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# The Dimension of the Equisingular Stratum of an Isolated Plane Curve Singularity

Theo de Jong

**Abstract**

Let $C$ be an isolated plane curve singularity. Zariski defined and studied equisingular deformations of $C$, see [12], [13], [14], and proved that this is the same as $\mu$–constant deformations. Wahl [10] showed that the functor of equisingular deformations is smooth, say of dimension $\mathrm{es}(C)$. This therefore is also the dimension of the $\mu$–constant stratum. In this paper we give a formula for $\mathrm{es}(C)$.

## 1 Statement of the Result

We consider an isolated plane curve singularity $C$, with local ring $\mathcal{O}_C$. Consider the following invariants:

(1) $\mathrm{es}(C)$: the dimension of the $\mu$–constant stratum of $C$. This is equal to the dimension of tangent space of the base space of a semi-universal equisingular deformation of $C$.

(2) $\tau(C)$: the Tjurina number. This is the dimension of tangent space of the base space of a semi-universal deformation of $C$. If $\mathcal{O}_C = \mathbb{C}\{x,y\}/(f)$, then $\tau(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

(3) $b_i$: the self-intersection numbers of $E_i$ on a minimal good embedded resolution of $C$, where $E_1, \ldots, E_k$ are the exceptional divisors.

(4) $\mathrm{IN}(C)$: the set of infinitely near points of $C$ (including the singularity itself); $m_P$ the multiplicity of the infinitely near point $P \in \mathrm{IN}(C)$.

(5) $\mathrm{eb}(C)$: the number of extra blowing-ups needed to come from a *minimal embedded* resolution of $C$ to a *minimal* **good** *embedded* resolution of $C$.

**Theorem 1.1.** *The following formula holds for all plane curve singularities $C$:*

$$\mathrm{es}(C) = \tau(C) + \mathrm{eb}(C) + \sum_{i=1}^{k}(b_i + 1) - \sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4).$$

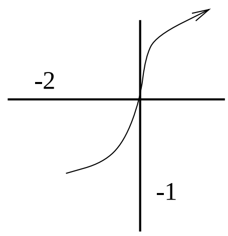
We consider some simple well-known examples.

**Example 1.2.** Let $C$ be the union of $n$ lines through the origin, say given by $x^n + y^n = 0$. We have that $\tau(C) = (n-1)^2$. The minimal (good) embedded resolution is gotten by blowing up once. Thus we get a $(-1)$–curve with $n$ lines intersecting. Thus $\mathrm{eb}(C) = 0$ (this is true for all singularities whose branches are all smooth), and $\sum_{i=1}^{k}(b_i + 1) = 0$. There is just one infinitely near point (the singularity itself), whose multiplicity is $n$. We get the formula

$$\mathrm{es}(C) = (n-1)^2 - \tfrac{1}{2}(n^2 + n - 4) = \binom{n-2}{2}.$$

Thus for two and three lines we get zero, and for four lines we get one (the cross-ratio).

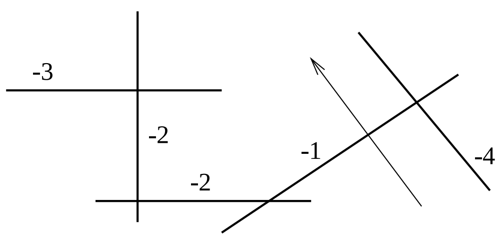
**Example 1.3.** We consider the irreducible curve singularity $C$ given by $x^4 + y^7 = 0$. The Tjurina number is equal to $(4-1) \cdot (7-1) = 18$. The minimal embedded resolution looks like




We get two infinitely near points, of multiplicity 4, and 3 respectively. Hence

$$\sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4) = 8 + 4 = 12.$$

The minimal good embedded resolution looks like



Thus we see that $\mathrm{eb}(C) = 3$, and $\sum(b_i + 1) = -7$. Hence

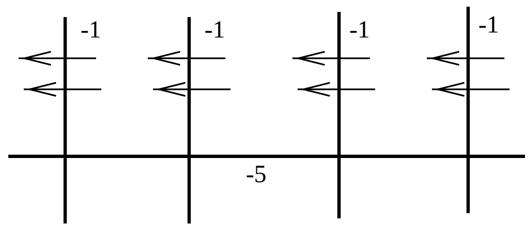
$$\mathrm{es}(C) = 18 + 3 - 7 - 12 = 2.$$

Of course, there is an easier way for the above examples to compute $\mathrm{es}(C)$. Namely, Wahl [10] proved that for $x^p + y^q = 0$, the number $\mathrm{es}(C)$ is equal to the number of monomials $x^i y^j$ with $i \leq p - 2$ and $j \leq q - 2$ such that $iq + jp \geq pq$. In case $p = q = n$, one easily computes this to be equal to $\binom{n-2}{2}$. We now consider the following example of Wahl, see [10] Example 6.8.

**Example 1.4.** Let the curve singularity $C$ be given by $f = (x^4 - y^4)^2 - x^{10} = 0$. Wahl computes the tangent space of the equisingular deformations to be $I/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ with

$$I = (x, y)^{10} + (x^6 y^2 - x^2 y^6) + \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

Its dimension one computes, for example with SINGULAR, [5] to be 6. Let us check this with our formula. The minimal (good) embedded resolution looks like




Hence $\mathrm{eb}(C) = 0$, and $\sum(b_i + 1) = -4$. There are 5 infinitely near points, one of multiplicity 8, and four of multiplicity 2. Hence $\sum_{P \in \mathrm{IN}(C)} \frac{1}{2}(m_P^2 + m_P - 4) = 38$. With the computer algebra system SINGULAR one computes $\tau(C) = 48$. Thus indeed $\mathrm{es}(C) = 6$.

For the proof of Theorem 1.1 we use the deformation theory of sandwiched singularities, as developed in the article [9]. One needs the knowledge of the first four sections and the Appendix of [9] in order to understand the present article.

Let $(C, l)$ be a decorated curve. Here $l$ assigns to each branch a natural number. We suppose that $l \gg 0$. Associated to $(C, l)$ there is a sandwiched singularity, $X(C, l)$. The idea is to compute the dimension of the Artin component of $X(C, l)$ in two different ways. First of all, one has the classical formula expressing it in the self-intersection numbers of the exceptional divisors of the minimal resolutions and the dimension of the equisingular

stratum $\mathrm{es}(X(C,l)$. We prove for the case $l \gg 0$ that $\mathrm{es}(X(C,l)) = \mathrm{es}(C)$, generalizing a result of Gustavsen, [4], who proved this for $C$ irreducible.
On the other hand, we can calculate, using the results of [9], the dimension of the Artin component in terms of the decoration $l$, the topological invariants of the curve singularity $C$ and the Tjurina number $\tau(C)$. Combining these results, the formula for $\mathrm{es}(C)$ drops out.

# 2 Equisingular Deformations of Certain Sandwiched Singularities.

We will consider *sandwiched* singularities. First consider a curve singularity $C = \cup_{i \in T} C_i$. We let $m(i)$ be the sum of the multiplicities of branch $i$ in the multiplicity sequence of the minimal embedded resolution of $C_i$, and $M(i)$ be the sum of the multiplicities of branch $i$ in the multiplicity sequence of the minimal good embedded resolution of $C_i$. Note the following:

**Lemma 2.1.**

$$\sum_{i \in T} M(i) - m(i) = \mathrm{eb}(C).$$

Let $l : T \longrightarrow \mathbb{N}$ be such that $l(i) \geq M(i) + 1$ for all $i$. The pair $(C, l)$ is called a *decorated curve*, see [9], 1.3. One gets the *sandwiched singularity* $X(C,l)$ as follows. First take a minimal good embedded resolution of $C$, and then do $l(i) - M(i)$ consecutive blowing-ups at the strict transform of the $i'th$ branch, thereby inducing a chain of $(-2)$ curves of length $l(i) - M(i)$. We get a modification of $\mathbb{C}^2$:

$$(Z(C,l), F) \longrightarrow (\mathbb{C}^2, 0).$$

Consider $E$, the subgraph of all irreducible components whose self-intersection is not $-1$. From the resolution process one sees that $E$ is connected, has negative definite intersection matrix, and so can be contracted by the result of Grauert–Mumford to a normal surface singularity $X(C,l)$, which one easily sees to be rational. This singularity is denoted by $X(C,l)$. The modification $(Z(C,l), F)$ we can get by blowing up a *complete ideal* $I(C,l)$.

The following theorem was proved by Gustavsen [4] for irreducible plane curve singularities $C$.

**Theorem 2.2.** *Let $X(C,l)$ and $X(C',l')$ be isomorphic sandwiched singularities. Suppose $l \gg 0$ and $l' \gg 0$. Then $C$ and $C'$ are isomorphic and*

$l = l'$. *(This is to be interpreted that for some isomorphism corresponding branches have the same number attached.)*

*Proof.* Let $\varphi$ be an isomorphism taking $X(C', l')$ to $X(C, l)$. By the universal property of blowing–up, this extends to an isomorphism between the minimal resolutions $\varphi : (X(C', l'), E') \longrightarrow (X(C, l), E)$. In particular the dual graphs of $X(C, l)$ and $X(C', l')$ are isomorphic. As both $l \gg 0$ and $l' \gg 0$ there are exactly $\#T$ very long chains of $(-2)$–curves in both $E$ and $E'$. Due to their length, they can be recognized. To each endpoint $G_i'$ of such a very long chain in $E'$ there is, by construction, a $(-1)$–curve $F_i$' in $Z(C', l')$ and a smooth curve $C_i'$ intersecting $F_i'$. Similar for $Z(C, l)$. The isomorphism $\varphi$ sends and $G_i'$ to a $G_i$, as the resolution graphs of $X(C', l')$ and $X(C, l)$ are isomorphic. By using the isomorphism $\varphi$, we may glue every $F_i'$ (and in it $C_i'$) to the curve $G_i$ in $X(C, l)$.
Without loss of generality, we may assume that both $x$ and $y$ are generic elements of $\mathcal{O}_{C'}$. By looking at the image under $\varphi$ of the divisors of the pull-back of $(x)$ and $(y)$ on $X(C', l')$ (which is the fundamental cycle of $Z(C'l')$ restricted to $E'$ plus a non-compact curve intersecting the first blown up curve), we see that the divisors of $\varphi(x), \varphi(y)$ are the divisors of two functions which generate the maximal ideal of $\mathcal{O}_C$. Thus $\varphi$ maps $C'$ to an isomorphic curve, which except for the $(-1)$–curves have the same resolution as $C$. Thus from now on we suppose that, and we assume that $C$ is given by $f = 0$, and $C'$ by $f' = 0$ for square free $f$ and $f'$, whose resolutions are, except from the $(-1)$–curves, identical. From the construction of the resolution graphs it now follows that $l = l'$.
There is an algorithm to get, from the resolution of a plane curve singularity, the resolution of every irreducible component. Following this algorithm, we see that in the resolution of any irreducible component $C_i$ of $C$ there is a very long chain of $(-2)$–curves. By the remarks above this is, except for the $(-1)$–curves, also the resolution for $C_i'$. By the formula for the intersection number, see for example [6] Theorem 5.4.8, the intersection number between $C$ and $C'$ increases if the chains of $(-2)$ curves, and thus $l(i)$ and $l'(i)$ become bigger. This intersection number is equal to both $\dim \mathbb{C}\{x, y\}/(f_i, f_i')$, and also to the vanishing order of $f_i'$ on $\widetilde{C}_i$, where $\widetilde{C}_i$ is the normalization of $C_i$. Thus we may assume that for all $i$ the vanishing order of $f_i'$ on $\widetilde{C}_i$ is at least $k \cdot c(i)$, for some large $k$, and where $c(i)$ is the $i'th$ conductor number of $C$, corresponding to the branch $C_i$. By definition of the conductor, every function which vanishes with order at least $c(i)$ on the normalization $\widetilde{C}_i$ for all $i$, is an element of the maximal ideal $(x, y)$ of $\mathcal{O}_C = \mathbb{C}\{x, y\}/(f)$. It follows that the class of $f' = \prod f_i'$ in $\mathcal{O}_C$ lies in $(x, y)^k$. Thus $uf - f' \in (x, y)^k$, where we now view $(x, y)$ as an ideal in $\mathbb{C}\{x, y\}$. By symmetry, $u$ is a unit.

By taking $k$ large, we may, by the finite determinacy theorem, assume that $uf$ and $f'$ are right equivalent. In particular, their zero sets $C$ and $C'$ are isomorphic. This is what we had to show. $\square$

We now study equisingular deformations. So we have an functor $\mathrm{ES}_C$ of equisingular deformation of the curve singularity $C$, and the functor $ES_{X(C,l)}$ of equisingular deformations of the sandwiched singularity $X(C,l)$. We quote the following result due to Gustavsen, see [4], Theorem 3.3.22.

**Theorem 2.3.** *There is a natural formally smooth map of functors*

$$\mathrm{ES}_C \longrightarrow \mathrm{ES}_{X(C,l)}\,.$$

This in particular, for any equisingular deformation of $X(C,l)$ we can find an equisingular deformation of $C$, mapping to it naturally, but this equisingular deformation of $C$ might not be unique, even on tangent spaces. The existence of the map $\mathrm{ES}_C \longrightarrow \mathrm{ES}_{X(C,l)}$ is quite obvious. An equisingular deformation induces a deformation of $Z(C,l)$, and thus a deformation of the resolution of $X(C,l)$. This thus gives an equisingular deformation of $X(C,l)$.
Our aim is to prove that for $l \gg 0$, we have the formally smooth map is in fact an isomorphism on tangent spaces.

**Proposition 2.4.** *Let $C$ be an isolated curve singularity, and $(C,l)$ be a decorated curve. Suppose that $l \gg 0$. Then the Zariski tangent spaces of the equisingular deformations of $X(C,l)$ and of $C$ are isomorphic. In particular for the dimensions we have*

$$\mathrm{es}(X(C,l)) = \mathrm{es}(C).$$

We use the theory developed in [9]. Let $C$ be given by $f=0$, and $c(i)$ be the conductor number on branch $C_i$. We consider a function $g(x,y)$ whose vanishing order on the normalization of $C_i$ is equal to $c(i)+l(i)$ for all $i \in T$. In [9] the following non–isolated surface singularity in $\mathbb{C}^3$ is considered

$$Y(C,l) = V(zf-g).$$

Properly speaking, the isomorphism class of $Y(C,l)$ depends on the choice of $g$. The normalization of $Y(C,l)$ is proved to be $X(C,l)$. The choice of $g$ is determined up to $f$. Let $\Sigma \subset \mathbb{C}^2$ be the space defined by the conductor $I$ of $C$. We will consider R.C.–deformation of the pair $(\Sigma, C)$. For the definition of R.C.–deformations, see for example [9], Appendix, and the references mentioned there. This functor is canonically isomorphic to deformations of the diagram $\widetilde{C} \longrightarrow C$, where $\widetilde{C} \longrightarrow C$ is the normalization. The space of infinitesimal deformation of R.C.–deformations of the pair $(\Sigma, C)$, we denote by $T^1(\Sigma, C)$. It was stated, but not properly shown in [9], Remark 3.18, that for $l \gg 0$ we have an exact sequence

$$0 \longrightarrow I^{ev}/(f, \Theta_\Sigma(g)) \longrightarrow T^1_{X(C,l)} \longrightarrow T^1(\Sigma, C) \longrightarrow 0. \tag{1}$$

Here $I^{ev} = \{g \in \mathbb{C}\{x,y\} : \mathrm{ord}(g_{|\widetilde{C}_i}) \geq l(i) + m(i) \text{ for all } i\}$ for $\widetilde{C}_i \longrightarrow C_i$ the normalization. Furthermore $\Theta_\Sigma$ are all derivations $\theta$ of $\mathbb{C}\{x,y\}$ satisfying $\theta(f) \subset (f)$, and $\theta(I) \subset I$. If $t_i$ is a parameter on $\widetilde{C}_i$, one identify $\Theta_\Sigma$ with the module generated by the $t_i\frac{\partial}{\partial t_i}$. Let us see how this sequence comes about. In [9], Proposition 3.7 it is shown that every infinitesimal deformation can be obtained as follows. Take an infinitesimal R.C.–deformation $(\Sigma_\varepsilon, C_\varepsilon)$ of $(\Sigma, C)$, and a $g_\varepsilon$ such that also $(\Sigma_\varepsilon, (g_\varepsilon = 0))$ is an R.C–deformation of $(\Sigma, (g = 0))$. Suppose that $\Sigma_\varepsilon$ is defined by the ideal $I_\varepsilon$, and $C_\varepsilon$ by $f_\varepsilon = 0$. We get a space $Y_\varepsilon$ defined by $zf_\varepsilon - g_\varepsilon = 0$, and the space $X_\varepsilon$ then is defined by the local ring

$$\mathrm{Hom}_{Y_\varepsilon}(I_\varepsilon, I_\varepsilon).$$

So $X_\varepsilon$ is obtained by "simultaneous normalization". In general, the choice of the isomorphism class of the R.C.–deformation of $(\Sigma, C)$ is not uniquely determined by the given infinitesimal deformation of $X(C,l)$. But we will see and have to show that in case $l \gg 0$ it in fact does. By linearity, it suffices to show that if we have a trivial infinitesimal deformation of $X(C,l)$, we are only allowed to deform $(\Sigma, C)$ trivially. Now a trivial first order family can be extended to a trivial family over a germ of a smooth curve $T$. By the results of [9], in particular Theorem 3.3 we can therefore extend the R.C–deformation $(\Sigma_\varepsilon, C_\varepsilon)$ and $(\Sigma_\varepsilon, (g_\varepsilon = 0))$ to deformations over this smooth curve. Thus we get an $I_T$ defining $\Sigma_T$, $f_T$ defining $C_T$, and some $g_T$, such that the local ring of the trivial family is given by $\mathrm{Hom}_{Y_T}(I_T, I_T)$, where $Y_T$ is given by $zf_T - g_T = 0$.

Writing $F_t = \sum_i t^i f_i$, we have that for all small $t$ the zero set of $F_t$ intersects on $Z(C,l)$ the $(-1)$–curve. This is because the induced deformation of $X(C,l)$ is trivialized. For fixed $k$, it follows, as in the proof of Theorem 2.2 that by taking $l \gg 0$, we may assume that $f_i \in (x,y)^{k+1}$ for all $i$. We take $k$ so big, that $(x,y)^{k+1} \subset (x,y)^2\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. We write $x_1 = x$, and $x_2 = y$ Thus we can write, formally, $\sum_i t^{i-1} f_i = \sum_i \xi_i(t)\frac{\partial f}{\partial x_i}$, with $\xi_i(t) \in (x,y)^2$. Hence

$$\frac{\partial F_T}{\partial t} = \sum_i t^{i-1} f_i = \sum_i \xi_i(t)\frac{\partial F_T}{\partial x_i} - \sum_j \sum_i \xi_i(t) t^j \frac{\partial f_j}{\partial x_i}.$$

Note that $\sum_j \sum_i \xi_i(t) t^j \frac{\partial f_j}{\partial x_i} \in (x,y)^{k+2}$. Iterating this procedure we get

$$\frac{\partial F_T}{\partial t} = \sum \overline{\xi}_i \frac{\partial F_T}{\partial x_i}.$$

for some formal $\overline{\xi}_i(t) \in (x,y)^2$. By Artin's Approximation Theorem, [1] we may even assume that the $\xi_i$ are analytic. By the Characterization of Local Analytic Triviality, see for example [6], Theorem 9.1.7, it follows that the deformation of $C_T$, given by $F_T = 0$, is trivial. By looking at the conductor, it follows that the family $\Sigma_T$ is trivial. In particular, the induced first order R.C.–deformation of $(\Sigma, C)$ is trivial. Hence we get a map

$$T^1_{X(C,l)} \longrightarrow T^1(\Sigma, C),$$

which one easily sees to be linear. As $l \gg 0$, it is also surjective, see [9], Proof of Theorem 3.3. The kernel one gets, as described in [9], as follows. Consider elements $g' \in \mathbb{C}\{x,y\}$ such that $g_\varepsilon := g + \varepsilon g'$ give an R.C–deformation of $(\Sigma, (g=0))$, by keeping $\Sigma$ fixed. As proved in [9], these are exactly the elements in $I^{ev}$. Then consider $Y_\varepsilon$ defined by $zf - g_\varepsilon = 0$. We get $X_\varepsilon$ as above. Obviously the $g'$ in $\Theta_\Sigma(g)$ give trivial deformations $Y_\varepsilon$ of $Y$, and thus trivial deformations $X_\varepsilon$ of $X$. As soon as $C$ is singular, and the class of $g'$ in $I^{ev}/(f, \Theta_\Sigma(g))$ is nonzero, this deformation is non-trivial. Then if it where, the deformation $Y_\varepsilon$ could be extended to a trivial family $Y_T$ given by $zf - g_T = 0$. But as the vanishing order of $g'$ on the normalization $\widetilde{C}_i$ of at least one branch $C_i$ at the point mapping to the singular point of $C$ is strictly smaller that $c(i) + l(i)$, this then also holds for a general $g_t$ for $t$ small. Thus the resolution graph of the normalization of $zf - g_t$ is different from the resolution graph of $X(C,l)$. In particular the deformation is not trivial, contradiction. Thus we proved the existence of the exact sequence (1).

*Proof of Proposition 2.4.* The proposition follows from the exact sequence (1). $\square$

**Remark 2.5.** Given the decorated curve $(C,l)$, the kernel of the map

$$\mathrm{ES}_C(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{ES}_{X(C,l)}(\mathbb{C}[\varepsilon])$$

can be computed explicitly. Namely, one searches for equisingular deformations of $C$, which deform $X(C,l)$ trivially. Given the projection $Y(C,l)$, say given by $zf - g = 0$, one can compute all R.C.–deformations of $Y(C,l)$ which deform $X(C,l)$ trivially. Consider generators $u_0 = 1, u_1, \dots, u_k$ of $\mathcal{O}_{X(C,l)}$ as $\mathcal{O}_{Y(C,l)}$–module. In [7], and [9], A.9, for any vector field $\theta$ on $\mathbb{C}^3$ an action of $u_i\theta$ on $zf - g$ is defined, which, by simultaneously normalizing, give all

infinitesimal trivial deformations of $X(C,l)$. These deformations in general, do not keep the form $zf-g$ fixed, that is, they are in general not of this the type $zf_\varepsilon - g_\varepsilon = 0$, for some deformation $f_\varepsilon$ and $g_\varepsilon$. But we can look at the subspace of deformations that do! Thus we get can compute all $f'$ so that there exist a $g'$ so that $z(f+\varepsilon f')-(g+\varepsilon g')$ that gives an R.C.–deformation of $Y(C,l)$ which gives trivial deformation of $X(C,l)$. These give equisingular deformations $f+\varepsilon f'$ of $C$. The totality of such $f'$ build an ideal in $\mathrm{ES}(\mathbb{C}[\varepsilon])$ giving the kernel of the map $\mathrm{ES}_C(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{ES}_{X(C,l)}(\mathbb{C}[\varepsilon])$.
We consider a simple example. Namely, for $C$, we take four lines through the origin, and for $l$ we take the function assigning two to each branch. The sandwiched singularity $X(C,l)$ is isomorphic to the cone over the rational normal curve of degree 5. As this singularity is taut, we get $\mathrm{es}(X(C,l))=0$. $Y(C,l)$ can be given by $z(y^4-x^4)-x^5=0$. As described in [7], the generators of $\mathcal{O}_{X(C,l)}$ as $\mathcal{O}_{Y(C,l)}$–module, say $1, u_1, u_2, u_3$, are given by the rows of the matrix:

$$\begin{pmatrix} zy & 0 & 0 & zx+x^2 \\ x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \end{pmatrix}$$

Note that the lower three rows gives a resolution of the conductor of $C$, and that $zf-g=z(y^4-x^4)-x^5=0$ is the determinant of this matrix. The columns give linear equations, so we have four of them:

$$\begin{aligned} L_1 &: \quad zy + xu_1 = 0 \\ L_2 &: \quad yu_1 + xu_2 = 0 \\ L_3 &: \quad yu_2 + xu_3 = 0 \\ L_4 &: \quad (zx+x^2) + yu_3 = 0. \end{aligned}$$

From these, the quadratic equations can be calculated:

$$\begin{aligned} u_1^2 &= zu_2 \\ u_2^2 &= z(z+x) \\ u_3^2 &= (z+x)u_2 \\ u_1u_2 &= zu_3 \\ u_1u_3 &= z(z+x) \\ u_2u_3 &= (z+x)u_1. \end{aligned}$$

Thus we get a total of ten equations, describing the cone over the rational normal curve of degree 5. The actions of $u_i\theta$ on these equations are totally determined by their values on the linear ones, but one needs the quadratic equations to compute them. We do not want to do the whole calculation in detail, but only look at the action of $u_3\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}$. We get the values

$$
\begin{aligned}
&L_1 \mapsto u_1u_3 - z^2 = xz\\
&L_2 \mapsto u_2u_3 - zu_1 = xu_1\\
&L_3 \mapsto u_2u_3 - u_3u_2 = 0\\
&L_4 \mapsto zu_2 + 2xu_2 - u_3^2 = xu_2.
\end{aligned}
$$

Thus we get the following infinitesimal deformation of the matrix:

$$
\begin{pmatrix}
z(y+\varepsilon x) & 0 & 0 & zx + x^2\\
x & y+\varepsilon x & 0 & 0\\
0 & x & y & \varepsilon x\\
0 & 0 & x & y
\end{pmatrix}
$$

which gives the R.C.–deformation:

$$
z(y^4 - x^4 + \varepsilon x^2y^2) - x^4.
$$

Thus we see that the non–trivial equisingular deformation $y^4 - x^4 + \varepsilon x^2y^2$ of $C$ maps to the trivial infinitesimal deformation of the cone over the rational normal curve of degree 5.

# 3 The Dimension of the Artin Component of Certain Sandwiched Singularities

First of all, we need the following result.

**Theorem 3.1 (Wahl, [11], Propositions 2.2 and 2.5).** *Let $X$ be a rational surface singularity, $p : \widetilde{X} \longrightarrow X$ be the minimal resolution, $E$ be the exceptional divisor of $p : \widetilde{X} \longrightarrow X$, and $E_1, \dots, E_s$ be the irreducible components of $E$. Let $b_i = E_i^2$. Then*

$$
h^1(\widetilde{X}, \Theta_{\widetilde{X}}) = -\sum_{i=1}^{s}(b_i + 1) + h^1(\widetilde{X}, \Theta(\log E)).
$$

*The vector space $H^1(\widetilde{X}, \Theta(\log E))$ is the Zariski tangent space to the base space of the equisingular deformations of $X$. Its dimension we denote by* $\mathrm{es}(X)$ *for short.*

The space $H^1(\widetilde{X}, \Theta_{\widetilde{X}})$ classifies the infinitesimal deformations of the resolutions. Its dimension is the dimension of the Artin component. We will give a different characterization of this dimension for sandwiched singularities $X(C,l)$, for the case $l \gg 0$. Note moreover that we can calculate $I^{ev}/(f, \Theta_\Sigma(g))$ on the normalization, as the conductor $I$ is contained in $I^{ev}$. Thus we get that

$$\dim_{\mathbb{C}} \left(I^{ev}/(f, \Theta_\Sigma(g)\right) = \sum_{i \in T} \left(l(i) - m(i)\right).$$

Hence from the exact sequence (1) it follows that

**Theorem 3.2.** *Let $(C,l)$ be a decorated curve, and suppose that $l \gg 0$. Let $\Sigma$ be the conductor of $C$. Then*

$$\dim_C(T^1_{X(C,l)}) = \sum_{i \in T} (l(i) - m(i)) + \dim_{\mathbb{C}}(T^1(\Sigma, C)).$$

We want to stress that in general for small $l$ the statement of the theorem is false.

We want to understand the deformations on the Artin component of $X(C,l)$. This has been described in [9], 4.13. One can decide whether a deformation occurs on the Artin component by looking at the corresponding R.C.–deformation of $(\Sigma, C)$. A general such one–parameter R.C.–deformation of $(\Sigma, C)$ is one, which has on a general fibre $q$ singular points, where $q$ is the number of infinitely near points of $C$. For each infinitely near point $P$ of $C$ of multiplicity $m_P$ we have on a general fibre of the deformation a singularity consisting of $m_P$ smooth branches intersecting transversely (which we call ordinary $m_p$–tuple point). Let $A$ be the closure of the stratum of the base space of R.C.–deformations of $(\Sigma, C)$ where the above mentioned deformations occur. All elements of $I^{ev}$ are unobstructed against these deformations. This follows immediately from the picture method, see [9]. As the Artin component is smooth, it follows that $A$ is smooth. We get:

**Proposition 3.3.** *Let $(C,l)$ be a decorated curve with $l \gg 0$. Let $A$ be the stratum described above. Then the dimension of the Artin component of $X(C,l)$ is equal to*

$$\dim(A) + \sum_{i \in T} \left(l(i) - m(i)\right).$$

We now compute $\dim(A)$. The result is:

**Proposition 3.4.** *Let $C$ be a plane curve singularity, and $A$ be the stratum described above. Then*

$$\dim(A) = \tau(C) - \sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4).$$

*Proof.* We have an R.C. deformation of $(\Sigma, C)$ over $A$ In particular we get a one–parameter deformation of $C$ over $A$. Let $B$ be the base space of a semi-universal deformation of $C$. It is a smooth space of dimension $\tau(C)$. By semi-universality we get a map of smooth spaces $A \longrightarrow B$. This map in general is not an immersion. Indeed, Buchweitz [2] showed that the kernel of the map $T^1(\Sigma, C) \longrightarrow T^1_C$ is equal to $m(C) - r(C)$, the multiplicity minus the number of branches. However, for a general point of $A$ we have that for all singularities $m_P(C) = r_P(C)$, so that the map $A \longrightarrow C$ is an immersion at a general point of $A$. Hence the image of $A$ at a general point is smooth of dimension $\dim(A)$. Thus, by openness of versality, it remains to compute for each infinitely near point $P$, the codimension of the stratum $A$ of the ordinary $m_P$–tuple point in the base space of the ordinary $m_P$–tuple point. This codimension is $\frac{1}{2}(m_P^2 + m_p - 4)$, being two less than the number of monomials of degree smaller than $m_p$. ☐

We can now prove the main theorem of the article.

*Proof of Theorem 1.1.* Given $C$, take a decorated curve $(C, l)$ with $l \gg 0$. Let $E_1, \ldots, E_s$ be the exceptional curves in the minimal resolution of $X(C, l)$. Let $b_i = -E_i^2$. By Wahl's result 3.1, the dimension of the Artin component of $X(C, l)$ is equal to

$$\mathrm{es}(X(C,l)) - \sum_{i=1}^{s} (b_i + 1).$$

On the other hand, combining 3.3 and 3.4 we get that this dimension is also equal to

$$\sum_{i \in T} (l(i) - m(i)) + \tau(C) - \sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4).$$

Using 2.4 we thus get

$$\mathrm{es}(C) = \tau(C) + \sum_{i=1}^{s} (b_i + 1) + \sum_{i \in T} (l(i) - m(i)) - \sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4).$$

It remains to show that $\sum_{i=1}^{s}(b_i + 1) + \sum_{i \in T}(l(i) - m(i)) = \mathrm{eb}(C) + \sum_{i=1}^{k}(b_i + 1)$. This is easy, as we know that we get the minimal resolution of $X(C, l)$ out of the minimal good resolution of $C$ by doing an extra $l(i) - M(i)$ blowing-ups, for each $i \in T$, thereby introducing a chain of $(-2)$–curves of length $l(i) - M(i)$. From $\mathrm{eb}(C) = \sum_{i \in T} M(i) - m(i)$, see 2.1 the result follows. ☐

**Remark 3.5.** It is not so difficult to see that for a decorated curve $(C, l)$ with $l \gg 0$ we have $m(X(C,l)) = m(C) + 1$. Note that we in fact computed the codimension of the Artin component in $T^1_{X(C,l)}$. With a little combinatorics one then proves that the invariant $c(X(C,l))$, introduced by Christophersen and Gustavsen [3], is equal to 0. This can also be proven directly, using the results of [3] and [4], chapter two. In particular, for $l \gg 0$ one gets a formula for the dimension of $T^2_{X(C,l)}$ by using the results of Christophersen and Gustavsen. By using a semicontinuity argument, this then also holds for all $X(C,l)$ with $m(X(C,l)) = m(C) + 1$. Also, in case $l \gg 0$, one can construct, using the picture method, see [9], a deformation of $X(C,l)$, where on a general fibre we have a cone over the rational normal curve of degree $m_P$ for each infinitely near point $P$ of $C$. Using a standard argument, as used in [8], we then get that the obstruction map is surjective, that is, the minimal number of equations for describing the base space of a semi-universal deformation of $X(C,l)$ is equal to the dimension of $T^2_{X(C,l)}$ if $l \gg 0$.

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