Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 207

The elastic-plastic torsion problem: a posteriori error estimates for approximate solutions

Michael Bildhauer, Martin Fuchs and Sergey Repin

Saarbrücken 2008

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 207
submitted: 19.03.2008

The elastic-plastic torsion problem: a posteriori error estimates for approximate solutions

Michael Bildhauer
Saarland University
Department of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
bibi@math.uni-sb.de

Martin Fuchs
Saarland University
Department of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Sergey Repin
V.A. Steklov Math. Inst.
St. Petersburg Branch
Fontanka 27
191011 St. Petersburg
Russia
repin@pdmi.ras.ru

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

The elastic-plastic torsion problem: a posteriori error estimates for approximate solutions

M. Bildhauer M. Fuchs S. Repin

AMS Subject Classification: 74 G 65, 74 C 05, 65 N 15, 65 K 10

Keywords: a posteriori error estimates, elastic-plastic torsion, deformation of bars, variational inequalities

Abstract

We consider the variational inequality, which describes the torsion problem for a long elasto-plastic bar. Using duality methods of the variational calculus, we derive a posteriori estimates of functional type that provide computable and guaranteed upper bounds of the energy norm of the difference between the exact solution and any function from the corresponding energy space that satisfies the Dirichlet boundary condition.

1 Introduction

Let Ω denote a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. We consider the classical torsion problem for a long elasto-plastic bar whose cross-section is the domain Ω. If the bar is made of an isotropic material, then the torsion problem is reduced to the following variational inequality with gradient constraint: find a function $u \in \mathbb{K}$ s.t.

$$\int_\Omega \nabla u \cdot \nabla(v-u)\,dx \geq \mu \int_\Omega (v-u)\,dx \quad \forall\, v \in \mathbb{K}, \tag{1.1}$$

where μ denotes a positive parameter and

$$\mathbb{K} := \{v \in V_0 : |\nabla v| \leq 1 \text{ a.e. on } \Omega\},$$

V_0 being the Sobolev space $\overset{\circ}{W}{}^1_2(\Omega)$ of functions with square-integrable weak derivatives and zero trace (see, e.g. [Ad]). Clearly (1.1) is equivalent to the minimization problem

$$J[w] := \int_\Omega \left[\frac{1}{2}|\nabla w|^2 - \mu w\right] dx \to \min \text{ in } \mathbb{K}\,. \tag{1.2}$$

For an explanation of the physical background and for the derivation of (1.1), (1.2) we refer the reader to the monograph [DL] of Duvaut and Lions. We also refer to the book [ET] Chapter IV, 3.4, of Ekeland and Temam. The analytical properties of the solution are discussed for example in the papers of Brezis [Br], of Brezis and Stampacchia [BS], of Caffarelli and Riviere [CR] and of Ting [Ti].

A complete regularity theory leading to the local boundedness of the second generalized derivatives of the solution of (1.1) is presented by Friedman in his book [Fr], where it is also outlined that (1.1) can be transformed into a variational inequality just requirering a pointwise constraint for the function itself. Numerical aspects of problem (1.1) can be found for instance in the books of Glowinski [Gl] and of Glowinski, Lions and Trémolierès [GLT].

In our paper we concentrate on a posteriori error estimates of functional type for the problem (1.1): if $u \in \mathbb{K}$ denotes the unique solution of (1.1) and if $v \in \mathbb{K}$ is arbitrary, then we want to establish an estimate of the form ($\|\cdot\|_{L^2}$ denoting the usual norm in the Lebesgue space of square-integrable scalar- or vector-valued functions)

$$\|\nabla u - \nabla v\|_{L^2} \leq \mathcal{M}(v, \mathcal{D}), \tag{1.3}$$

where $\mathcal{D}$ stands for the set of known data (e.g. Ω, μ) and where $\mathcal{M}$ is a non-negative functional depending on these data, on v and additionally depending on parameters, which are under our disposal. $\mathcal{M}$ should satisfy the following requirements:

i.) $\mathcal{M}(v, \cdot)$ is easy to calculate for any approximation v;

ii.) estimate (1.3) is consistent in the sense that

$$\begin{cases} \mathcal{M}(v, \mathcal{D}) = 0 & \text{if and only if } \ v = u; \\ \mathcal{M}(v_k, \mathcal{D}) \to 0 & \text{if } \ \|\nabla v_k - \nabla u\|_{L^2} \to 0; \end{cases}$$

iii.) $\mathcal{M}$ provides a realistic upper bound for the deviation $\|\nabla u - \nabla v\|_{L^2}$.

Here iii) means that for obtaining (1.3) one carefully tries to avoid overestimation so that (1.3) can be used for a reliable verification of approximative solutions obtained by various numerical methods.
Estimate (1.3) does not refer to a concrete numerical scheme - its derivation is based on purely functional grounds such as basic principles of duality theory. We like to mention that functional a posteriori error estimates have been established for various problems in continuum and fluid mechanics (see the book [NR], the recent overview paper [Re3], and the papers [BFR1], [Re2] devoted to important classes of viscous flow problems). Also we note that such estimates have been derived for other problems related to variational inequalities in [BF], [BFR2], [Re1] and [Re4].

Our paper is organized as follows: in Section 2 we collect some preliminary material mainly concerned with a suitable perturbation of the variational inequality (1.1) and its dual variant. Section 3 contains error estimates of the form (1.3) for approximations v from the class $\mathbb{K}$, and in Section 4 we briefly indicate the situation for more general approximations.

2 Perturbation of the variational inequality

Let $X := L^2(\Omega;\mathbb{R}^2)$. Then the functional J defined in (1.2) can be written as

$$\begin{aligned} J[w] &= \sup_{\tau^*\in X} \int_\Omega \big[\nabla w\cdot\tau^* - \frac{1}{2}|\tau^*|^2 - \mu w\big]\,dx \\ &=: \sup_{\tau^*\in X} \ell(w,\tau^*), \end{aligned}$$

and if we introduce the problem dual to (1.2)

$$J^*[\tau^*] := \inf_{w\in\mathbb{K}} \ell(w,\tau^*) \to \max \text{ on } X, \tag{2.1}$$

then standard arguments from duality theory (see, e.g., [ET]) imply

$$J[u] = J^*[\sigma^*], \tag{2.2}$$

u and σ^* denoting the unique solutions of (1.2) and (2.1), respectively. Let $\Lambda := \{\rho \in L^2(\Omega) : \rho \geq 0 \text{ a.e. on } \Omega\}$ and choose $\rho\in\Lambda$, $\xi^*\in X$ s.t. $|\xi^*|\leq 1$ a.e. on Ω. W.r.t. these parameters we consider the following perturbation of (1.2)

$$J_{\rho,\xi^*}[w] := J[w] - \int_\Omega \rho(1-\xi^*\cdot\nabla w)\,dx \to \min \text{ in } V_0 \tag{2.3}$$

with unique solution $u_{\rho,\xi^*}\in V_0$. The relation between the functionals J and J_{ρ,ξ^*} is the following:

$$\sup_{\rho\in\Lambda,|\xi^*|\leq 1} J_{\rho,\xi^*}[w] = \begin{cases} J[w], & \text{if } w\in\mathbb{K} \\ +\infty, & \text{if } w\notin\mathbb{K}. \end{cases} \tag{2.4}$$

In fact, to prove (2.4) we note that for $w\in V_0$

$$\sup_{\rho\in\Lambda,|\xi^*|\leq 1} J_{\rho,\xi^*}[w] = J[w] - \inf_{\rho\in\Lambda,|\xi^*|\leq 1} \int_\Omega \rho(1-\xi^*\cdot\nabla w)\,dx,$$

and if $|\nabla w| > 1$ on a set $E\subset\Omega$ with $|E|>0$ ($|E| :=$ Lebesgue measure of E), we let

$$\xi^* := \begin{cases} 0 & \text{on } \Omega - E \\ \nabla w/|\nabla w| & \text{on } E \end{cases}$$

so that $1-\xi^*\cdot\nabla w = 1-|\nabla w| < 0$ on E. This implies

$$\inf_{\rho\in\Lambda} \int_\Omega \rho(1-\xi^*\cdot\nabla w)\,dx = -\infty.$$

On the other hand, if $w \in \mathbb{K}$, then $1 - \xi^* \cdot \nabla w \geq 0$ for all $\xi^* \in X, |\xi^*| \leq 1$, and therefore

$$\inf_{\rho \in \Lambda, |\xi^*| \leq 1} \int_\Omega \rho(1 - \xi^* \cdot \nabla w)\, dx = 0,$$

which finally gives (2.4).
Letting

$$L(w, \tau^*, \rho, \xi^*) := \ell(w, \tau^*) - \int_\Omega \rho(1 - \xi^* \cdot \nabla w)\, dx,$$

$w \in V_0$, $\rho \in \Lambda$, τ^*, $\xi^* \in X$, $|\xi^*| \leq 1$, the problem dual to (2.3) takes the form

$$J^*_{\rho,\xi^*}[\tau^*] := \inf_{w \in V_0} L(w, \tau^*, \rho, \xi^*) \to \max \text{ in } X, \tag{2.5}$$

and (2.5) can be seen as a perturbed variant of (2.1). If we denote by $\sigma^*_{\rho,\xi^*} \in X$ the unique solution of (2.5), then we have in extension of (2.2)

$$J_{\rho,\xi^*}[u_{\rho,\xi^*}] = J^*_{\rho,\xi^*}[\sigma^*_{\rho,\xi^*}]. \tag{2.6}$$

We further note that

$$J_{\rho,\xi^*}[u_{\rho,\xi^*}] \leq J[u] \tag{2.7}$$

holds. This follows from

$$\begin{aligned} \inf_{v \in V_0} J_{\rho,\xi^*}[v] &\leq \inf_{w \in \mathbb{K}} J_{\rho,\xi^*}[w] \\ &= \inf_{w \in \mathbb{K}} \Big[J[w] - \int_\Omega \underbrace{\rho(1 - \xi^* \cdot \nabla w)}_{\geq 0}\, dx \Big] \leq \inf_{w \in \mathbb{K}} J[w]. \end{aligned}$$

Finally we consider $\tau^* \in X$ such that $J^*_{\rho,\xi^*}[\tau^*] > -\infty$. Then we must have

$$\int_\Omega \Big[\nabla w \cdot \tau^* - \mu w + \rho \xi^* \cdot \nabla w \Big]\, dx = 0 \quad \forall\, w \in V_0, \tag{2.8}$$

and we denote with Q^*_{ρ,ξ^*} the set of all $\tau^* \in X$ for which (2.8) is satisfied.

3 Error estimates for approximations from the constrained class $\mathbb{K}$

We begin with the following result

Theorem 3.1. *Let $u \in \mathbb{K}$ denote the unique solution of problem (1.1). Then, for any $v \in \mathbb{K}$, for all $\eta^* \in X$, and for all $\rho \in \Lambda$, $\xi^* \in X$, $|\xi^*| \leq 1$, it holds*

$$\begin{aligned} \big\| \nabla u - \nabla v \big\|^2_{L^2} \;\leq\; & 2\Big[D\big[v, \eta^*\big] + d_{\rho,\xi^*}(\eta^*) \Big[\frac{1}{2} d_{\rho,\xi^*}(\eta^*) + \|\nabla v - \eta^*\|_{L^2} \Big] \\ & + \int_\Omega \rho(1 - \xi^* \cdot \nabla v)\, dx \Big]. \end{aligned} \tag{3.1}$$

Here we have abbreviated:

$$\begin{aligned} D[w,\eta^*] &:= \int_\Omega \Big[\frac{1}{2}|\nabla w|^2 + \frac{1}{2}|\eta^*|^2 - \nabla w\cdot\eta^*\Big]\,dx,\\ d_{\rho,\xi^*}(\eta^*) &:= \inf\nolimits_{\tau^*\in Q^*_{\rho,\xi^*}} \|\eta^*-\tau^*\|_{L^2}, \quad w\in V_0, \quad \eta^*\in X. \end{aligned}$$

Remark 3.1. *Clearly all the terms appearing on the r.h.s. of (3.1) are non-negative. Suppose now that the r.h.s. of (3.1) vanishes. Then we must have the basic relations*

$$\eta^* = \nabla v, \quad \eta^*\in Q^*_{\rho,\xi^*}, \quad \rho(1-\xi^*\cdot\nabla v) = 0,$$

which are necessary and sufficient conditions that define the exact solution of the torsion problem and also its gradient. In fact, it is easy to see that the validity of the above relations implies that v is the solution of the variational inequality (1.1): for any $w\in\mathbb{K}$ we get

$$\begin{aligned} \int_\Omega \nabla v\cdot\nabla(w-v)\,dx &= \int_\Omega \eta^*\cdot\nabla(w-v)\,dx\\ &= \int_\Omega \mu(w-v)\,dx - \int_\Omega \rho\xi^*\cdot\nabla(w-v)\,dx\\ &= \int_\Omega \mu(w-v)\,dx - \int_\Omega \rho\xi^*\cdot\nabla w\,dx + \int_\Omega \rho\xi^*\cdot\nabla v\,dx\\ &= \int_\Omega \mu(w-v)\,dx - \int_\Omega \rho\xi^*\cdot\nabla w\,dx + \int_\Omega \rho\,dx\\ &\geq \int_\Omega \mu(w-v)\,dx, \end{aligned}$$

since $\rho - \rho\xi^*\cdot\nabla w \geq 0$. Thus, v solves (1.1) and therefore $v = u$.

Proof of Theorem 3.1. Let us fix $v\in\mathbb{K}, \rho\in\Lambda$ and $\xi^*\in X, |\xi^*|\leq 1$. Then we have using (1.1)

$$\begin{aligned} J[v]-J[u] &= \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \mu(v-u)\,dx\\ &= \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \nabla u\cdot(\nabla v-\nabla u)\,dx\\ &\quad + \int_\Omega \nabla u\cdot(\nabla v-\nabla u)\,dx - \int_\Omega \mu(v-u)\,dx\\ &\geq \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \nabla u\cdot(\nabla v-\nabla u)\,dx\\ &= \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \nabla u\cdot\nabla v\,dx\\ &= \frac{1}{2}\int_\Omega |\nabla u-\nabla v|^2\,dx, \end{aligned}$$

i.e. by (2.7) and (2.6)

$$\begin{aligned}\frac{1}{2}\|\nabla u-\nabla v\|_{L^2}^2 &\le J[v]-J[u]\\ &\le \Big[J[v]-J_{\rho,\xi^*}[u_{\rho,\xi^*}]\Big]\\ &= 2\Big[J[v]-J^*_{\rho,\xi^*}[\sigma^*_{\rho,\xi^*}]\Big].\end{aligned}$$

Hence we obtain for $\tau^*\in X^*$

$$\|\nabla u-\nabla v\|_{L^2}^2 \le 2\Big[J[v]-J^*_{\rho,\xi^*}[\tau^*]\Big]. \tag{3.2}$$

If $\tau^*\in Q^*_{\rho,\xi^*}$, then we must have by (2.8)

$$\begin{aligned}J[v]-J^*_{\rho,\xi^*}[\tau^*] &= \int_\Omega \Big[\frac{1}{2}|\nabla v|^2-\mu v+\frac{1}{2}|\tau^*|^2+\rho\Big]\,dx\\ &= \int_\Omega \Big[\frac{1}{2}|\nabla v|^2+\frac{1}{2}|\tau^*|^2-\tau^*\cdot\nabla v\Big]\,dx\\ &\quad+\int_\Omega \Big[\rho+\tau^*\cdot\nabla v-\mu v\Big]\,dx\\ &= D[v,\tau^*]+\int_\Omega \rho(1-\tau^*\cdot\nabla v)\,dx,\end{aligned}$$

and (3.2) turns into

$$\|\nabla u-\nabla v\|_{L_2}^2 \le 2\Big[D[v,\tau^*]+\int_\Omega \rho(1-\xi^*\cdot\nabla v)\,dx\Big] \tag{3.3}$$

valid for all $v\in\mathbb{K}$, $\tau^*\in Q^*_{\rho,\xi^*}$, $\rho\in\Lambda$, $\xi^*\in X$, $|\xi^*|\le 1$. Obviously (3.3) is the desired inequality (3.1) if we choose $\eta^*\in Q^*_{\rho,\xi^*}$.
Next let v, ρ, ξ^* and τ^* as in (3.3) and consider $\eta^*\in X$. (3.3) implies

$$\begin{aligned}\big\|\nabla u-\nabla v\big\|_{L^2}^2 &\le 2D[v,\eta^*]+2\int_\Omega \Big[\frac{1}{2}|\tau^*|^2-\frac{1}{2}|\eta^*|^2-\nabla v\cdot(\tau^*-\eta^*)\Big]\,dx\\ &\quad +2\int_\Omega \rho(1-\xi^*\cdot\nabla v)\,dx,\end{aligned} \tag{3.4}$$

moreover we have for a, b, $c\in\mathbb{R}^2$

$$\frac{1}{2}|a|^2-\frac{1}{2}|b|^2-c\cdot(a-b)=\frac{1}{2}|a-b|^2+(a-b)\cdot(b-c),$$

so that by Hölder's inequality

$$\begin{aligned}&\int_\Omega \Big[\frac{1}{2}|\tau^*|^2-\frac{1}{2}|\eta^*|^2-\nabla v\cdot(\tau^*-\eta^*)\Big]\,dx\\ &\quad\le \|\tau^*-\eta^*\|_{L^2}\Big[\frac{1}{2}\|\tau^*-\eta^*\|_{L^2}+\|\eta^*-\nabla v\|_{L^2}\Big].\end{aligned}$$

Inserting this inequality into (3.4) and passing to the infimum w.r.t. $\tau^* \in Q^*_{\rho,\xi^*}$, estimate (3.1) is established. $\square$

Now we derive a bound for the quantity $d_{\rho,\xi^*}(\eta^*)$ assuming that $\eta^* \in X$ is such that $\operatorname{div}\eta^*$ belongs to the space $L^2(\Omega)$. Then

$$\inf_{\tau^* \in Q^*_{\rho,\xi^*}} \frac{1}{2}\|\eta^* - \tau^*\|^2_{L^2} = -\sup_{\varkappa^* \in \overline{Q^*_{\rho,\xi^*}}} \Big[-\frac{1}{2}\|\varkappa^*\|^2_{L^2}\Big],$$

where

$$\overline{Q^*_{\rho,\xi^*}} := \Big\{\varkappa^* \in X : \int_\Omega \Big[\nabla w \cdot \varkappa^* - (\mu + \operatorname{div}\eta^*)w + \rho\xi^* \cdot \nabla w\Big]\,dx = 0 \quad \forall\, w \in V_0\Big\},$$

and by duality we obtain

$$\begin{aligned}
&\sup_{\varkappa^* \in \overline{Q^*_{\rho,\xi^*}}} \Big[-\frac{1}{2}\|\varkappa^*\|^2_{L^2}\Big] \\
&= \inf_{w \in V_0} \Big[\int_\Omega \frac{1}{2}|\nabla w|^2\,dx + \int_\Omega \rho\xi^* \cdot \nabla w\,dx - \int_\Omega (\mu + \operatorname{div}\eta^*)w\,dx\Big] \\
&= \inf_{w \in V_0} \Big[\frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_\Omega w\Big(\mu + \operatorname{div}\eta^* + \operatorname{div}(\rho\xi^*)\Big)\,dx\Big],
\end{aligned}$$

where in the last step we assume that $\operatorname{div}(\rho\xi^*)$ is an element of $L^2(\Omega)$. In this case we get

$$\begin{aligned}
\inf_{w \in V_0}\Big[\dots\Big] &\geq \inf_{w \in V_0}\Big[\frac{1}{2}\|\nabla w\|^2_{L^2} - \|w\|_{L^2}\|\mu + \operatorname{div}(\eta^* + \rho\xi^*)\|_{L^2}\Big] \\
&\geq \inf_{w \in V_0}\Big[\frac{1}{2}\|\nabla w\|^2_{L^2} - C(\Omega)\|\nabla w\|_{L^2}\|\mu + \operatorname{div}(\eta^* + \rho\xi^*)\|_{L^2}\Big] \\
&= \inf_{t \geq 0}\Big[\frac{1}{2}t^2 - C(\Omega)t\|\mu + \operatorname{div}(\eta^* + \rho\xi^*)\|_{L^2}\Big],
\end{aligned}$$

where $C(\Omega)$ is the constant from Friedrichs' inequality, i.e. we have

$$\|v\|_{L^2} \leq C(\Omega)\|\nabla v\|_{L^2} \quad \forall\, v \in V_0$$

for a positive constant depending on the domain. If for example Ω is contained in a disc $B_R(x_0)$, then according to [Mo], Theorem 3.2.1, it holds $C(\Omega)^2 \leq \frac{1}{2}R^2$. Alternatively we may quote [GT], (7.44), to see $C(\Omega)^2 \leq \frac{1}{\pi}|\Omega|$. The value of $\inf_{t\geq 0}[\dots]$ is attained at $t_0 := C(\Omega)\|\mu + \operatorname{div}(\eta^* + \rho\xi^*)\|_{L^2}$, thus

$$\inf_{t \geq 0}\Big[\dots\Big] = -\frac{1}{2}C(\Omega)^2\|\mu + \operatorname{div}(\eta^* + \rho\xi^*)\|^2_{L^2},$$

and we have shown

Theorem 3.2. *Suppose that we are given $\rho \in \Lambda$, $\xi^* \in X$, $|\xi^*| \leq 1$, with the property* $\operatorname{div}(\rho\xi^*) \in L^2(\Omega)$. *Then, if $\eta^* \in X$ satisfies $\operatorname{div}\eta^* \in L^2(\Omega)$, we have the estimate*

$$\inf_{\tau^* \in Q_{\rho,\xi^*}} \|\eta^* - \tau^*\|_{L^2} \leq C(\Omega)\|\mu + \operatorname{div}(\eta^* + \rho\xi^*)\|_{L^2}, \tag{3.5}$$

and in inequality (3.1) we can replace the quantity $d_{\rho,\xi^*}(\eta^*)$ by the r.h.s. of (3.5).

Let us a look at a special case: suppose that we are given $v \in \mathbb{K}$. We let

$$\xi^* := \eta^* := \nabla v$$

and select $\rho \in \Lambda$ s.t. $\rho \equiv 0$ on $\big[|\nabla v| < 1\big]$. From (3.1) and (3.5) we get

$$\|\nabla u - \nabla v\|_{L^2}^2 \leq 4C(\Omega)^2 \|\mu + \operatorname{div}(\nabla v + \rho\nabla v)\|_{L^2}^2. \tag{3.6}$$

Suppose that the r.h.s. of (3.6) vanishes, i.e.

$$\mu = -\operatorname{div}(\nabla v + \rho\nabla v).$$

Then we have for all $w \in \mathbb{K}$ (recall $\rho \geq 0$)

$$\begin{aligned}
\int_\Omega \nabla v \cdot \nabla(w - v)\,dx &= -\int_\Omega \operatorname{div}\nabla v(w-v)\,dx \\
&= \int_\Omega \mu(w-v)\,dx + \int_\Omega \operatorname{div}(\rho\nabla v)(w-v)\,dx \\
&= \int_\Omega \mu(w-v)\,dx - \int_\Omega \rho\nabla v \cdot (\nabla w - \nabla v)\,dx \\
&= \int_\Omega \mu(w-v)\,dx + \int_{[|\nabla v|=1]} \rho(1 - \nabla v \cdot \nabla w)\,dx \\
&\geq \int_\Omega \mu(w-v)\,dx,
\end{aligned}$$

which immediately implies $v = u$.

4 Error estimates for more general approximations

Let $w \in V_0$ such that $|\nabla w| \in L^\infty(\Omega)$. Then

$$\bar{v} := \min(1, 1/\|\nabla w\|_{L^\infty})w$$

belongs to the class $\mathbb{K}$ and therefore

$$\begin{aligned}
\inf_{v\in\mathbb{K}} \|\nabla v - \nabla w\|_{L^2}^2 &\leq \|\nabla\bar{v} - \nabla w\|_{L^2}^2 \\
&= \Big[\min\big(1, 1/\|\nabla w\|_{L^\infty}\big) - 1\Big]^2 \|\nabla w\|_{L^2}^2 \\
&= \left\{\begin{array}{cl} 0, & \text{if } \|\nabla w\|_{L^\infty} \leq 1 \\ \big[\|\nabla w\|_{L^\infty} - 1\big]^2, & \text{if } \|\nabla w\|_{L^\infty} > 1 \end{array}\right\} \frac{\|\nabla w\|_{L^2}^2}{\|\nabla w\|_{L^\infty}^2} \\
&\leq \Theta\big(\|\nabla w\|_{L^\infty}\big)\Big[\|\nabla w\|_{L^\infty} - 1\Big]^2 |\Omega|,
\end{aligned}$$

where $\Theta\colon [0,\infty) \to [0,\infty)$,

$$\Theta(t) := \begin{cases} 0, & \text{if} \quad t \leq 1 \\ 1, & \text{if} \quad t > 1. \end{cases}$$

Using this estimate in (3.1) we get

Theorem 4.1. *Let $u \in \mathbb{K}$ denote the unique solution of (1.1). Then, for any $w \in V_0$, $|\nabla w| \in L^\infty(\Omega)$, for all $\rho \in \Lambda$, $\xi^* \in X$, $|\xi^*| \leq 1$, and for any $\eta^* \in X$ it holds*

$$\begin{aligned} \|\nabla u - \nabla w\|_{L^2}^2 \;\leq\;& 4\Bigg[D[w,\eta^*] + d_{\rho,\xi^*}(\eta^*)\Big[\frac{1}{2} d_{\rho,\xi^*}(\eta^*) + \|\nabla w - \eta^*\|_{L^2}\Big] \\ &+ \int_\Omega \rho(1 - \xi^* \cdot \nabla w)\,dx\Bigg] + 4\Theta\big(\|\nabla w\|_\infty\big)\big[\|\nabla w\|_\infty - 1\big]^2 |\Omega| \\ &+ 4\Theta\big(\|\nabla w\|_\infty\big)\big|\|\nabla w\|_{L^\infty} - 1\big||\Omega|^{1/2}\Big[d_{\rho,\xi^*}(\eta^*) + \|\nabla w - \eta^*\|_{L^2} + \|\rho\|_{L^2}\Big], \end{aligned} \tag{4.1}$$

where D and d_{ρ,ξ^*} are as in Theorem 3.1. Moreover, if $\operatorname{div}\eta^*, \operatorname{div}(\rho\xi^*) \in L^2(\Omega)$, then d_{ρ,ξ^*} can be bounded according to Theorem 3.2.

Proof of Theorem 4.1. We have by (3.1)

$$\begin{aligned} \|\nabla u - \nabla w\|_2^2 \;\leq\;& 2\big[\|\nabla u - \nabla w\|_2^2 + \|\nabla v - \nabla w\|_2^2\big] \\ \leq\;& 2\|\nabla v - \nabla w\|_2^2 + 4\Bigg[D[v,\eta^*] + d_{\rho,\xi^*}(\eta^*)\Big[d_{\rho,\xi^*}(\eta^*)\frac{1}{2} + \|\nabla v - \eta^*\|_2\Big] \\ &+ \int_\Omega \rho(1 - \xi^* \cdot \nabla v)\,dx\Bigg] \\ \leq\;& 2\|\nabla v - \nabla w\|_2^2 + 4\|\rho\|_2\|\nabla w - \nabla v\|_2 \\ &+4\Bigg[D[v,\eta^*] + d_{\rho,\xi^*}(\eta^*)\Big[d_{\rho,\xi^*}(\eta^*)\frac{1}{2} + \|\nabla v - \eta^*\|_2\Big] \\ &+ \int_\Omega \rho(1 - \xi^* \cdot \nabla w)\,dx\Bigg] \\ \leq\;& 2\|\nabla v - \nabla w\|_2^2 + 4\big[\|\rho\|_2 + d_{\rho,\xi^*}(\eta^*)\big]\|\nabla w - \nabla v\|_2 + \\ &+4\Bigg[D[v,\eta^*] + d_{\rho,\xi^*}(\eta^*)\Big[d_{\rho,\xi^*}(\eta^*)\frac{1}{2} + \|\nabla w - \eta^*\|_2\Big] \\ &+ \int_\Omega \rho(1 - \xi^* \cdot \nabla w)\,dx\Bigg], \end{aligned}$$

and if we estimate

$$\begin{aligned} D[v,\eta^*] &= D[w,\eta^*] + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_\Omega \eta^*\cdot(\nabla v - \nabla w)\,dx \\ &= \frac{1}{2}\|\nabla v - \nabla w|^2\,dx + \int_\Omega (\nabla v - \nabla w)\cdot(\nabla w - \eta^*)\,dx \\ &\le \frac{1}{2}\|\nabla v - \nabla w\|_2^2 + \|\nabla v - \nabla w\|_2 \|\nabla w - \eta^*\|_2, \end{aligned}$$

then we finally obtain

$$\begin{aligned} \|\nabla u - \nabla w\|_2^2 \;\le\; & 4\|\nabla v - \nabla w\|_2^2 + 4\|\nabla v - \nabla w\|_2\big[\|\rho\|_2 + d_{\rho,\xi^*}(\eta^*) + \|\nabla w - \eta^*\|_2\big] \\ & +4\Bigg[D[w,\eta^*] + d_{\rho,\xi^*}(\eta^*)\Big[\frac{1}{2}d_{\rho,\xi^*}(\eta^*) + \|\nabla w - \eta^*\|_2\Big] \\ & + \int_\Omega \rho(1-\xi^*\cdot\nabla w)\,dx\Bigg], \end{aligned}$$

which proves (4.1). □

Remark 4.1. *Since ∇w is only bounded, the quantity $\int_\Omega \rho(1-\xi^*\cdot\nabla w)\,dx$ is not necessarily ≥ 0. We may therefore replace this item on the r.h.s. of (4.1) by the non-negative one*

$$\int_\Omega \rho\Big(1 - \xi^*\cdot\nabla w \min(1, 1/\|\nabla w\|_\infty)\Big)\,dx,$$

which means that we have to add the quantity

$$4\left|\int_\Omega \rho\xi^*\cdot\nabla w\Big[\min\big(1, 1/\|\nabla w\|_{L^\infty}\big) - 1\Big]\,dx\right|$$

to the r.h.s. of (4.1). This term is bounded from above by

$$4\|\rho\|_{L^2}\Theta\Big(\|\nabla w\|_{L^\infty}\Big)\Big|\|\nabla w\|_{L^\infty} - 1\Big||\Omega|^{1/2}$$

so that it is enough to replace $\|\rho\|_{L^2}$ in the last line of (4.1) by $2\|\rho\|_{L^2}$.

References

[Ad] Adams R. A., Sobolev spaces. Academic Press, New York-San Francisco-London 1975.

[BF] Bildhauer, M., Fuchs, M., Error estimates for obstacle problems of higher order. Zapiski. Nauchn. Semin. POMI 348 (2007), 5–18.

[BFR1] Bildhauer, M., Fuchs, M., Repin, S., A posteriori error estimates for stationary slow flows of power-law fluids. J. Non-Newtonian Fluid Mech. 142 (2007), 112–122.

[BFR2] Bildhauer, M., Fuchs, M., Repin, S., Duality based a posteriori error estimates for higher order variational inequalities with power growth functionals. To appear in Ann. Acad. Sci. Fenn. Math.

[Br] Brezis, H., Multiplicateur de Lagrange en torsion elastoplastique. Arch. Rat. Mech. Anal. 41 (1971), 254-265.

[BS] Brezis, H., Stampacchia, G., Sur la régularité de la solution d'inéquations elliptiques. Bull. Soc. Math. France 96 (1968), 153-180.

[CR] Caffarelli, L. A., Riviere, N. M., On the lipschitz character of the stress tensor when twisting an elastic plastic bar. Arch. Rat. Mech. Anal. 69 (1979), 31-36.

[DL] Duvaut, G., Lions, J. L., Les inequations en mecanique et en physique. Dunod, Paris 1972.

[ET] Ekeland, I., Temam R., Convex analysis and variational problems. North Holland, Amsterdam 1976.

[Fr] Friedman, A., Variational principles and free boundary problems. Wiley and Sons, New York 1982.

[GT] Gilbarg, D., Trudinger, N. S., Elliptic partial differential equations of second order. Springer Verlag, New York 1998.

[Gl] Glowinski, R., Numerical methods for nonlinear variational problems. Springer Verlag, New York 1982.

[GLT] Glowinsksi, R., Lions, J. L., Trémolierès, R., Analyse numérique des inéqualitions variationels. Dunod, Paris 1976.

[Mo] Morrey, C. B., Multiple integrals in the calculus of variations. Springer Verlag, New York 1966.

[NR] Neittaanmäki, P., Repin, S., Reliable methods for computer simulation, error control and a posteriori estimates. Elsevier, New York 2004.

[Re1] Repin, S., Estimates of deviations from exact solutions of elliptic variational inequalities. Zapiski. Nauchn. Semin. POMI 271 (2000), 188–203.

[Re2] Repin, S., Estimates of deviations from exact solutions for some boundary-value problems with incompressibility condition. Algebra and Analiz 16 (2004), 124–161.

[Re3] Repin, S., A posteriori error estimation for partial differential equations. In Lectures on Advanced Computational Methods in Mechanics, Eds J. Kraus and U. Langer, Radon Series Comp. Appl. Math., 161–226, DeGruyter, 2007.

[Re4] Repin, S., Functional a posteriori estimates for elliptic variational inequalities. Zapiski. Nauchn. Semin. POMI 348 (2007), 147–164.

[Ti] Ting, T. W., Elastic-plastic torsion, Arch. Rational Mech. Anal. 24 (1969), 228-244.