) \hat{\otimes}_{\mathcal{O}(U)} K^\bullet\big(z - T, \mathcal{O}(U, X)\big) = K^\bullet\big(z - T, \mathcal{O}(V, X)\big). \end{aligned}$$

It is well known (see Chapter 5.1 in [4]) that

$$H^p\big(\text{Tot } (K_2)\big) \cong \begin{cases} 0 & ; \quad p \neq n \\ X & ; \quad p = n \end{cases}$$

as topological $\mathcal{O}(U)$-modules. One obtains induced short exact sequences

$$0 \to \text{Tot } (K_1) \to \text{Tot } (K_2) \to \text{Tot } (K_3) \to 0$$

between the corresponding total complexes and between double complexes

$$0 \to \tilde{K}_1 \to \tilde{K}_2 \to \tilde{K}_3 \to 0,$$

where $\tilde{K}_i = \text{Tot } (K_i) \hat{\otimes}_{\mathcal{O}(U)} K^\bullet\big(f, \mathcal{O}(U)\big)$ for $i = 1, 2, 3$. Up to the sign, the columns of the double complex $\tilde{K}_1$ are direct sums of complexes of the form $K^\bullet\big(f, \mathcal{O}(W, X)\big)$, where W is a Stein open subset of $U \setminus \{0\} = U \setminus f^{-1}(\{0\})$. Hence, as the total complex of a double complex with exact columns, the complex Tot $(\tilde{K}_1)$ is exact. Therefore the cochain map $r : \text{Tot } (\tilde{K}_2) \to \text{Tot } (\tilde{K}_3)$ is a quasi-isomorphism. Standard double complex arguments (Lemma A 2.6 in [4]) show that there are vector space isomorphisms

$$\begin{aligned} H^p\big(f(T), X\big) &\cong H^p\big(f, H^n(\text{Tot } (K_2))\big) \\ &\cong H^{p+n}\big(\text{Tot } (\tilde{K}_2)\big) \cong H^{p+n}\big(\text{Tot } (\tilde{K}_3)\big). \end{aligned}$$

To complete the proof, observe that the double complex $\tilde{K}_3$ has the form

$$\begin{aligned} &K^\bullet\big(z - T, \mathcal{O}(V, X)\big) \hat{\otimes}_{\mathcal{O}(U)} K^\bullet\big(f, \mathcal{O}(U)\big) \\ \cong\ &K^\bullet\big(z - T, \mathcal{O}(V, X)\big) \hat{\otimes}_{\mathcal{O}(V)} K^\bullet\big(f, \mathcal{O}(V)\big). \end{aligned}$$

Since by hypothesis all columns of the double complex $\tilde{K}_3$ are exact in degree $p \neq n$, the same double complex result used above (Lemma A 2.6 in [4]) yields vector space isomorphisms

$$H^{p+n}\big(\text{Tot } (\tilde{K}_3)\big) \cong H^p\big(z - T, H^n(f, \mathcal{O}(V, X))\big)$$

and thus completes the proof. □

Let V be a Stein open neighbourhood of $0 \in \mathbb{C}^n$. Using the well-known fact that, for $k = 1, \ldots, n$, a function $f \in \mathcal{O}(V)$ belongs to $\sum_{\nu=1}^k z_\nu \mathcal{O}(V)$ if and only if f vanishes on the set $\{z \in V; z_1 = \ldots = z_k = 0\}$, one easily obtains that $(z_1, \ldots, z_n)$ is an $\mathcal{O}(V)$-regular sequence, that is, z_i is a non-zero divisor on $\mathcal{O}(V)/\big(z_1\mathcal{O}(V) + \ldots + z_{i-1}\mathcal{O}(V)\big)$ for $i = 1, \ldots, n$.

Let $k = (k_1, \ldots, k_n)$ be an n-tuple of positive integers. Then the sequence $z^k = (z_1^{k_1}, \ldots, z_n^{k_n})$ remains $\mathcal{O}(V)$-regular (Theorem 5.3 in [10]). It follows that (Proposition IV.2 in [11])

$$H^p\big(z^k, \mathcal{O}(V)\big) = \{0\} \qquad (p = 0, \ldots, n-1).$$

For an arbitrary $\mathcal{O}(\mathbb{C}^n)$-module M, we shall denote by

$$(z^k)M = \sum_{\nu=1}^{n} z_\nu^{k_\nu} M$$

the $\mathcal{O}(\mathbb{C}^n)$-submodule determined by the ideal $(z^k) \subset \mathcal{O}(\mathbb{C}^n)$. If $f \in \mathcal{O}_0^X$ is the germ of an analytic Banach-space valued function defined near $z = 0$, then we shall write

$$f_\alpha = (\partial^\alpha f)(0)/\alpha! \qquad (\alpha \in \mathbb{N}^n)$$

for the Taylor coefficients of f at 0. To simplify the notation we use the abbreviation

$$I_k = \{\alpha \in \mathbb{N}^n;\ \alpha_\nu < k_\nu \text{ for } \nu = 1, \ldots, n\}.$$

In the particular case where V is an open polydisc or ball with centre $0 \in \mathbb{C}^n$, one can easily compute the n-th cohomology groups of the Koszul complex $K^\bullet\big(z^k, \mathcal{O}(V, X)\big)$.

Lemma 1.2 *Let X be a Banach space and let $V \subset \mathbb{C}^n$ be an open polydisc or ball with centre $0 \in \mathbb{C}^n$. Then, for every tuple $k = (k_1, \ldots, k_n)$ of positive integers, we have*

$$(z^k)\mathcal{O}(V, X) = \{f \in \mathcal{O}(V, X);\ f_\alpha = 0 \textit{ for all } \alpha \in I_k\}.$$

Proof. Obviously, the left-hand side is contained in the set on the right. Conversely, if f belongs to the set on the right, then we obtain the decomposition

$$f(z) = \sum_{\nu=1}^{n} z_\nu^{k_\nu} \sum_{\alpha \in A_\nu} f_\alpha z^{\alpha - k_\nu e_\nu},$$

where $A_\nu \subset \mathbb{N}^n$ consists of all multiindices α with $\alpha_i < k_i$ for $i < \nu$ and $\alpha_\nu \geq k_\nu$. □

For an index tuple k as above, define

$$\mathcal{V}_k = \{p \in \mathbb{C}[z];\ p_\alpha = 0 \text{ for all } \alpha \in \mathbb{N}^n \setminus I_k\}.$$

Then, in the setting of the preceding lemma, we obtain obvious vector space isomorphisms

$$H^n\big(z^k, \mathcal{O}(V, X)\big) \cong H^n(z^k, \mathcal{O}_0^X) \cong \mathcal{V}_k \otimes X.$$

The proof of our main result will be based on the following particular case of Theorem 1.1.

Corollary 1.3 *Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a complex Banach space X such that $0 \in \sigma(T) \setminus \sigma_e(T)$. Then, for every open ball or polydisc $V \subset \mathbb{C}^n$ with centre $0 \in \mathbb{C}^n$ and all families $k = (k_1, \ldots, k_n)$ of positive integers k_i, there are vector space isomorphisms*

$$H^p(T^k, X) \cong H^p\big(z - T, H^n(z^k, \mathcal{O}(V, X))\big) \cong H^p\big(z - T, \mathcal{O}_0^X/(z^k)\mathcal{O}_0^X\big)$$

for $p = 0, \ldots, n$.

2 Fredholm complexes

Let $\Omega \subset \mathbb{C}^n$ be an open neighbourhood of $0 \in \mathbb{C}^n$ and let $M^\bullet = (M^p, d^p)_{p=0}^n$ be a finite analytically parametrized complex of Banach spaces M^p on Ω such that $\dim H^p\big(d^\bullet(0), M^\bullet\big) < \infty$ for $p = 0, \ldots, n$. It is well known (see Proposition 9.4.5 and Remark 9.4.6 in [4]) that there exist an analytically parametrized complex $L^\bullet = (L^p, u^p)_{p=0}^n$ of finite-dimensional vector spaces L^p on a possibly smaller open neighbourhood U of $0 \in \mathbb{C}^n$ and a family $h = (h^p)_{p=0}^n$ of holomorphic mappings

$$h^p \in \mathcal{O}\big(U, L(L^p, M^p)\big)$$

such that, for each point $z \in U$, the resulting maps

$$L^\bullet \xrightarrow{h^\bullet(z)} M^\bullet$$

are quasi-isomorphisms. Equivalently, the mapping cone $C^\bullet = (C^p, \alpha^p)_{p=0}^n$ of h, that is, the complex with spaces $C^p = M^p \oplus L^{p+1}$ and coboundaries $\alpha^p(z) : C^p \to C^{p+1}$ given by

$$\alpha^p(z)(x, y) = (d^p(z)x + (-1)^{p+1}h^{p+1}(z)y, u^{p+1}(z)y)$$

is pointwise exact on U.

As before, for a given Banach space E, let us denote by $\mathcal{O}_z^E$ the stalk of germs of all analytic E-valued functions defined near z. Let $\mathfrak{m}$ be the maximal ideal in the Noetherian local ring $\mathcal{O}_0$ of all scalar-valued convergent power series at $z = 0$, and let (z^k) be the ideal in $\mathcal{O}_0$ generated by the finite system $(z_1^{k_1}, \ldots, z_n^{k_n})$ for $k \in \mathbb{N}^n$ arbitrary. The original complex $(\mathcal{O}_0^{M^\bullet}, d^\bullet)$ is quasi-isomorphic to the complex

$$\mathcal{L}^\bullet : 0 \longrightarrow \mathcal{O}_0^{L^0} \xrightarrow{u^0} \mathcal{O}_0^{L^1} \xrightarrow{u^1} \ldots \xrightarrow{u^{n-2}} \mathcal{O}_0^{L^{n-1}} \xrightarrow{u^{n-1}} \mathcal{O}_0^{L^n} \longrightarrow 0$$

of finitely generated $\mathcal{O}_0$-modules. More precisely, the family $h = (h^p)_{p=0}^n$ induces isomorphisms of cohomology

$$H^p(u^\bullet, \mathcal{L}^\bullet) \xrightarrow{\sim} H^p(d^\bullet, \mathcal{O}_0^{M^\bullet})$$

for $p = 0, \dots, n$. Using an exactness result for analytically parametrized complexes, we can improve this observation.

Lemma 2.1 *Let $k = (k_1 \dots, k_n)$ be a family of positive integers. Then the cochain map $h = (h^p)_{p=0}^n$ induces isomorphisms of cohomology*

$$H^p\big(u^\bullet, \mathcal{L}^\bullet/(z^k)\mathcal{L}^\bullet\big) \xrightarrow{\sim} H^p\big(d^\bullet, \mathcal{O}_0^{M^\bullet}/(z^k)\mathcal{O}_0^{M^\bullet}\big)$$

for $p = 0, \dots, n$.

Proof. Since the mapping cone of the cochain map

$$L^\bullet/(z^k)\mathcal{L}^\bullet \xrightarrow{h} \mathcal{O}_0^{M^\bullet}/(z^k)\mathcal{O}_0^{M^\bullet}$$

can be identified with the complex $\mathcal{O}_0^{C^\bullet}/(z^k)\mathcal{O}_0^{C^\bullet}$, it suffices to show that the latter complex is exact. But we know that $(C^\bullet, \alpha^\bullet(0))$ is an exact complex of Banach spaces. Then by Lemma 2.1.5 in [4] there is a real number $r_0 > 0$ such that, for each open polydisc $V = P_r(0)$ with centre 0 and radius $0 < r < r_0$, the complex $(\mathcal{O}(V, C^\bullet), \alpha^\bullet)$ is exact. More precisely, choose r_0 small enough, as in the proof of Lemma 2.1.5 from [4]. Consider $r \in \mathcal{O}(V, C^p)$ and $g = \sum_{\alpha\in\mathbb{N}^n} g_\alpha z^\alpha \in (z^k)\mathcal{O}(V, C^{p+1})$ with

$$g = \alpha^p r.$$

Then $g_\alpha = 0$ for all $\alpha \in I_k$, and it is easily checked that the inductive construction from the proof of Lemma 2.1.5 in [4] can be used to define a family of coefficients $(f_\alpha)_{|\alpha|=j}$ $(j \in \mathbb{N})$ in C^p such that the power series $f = \sum_{\alpha\in\mathbb{N}^n} f_\alpha z^\alpha$ defines a solution of the equation $\alpha^p f = g$ in $\mathcal{O}(V, C^p)$ with $f_\alpha = 0$ for every $\alpha \in I_k$. Since $\alpha^p(r - f) = 0$, it follows that

$$r \in (z^k)\mathcal{O}(V, C^p) + \alpha^{p-1}\mathcal{O}(V, C^{p-1}).$$

Thus we have proved the exactness of the complex

$$\big(\mathcal{O}(V, C^\bullet)/(z^k)\mathcal{O}(V, C^\bullet), \alpha^\bullet\big)$$

on each open polydisc $V = P_r(0)$ with sufficiently small $r > 0$. Hence $\mathcal{O}_0^{C^\bullet}/(z^k)\mathcal{O}_0^{C^\bullet}$ is exact, and the proof is complete. □

Let us define submodules

$$\mathcal{N}^p = \text{Ker } u^p \subset \mathcal{L}^p, \quad \mathcal{Z}^p = \text{Im } u^{p-1} \subset \mathcal{L}^p.$$

The short exact sequences

$$0 \longrightarrow (z^k)\mathcal{L}^\bullet \longrightarrow \mathcal{L}^\bullet \longrightarrow \mathcal{L}^\bullet/(z^k)\mathcal{L}^\bullet \longrightarrow 0$$

induce long exact cohomology sequences

$$\begin{array}{cccccc} 0 & \longrightarrow & H^0\big((z^k)\mathcal{L}^\bullet)\big) & \stackrel{j_{0,k}}{\longrightarrow} & H^0(\mathcal{L}^\bullet) & \stackrel{q_{0,k}}{\longrightarrow} & H^0\big(\mathcal{L}^\bullet/(z^k)\mathcal{L}^\bullet\big) \\ & \longrightarrow & H^1\big((z^k)\mathcal{L}^\bullet\big) & \stackrel{j_{1,k}}{\longrightarrow} & \dots & & \dots \\ & & \dots & & \dots & & \dots \\ & \longrightarrow & H^n\big((z^k)\mathcal{L}^\bullet\big) & \stackrel{j_{n,k}}{\longrightarrow} & H^n(\mathcal{L}^\bullet) & \stackrel{q_{n,k}}{\longrightarrow} & H^n\big(\mathcal{L}^\bullet/(z^k)\mathcal{L}^\bullet\big) \longrightarrow 0. \end{array}$$

Since the spaces $H^p(\mathcal{L}^\bullet)$ are finitely generated modules over the local ring $\mathcal{O}_0$, the Samuel multiplicities of $H^p(\mathcal{L}^\bullet)$ is well defined and can be calculated by using the limit formula of Lech [9] (Theorem 2)

$$c_p = \lim_{\min k\to\infty} \dim\big(H^p(\mathcal{L}^\bullet)/(z^k)H^p(\mathcal{L}^\bullet)\big)/k_1\dots k_n \qquad (p = 0,\dots,n).$$

To apply the results of Section 1 we need a different variant of this limit formula.

Theorem 2.2 *In the above setting, we obtain the representations*

$$c_p = \lim_{\min k\to\infty} \dim H^p\big(\mathcal{L}^\bullet/(z^k)\mathcal{L}^\bullet\big)/k_1\dots k_n$$

for $p = 0,\dots,n$.

Proof. Fix a number $p \in \{0,\dots,n\}$. Define

$$c_{p,k} = \dim H^p(\mathcal{L}^\bullet)/(z^k)H^p(\mathcal{L}^\bullet), \quad b_{p,k} = \dim H^p\big(\mathcal{L}^\bullet/(z^k)\mathcal{L}^\bullet\big)$$

for $k \in \mathbb{N}^n$. Note that Ker $j_{p,k} = (z^k)\mathcal{L}^p \cap \mathcal{Z}^p/(z^k)\mathcal{Z}^p$ and that

$$(z^k)H^p(\mathcal{L}^\bullet) \ \subset \ \mathrm{Im}\ j_{p,k}$$

for all $k \in \mathbb{N}^n$. Elementary linear algebra shows that

$$\begin{aligned} c_{p,k} &= \dim H^p(\mathcal{L}^\bullet)/\mathrm{Im}\ j_{p,k} \ + \ \dim \mathrm{Im}\ j_{p,k}/(z^k)H^p(\mathcal{L}^\bullet) \\ &= b_{p,k} - \dim \mathrm{Ker}\ j_{p+1,k} \ + \ \dim \mathrm{Im}\ j_{p,k}/(z^k)H^p(\mathcal{L}^\bullet) \end{aligned}$$

for all $k \in \mathbb{N}^n$. Using the short exact sequences

$$0 \longrightarrow \frac{\mathcal{Z}^p}{(z^k)\mathcal{L}^p \cap \mathcal{Z}^p} \longrightarrow \frac{\mathcal{L}^p}{(z^k)\mathcal{L}^p} \longrightarrow \frac{(\mathcal{L}^p/\mathcal{Z}^p)}{(z^k)(\mathcal{L}^p/\mathcal{Z}^p)} \longrightarrow 0$$

and the additivity of the Samuel multiplicity for finitely generated modules over the Noetherian local ring $\mathcal{O}_0$ ([11], Proposition II.10), we obtain that both limits

$$\lim_{\min k\to\infty} \frac{\dim \mathcal{Z}^p/(z^k)\mathcal{L}^p \cap \mathcal{Z}^p}{k_1 \dots k_n} = \lim_{\min k\to\infty} \frac{\dim \mathcal{Z}^p/(z^k)\mathcal{Z}^p}{k_1 \dots k_n}$$

calculate the Samuel multiplicity of the $\mathcal{O}_0$- module $\mathcal{Z}^p$. Thus we find that

$$\lim_{\min k\to\infty} \dim \operatorname{Ker}\, j_{p,k}/k_1 \dots k_n = 0$$

Since $\operatorname{Im}\, j_{p,k} = \big(\mathcal{Z}^p + (z^k)\mathcal{L}^p \cap \mathcal{N}^p\big)/\mathcal{Z}^p$, there are canonical short exact sequences

$$0 \longrightarrow \frac{\mathcal{Z}^p}{(z^k)\mathcal{L}^p \cap \mathcal{Z}^p} \longrightarrow \frac{\mathcal{N}^p}{(z^k)\mathcal{L}^p \cap \mathcal{N}^p} \longrightarrow \frac{H^p(\mathcal{L}^\bullet)}{\operatorname{Im}\, j_{p,k}} \longrightarrow 0.$$

Using the additivity of the Samuel multiplicity a second time, we conclude that

$$\lim_{\min k\to\infty} \frac{\dim(H^p(\mathcal{L}^\bullet)/\operatorname{Im}\, j_{p,k})}{k_1 \dots k_n} = c_p.$$

This observation completes the proof. □

In the particular case $p = n$, Theorem 2.2 can be improved. Indeed it is elementary to check that in this case even the equality $\operatorname{Im}\, j_{n,k} = (z^k)H^n(\mathcal{L}^\bullet)$ holds. Hence we obtain that

$$\dim H^n(\mathcal{L}^\bullet)/(z^k)H^n(\mathcal{L}^\bullet) = \dim H^n\big(\mathcal{L}^\bullet/(z^k)\mathcal{L}^\bullet\big)$$

for all $k \in \mathbb{N}^n$.

Let us specialize our results to the case where $M^\bullet = (M^p, d^p)_{p=0}^n$ is the Koszul complex $K^\bullet(z - T, X)$ of a Fredholm tuple $T \in L(X)^n$ of commuting bounded operators on a complex Banach space X. We begin by choosing an analytically parametrized complex $L^\bullet = (L^p, u^p)_{p=0}^n$ of finite-dimensional vector spaces on an open neighbourhood U of $0 \in \mathbb{C}^n$ which is quasi-isomorphic to $K^\bullet(z - T, X)$ in the sense explained above. Then the cohomology sheaves

$$\mathcal{H}^p = H^p(z - T, \mathcal{O}_U^X) \cong H^p(u^\bullet, \mathcal{O}_U^{L^\bullet}) \qquad (p = 0, \dots, n)$$

of the associated complexes of $\mathcal{O}_U$-modules are coherent analytic sheaves on U. As an application of Theorem 2.2 and Corollary 1.3 we show that the Samuel multiplicities c_p of the stalks $\mathcal{H}_0^p$ can be expressed in terms of the cohomology groups $H^p(T^k, X)$ of the powers T^k of the given Fredholm tuple T.

Corollary 2.3 *For a Fredholm tuple $T \in L(X)^n$ of commuting bounded operators on a complex Banach space X, the Samuel multiplicities c_p of the stalks of the cohomology sheaves $\mathcal{H}^p = H^p(z-T, \mathcal{O}^X_{\mathbb{C}^n})$ at $z = 0$ can be calculated as*

$$c_p = \lim_{\min k \to \infty} \frac{\dim H^p(T^k, X)}{k_1 \dots k_n} .$$

Proof. By combining Theorem 2.2 and Corollary 1.3, and by using the cohomology isomorphisms explained in the sections leading to Theorem 2.2, we obtain the following chain of equalities

$$\begin{aligned} c_p &= \lim_{\min k \to \infty} \frac{\dim H^p(z-T, \mathcal{O}^X_0)/(z^k)H^p(z-T, \mathcal{O}^X_0)}{k_1 \dots k_n} \\ &= \lim_{\min k \to \infty} \frac{\dim H^p(\mathcal{L}^\bullet)/(z^k)H^p(\mathcal{L}^\bullet)}{k_1 \dots k_n} \\ &= \lim_{\min k \to \infty} \frac{\dim H^p(\mathcal{L}^\bullet/(z^k)\mathcal{L}^\bullet)}{k_1 \dots k_n} \\ &= \lim_{\min k \to \infty} \frac{\dim H^p(z-T, \mathcal{O}^X_0/(z^k)\mathcal{O}^X_0)}{k_1 \dots k_n} \\ &= \lim_{\min k \to \infty} \frac{\dim H^p(T^k, X)}{k_1 \dots k_n} \end{aligned}$$

The second equality follows from the fact that the isomorphisms $H^p(\mathcal{L}^\bullet) \to H^p(z-T, \mathcal{O}^X_0)$, explained above, and their inverses are isomorphisms of $\mathcal{O}_0$-modules. □

To bring our results in a more concrete and applicable form, we look for a different interpretation of the Samuel multiplicities c_p of the cohomology sheaves $\mathcal{H}^p(z-T, \mathcal{O}^X)$ at $z = 0$.

Let $V \subset \mathbb{C}^n$ be a connected open neighbourhood of $0 \in \mathbb{C}^n$, and let $\mathcal{F}$ be a coherent analytic sheaf on V. The set S of all points $z \in V$ for which $\mathcal{F}$ is not locally free at z is a proper analytic subset of V, and the complement of S in V is connected ([7], Theorem 4.4). By definition the rank $\mathrm{rk}_V(\mathcal{F})$ of the coherent sheaf $\mathcal{F}$ on V is the constant value of $\mathrm{rk}_z(\mathcal{F})$ for $z \in V \setminus S$. This number is independent of the choice of V and is usually referred to as the rank $\mathrm{rk}_0(\mathcal{F})$ of $\mathcal{F}$ at $z = 0$. By shrinking V, if necessary, one can achieve in addition that $\mathcal{F}$ has a finite resolution

$$0 \to \mathcal{O}_V^{pr} \to \mathcal{O}_V^{pr_1} \to \dots \to \mathcal{O}_V^{p_1} \to \mathcal{F} \to 0$$

by free $\mathcal{O}_V$-modules ([8], Theorem VI.F.5). Since the Samuel multiplicity for finitely generated modules over the Noetherian local ring $\mathcal{O}_0$ is additive

([11], Proposition II.10) and since the Samuel multiplicity of a free $\mathcal{O}_V$-module coincides with its rank, it follows that

$$c(\mathcal{F}_z) = \sum_{i=1}^{r}(-1)^i c(\mathcal{O}_V^{p_i}) = \sum_{i=1}^{r}(-1)^i p_i = \mathrm{rk}_0(\mathcal{F})$$

for $z \in V$.

Theorem 2.4 *Let $T \in L(X)^n$ be a Fredholm tuple of commuting bounded Banach-space operators and let $U \subset \rho_e(T)$ be a connected open neighbourhood of $0 \in \mathbb{C}^n$. Then there is a proper analytic subset $S \subset U$ such that*

$$\dim H^p(z - T, X) = \lim_{\min k \to \infty} \frac{\dim H^p(T^k, X)}{k_1 \dots k_n} = \inf_k \frac{\dim H^p(T^k, X)}{k_1 \dots k_n}$$

for $p = 0, \dots, n$ and $z \in U \setminus S$.

Proof. Since $U \subset \rho_e(T)$, the arguments preceding Lemma 2.1 imply that the cohomology sheaves

$$\mathcal{H}^p = H^p(z - T, \mathcal{O}_U^H) \cong H^p(u^\bullet, \mathcal{O}_U^{L^\bullet}) \qquad (p = 0, \dots, n)$$

are coherent analytic sheaves on U. By Proposition 9.4.5 in [4], there are proper analytic subsets $S_p \subset U$ such that the functions

$$z \mapsto \dim H^p(z - T, X) \qquad (p = 0, \dots, n)$$

have constant values d_p on $U \setminus S_p$ and such that

$$\dim H^p(z - T, X) > d_p \qquad (p = 0, \dots, n,\ z \in S_p).$$

As shown in the proof of Proposition 10.3.3 from [4], the number d_p is the rank of the coherent sheaf $\mathcal{H}^p$ on U. Using Corollary 2.3 and the subsequent remarks, we obtain that

$$d_p = \mathrm{rk}_0(\mathcal{H}^p) = c(\mathcal{H}_0^p) = \lim_{\min k \to \infty} \frac{\dim H^p(T^k, X)}{k_1 \dots k_n}.$$

Let $k = (k_1, \dots, k_n)$ be a family of positive integers. Using the fact that the function $\dim H^p(w - T^k, X)$ is upper-semicontinuous in w ([4], Proposition 9.4.5), we can choose a real number $r > 0$ such that

$$\dim H^p(w - T^k, X) \leq \dim H^p(T^k, X)$$

for all w with $\|w\| < r$ and every $p = 0, \ldots, n$. After shrinking r we may suppose that $\{z \in \mathbb{C}^n; z^k = w\} \subset U$ for $\|w\| < r$. Fix a point $w \in (\mathbb{C}^n \setminus \{0\})^n$ with $\|w\| < r$. Then the proof of Theorem 10.3.13 in [4] shows that

$$\begin{aligned} \dim H^p(T^k, X) &\geq \dim H^p(w - T^k, X) \\ &= \textstyle\sum_{z^k=w} \dim H^p(z - T, X) \geq k_1 \ldots k_n \, d_p. \end{aligned}$$

This observation completes the proof of Theorem 2.4. □

By Theorem 2.4 the numbers

$$c_p(T) = \lim_{\min k \to \infty} \frac{\dim H^p(T^k, X)}{k_1 \ldots k_n} \qquad (p = 0, \ldots, n)$$

calculate the stabilized dimensions of the pth order cohomology groups of the Koszul complexes $K^\bullet(z - T, X)$ of a Fredholm tuple $T \in L(X)^n$.

Corollary 2.5 *Let $T \in L(X)^n$ be a Fredholm tuple on a Banach space X. Then for any family of non-negative integers $s_1, \ldots, s_n$, we obtain that*

$$c_p(T^s) = s_1 \cdot s_2 \cdot \ldots \cdot s_n \, c_p(T) \qquad (p = 0, \ldots, n).$$

Proof. If $s_i = 0$ for some index $i \in \{1, \ldots, n\}$, then $c_p(T^s) = 0$ and the assertion holds. If $s_1, \ldots, s_n \geq 1$, then the observation that

$$\begin{aligned} c_p(T^s) &= \lim_{k \to \infty} \tfrac{\dim H^p(T^{k,s}, X)}{k^n} \\ &= s_1 \ldots s_n \lim_{k \to \infty} \tfrac{\dim H^p(T^{ks}, X)}{s_1 \ldots s_n k^n} = s_1 \ldots s_n \, c_p(T) \end{aligned}$$

completes the proof. □

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