Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 206

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Saarbrücken 2008

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 206
submitted: 28.02.2008

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Abstract

Inspired by the work of Marcellini and Papi [MP] we consider local minima u: $\mathbb{R}^n \supset \Omega \to \mathbb{R}^M$ of variational integrals of the form $\int_\Omega h(|\nabla u|)\,\mathrm{d}x$ and prove interior gradient bounds under rather general assumptions on h working with the additional hypothesis that u is locally bounded. Our requirements imposed on the density h do not involve the dimension n.

1 Introduction

In our note we discuss the Lipschitz regularity of vector-valued functions u: $\Omega \to \mathbb{R}^M$ (from a suitable weak function space) defined on an open set $\Omega \subset \mathbb{R}^n$ which locally minimize the variational integral

$$I[u,\Omega] = \int_\Omega H(\nabla u)\,\mathrm{d}x \tag{1.1}$$

for a strictly convex density H: $\mathbb{R}^{nM} \to [0,\infty)$. For vector-valued minimizers in general only almost everywhere regularity results are available, and the reader will find a (historical) overview on this phenomenon of partial regularity together with the most important contributions in the monographs of Giaquinta [Gia] and of Giusti [Giu]. In order to exclude the occurrence of singular points, we concentrate on the case

$$H(Z) := h(|Z|)\,, \quad Z \in \mathbb{R}^{nM}\,, \tag{1.2}$$

for a function h: $[0,\infty) \to [0,\infty)$ whose properties will be specified below. The restriction (1.2) is motivated by the works of many prominent authors: the case $h(t) = t^p$ with $p \geq 2$ was considered first by Uhlenbeck [Uh] (with extensions due to [GM]), and later on much attention has been paid to so-called “general growth conditions” mainly by Marcellini [Ma1], [Ma2], [Ma3], who includes integrands of exponential growth like $h(t) = \exp(t^p)$ with $p \geq 2$. The case of nearly linear growth, i.e. the model $h(t) = t\ln(1+t)$, has been the subject of the paper [MS] of Mingione and Siepe.

Of course our list is not complete, but the reader will find further references in the papers of these authors. Roughly speaking, for the above mentioned examples of functions h it is possible to show that $|\nabla u| \in L^\infty_{loc}$ holds for a local minimizer u, from which $C^{1,\alpha}$ (and even C^∞) regularity of u can be deduced by standard arguments.

Very recently Marcellini and Papi [MP] published an interesting paper addressing the regularity problem for local minimizers of (1.1) working with hypothesis (1.2) and exhibiting conditions on h which include various kinds of growth. Inspired by this work we

AMS Subject Classification: 49N60, 35J50, 35D10
Keywords: variational problems, vector functions, regularity, nonstandard growth

will impose the following conditions on the function h: consider h: $[0,\infty) \to [0,\infty)$ of class C^2 such that

$$\begin{gathered} h \text{ is strictly increasing and } h''(t) > 0 \text{ for all } t > 0 \text{ together with} \\ \lim_{t\to 0} \frac{h(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{h(t)}{t} = \infty\,. \end{gathered} \tag{H1}$$

(Note that the first requirement in (H1) is a consequence of the second and the third one.) From (H1) it follows that

$$h'(t) > 0 \ \text{ for all } t > 0 \quad \text{and} \quad h'(0) = 0\,. \tag{1.3}$$

Moreover, according to (H1), h is a N-function in the sense of [Ad], Section 8.2. We therefore call a function u from the local Orlicz-Sobolev class $W^1_{h,loc}(\Omega;\mathbb{R}^M)$ a local minimizer of the energy I from (1.1) if

$$I[u,\tilde{\Omega}] \le I[v,\tilde{\Omega}]$$

is true for all $v \in W^1_{h,loc}(\Omega;\mathbb{R}^M)$ with $\operatorname{spt}(u-v) \subset \tilde{\Omega}$, where $\tilde{\Omega}$ is any subdomain of Ω s.t. $\tilde{\Omega} \Subset \Omega$.

Our next requirement is:

$$\text{there exist } \bar{k} > 0,\ t_0 \ge 0 \text{ s.t.} \quad h(2t) \le \bar{k}h(t) \quad \text{for all } \ t \ge t_0\,. \tag{H2}$$

This means that h satisfies a Δ_2-condition near infinity from which we deduce the existence of an exponent $m \ge 1$ such that

$$h(t) \le C(t^m + 1) \tag{1.4}$$

holds for all $t \ge 0$ with a suitable constant $C \ge 0$. In particular (H2) excludes exponential growth.

Finally we suppose that

$$\begin{gathered} \text{there exist } \bar{\varepsilon},\ \bar{h} > 0,\ T_0,\ \kappa,\ \mu \ge 0 \text{ such that for all } t \ge T_0 \\ \bar{\varepsilon}\frac{h'(t)}{t}(1+t^2)^{-\frac{\mu}{2}} \le h''(t) \le \bar{h}(1+t^2)^{-\frac{\kappa}{2}}h(t)\,. \end{gathered} \tag{H3}$$

(H1) implies the inequality

$$\frac{t}{2}h'(t/2) \le \int_{t/2}^t h'(s)\,\mathrm{d}s \le h(t) = \int_0^t h'(s)\,\mathrm{d}s \le th'(t)$$

which on account of (H2) gives the estimate

$$\frac{1}{\bar{k}}th'(t) \le h(t) \le th'(t)$$

at least for $t \geq 2t_0$. Clearly it is also possible to replace T_0 by a number larger than $\max\{1, 2t_0\}$. Thus, under the hypotheses (H1) and (H2), (H3) can be replaced by the equivalent requirement

$$\begin{gathered}\text{there exist } \bar{\varepsilon},\ \bar{h} > 0,\ T_0,\ \kappa,\ \mu \geq 0 \text{ such that for all } t \geq T_0 \\ \bar{\varepsilon}\frac{h'(t)}{t}t^{-\mu} \leq h''(t) \leq \bar{h}t^{2-\kappa}\frac{h'(t)}{t}\,. \end{gathered} \tag{$\widetilde{\mathrm{H3}}$}$$

In particular the bound $\kappa \leq 2 + \mu$ is a consequence of our hypotheses.

Note that the functions $h_s(t) := t^s \ln(1+t)$, $t \geq 0$, $s \geq 1$, satisfy (H1)–(H3) for any choice of $\mu > 0$ and $\kappa \leq 2$ on the whole intervall $(0, \infty)$ with suitable constants $\bar{\varepsilon}$, $\bar{h} > 0$ depending on s. Moreover we can can cover integrands h oscillating between two powers as introduced for example in formula (2.10) of [MP] and which can also be found in the work [DMP]. For a precise statement we refer to Remark 1.1 and to the Appendix.

If the integrand $H(Z)$, $Z \in \mathbb{R}^{nM}$, is defined according to (1.2), then we have for all Y, $Z \in \mathbb{R}^{nM}$ the estimate

$$\min\Big\{\frac{h'(|Z|)}{|Z|}, h''(|Z|)\Big\}|Y|^2 \leq D^2H(Z)(Y,Y) \leq \max\Big\{\frac{h'(|Z|)}{|Z|}, h''(Z)\Big\}|Y|^2 \tag{1.5}$$

and from (H1) and (1.3) it follows that $D^2H(Z)$ is strictly positive for all $Z \in \mathbb{R}^{nM} - \{0\}$ which gives the strict convexity of H: $\mathbb{R}^{nM} \to [0, \infty)$. On the other hand, the convexity of h together with (1.4) implies that $h'(t)$ grows at most as t^{m-1}, hence (H3) and (1.5) show the existence of an exponent q (w.l.o.g. $q \geq 2$) s.t.

$$D^2H(Z)(Y,Y) \leq \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2 \tag{1.6}$$

holds for all Y, $Z \in \mathbb{R}^{nM}$ with a suitable constant $\Lambda > 0$. However, the reader should note that (H3) together with (1.5) does not guarantee the uniform ellipticity of D^2H and that in general we do not have $q = m$.

Let us now state our main result:

Theorem 1.1. *Suppose that (H1)–(H3) are valid with*

$$2\mu < \kappa\,.$$

Consider a local minimizer $u \in W^1_{h,loc}(\Omega; \mathbb{R}^M)$ of the functional I defined in (1.1) with H from (1.2). Suppose further that u is locally bounded. Then $|\nabla u|$ is in the space $L^\infty_{loc}(\Omega)$.

Remark 1.1. *Theorem 1.1 shows that the results known in the literature are not sharp for energy densities satisfying the structure condition (1.2): typical examples of functions h we have in mind "alternate" between two powers t^p and t^s with given exponents $p < s$.*

More precisely, in the Appendix we will construct an example of a function h for which (H1)–(H3) hold with $2\mu < \kappa$ and for which the condition of (p, s)-ellipticity, i.e.

$$\lambda |Z|^p \le D^2 H(Z) \le \Lambda |Z|^s \quad \textit{for all } |Z| \gg 1\,,$$

is satisfied but only with exponents p, s such that $s > p+2$ which means that the regularity of locally bounded local minimizers does not follow along the lines of [Bi], Section 5.2, or from Theorem 2 in [BF].

Remark 1.2. *a) W.r.t. our structure condition (1.2) the hypothesis $u \in L^\infty_{loc}(\Omega; \mathbb{R}^M)$ is rather natural: in fact, if we consider the global minimization problem for boundary values $u_0 \in L^\infty(\Omega; \mathbb{R}^M)$ on a bounded Lipschitz domain Ω, then the maximum-principle of [DLM] implies the boundedness of the minimizer.*

b) Suppose that the function h has the property

$$\int_1^\infty \frac{h^{-1}(s)}{s^{\frac{n+1}{n}}}\,\mathrm{d}x < \infty\,, \tag{1.7}$$

h^{-1} being the inverse. Theorem 8.35 in [Ad] then gives the local boundedness of functions from $W^1_{h,loc}(\Omega; \mathbb{R}^M)$. Transforming the integral and using the fact that $h(t)$ and $th'(t)$ have the same behaviour, (1.7) is seen to be equivalent to

$$\int_1^\infty \frac{1}{\sqrt[n]{h(t)}}\,\mathrm{d}t < \infty\,, \tag{1.8}$$

which means that in a certain sense $h(t)$ grows faster than t^n as $t \to \infty$. So if we have (1.8), then $u \in L^\infty_{loc}(\Omega; \mathbb{R}^M)$ is automatically true. Let us look at the case that we just know

$$h(t) \ge \textit{const}\, t^n \tag{1.9}$$

for large values of t, and consider a local minimizer u being not necessarily bounded. Then, for all $x \in \Omega$ and almost all $r > 0$ s.t. $B_r(x) \subset \Omega$, we get $u \in C^0(\partial B_r(x); \mathbb{R}^M)$ and therefore $u \in L^\infty(B_r(x); \mathbb{R}^M)$ by the maximum-principle of [DLM]. Thus (1.9) implies the boundedness of u on suitable balls around each point $x \in \Omega$, and this is enough for carrying out the proof of Theorem 1.1 which means that in the beginning of Section 2 we require $B_{2R}(x_0) \subset \Omega$ for a "good" radius $2R$ getting $u \in L^\infty(B_R(x_0); \mathbb{R}^M)$ as well as part b) of Lemma 2.1.

Remark 1.3. *Note that the condition $2\mu < \kappa$ automatically gives $\mu < 2$ since (H3) implies $\kappa \le 2 + \mu$.*

Remark 1.4. *Theorem 1.1 corresponds to Theorem A of [MP], and our condition ($\widetilde{\mathrm{H3}}$) looks quite similar to (2.9) of [MP]. We like to emphasize that in contrast to [MP] our exponents κ, μ do not depend on the dimension n since we work with locally bounded local minima. This gives in particular in higher dimensions much better results. On the other hand, no Δ_2-condition has to be imposed in [MP].*

Remark 1.5. *Of course Theorem 1.1 extends to integrands of linear growth which satisfy appropriate versions of (H1)–(H3), provided local minimizers from the class $W^1_{1,loc}(\Omega;\mathbb{R}^M)$ are considered. We thereby obtain a variant of Theorem B from the paper [MP]. Since the existence of local or global minimizers of variational problems with linear growth in general cannot be expected in subclasses of $W^1_{1,loc}(\Omega;\mathbb{R}^M)$, it seems to be more natural to extend first the functional I from (1.1) to the space $BV_{loc}(\Omega;\mathbb{R}^M)$ and then to study the regularity of local BV-minimizers. This requires rather subtile considerations involving both measure theoretic arguments and a refined look at convex analysis. Note that on account of our weak assumptions on h even uniqueness results for the dual solution are not known. We therefore decided to present this material in a separate paper.*

Our paper is organized as follows: in Section 2 we introduce a sequence of regularized problems and collect some auxiliary results. Section 3 contains a higher integrability result for $|\nabla u|$ based on an integration-by-parts argument (first used by Choe [Ch] for the case $M = 1$) combined with an iteration process. Theorem 1.1 then is established in Section 4 using DeGiorgi's technique. In the Appendix we describe the class of examples mentioned in Remark 1.1.

2 Some auxiliary results

From now on we assume the validity of (H1)–(H3) and consider a local $I[\cdot,\Omega]$-minimizer $u \in W^1_{h,loc}(\Omega;\mathbb{R}^M)$ with the additional property $u \in L^\infty_{loc}(\Omega;\mathbb{R}^M)$. For $\varepsilon > 0$ let $(u)_\varepsilon$ denote the mollification of u with small radius and choose a ball $B := B_R(x_0)$ with compact closure in Ω. Moreover, we fix an exponent $\tilde{q} > q$ ($q \geq 2$ is defined in (1.6)) and let

$$\delta = \delta(\varepsilon) := \Big[1 + \varepsilon^{-1} + \|\nabla(u)_\varepsilon\|^{2\tilde{q}}_{L^{\tilde{q}}(B)}\Big]^{-1}.$$

Then an appropriate regularization of our original problem is given by ($H_\delta(Z) := \delta(1 + |Z|^2)^{\tilde{q}/2} + H(Z)$, $Z \in \mathbb{R}^{nM}$)

$$I_\delta[w,B] := \int_B H_\delta(\nabla w)\,\mathrm{d}x \to \min \quad \text{in } (u)_\varepsilon + \overset{\circ}{W}{}^1_{\tilde{q}}(B;\mathbb{R}^M)$$

with unique solution u_δ. In fact, we have the following properties of $\{u_\delta\}$:

Lemma 2.1. *a) If we let $\varepsilon \to 0$, then:*

$$u_\delta \rightharpoondown u \quad \textit{in } W^1_1(B;\mathbb{R}^M)\,;$$
$$\delta \int_B (1 + |\nabla u_\delta|^2)^{\frac{\tilde{q}}{2}}\,\mathrm{d}x \to 0\,;$$
$$\int_B H(\nabla u_\delta)\,\mathrm{d}x \to \int_B H(\nabla u)\,\mathrm{d}x\,.$$

b) $\|u_\delta\|_{L^\infty(B)}$ is bounded independent of ε.

c) ∇u_δ *is in the space* $L^\infty_{loc} \cap W^1_{2,loc}(B;\mathbb{R}^{nM})$*.*

Proof. a) From the definition of u_δ it follows that

$$\sup_{\varepsilon>0} \|u_\delta\|_{W^1_h(B)} < \infty\,,$$

hence $u_\delta \rightharpoondown: \bar{u}$ in $W^1_1(B;\mathbb{R}^M)$ as $\varepsilon \to 0$ for some function $\bar{u}$ from this space. But the lower-semicontinuity of $I[\cdot, B]$ together with the strict convexity of H implies $\bar{u} = u$.

b) This is a consequence of the maximum-principle established in [DLM].

c) Since $\tilde{q} > q$ we can quote [GM], Theorem 5.1, for the local boundedness of ∇u_δ. Well-known arguments presented for example in [Ca] imply the weak differentiability of ∇u_δ. □

Lemma 2.2. *(Caccioppoli-type inequality) Let* $s \geq 0$ *and* $\eta \in C^\infty_0(B)$*,* $0 \leq \eta \leq 1$*. For* $\varkappa > 1$ *consider the set (*$\Gamma_\delta := 1 + |\nabla u_\delta|^2$*)*

$$B^\varkappa := \{x \in B : \Gamma_\delta > \varkappa\}\,.$$

Then there is a positive constant $c = c(s)$ *independent of* ε *and* $\varkappa$ *such that (summation w.r.t.* $\alpha = 1, \dots, n$*)*

$$\int_{B^{2\varkappa}} D^2H_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta)\Gamma_\delta^{\frac{s}{2}}\eta^2 \,\mathrm{d}x \leq c(s)\int_{B^\varkappa} |D^2H_\delta(\nabla u_\delta)|\Gamma_\delta^{\frac{s+2}{2}}|\nabla\eta|^2\,\mathrm{d}x\,. \quad (2.1)$$

Proof. For the case $M = 1$ inequality (2.1) is presented in [Bi], Lemma 5.20; we also refer to [BFM]. But since $H_\delta(Z) = h_\delta(|Z|)$, we can repeat these calculations starting from the identity

$$0 = \int_B D^2H_\delta(\nabla u_\delta)\big(\partial_\alpha \nabla u_\delta, \nabla\big[\eta^2 \partial_\alpha u_\delta \Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{s/2}\big]\big)\,\mathrm{d}x\,,$$

where $\Phi_\varkappa(t) := \tilde{\Phi}(t/\varkappa)$, and where $\tilde{\Phi}$: $[0,\infty) \to [0,\infty)$ satisfies

$$\tilde{\Phi}(t) = \begin{cases} 0\,, & t \in [0,1]\,, \\ 1\,, & t \geq 2 \end{cases}$$

together with $\tilde{\Phi}' \geq 0$. (For instance following [Bi], formula (32), p.62, it is evident that our arguments really rely on the structure condition $H_\delta(Z) = h_\delta(|Z|)$.) □

Remark 2.1. *If in addition to (1.6) we have the "Hölder condition" (1.4) of [GM] for our integrand* H *(letting* $m = q$ *in (1.4) of [GM]), then we can choose* $\tilde{q} := q$*, and Theorem 3.1 of [GM] implies* $u_\delta \in C^{1,\alpha}(B;\mathbb{R}^M)$ *for a suitable* $\alpha > 0$*. Since we do not want to put this extra assumption on the function* H*, we decided to perturb* H *with the* $\tilde{q}$*-power of* $|Z|$*, which means that the resulting density* H_δ *is asymptotically regular in the sense of [GM], Section 5.*

3 Higher integrability of $|\nabla u|$

Let the assumptions of Theorem 1.1 hold. Referring to Section 2 we let

$$\begin{aligned} h_\delta(t) &:= \delta(1+t^2)^{\frac{\bar q}{2}} + h(t)\,, \quad t \geq 0\,,\\ H_\delta(Z) &:= h_\delta(|Z|)\,, \quad Z \in \mathbb{R}^{nM}\,, \end{aligned}$$

and consider the corresponding approximations u_δ. Let us now fix numbers $\varkappa > 0$ and $s \geq 0$, and define $\Phi_\varkappa$ as done after Lemma 2.2. Finally we choose $\eta \in C_0^\infty(B)$, $0 \leq \eta \leq 1$, and proceed similar to [Bi], proof of Theorem 5.21, observing that the following calculations are justified on account of Lemma 2.1, c). We have (summation w.r.t. $\alpha = 1, \dots, n$)

$$\begin{aligned} &\int_B \eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x\\ &= \int_B \eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x + \int_B \partial_\alpha u_\delta \cdot \partial_\alpha u_\delta \eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\\ &=: T_1 + T_2\,, \end{aligned} \tag{3.1}$$

$$\begin{aligned} T_2 &= -\int_B u_\delta \cdot \partial_\alpha\big[\partial_\alpha u_\delta \eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{s/2}\big]\,\mathrm{d}x\\ &\leq c\Bigg[\int_B |\nabla\eta|\eta h(|\nabla u_\delta|)|\nabla u_\delta|\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\\ &\quad + \int_B \eta^2|\nabla^2 u_\delta| h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\\ &\quad + \int_B \eta^2|\nabla u_\delta| h'(|\nabla u_\delta|)|\nabla^2 u_\delta|\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\\ &\quad + \int_B \eta^2|\nabla u_\delta|^2 h(|\nabla u_\delta|)\Phi'_\varkappa(\Gamma_\delta)|\nabla^2 u_\delta|\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\Bigg]\\ &=: c[T_3 + T_4 + T_5 + T_6]\,, \end{aligned} \tag{3.2}$$

where here and in what follows c always denotes a positive constant independent of ε but possibly depending on s (and later) on $\varkappa$. Young's inequality implies for any $\tau > 0$

$$T_3 \leq \tau \int_B \eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x + c(\tau)\int_B |\nabla\eta|^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\,,$$

and for $\tau \ll 1$ the τ-integral can be absorbed in the l.h.s. of (3.1). Therefore we get from (3.1) and (3.2)

$$\begin{aligned} &\int_B \eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x\\ &\leq c\left[\int_B (\eta^2 + |\nabla\eta|^2) h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x + T_4 + T_5 + T_6\right]. \end{aligned} \tag{3.3}$$

Similarly we apply Young's inequality to T_4:

$$T_4 \le \tau \int_B \eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x + c(\tau)\int_B \eta^2|\nabla^2 u_\delta|^2 h(|\nabla u_\delta|)\Gamma_\delta^{\frac{s-2}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\,.$$

For handling T_5 we recall the inequality (see the formula after (H3))

$$\xi h'(\xi) \le h(2\xi) \le \bar{k}h(\xi)\,,$$

provided $\xi \ge t_0$. This implies

$$h'(|\nabla u_\delta|) \le \bar{k}\frac{1}{|\nabla u_\delta|}h(|\nabla u_\delta|) \tag{3.4}$$

on $[|\nabla u_\delta| \ge t_0]$, but since in T_5 we only have to consider the set $[\Gamma_\delta \ge \varkappa]$, we can assume the validity of (3.4) for $\varkappa \ge \varkappa(t_0)$ sufficiently large. From (3.4) we therefore get

$$T_5 \le c\int_B h(|\nabla u_\delta|)\eta^2|\nabla^2 u_\delta|\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x = cT_4\,,$$

and this quantity has already been discussed. Returning to (3.3) we see

$$\begin{aligned}
&\int_B \eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x\\
&\le\; c\Bigg[\int_B(\eta^2+|\nabla\eta|^2)h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x + \int_B \eta^2|\nabla^2 u_\delta|^2 h(|\nabla u_\delta|)\Gamma_\delta^{\frac{s-2}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\\
&\qquad + \int_B \eta^2|\nabla u_\delta|^2 h(|\nabla u_\delta|)\Phi'_\varkappa(\Gamma_\delta)|\nabla^2 u_\delta|\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\Bigg] =: c\big[S_1+S_2+S_3\big]\,.
\end{aligned}\tag{3.5}$$

In S_1 and S_2 the domain of integration is the set $B^\varkappa = \{x \in B : \Gamma_\delta(x) > \varkappa\}$, and by enlarging $\varkappa$ (if necessary) we may assume $|\nabla u_\delta| \ge T_0$ on $B^\varkappa$. By (H3) we therefore have on $B^\varkappa$

$$\bar{\varepsilon}|\nabla u_\delta|^{-1}h'(|\nabla u_\delta|)\Gamma_\delta^{-\frac{\mu}{2}} \le h''(|\nabla u_\delta|) \le \bar{h}\Gamma_\delta^{-\frac{\kappa}{2}}h(|\nabla u_\delta|)\,. \tag{3.6}$$

Using $h(t) \le th'(t)$ (recall the second formula after (H3)) we deduce

$$\begin{aligned}
S_2 &\le \int_B \eta^2|\nabla^2 u_\delta|^2|\nabla u_\delta|h'(|\nabla u_\delta|)\Gamma_\delta^{\frac{s-2}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\\
&\le \int_B \eta^2|\nabla^2 u_\delta|^2\frac{h'(|\nabla u_\delta|)}{|\nabla u_\delta|}\Gamma_\delta^{\frac{s}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\\
&\le \int_B \eta^2 D^2H(\nabla u_\delta)(\partial_\alpha\nabla u_\delta,\partial_\alpha\nabla u_\delta)\Gamma_\delta^{\frac{s+\mu}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\,,
\end{aligned}$$

where the last inequality follows from (3.6) and (1.5). In S_3 we actually have to integrate over the set $[\varkappa \le \Gamma_\delta \le 2\varkappa]$, and for this reason we have for κ sufficiently large with constants depending on κ

$$\begin{aligned}
S_3 &\le c\int_{[\varkappa\le\Gamma_\delta\le 2\varkappa]}\eta^2|\nabla^2 u_\delta|\,\mathrm{d}x\\
&\le c\left[1+\int_{[\varkappa\le\Gamma_\delta\le 2\varkappa]}\eta^2|\nabla^2 u_\delta|^2\,\mathrm{d}x\right]\\
&\le c\left[1+\int_{[\varkappa\le\Gamma_\delta\le 2\varkappa]}\eta^2 D^2H(\nabla u_\delta)(\partial_\alpha\nabla u_\delta,\partial_\alpha\nabla u_\delta)\Gamma_\delta^{\frac{s+\mu}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\right]\\
&\le c\left[1+\int_B\eta^2 D^2H(\nabla u_\delta)(\partial_\alpha\nabla u_\delta,\partial_\alpha\nabla u_\delta)\Gamma_\delta^{\frac{s+\mu}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\right].
\end{aligned}$$

Together with (3.5) we have shown that

$$\begin{aligned}
&\int_B\eta^2 h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x\\
&\quad\le c\Bigg[1+\int_B(\eta^2+|\nabla\eta|^2)h(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\\
&\qquad+\int_B D^2H(\nabla u_\delta)(\partial_\alpha\nabla u_\delta,\partial_\alpha\nabla u_\delta)\eta^2\Gamma_\delta^{\frac{s+\mu}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\Bigg].
\end{aligned}\tag{3.7}$$

Let $g_\delta(t):=\delta(1+t^2)^{\tilde{q}/2}$, $G_\delta(Z):=g_\delta(|Z|)$ for $t\ge 0$ and $Z\in\mathbb{R}^{nM}$. Then the same calculations lead to

$$\begin{aligned}
&\int_B\eta^2 g_\delta(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x\\
&\quad\le c\Bigg[1+\int_B(\eta^2+|\nabla\eta|^2)g_\delta(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\\
&\qquad+\int_B D^2G_\delta(\nabla u_\delta)(\partial_\alpha\nabla u_\delta,\partial_\alpha\nabla u_\delta)\eta^2\Gamma_\delta^{\frac{s}{2}}\Phi_\varkappa(\Gamma_\delta)\,\mathrm{d}x\Bigg].
\end{aligned}\tag{3.8}$$

Let us give a comment on (3.8): first we replace g_δ by $\tilde{g}_\delta:=g_\delta(t)-\delta$ in order to have $\tilde{g}_\delta(0)=0$. Then we get (3.5) with h replaced by $\tilde{g}_\delta$. Next we observe

$$c|\nabla u_\delta|^{-1}g_\delta'(|\nabla u_\delta|)\le g_\delta''(|\nabla u_\delta|)\le C\Gamma_\delta^{-1}\tilde{g}_\delta(|\nabla u_\delta|)\tag{3.9}$$

on $B^\varkappa$, hence we obtain (3.8) with $\tilde{g}_\delta$ on the l.h.s. But of course

$$\delta\int_B\eta^2\Phi_\varkappa(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x$$

is bounded from above by

$$\int_B\eta^2 g_\delta(|\nabla u_\delta|)\Phi_\varkappa(|\nabla u_\delta|)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\,,$$

and (3.8) follows.

Now we combine (3.7) and (3.8): by (2.1) we have

$$\begin{aligned}
&\int_B \eta^2 h_\delta(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x \\
&\leq c\Bigg[1+\int_B(\eta^2+|\nabla\eta|^2)h_\delta(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x \\
&\qquad +\int_B D^2H_\delta(\nabla u_\delta)(\partial_\alpha\nabla u_\delta,\partial_\alpha\nabla u_\delta)\eta^2\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+\mu}{2}}\,\mathrm{d}x\Bigg] \\
&\leq c\Bigg[1+\int_B(\eta^2+|\nabla\eta|^2)h_\delta(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x \\
&\qquad +\int_{B^{\kappa/2}}|D^2H_\delta(\nabla u_\delta)||\nabla\eta|^2\Gamma_\delta^{\frac{s+2+\mu}{2}}\,\mathrm{d}x\Bigg],
\end{aligned}$$

hence by (3.6) and (3.9) (combined with $|D^2H_\delta(Z)| \leq |D^2G_\delta(Z)| + |D^2H(Z)| \leq \max\{g_\delta'(|Z|)/|Z|, g_\delta''(|Z|)\} + \max\{h'(|Z|)/|Z|, h''(|Z|)\}$)

$$\begin{aligned}
&\int_B \eta^2 h_\delta(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x \\
&\leq c\Bigg[1+\int_B(\eta^2+|\nabla\eta|^2)h_\delta(|\nabla u_\delta|)\Phi_\varkappa(\Gamma_\delta)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x \\
&\qquad +\int_{B^{\varkappa/2}}|\nabla\eta|^2 g_\delta(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+\mu}{2}}\,\mathrm{d}x+\int_{B^{\varkappa/2}}|\nabla\eta|^2 h(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+2+2\mu-\kappa}{2}}\,\mathrm{d}x\Bigg],
\end{aligned}$$

and since $\varkappa$ is a fixed value we can state:

$$\begin{aligned}
&\int_B \eta^2 g_\delta(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x+\int_B \eta^2 h(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x \\
&\leq c\Bigg[1+\int_B(\eta^2+|\nabla\eta|^2)g_\delta(|\nabla u_\delta|)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x+\int_B(\eta^2+|\nabla\eta|^2)h(|\nabla u_\delta|)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x \\
&\qquad +\int_B|\nabla\eta|^2 g_\delta(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+\mu}{2}}\,\mathrm{d}x+\int_B|\nabla\eta|^2 h(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+2+2\mu-\kappa}{2}}\,\mathrm{d}x\Bigg].
\end{aligned}$$

Let us replace η by η^l for $l \in \mathbb{N}$ large. Recalling $\mu < 2$, $2\mu < \kappa$ we have for suitable α_1, α_2, β_1, β_2

$$\begin{aligned}
\eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^{\frac{s+\mu}{2}} &\leq \tau\eta^{\alpha_1}\Gamma_\delta^{\frac{s+2}{2}}+c(\tau)|\nabla\eta|^{\beta_1} \\
\eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^{\frac{s+2+2\mu-\kappa}{2}} &\leq \tau\eta^{\alpha_2}\Gamma_\delta^{\frac{s+2}{2}}+c(\tau)|\nabla\eta|^{\beta_2}
\end{aligned}$$

so that we can split the last two integrals on the r.h.s. of the above inequality and obtain for $\tau \ll 1$ and l very large

$$\begin{aligned}
&\int_B \eta^{2l} g_\delta(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x + \int_B \eta^{2l} h(|\nabla u_\delta|)\Gamma_\delta^{\frac{s+2}{2}}\,\mathrm{d}x \\
&\quad \leq \; c\left[1 + \int_{\operatorname{spt}\eta} g_\delta(|\nabla u_\delta|)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} h(|\nabla u_\delta|)\Gamma_\delta^{\frac{s}{2}}\,\mathrm{d}x\right]. \qquad (3.10)
\end{aligned}$$

(3.10) is valid for all $s \geq 0$ with c depending on s and on $\|\nabla\eta\|_\infty$. Lemma 2.1 implies for any $\rho < R$

$$\int_{B_\rho} \big[g_\delta(|\nabla u_\delta|) + h(|\nabla u_\delta|)\big]\,\mathrm{d}x \leq c(\rho) < \infty$$

with $c(\rho)$ independent of ε, thus by (3.10)

$$\int_{B_\rho} \Gamma_\delta\big[g_\delta(|\nabla u_\delta|) + h(|\nabla u_\delta|)\big]\,\mathrm{d}x \leq c(\rho) < \infty\,,$$

and iteration of (3.10) shows

$$\int_{B_\rho} \Gamma_\delta^{\frac{s}{2}}\big[g_\delta(|\nabla u_\delta|) + h(|\nabla u_\delta|)\big]\,\mathrm{d}x \leq c(s,\rho) < \infty$$

for any $s \geq 0$. (H1) implies

$$\int_{B_\rho} |\nabla u_\delta|^s\,\mathrm{d}x \leq c(s,\rho) < \infty$$

for any finite s, and by Lemma 2.1 the same is true for ∇u. $\square$

4 Proof of Theorem 1.1

We use the same notation as in the previous section. Let $r < R$ and $\eta \in C_0^\infty(B_r(x_0))$. We further take a number $k > 0$ and define

$$A(k,r) := \{x \in B_r(x_0) :\, \Gamma_\delta > k\}\,,$$

where the dependence of $A(k,r)$ on the parameter $\delta = \delta(\varepsilon)$ is not explicitely stated. In order to prove our claim we apply an appropriate variant of the DeGiorgi technique as it is also done in [Bi], proof of Theorem 5.22. We have (summation w.r.t. $\alpha = 1, \dots, n$)

$$0 = \int_{B_r(x_0)} D^2 H_\delta(\nabla u_\delta)\big(\partial_\alpha \nabla u_\delta, \nabla\big[\eta^2 \partial_\alpha u_\delta \max(\Gamma_\delta - k, 0)\big]\big)\,\mathrm{d}x\,,$$

which follows by differentiating the Euler equation satisfied by u_δ and using the test-vector $\eta^2\partial_\alpha u_\delta \max(\Gamma_\delta - k, 0)$, whose admissibility is guaranteed by Lemma 2.1. It follows

$$\begin{aligned}
&\int_{A(k,r)} D^2H_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta)(\Gamma_\delta - k)\eta^2 \,\mathrm{d}x \\
&\quad + \int_{A(k,r)} D^2H_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u_\delta \otimes \nabla\Gamma_\delta)\eta^2 \,\mathrm{d}x \\
&\qquad = -2\int_{A(k,r)} D^2H_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u_\delta \otimes \nabla\eta)\eta(\Gamma_\delta - k) \,\mathrm{d}x\,.
\end{aligned}$$

We drop the first integral on the l.h.s. and observe that the second term on the l.h.s. is equal to

$$\int_{A(k,r)} a_{\alpha\beta}\partial_\alpha\Gamma_\delta\partial_\beta\Gamma_\delta\eta^2 \,\mathrm{d}x\,,$$

$$a_{\alpha\beta} := \frac{1}{2}\delta_{\alpha\beta}\frac{h_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|} + \frac{1}{2}\Big[h_\delta''(|\nabla u_\delta|) - \frac{h_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|}\Big]\frac{\partial_\alpha u_\delta \cdot \partial_\beta u_\delta}{|\nabla u_\delta|^2}\,. \tag{4.1}$$

Using

$$D^2H_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u_\delta \otimes \nabla\eta) = a_{\alpha\beta}\partial_\alpha\Gamma_\delta\partial_\beta\eta$$

we get the inequality

$$\int_{A(k,r)} \eta^2 a_{\alpha\beta}\partial_\alpha\Gamma_\delta\partial_\beta\Gamma_\delta \,\mathrm{d}x \le -2\int_{A(k,r)} a_{\alpha\beta}\partial_\alpha\Gamma_\delta\partial_\beta\eta\eta(\Gamma_\delta - k)\,\mathrm{d}x\,. \tag{4.2}$$

On the r.h.s. of (4.2) we can apply the Cauchy-Schwarz inequality to the symmetric form induced by $(a_{\alpha\beta}) = (a_{\beta\alpha})$ which in combination with Young's inequality implies the estimate (with c being independent of ε)

$$\int_{A(k,r)} \eta^2 a_{\alpha\beta}\partial_\alpha\Gamma_\delta\partial_\beta\Gamma_\delta \,\mathrm{d}x \le c\int_{A(k,r)} a_{\alpha\beta}\partial_\alpha\eta\partial_\beta\eta(\Gamma_\delta - k)^2\,\mathrm{d}x\,. \tag{4.3}$$

Here we like to remark that the coefficients $a_{\alpha\beta}$ defined in (4.1) satisfy the inequality

$$\begin{aligned}
&\frac{1}{2}\min\Big[\frac{h_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|}, h_\delta''(|\nabla u_\delta|)\Big]|\tau|^2 \\
&\quad \le\ a_{\alpha\beta}\tau_\alpha\tau_\beta\ \le\ \frac{1}{2}\max\Big[\frac{h_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|}, h_\delta''(|\nabla u_\delta|)\Big]|\tau|^2\,,
\end{aligned} \tag{4.4}$$

$\tau \in \mathbb{R}^n$. Now we follow [Bi], proof of Lemma 5.23: let $B_r \subset B_{\hat r} \subset B_R$ (balls with center x_0) and consider $\eta \ge 0$, $n \equiv 1$ on B_r, $\operatorname{spt}\eta \subset B_{\hat r}$, $|\nabla\eta| \le c/(\hat r - r)$. Then we have

$$\int_{A(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}}\,\mathrm{d}x \le c\Big[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\Big]\,, \tag{4.5}$$

where

$$
\begin{aligned}
I_1^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat r)} |\nabla\eta|(\Gamma_\delta - k)\,\mathrm{d}x\right]^{\frac{n}{n-1}} \le c(\hat r - r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} (\Gamma_\delta - k)\,\mathrm{d}x\right]^{\frac{n}{n-1}}, \\
I_2^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat r)} \eta|\nabla\Gamma_\delta|\,\mathrm{d}x\right]^{\frac{n}{n-1}}.
\end{aligned}
$$

Note that (4.5) follows from

$$
\int_{A(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}}\,\mathrm{d}x \le \int_{B_{\hat r}} [\eta(\Gamma_\delta - k)^+]^{\frac{n}{n-1}}\,\mathrm{d}x\,,
$$

if we apply Sobolev's inequality on the r.h.s. We want to use (4.3) in order to estimate I_2. To this purpose we need control on the quantities $\min\{\dots\}$, $\max\{\dots\}$ occuring in (4.4): in I_2 the domain of integration is $B_{\hat r} \cap [\Gamma_\delta > k]$, and we assume that $k \ge k(T_0)$ in order to have (H3) on the relevant set. Then it holds

$$
\begin{aligned}
h_\delta'(|\nabla u_\delta|)|\nabla u_\delta|^{-1} &= g_\delta'(|\nabla u_\delta|)|\nabla u_\delta|^{-1} + h'(|\nabla u_\delta|)|\nabla u_\delta|^{-1} \\
&\le c\big[g_\delta'(|\nabla u_\delta|)|\nabla u_\delta|^{-1} + \Gamma_\delta^{\frac{\mu}{2}} h''(|\nabla u_\delta|)\big] \\
&\le c\big[g_\delta'(|\nabla u_\delta|)|\nabla u_\delta|^{-1} + \Gamma_\delta^{\frac{\mu-\kappa}{2}} h(|\nabla u_\delta|)\big] \\
&\le c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + \Gamma_\delta^{\frac{\mu-\kappa}{2}} h(|\nabla u_\delta|)\big]
\end{aligned}
$$

and

$$
h_\delta''(|\nabla u_\delta|) \le c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + \Gamma_\delta^{-\frac{\kappa}{2}} h(|\nabla u_\delta|)\big]\,,
$$

hence

$$
\max\Big[\frac{h_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|}, h_\delta''(|\nabla u_\delta|)\Big] \le c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + \Gamma_\delta^{\frac{\mu-\kappa}{2}} h(|\nabla u_\delta|)\big] \tag{4.6}
$$

and in the same way

$$
\begin{aligned}
h_\delta'(|\nabla u_\delta|)|\nabla u_\delta|^{-1} &\ge c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + |\nabla u_\delta|^{-1} h'(|\nabla u_\delta|)\big] \\
&\ge c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + \Gamma_\delta^{-1} h(|\nabla u_\delta|)\big]
\end{aligned}
$$

as well as

$$
\begin{aligned}
h_\delta''(|\nabla u_\delta|) &\ge c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + h''(|\nabla u_\delta|)\big] \\
&\ge c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + \Gamma_\delta^{-\frac{\mu}{2}} |\nabla u_\delta|^{-1} h'(|\nabla u_\delta|)\big] \\
&\ge c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + \Gamma_\delta^{-\frac{\mu+2}{2}} h(|\nabla u_\delta|)\big]\,.
\end{aligned}
$$

This gives

$$
\min\Big[\frac{h_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|}, h_\delta''(|\nabla u_\delta|)\Big] \ge c\big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + \Gamma_\delta^{-\frac{\mu+2}{2}} h(|\nabla u_\delta|)\big]\,. \tag{4.7}
$$

We have by Hölder's inequality

$$
\begin{aligned}
I_2^{\frac{n}{n-1}} &= \left[\int_{A(k,\hat{r})} \eta|\nabla\Gamma_\delta| h(|\nabla u_\delta|)^{\frac{1}{2}}\Gamma_\delta^{-\frac{\mu+2}{4}}\Gamma_\delta^{\frac{\mu+2}{4}} h(|\nabla u_\delta|)^{-\frac{1}{2}}\,\mathrm{d}x\right]^{\frac{n}{n-1}} \\
&\le \left[\int_{A(k,\hat{r})} \eta^2|\nabla\Gamma_\delta|^2 h(|\nabla u_\delta|)\Gamma_\delta^{-\frac{\mu+2}{2}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}\left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{\mu+2}{2}} h(|\nabla u_\delta|)^{-1}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}
\end{aligned}
$$

and by (4.3), (4.4), (4.7)

$$
\begin{aligned}
I_2^{\frac{n}{n-1}} &\le \left[\int_{A(k,\hat{r})} \eta^2 a_{\alpha\beta}\partial_\alpha\Gamma_\delta\partial_\beta\Gamma_\delta\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}\left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{\mu+2}{2}} h(|\nabla u_\delta|)^{-1}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}} \\
&\le c\left[\int_{A(k,\hat{r})} a_{\alpha\beta}\partial_\alpha\eta\partial_\beta\eta(\Gamma_\delta-k)^2\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}\left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{\mu+2}{2}} h(|\nabla u_\delta|)^{-1}\right]^{\frac{1}{2}\frac{n}{n-1}}.
\end{aligned}
$$

(4.4) and (4.6) give

$$
\begin{aligned}
I_2^{\frac{n}{n-1}} \le\ & c(\hat{r}-r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat{r})} (\Gamma_\delta-k)^2\big[\Gamma_\delta^{-1}g_\delta(|\nabla u_\delta|)+\Gamma_\delta^{\frac{\mu-\kappa}{2}}h(|\nabla u_\delta|)\big]\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}} \\
&\left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{\mu+2}{2}} h(|\nabla u_\delta|)^{-1}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}.
\end{aligned} \tag{4.8}
$$

Again from Hölder's inequality we get

$$
\begin{aligned}
I_1^{\frac{n}{n-1}} &\le c(\hat{r}-r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat{r})} (\Gamma_\delta-k)\,\mathrm{d}x\right]^{\frac{n}{n-1}} \\
&\le c(\hat{r}-r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat{r})} (\Gamma_\delta-k)^2\big[\Gamma_\delta^{-1}g_\delta(|\nabla u_\delta|)+\Gamma_\delta^{\frac{\mu-\kappa}{2}}h(|\nabla u_\delta|)\big]\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}} \\
&\quad\left[\int_{A(k,\hat{r})} \big[\Gamma_\delta^{-1}g_\delta(|\nabla u_\delta|)+\Gamma_\delta^{\frac{\mu-\kappa}{2}}h(|\nabla u_\delta|)\big]^{-1}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}.
\end{aligned}
$$

Clearly

$$
\big[\Gamma_\delta^{-1}g_\delta(|\nabla u_\delta|)+\Gamma_\delta^{\frac{\mu-\kappa}{2}}h(|\nabla u_\delta|)\big]^{-1} \le \Gamma_\delta^{\frac{\kappa-\mu}{2}}h(|\nabla u_\delta|)^{-1} \le \Gamma_\delta^{\frac{\mu+2}{2}}h(|\nabla u_\delta|)^{-1},
$$

which follows from $\kappa \leq \mu + 2$. Thus the r.h.s. of (4.8) also is an upper bound for $I_1^{n/(n-1)}$ and returning to (4.5) it is shown that

$$\begin{aligned}
&\int_{A(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}}\,\mathrm{d}x \\
&\leq \; c(\hat{r} - r)^{-\frac{n}{n-1}} \left[\int_{A(k,\hat{r})} (\Gamma_\delta - k)^2 \big[\Gamma_\delta^{-1} g_\delta(|\nabla u_\delta|) + \Gamma_\delta^{\frac{\mu-\kappa}{2}} h(|\nabla u_\delta|)\big]\,\mathrm{d}x \right]^{\frac{1}{2}\frac{n}{n-1}} \\
&\quad \left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{\mu+2}{2}} h(|\nabla u_\delta|)^{-1}\,\mathrm{d}x \right]^{\frac{1}{2}\frac{n}{n-1}} . \qquad (4.9)
\end{aligned}$$

We recall (1.4) and choose an exponent $q^* > 1$ such that

$$(1+t^2)^{-1} g_\delta(t) + (1+t^2)^{\frac{\mu-\kappa}{2}} h(t) \leq C(1+t^2)^{\frac{q^*-2}{2}} \qquad (4.10)$$

for all $t \geq 1$, C being independent of ε. At the same time we have $h(t) \geq Ct$ for all $t \geq 1$, for another constant, hence there exists an exponent μ^* such that

$$h(t)^{-1}(1+t^2)^{\frac{\mu+2}{2}} \leq C(1+t^2)^{\frac{\mu^*}{2}} \qquad (4.11)$$

for all $t \geq 1$. If we use (4.10) and (4.11) in (4.9), then (4.9) exactly takes the form of inequality (24) in Lemma 5.23 of [Bi] (with q, μ replaced by q^*, μ^*). Then – without further changes – we can follow the calculations from p.158 of [Bi] (using Section 3) to get uniform local boundedness of ∇u_δ which completes the proof. □

Appendix. An example of a function h satisfying (H1)–(H3) with $2\mu < \kappa$ and for which "(p, s)-ellipticity" with $s < p + 2$ does not hold

It remains to give an explicit construction of an energy density h as indicated in Remark 1.1. The idea is that we have piecewise the "usual" relations $h \approx th' \approx t^2 h''$ s.t. Theorem 1.1 applies. On the other hand, h does not satisfy a global uniform power growth estimate and it is not possible to find uniform global exponents s.t. results similar to [Bi], Section 5.2, apply.

Suppose that $r > 2$ and let

$$[0,\infty) = \bigcup_{i=1}^{\infty} I_i\,, \qquad I_i := [a_{i-1}, a_i]\,,$$

where $0 = a_0 < a_1 < \dots < a_{i-1} < a_i < \dots$ and where $a_i \ll a_{i+1}$. We define the function

$$g(t) := \begin{cases} c_i t & \text{on } \ I_i\,, \quad \text{if } i \text{ is odd}\,, \\ c_i t^{r-1} & \text{on } \ I_i\,, \quad \text{if } i \text{ is even}\,, \end{cases}$$

where $c_1 := 1$ and where $c_i > 0$, $i \geq 2$, are such that $g \in C^0([0,\infty))$. Finally we let

$$h(t) := \int_0^t g(s)\,\mathrm{d}s$$

being of class $C^{1,1}([0,A])$ for any $A > 0$. In order to have "$h \in C^2$" it will be necessary to replace h by its *local* mollification around points a_i with small radii depending on i.

The validity of (H1) for h is immediate. For (H2) we observe that by the definition of g

$$\tilde{g}: \ t \mapsto \frac{g(t)}{t^{r-1}} \qquad (t > 0)$$

is decreasing on each intervall I_i, and since g is continuous, $\tilde{g}$ is decreasing on $(0,\infty)$, in particular

$$\tilde{g}(2t) \leq \tilde{g}(t) \quad \text{ for all } \ t > 0\,,$$

hence $g(2t) \leq 2^{r-1} g(t)$. This implies

$$h(2t) = \int_0^{2t} g(s)\,\mathrm{d}s = \int_0^t 2g(2s)\,\mathrm{d}s \leq 2^r \int_0^t g(s)\,\mathrm{d}s = 2^r h(t)\,,$$

and we have (H2) for h.

For $t \neq a_i$ it holds

$$\frac{1}{t} h'(t) = \begin{cases} h''(t) & \text{on } \ I_i\,, \quad \text{if } i \text{ is odd}\,, \\ h''(t)/(r-1) & \text{on } \ I_i\,, \quad \text{if } i \text{ is even}\,, \end{cases}$$

so that ($\widetilde{\mathrm{H3}}$) is true a.e. with $\mu = 0$ and $\kappa = 2$. Thus the growth of $D^2H(Z)$ is exactly measured in terms of $h'(t)/t$, $t = |Z|$.

From (H1), (H2) we deduce as usual

$$cth'(t) \leq h(t) \leq Cth'(t)\,,$$

hence by (1.5)

$$\tilde{c}\frac{h(|Z|)}{|Z|^2}|Y|^2 \leq D^2H(Z)(Y,Y) \leq \tilde{C}\frac{h(|Z|)}{|Z|^2}|Y|^2 \tag{A.1}$$

for all Y, $Z \in \mathbb{R}^{nM}$, $|Z| \geq 1$, $|Z| \neq a_i$.

Let us fix a very small number $\delta > 0$. We claim that the sequence $\{a_i\}$ can be chosen in such a way that

$$\frac{h(\tilde{a}_i)}{\tilde{a}_i^{r-\delta}} \geq i\,, \quad \text{for even } i\,, \tag{A.2}$$

$$\frac{h(\tilde{a}_i)}{\tilde{a}_i^{2+\delta}} \leq \frac{1}{i}\,, \quad \text{for odd } i\,. \tag{A.3}$$

Here we pass from a_i to $\tilde{a}_i := a_i/2 \gg a_{i-1}$ in order to avoid difficulties during the mentioned smoothing procedure. Recalling $c_1 = 1$ we proceed by induction:

$$h(\tilde{a}_1) = \int_0^{\tilde{a}_1} c_1 t \, \mathrm{d}t = \frac{\tilde{a}_1^2}{2} \quad \Rightarrow \quad \frac{h(\tilde{a}_1)}{\tilde{a}_1^{2+\delta}} = \frac{1}{2}\tilde{a}_1^{-\delta} \leq 1\,,$$

if $\tilde{a}_1 \geq (1/2)^{1/\delta}$,

$$h(\tilde{a}_2) = h(a_1) + \int_{a_1}^{\tilde{a}_2} c_2 t^{r-1} \, \mathrm{d}t =: K_1(a_1) + K_2 \tilde{a}_2^r \quad \Rightarrow \quad \frac{h(\tilde{a}_2)}{\tilde{a}_2^{r-\delta}} = K_1 \tilde{a}_2^{\delta - r} + K_2 \tilde{a}_2^{\delta} \geq 2\,,$$

if $a_2 \gg 1$. Thus we have (A.2) and (A.3) for $i = 1, 2$. The inductive step is the same calculation.

We now claim

$$\begin{gathered}\text{there exist } \lambda,\, \Lambda > 0 \text{ s.t. for all } |Z| \geq 1,\, |Z| \neq a_i, \\ \lambda \leq D^2 H(Z) \leq \Lambda |Z|^{r-2}\end{gathered} \tag{A.4}$$

and

$$\begin{gathered}\text{there } \textbf{do not} \text{ exist } \bar{\lambda},\, \bar{\Lambda} > 0 \text{ s.t. for all } |Z| \geq 1,\, |Z| \neq a_i, \\ \bar{\lambda} |Z|^{\delta} \leq D^2 H(Z) \leq \bar{\Lambda} |Z|^{r-2-\delta}\,.\end{gathered} \tag{A.5}$$

Inequality (A.4) shows that H is of anisotropic $(2, r)$-growth, and (A.5) means that the upper and lower growth rates are nearly optimal.

ad (A.4): we showed that $t \mapsto h'(t)/t^{r-1}$ is decreasing, in particular

$$\frac{h'(t)}{t^{r-1}} \leq h'(1) \text{ for all } t \geq 1 \quad \Rightarrow \quad \frac{h'(t)}{t} \leq const\, t^{r-2} \text{ for all } t \geq 1\,,$$

and since $D^2 H(Z)$ behaves like $h'(|Z|)/|Z|$, the second inequality of (A.4) follows. Note that $t \mapsto h'(t)/t$ increases (immediate from the definition of g), hence $h'(t)/t \geq h'(1) = 1$ for $t \geq 1$, which gives the first part of (A.4).

ad (A.5): suppose that we can find $\bar{\lambda}$, $\bar{\Lambda}$ s.t. the estimates hold. From (A.1) it then follows

$$\tilde{\lambda} t^{\delta} \leq \frac{1}{t^2} h(t) \leq \tilde{\Lambda} t^{r-2-\delta}$$

for all $t > 1$ with $\tilde{\lambda}$, $\tilde{\Lambda} > 0$, in particular

$$\frac{h(\tilde{a}_i)}{\tilde{a}_i^{2+\delta}} \geq \tilde{\lambda}\,, \qquad \frac{h(\tilde{a}_i)}{\tilde{a}_i^{r-\delta}} \leq \tilde{\Lambda}$$

for all i, which is in contradiction to (A.1) and (A.2).

Finally we observe that r can be chosen arbitrary large, which means that the condition "$s < p+2$" implying the regularity of bounded local minima of (p, s)-elliptic integrals is violated. On the contrary, the hypotheses of Theorem 1.1 (with "$2\mu < \kappa$") are clearly satisfied, and we can deduce the regularity of local minimizers $u \in L^{\infty}_{loc}(\Omega; \mathbb{R}^M)$.

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