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Abstract

We investigate the interior regularity of minimizers for an obstacle problem of higher order that can be seen as a model for the behaviour of a plate subject to a rather general constitutive law including nonlinear elastic materials.

If we consider a plate that must stay over an obstacle, then this problem formulated for the linear elastic case can be solved by discussing a suitable variational inequality or equivalently by minimizing the energy

$$J[u,\Omega] = \int_\Omega |\nabla^2 u|^2\, dx + \text{ lower order terms} \tag{1}$$

among all functions $u : \overline{\Omega} \to \mathbb{R}$ from a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, which satisfy the boundary condition

$$u = 0 = \partial_\nu u \text{ on } \partial\Omega \tag{2}$$

and respect the obstacle $\Phi : \overline{\Omega} \to \mathbb{R}$ in the sense that

$$u \geq \Phi \text{ on } \overline{\Omega}\,. \tag{3}$$

Of course (2) and (3) should be compatible, which can be achieved by the requirement $\Phi \leq 0$ on $\partial\Omega$. In (1) $\nabla^2 u$ is the matrix $(\partial_\alpha\partial_\beta u)_{1\leq\alpha,\beta\leq 2}$ of the second generalized derivatives, and in (2) ν denotes the exterior normal to $\partial\Omega$. Note that (2) corresponds to the fact that the plate is clamped at the boundary. The domain Ω represents the undeformed state of the plate, and our model energy (1) describes the situation if some outer forces are applied acting in vertical direction.

The mathematical background together with the mechanical framework is discussed in the monographs of Ciarlet and Rabier [CR], of Friedman [Fr], of Necǎs and Hlávácek [NH] and of Chudinovich and Costanda [CC], further details and additional references are contained in Zeidler's book [Ze]. In particular, it is outlined in these textbooks how to get a suitable weak formulation of the problem: let $\mathbb{K} := \{w \in \overset{\circ}{W}{}^2_2(\Omega) : w \geq \Phi$ on $\Omega\}$, where $\overset{\circ}{W}{}^2_2(\Omega)$ denotes the standard Sobolev space as introduced for example in [Ad]. Then there exists a unique $J[\cdot,\Omega]$-minimizer within the class $\mathbb{K}$, which according to Theorem 10.6 of [Fr] is in the space $C^2(\Omega)$, provided $\Phi \in C^2(\overline{\Omega})$.

The purpose of our note is to prove regularity results of this type, when the quantity $|\nabla^2 u|^2$ occurring in (1) is replaced by a more general expression like $f(\nabla^2 u)$ corresponding to different nonlinear elastic laws. Here $f : \mathbb{S}^2 \to [0, \infty)$ is a convex function defined on the space $\mathbb{S}^2$ of symmetric (2×2)-matrices satisfying suitable growth conditions. Of course the definition of the class $\mathbb{K}$ has to be adjusted, which means that we have to give weak variants of (2) and (3) in appropriate function spaces. Then a (weak) minimizer of

$$I[u, \Omega] := \int_\Omega f(\nabla^2 u)\, dx \tag{4}$$

(for technical simplicity we drop any lower order terms) within $\mathbb{K}$ is of class $C^{1,\alpha}(\Omega)$ for all $0 < \alpha < 1$ in each of the following cases:

(i) (subquadratic power growth) $f(\varepsilon) = (1 + |\varepsilon|^2)^{p/2}$, $\varepsilon \in \mathbb{S}^2$, for some exponent $1 < p \le 2$ (see ([FLM]);

(ii) general subquadratic growth of f including $f(\varepsilon) = |\varepsilon| ln(1 + |\varepsilon|)$ (compare [BF1]);

(iii) (anisotropic power growth) we have $\lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}} |\sigma|^2 \le D^2 f(\varepsilon)(\sigma, \sigma) \le \Lambda(1 + |\varepsilon|^2)^{\frac{q-2}{2}} |\sigma|^2$ with constants $\lambda, \Lambda > 0$ and for exponents $1 < p < q < \infty$ such that (see [BF2], Remark 2.2)

$$q < 2p\,. \tag{5}$$

We remark that the exceptional case that f is of linear growth modelling perfect elastoplastic plates has been studied by Seregin in his deep paper [Se] and that the logarithmic variant from (ii) can be seen as an approximation of the linear situation.

Let us now formulate our hypothesis, which are inspired by the work of Marcellini (see, e.g. [Ma1], [Ma2], [Ma3] and [MP]) on variational problems with nonstandard growth: let

$$f(\varepsilon) = h(|\varepsilon|), \varepsilon \in \mathbb{S}^2\,, \tag{6}$$

for a function $h : [0, \infty) \to [0, \infty)$ of class C^2 for which the following assumptions are valid:

(A1) h is strictly increasing and $h''(t) > 0$ for all $t > 0$ together with $\lim\limits_{t \downarrow 0} \frac{h(t)}{t} = 0$ and $\lim\limits_{t \to \infty} \frac{h(t)}{t} = \infty\,.$

Here the first requirement in (A1) is a consequence of the second and the third one. Note also that $h'(t) > 0$ for all $t > 0$ and $h'(0) = h(0) = 0$.

(A2) There exists $\overline{k} > 0$ such that $h(2t) \le \overline{k}\, h(t)$ for all $t \ge 0\,.$

It is easy to check that the Δ_2-condition (A2) implies the existence of an exponent $m > 1$ such that

$$h(t) \le C(t^m + 1),\ t \ge 0\,, \tag{7}$$

for a suitable constant C. Moreover, (A1) combined with (A2) implies the inequality

$$\frac{1}{k}\, th'(t) \le h(t) \le th'(t)\ \forall\ t \ge 0\,. \tag{8}$$

Finally we suppose that

$$\text{(A3)}\qquad \begin{array}{l} \text{there exist } \overline{\varepsilon}, \overline{h} > 0 \text{ and } \overline{\alpha} \ge 0 \text{ such that} \\ \overline{\varepsilon}\frac{h'(t)}{t} \le h''(t) \le \overline{h}(1+t^2)^{\frac{\overline{\alpha}}{2}}\,\frac{h'(t)}{t} \quad \text{for all } t > 0\,. \end{array}$$

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Note that the functions $t \mapsto t^s ln(1+t)$, $t \ge 0$, $s > 1$, satisfy (A1 - 3) with suitable constants $\overline{\varepsilon}, \overline{k}, \overline{h}$ and $\overline{\alpha}$ depending on the value of s. Clearly $t \mapsto t^p$ is admissible for any $p \ge 2$, whereas for $p > 1$ we can include $t \mapsto (1+t^2)^{\frac{p}{2}} - 1$. Moreover, if we consider arbitrary exponents $1 < p < q < \infty$, for which (5) is violated, then it is easy to construct integrands $f(\varepsilon) = h(|\varepsilon|)$, which satisfy the ellipticity inequality stated in (iii) exactly for this choice of p and q and for which (A1 - 3) are satisfied. To these examples we can not apply the regularity result of [BF2] but the interior smoothness of minimizers will be a consequence of Theorem 1 stated below.

Concerning Φ we require for simplicity that

$$\Phi \in C^4(\overline{\Omega}) \text{ and } \Phi|_{\partial\Omega} < 0\,. \tag{9}$$

If h satisfies (A1 - 3), we let

$$\mathbb{K}_\Phi := \{w \in \overset{\circ}{W}{}^2_h(\Omega) : w \ge \Phi \text{ on } \Omega\}\,,$$

$\overset{\circ}{W}{}^2_h(\Omega)$ being the Orlicz-Sobolev class as defined for example in [Ad]. Note that from the first inequality in (A3) it follows that ($p := 1 + \overline{\varepsilon}$)

$$h(t) \ge a\, t^p - b,\ t \ge 0\,, \tag{10}$$

with constants $a > 0$, $b \ge 0$, hence

$$\overset{\circ}{W}{}^2_h(\Omega) \subset \overset{\circ}{W}{}^2_p(\Omega) \subset C^0(\overline{\Omega})$$

on account of Sobolev's embedding theorem. As remarked in [FLM] we can find a function $w \in C_0^\infty(\Omega)$ s.t. $w \ge \Phi$, in particular we have $\mathbb{K}_\Phi \ne \emptyset$ and therefore the variational problem (with f from (6))

$$I[u,\Omega] = \int_\Omega f(\nabla^2 u)\, dx \to \min \text{ in } \mathbb{K}_\Phi \tag{$\mathcal{P}$}$$

is well-posed. The existence of a unique solution $u \in \mathbb{K}_\Phi$ follows exactly along the lines of [FO], Theorem 3.1, and this solution is different from zero, if $\Phi(x) > 0$ for some point $x \in \Omega$. Of course we could also add lower order terms to the functional $I[\cdot, \Omega]$ but this will not lead to a deeper insight.

We have the following regularity result:

THEOREM 1. *Let (A1 - 3) and (9) hold. Suppose that $u \in \mathbb{K}_\Phi$ is the solution of problem $(\mathcal{P})$ with f satisfying (6). Then u is of class $W^2_{t,\mathrm{loc}}(\Omega)$ for any finite t, in particular we have $u \in C^{1,\mu}(\Omega)$ for all $\mu < 1$.*

REMARK 1. *We do not know if $|\nabla^2 u| \in L^\infty_{\mathrm{loc}}(\Omega)$ or even $u \in C^2(\Omega)$ are true.*

REMARK 2. *Our result easily extends to vectorial functions $u : \Omega \to \mathbb{R}^M$ with componentwise constraints $u^i \geq \Phi^i$ for $i = 1, \dots, M$.*

REMARK 3. *There is no restriction on the size of the parameter $\overline{\alpha}$ occurring in (A3). However, if we want to weaken (9) in the sense that only $\Phi \in C^3(\overline{\Omega})$ (with $\Phi|_{\partial\Omega} < 0$) is required, then we need the bound $\overline{\alpha} \leq 2$: for $\Phi \in C^3(\overline{\Omega})$ inequality (21) below has to be replaced in an obvious way by an alternative variant and a substitute for (22) can only be established if $\overline{\alpha} \leq 2$.*

Proof of Theorem 1

We divide our arguments into several steps.

Step 1. approximation

¿From (6) we deduce for all $\varepsilon, \sigma \in \mathbb{S}^2$

$$\begin{aligned} &\min \left\{ \frac{h'(|\varepsilon|)}{|\varepsilon|},\ h''(|\varepsilon|) \right\} |\sigma|^2 \leq D^2 f(\varepsilon)(\sigma, \sigma) \leq \\ &\max \left\{ \frac{h'(|\varepsilon|)}{|\varepsilon|},\ h''(|\varepsilon|) \right\} |\sigma|^2 , \end{aligned}$$

so that by assumption (A3) is follow (w.l.o.g. $\overline{\varepsilon} \leq 1, \overline{h} \geq 1$)

$$\overline{\varepsilon}\, \frac{h'(|\varepsilon|)}{|\varepsilon|} |\sigma|^2 \leq D^2 f(\varepsilon)(\sigma, \sigma) \leq \overline{h}(1 + |\varepsilon|^2)^{\frac{\overline{\alpha}}{2}}\, \frac{h'(|\varepsilon|)}{|\varepsilon|} |\sigma|^2 . \tag{11}$$

We also remark that from (7) and the convexity of h we have

$$|Df(\varepsilon)| = h'(|\varepsilon|) \leq c \left(|\varepsilon|^{m-1} + 1 \right) . \tag{12}$$

Combining (11) and (12) it is immediate that we can find an exponent $q \geq 2$ such that

$$D^2 f(\varepsilon)(\sigma, \sigma) \leq c \left(1 + |\varepsilon|^2 \right)^{\frac{q-2}{2}} |\sigma|^2 \ \forall \varepsilon, \sigma \in \mathbb{S}^2 , \tag{13}$$

where c always denotes a positive constant whose value may change from line to line. For $\delta \in (0,1]$ we let $h_\delta(t) := \delta\left[(1+t^2)^{q/2}-1\right] + h(t)$ and $f_\delta(\varepsilon) = h_\delta(|\varepsilon|)$. We then consider the variational problem

$$(\mathcal{P}_\delta) \qquad I_\delta[w,\Omega] := \int_\Omega f_\delta(\nabla^2 w)\,dx \to \min \text{ in } \overset{\circ}{W}{}^2_{q,\Phi}(\Omega)\,,$$

where we have set

$$\overset{\circ}{W}{}^2_{q,\Phi}(\Omega) := \left\{ w \in \overset{\circ}{W}{}^2_q(\Omega) : w \geq \Phi \text{ on } \Omega \right\}.$$

Let u_δ denote the unique solution to $(\mathcal{P}_\delta)$. We fix a subdomain $\Omega' \Subset \Omega$ and a function $\eta \in C_0^\infty(\Omega')$ such that $0 \leq \eta \leq 1$. For a coordinate direction $e_\gamma, \gamma = 1,2$, and for $\rho \neq 0$ we define the difference quotient

$$\Delta_\rho w(x) := \frac{1}{\rho}\left(w(x+\rho e_\gamma) - w(x)\right)$$

of a function w in direction e_γ. For $t > 0$ such that $\rho^{-2}t < 1/2$ we finally let

$$v_t^\delta := u_\delta + t\Delta_{-\rho}\left(\eta^6 \Delta_\rho[u_\delta - \Phi]\right) =: u_\delta + t\varphi\,.$$

¿From

$$\begin{aligned} v_t^\delta(x) &= \Phi(x) + \left[1 - \frac{t}{\rho^2}\eta^6(x-\rho e_\gamma) - \frac{t}{\rho^2}\eta^6(x)\right](u-\Phi)(x) \\ &\quad + \frac{t}{\rho^2}\eta^6(x-\rho e_\gamma)(u-\Phi)(x-\rho e_\gamma) \\ &\quad + \frac{t}{\rho^2}\eta^6(x)(u-\Phi)(x+\rho e_\gamma) \end{aligned}$$

it follows that $v_t^\delta \in \overset{\circ}{W}{}^2_{q,\Phi}(\Omega)$ together with $\mathrm{spt}(u_\delta - v_t^\delta) \subset \Omega'$, provided ρ is sufficiently small. The minimality of u_δ gives

$$\frac{1}{t}\int_\Omega \left[f_\delta(\nabla^2 u_\delta + t\nabla^2\varphi) - f_\delta(\nabla^2 u_\delta)\right]\,dt \geq 0\,,$$

and exactly as in [FLM], (3.1), we may pass to the limit $t \downarrow 0$ to obtain

$$(14) \qquad \int_\Omega \Delta_\rho\left[Df_\delta(\nabla^2 u_\delta)\right] : \nabla^2\left[\eta^6\Delta_\rho(u_\delta - \Phi)\right]\,dx \leq 0\,.$$

Note that the derivation of (14) clearly uses the fact that f_δ is a power growth integrand with exponent q as it follows from (13) and the definition of f_δ. We have

$$\begin{aligned} \nabla^2(\eta^6\Delta_\rho u_\delta) &= \eta^6\nabla^2(\Delta_\rho u_\delta) \\ &\quad + \left(\partial_\alpha\partial_\beta\eta^6\Delta_\rho u_\delta + \partial_\alpha\eta^6\partial_\beta\Delta_\rho u_\delta + \partial_\beta\eta^6\partial_\alpha\Delta_\rho u_\delta\right)_{1\leq\alpha,\beta\leq 2} \\ &=: \eta^6\nabla^2(\Delta_\rho u_\delta) + T_\rho^\delta\,, \end{aligned}$$

hence by (14)

$$\int_\Omega \Delta_\rho \left[Df_\delta(\nabla^2 u_\delta) \right] : \nabla^2(\Delta_\rho u_\delta)\eta^6 \, dx \tag{15}$$
$$\leq \int_\Omega \Delta_\rho \left[Df_\delta(\nabla^2 u_\delta) \right] : \left[\nabla^2(\eta^6 \Delta_\rho \Phi) - T_\rho^\delta \right] \, dx \, .$$

Introducing the parameter dependent bilinear form

$$\begin{aligned} B_x(\varepsilon, \sigma) := & \int_0^1 D^2 f_\delta \left(\xi_t(x) \right) (\varepsilon, \sigma) \, dt \, , \\ \xi_t := & \quad \nabla^2 u_\delta + t\rho \Delta_\rho (\nabla^2 u_\delta) \, , \end{aligned}$$

(15) can be written as

$$\int_\Omega \eta^6 B_x(\Delta_\rho \nabla^2 u_\delta, \Delta_\rho \nabla^2 u_\delta) \, dx$$
$$\leq \int_\Omega B_x \left(\Delta_\rho \nabla^2 u_\delta, \nabla^2(\eta^6 \Delta_\rho \Phi) - T_\rho^\delta \right) \, dx \, ,$$

and an application of the Cauchy-Schwarz inequality gives

$$\int_\Omega \eta^6 B_x(\Delta_\rho \nabla^2 u_\delta, \Delta_\rho \nabla^2 u_\delta) \, dx \tag{16}$$
$$\leq c(\eta) \int_{\operatorname{spt} \eta} |B_x| \left(|\Delta_\rho \Phi|^2 + |\nabla \Delta_\rho \Phi|^2 + |\nabla^2 \Delta_\rho \Phi|^2 \right.$$
$$\left. + |\Delta_\rho u_\delta|^2 + |\nabla \Delta_\rho u_\delta|^2 \right) \, dx \, .$$

Using the definition of B_x, the $(q-2)$-growth of $D^2 f_\delta$ as well as elementary properties of difference quotients and the fact that Φ is in $C^3(\overline{\Omega})$, Hölder's inequality implies:

$$\text{r.h.s. of (16) } \leq c(\eta, \delta) < \infty$$

uniformly in ρ, hence by (16)

$$\int_\Omega \eta^6 B_x(\Delta_\rho \nabla^2 u_\delta, \Delta_\rho \nabla^2 u_\delta) \, dx \leq c(\eta, \delta) \, . \tag{17}$$

Obviously $B_x(\varepsilon, \varepsilon) \geq \delta |\varepsilon|^2$ and (17) gives $|\nabla^3 u_\delta| \in L^2_{\text{loc}}(\Omega)$, in particular we have

$$\Delta_\rho \nabla^2 u_\delta \to \partial_\gamma \nabla^2 u_\delta \text{ in } L^2_{\text{loc}}(\Omega) \text{ and a.e.} \tag{18}$$

Combining (18) with Fatou's Lemma, we deduce from (17) that (from now on summation w.r.t. $\gamma = 1, 2$)

$$w_\delta := D^2 F_\delta(\nabla^2 u_\delta)(\partial_\gamma \nabla^2 u_\delta, \partial_\gamma \nabla^2 u_\delta) \in L^1_{\mathrm{loc}}(\Omega)\,. \tag{19}$$

Note that (up to now) (19) is not uniform in δ. X-AntiVirus: checked by AntiVir MailGuard (Version: 8.0.0.18; AVE: 8.1.0.30; VDF: 7.0.3.177)

Step 2. a (uniform) bound for $\frac{h'_\delta(|\nabla^2 u_\delta|)}{|\nabla^2 u_\delta|}|\nabla^3 u_\delta|^2$

We return to (15) and observe that by (18) together with (11) and Fatou's Lemma we have

$$\overline{\varepsilon} \int\limits_\Omega \eta^6 \frac{h'_\delta(|\nabla^2 u_\delta|)}{|\nabla^2 u_\delta|} |\nabla^3 u_\delta|^2 \, dx \le \liminf_{\rho \to 0} \, \{\text{r.h.s. of (15)}\}\,. \tag{20}$$

In order to handle the r.h.s. of (20) we can use the calculations starting with (3.20) from [BF3]: replacing F, v, $\varepsilon(v)$, s in this reference by f_δ, u_δ, $\nabla^2 u_\delta$, q we see

$$\begin{aligned} &\liminf_{\rho \to 0} \, \{\text{r.h.s. of (15)}\} \\ &\quad = \int\limits_\Omega \partial_\gamma \left[D f_\delta(\nabla^2 u_\delta) \right] : \left[\nabla^2(\eta^6 \partial_\gamma \Phi) - S^\delta_\gamma \right] dx\,, \\ S^\delta_\gamma \; &:= \; \left(\partial_\alpha \partial_\beta \eta^6 \partial_\gamma u_\delta + \partial_\alpha \eta^6 \partial_\beta \partial_\gamma u_\delta + \partial_\beta \eta^6 \partial_\alpha \partial_\gamma u_\delta \right)_{1 \le \alpha, \beta \le 2}\,, \end{aligned}$$

and therefore (20) yields after integration by parts

$$\begin{aligned} &\overline{\varepsilon} \int\limits_\Omega \eta^6 \frac{h'_\delta(|\nabla^2 u_\delta|)}{|\nabla^2 u_\delta|} |\nabla^3 u_\delta|^2 \, dx \\ &\quad \le \; - \int\limits_\Omega D f_\delta(\nabla^2 u_\delta) : \partial_\gamma \left[\nabla^2(\eta^6 \partial_\gamma \Phi) - S^\delta_\gamma \right] dx\,. \end{aligned} \tag{21}$$

Note that this integration by parts is justified since the "critical term" occurring in $Df(\nabla^2 u_\delta) : \partial_\gamma[\ldots]$ is of the type

$$|D f_\delta(\nabla^2 u_\delta)| |\nabla^3 u_\delta|$$

which according to (12) can be estimated as follows:

$$
\begin{aligned}
&|Df_\delta(\nabla^2 u_\delta)||\nabla^3 u_\delta| \\
&= \sqrt{h'_\delta(|\nabla^2 u_\delta|)/|\nabla^2 u_\delta|}\,\sqrt{h'_\delta(|\nabla^2 u_\delta|)|\nabla^2 u_\delta|}\;|\nabla^3 u_\delta| \\
&\overset{(8)}{\leq} c\,\sqrt{h'_\delta(|\nabla^2 u_\delta|)/|\nabla^2 u_\delta|}\,\sqrt{h_\delta(|\nabla^2 u_\delta|)}\;|\nabla^3 u_\delta| \\
&\leq \tau\,\frac{h'_\delta(|\nabla^2 u_\delta|)}{|\nabla^2 u_\delta|}|\nabla^3 u_\delta|^2 + c(\tau)h_\delta(|\nabla^2 u_\delta|)\,.
\end{aligned}
$$

Here we have used Young's inequality. Since w_δ from (19) is an upper bound for the τ-term and since we have (19), we see that (21) is true. At the same time the above inequality shows that if we take care of the test function in front of $\nabla^3 u_\delta$ and choose τ small enough, we arrive at

$$
\begin{aligned}
(22)\qquad &\int_\Omega \eta^6 \frac{h'_\delta(|\nabla^2 u_\delta|)}{|\nabla^2 u_\delta|}|\nabla^3 u_\delta|^2\,dx \\
&\leq c(\eta)\int_\Omega h'_\delta(|\nabla^2 u_\delta|)\left[|\nabla u_\delta| + |\nabla^2 u_\delta| + \sum_{i=1}^4 |\nabla^i \Phi|\right],
\end{aligned}
$$

$c(\eta)$ being independent of δ. Since h_δ is a N-function, we have for s, $t \geq 0$

$$
h'_\delta(t)s \leq h^*_\delta(h'_\delta(t)) + h_\delta(s) = th'_\delta(t) - h_\delta(t) + h_\delta(s) \overset{(8)}{\leq} ch_\delta(t) + h_\delta(s)\,.
$$

Applying this estimate with $t = |\nabla^2 u_\delta|$ and for appropriate choices of s, (22) leads to

$$
\begin{aligned}
(23)\qquad &\int_\Omega \eta^6 \frac{h'_\delta(|\nabla^2 u_\delta|)}{|\nabla^2 u_\delta|}|\nabla^3 u_\delta|^2\,dx \\
&\leq c(\eta,\Phi)\int_\Omega \big(h_\delta(|\nabla u_\delta|) + h_\delta(|\nabla^2 u_\delta|) + 1\big)\,dx\,,
\end{aligned}
$$

where obviously all integrals involving Φ have been estimated in a rough way. We emphasize that $c(\eta,\Phi)$ does not depend on δ.
Referring to Step 3 we will now use that

$$
(24)\qquad \sup_{0<\delta<1}\int_\Omega \big(h_\delta(|\nabla u_\delta|) + h_\delta(|\nabla^2 u_\delta|)\big)\,dx < \infty\,.
$$

Let us introduce the function

$$\Psi_\delta := \int_0^{|\nabla^2 u_\delta|} \left(\frac{h'_\delta(t)}{t}\right)^{1/2} dt\,.$$

Then we have

$$|\nabla\Psi_\delta|^2 \le \frac{h'(|\nabla^2 u_\delta|)}{|\nabla^2 u_\delta|}|\nabla^3 u_\delta|^2\,,$$

hence by (23) and (24)

$$\int_{\Omega'} |\nabla\Psi_\delta|^2\,dx \le c(\Omega',\Phi) \tag{25}$$

for any subdomain $\Omega' \Subset \Omega$. At the same time we have a.e. on the set $[|\nabla^2 u_\delta| \ge 1]$

$$\begin{aligned}\Psi &\overset{(A3)}{\le} \int_0^1 (h''_\delta(t)/\overline{\varepsilon})^{1/2} + \int_1^{|\nabla^2 u_\delta|} \left(\frac{h'_\delta(t)}{t}\right)^{1/2} dt\\ &\le c + (|\nabla^2 u_\delta| - 1)h'_\delta(|\nabla^2 u_\delta|)^{1/2}\\ &\le c\left(1 + |\nabla^2 u_\delta| + |\nabla^2 u_\delta| h'_\delta(|\nabla^2 u_\delta)\right)\,,\end{aligned}$$

whereas $\Psi \le c$ a.e. on $[|\nabla^2 u_\delta| \le 1]$. Using (8) and (10) we get

$$\int\limits_\Omega \Psi_\delta\,dx \le c\left(1 + \int\limits_\Omega h_\delta(|\nabla^2 u_\delta|)\,dx\right)\,,$$

so that together with (25)

$$\|\Psi_\delta\|_{W^1_1(\Omega')} \le c(\Omega',\Phi)$$

for all $\Omega' \Subset \Omega$. But then Sobolev's embedding theorem shows

$$\|\Psi_\delta\|_{L^2(\Omega')} \le c(\Omega',\Phi)\,,$$

and we may quote (25) one more time to obtain.

$$\|\Psi_\delta\|_{W^1_2(\Omega')} \le c(\Omega',\Phi)\,. \tag{26}$$

Another application of Sobolev's embedding theorem in combination with (26) yields

$$\int_{\Omega'} \Psi_\delta^t\,dx \le c(\Omega',\Phi,t) \tag{27}$$

for any finite t. Now we observe that

$$\begin{aligned}
\Psi_\delta &\geq \int_{|\nabla^2 u_\delta|/2}^{|\nabla^2 u_\delta|} \left(\frac{h'_\delta(t)}{t}\right)^{1/2} dt \\
&\overset{(A1)}{\geq} \frac{1}{2}|\nabla^2 u_\delta| h'_\delta(|\nabla^2 u_\delta|/2)^{1/2} |\nabla^2 u_\delta|^{-1/2} \\
&= \big(h'_\delta(|\nabla^2 u_\delta|/2)|\nabla^2 u_\delta|/2\big)^{-1/2} \, 1/\sqrt{2} \\
&\overset{(8),(A2)}{\geq} c h_\delta(|\nabla^2 u_\delta|)^{1/2}
\end{aligned}$$

and see from (10) that the above estimate together with (27) leads to

$$\int_{\Omega'} |\nabla^2 u_\delta|^t \, dx \leq c(\Omega', \Phi, t) \tag{28}$$

for all $t < \infty$ and any $\Omega' \Subset \Omega$. It therefore remains to verify (24) and to discuss in which sense we have convergence of $\{u_\delta\}$ towards u. This will be done in X-AntiVirus: checked by AntiVir MailGuard (Version: 8.0.0.18; AVE: 8.1.0.30; VDF: 7.0.3.177)

Step 3. passage to the limit $\delta \searrow 0$ and conclusions

We fix some $u_0 \in \overset{\circ}{W}{}^2_{q,\Phi}(\Omega)$ and get from the $I_\delta[\cdot, \Omega]$-minimality of u_δ

$$I_\delta[u_\delta, \Omega] \leq I_\delta[u_0, \Omega] \leq I_1[u_0, \Omega] =: c_1 \, ,$$

so that $I[u_\delta, \Omega] \leq c_1$ for all $0 < \delta \leq 1$. Since $u_\delta \in \overset{\circ}{W}{}^2_h(\Omega)$ we can apply Poincaré's inequality (see, e.g. [FO], Lemma 2.4) two times to see

$$\|u_\delta\|_{W^2_h(\Omega)} \leq c_2 \, , \tag{29}$$

which immediately leads to (24). At the same time (29) together with (10) implies $u_\delta \rightharpoondown: \overline{u}$ weakly in $W^2_p(\Omega)$. Obviously $\overline{u} \in \overset{\circ}{W}{}^2_p(\Omega)$ and also (by lower semicontinuity) $\overline{u} \in W^2_h(\Omega)$, but a variant of Theorem 2.1 from [FO] gives $\overline{u} \in \overset{\circ}{W}{}^2_h(\Omega)$, thus $\overline{u} \in \mathbb{K}_\Phi$, since we clearly have $\overline{u} \geq \Phi$. The lower semicontinuity of $I[\cdot, \Omega]$ shows

$$\begin{aligned}
I[\overline{u}, \Omega] &\leq \liminf_{\delta \searrow 0} I[u_\delta, \Omega] \\
&\leq \liminf_{\delta \searrow 0} I_\delta[u_\delta, \Omega] \leq \liminf_{\delta \searrow 0} I_\delta[v, \Omega] = I[v, \Omega]
\end{aligned}$$

for all $v \in \overset{\circ}{W}{}^2_{q,\Phi}(\Omega)$, where again the $I_\delta[\cdot, \Omega]$-minimality of u_δ has been used. But $\overset{\circ}{W}{}^2_{q,\Phi}(\Omega)$ is dense in $\mathbb{K}_\Phi$, which follows from an adjustment of Lemma 2.3 from [FLM] to the present setting, hence $\overline{u}$ is a solution of $(\mathcal{P})$ and therefore $u = \overline{u}$. Now (28) combined with $u_\delta \rightharpoondown u$ in $W^2_p(\Omega)$ gives the claim of Theorem 1. We remark that Lemma 2.3 of [FLM] is based on a deep result of Adams and Hedberg [AH], Theorem 9.1.3, but it is possible to avoid this rather delicate tool by working with a slightly different regularisation, which means that $(\mathcal{P}_\delta)$ is formulated with respect to a suitable perturbed obstacle function Φ_δ. Then as before it follows $\overline{u} = u$ and all properties of the sequence $\{u_\delta\}$ remain unchanged. The reader will find the details in Section 2 of [BF1].

□

REMARK 4. *Suppose that $\overline{\varepsilon} = 1$ in (A3). Then the estimates following (27) together with (10) show that $\Psi_\delta \geq A|\nabla^2 u_\delta| - B$ with constants $A > 0, B \geq 0$. (26) combined with Trudinger's inequality (see [GT], Theorem 7.15) implies*

$$\int_{\Omega'} \exp(\Psi_\delta)\, dx \leq c(\Omega')\,,$$

hence

$$\int_{\Omega'} \exp\left(|\nabla^2 u_\delta|\right)\, dx \leq c(\Omega')$$

for any subdomain Ω'. By lower semicontinuity it follows in addition to the result of Theorem 1 that $\exp(|\nabla^2 u|)$ *is in the space* $L^1_{\mathrm{loc}}(\Omega)$.

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