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Abstract

We discuss variational integrals with density having linear growth on spaces of vector valued BV-functions and prove $\mathrm{Im}(u) \subset K$ for minimizers u provided that the boundary data take their values in the closed convex set K assuming in addition that the integrand satisfies natural structure conditions.

Given a closed convex set $K \subset \mathbb{R}^N$, we say that minimizers of some variational problem have the convex hull property if they are contained in K in a sense to be made precise provided this is true for their boundary data. A prominent example is given by mass minimizing integer multiplicity m-currents T with compact support, where $m \leq N$ and where the comparison currents S are such that $\partial S = T_0$ for a $(m-1)$-current T_0 with compact support and $\partial T_0 = 0$. Then the support of T is contained in the convex hull of $\operatorname{spt} T_0$, which is a consequence of the monotonicity formula for stationary varifolds. We refer the reader to [Si], 19.2 Theorem and 34.2 Remarks. Let us now pass to the setting of variational integrals

$$I[u,\Omega] = \int_\Omega f(\nabla u)\,\mathrm{d}x$$

defined for functions u: $\mathbb{R}^n \supset \Omega \to \mathbb{R}^N$, Ω denoting a bounded Lipschitz domain. Suppose that we are given a function u_0 such that

$$u_0 \in W^1_1(\Omega;\mathbb{R}^N)\,, \qquad u_0(x) \in K \ \text{ a.e.}\,, \tag{1}$$

where $W^1_1(\Omega;\mathbb{R}^N)$ is the Sobolev space of vector-valued mappings (see, e.g., [Ad]). Let us further assume that $f(Z) = h(|Z|)$ with

$$h:\ [0,\infty) \to [0,\infty) \ \text{ strictly increasing and convex}\,. \tag{2}$$

Then, if $u \in W^1_1(\Omega;\mathbb{R}^N)$ minimizes $I[\cdot,\Omega]$ w.r.t. the boundary data u_0, i.e.

$$\left.\begin{array}{l} I[u,\Omega] < \infty\,, \qquad u-u_0 \in W^1_1(\Omega;\mathbb{R}^N) \quad \text{and} \\ I[u,\Omega] \leq I[v,\Omega] \quad \text{for all } \ v \in u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)\,, \end{array}\right\}$$

it follows that $u(x) \in K$ for almost any $x \in \Omega$. A simple proof is given by the following observation: let Φ: $\mathbb{R}^N \to K$ denote the nearest-point-projection being Lipschitz with $\mathrm{Lip}(\Phi) = 1$. From [AFP], comments given at the beginning of the proof of Theorem 3.96, we see that $v = \Phi(u)$ is admissible and satisfies $|\nabla v| \leq \mathrm{Lip}(\Phi)|\nabla u| = |\nabla u|$. Using the properties of h stated in (2) combined with $|\nabla v| \leq |\nabla u|$, we get from the minimality of u that $I[u,\Omega] = I[v,\Omega]$, and as it is outlined below, this will lead to $\nabla u = \nabla v$, hence $u = v$ and in conclusion $u \in K$ a.e. We remark first that a related maximum principle is due to D'Ottavio, Leonetti and Musciano [DLM], and second that a similar argument

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together with a proof of the chain rule in the Lipschitz setting has been presented in [BF1]. However, the reader should note at this stage that a much more general chain rule formula implying $|\nabla(\Phi \circ u)| \le \mathrm{Lip}(\Phi)|\nabla u|$ is due to Ambrosio and Dal Maso [ADM1]. As a matter of fact the existence of a minimizer u in a suitable Sobolev class requires that h is of superlinear growth, and therefore in general can not be guaranteed if in addition to (2) the function h satisfies

$$\bar{c} := \lim_{t\to\infty} \frac{h(t)}{t} \quad \text{exists in } \ (0,\infty)\,, \tag{3}$$

which means that now h is just of linear growth.

W.l.o.g. we will also assume that $h(0) = 0$. Based on ideas of De Giorgi (see the recent book [Gio] for an overview on his work), of Giusti [Giu], of Giaquinta, Modica, Souček [GMS], of Goffman and Serrin [GS], of Ambrosio and Dal Maso [ADM2] and of Buttazzo [Bu] it is possible to introduce suitable concepts of generalized solutions to the problem

$$I[u,\Omega] = \int_\Omega h(|\nabla u|)\,\mathrm{d}x \to \min \quad \text{ in } \ u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)\,. \tag{$\mathcal{P}$}$$

Let

$$\mathcal{M} \;:=\; \big\{u \in BV(\Omega;\mathbb{R}^N) : u \text{ is a } L^1\text{-cluster point of a minimizing sequence of problem } (\mathcal{P})\big\}$$

and define $K[\cdot,\Omega]$: $BV(\Omega;\mathbb{R}^N) \to \mathbb{R}$,

$$K[u,\Omega] := \int_\Omega h(|\nabla^a u|)\,\mathrm{d}x + \bar{c}|\nabla^s u|(\Omega) + \int_{\partial\Omega} \bar{c}|(u_0 - u)\otimes\mathcal{N}|\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

where $BV(\Omega;\mathbb{R}^N)$ is the space of functions of bounded variation (see [AFP] or [Giu]), $\mathcal{N}$ is the exterior normal of $\partial\Omega$ and where we have used the decomposition of the vector measure ∇u in its absolutely continuous part $\nabla^a u \llcorner \mathcal{L}^n$ and its singular part $\nabla^s u$. According to a theorem of Besicovitch ([AFP], Theorem 2.22) we have $\nabla^a u \in L^1(\Omega;\mathbb{R}^{nN})$ and

$$\nabla^a u(x) = \lim_{\rho\downarrow 0} \frac{\nabla u(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} \tag{4}$$

holds for $\mathcal{L}^n$-a.a. $x \in \Omega$. Note that on account of (3) the recession function

$$f_\infty(Z) := \lim_{t\to 0} \frac{f(tZ)}{t}\,, \qquad Z \in \mathbb{R}^{nN}\,,$$

equals $\bar{c}|Z|$, hence we have the more familiar formula

$$\begin{aligned} K[u,\Omega] \;&=\; \int_\Omega f(\nabla^a u)\,\mathrm{d}x + \int_\Omega f_\infty\Big(\frac{\nabla^s u}{|\nabla^s u|}\Big)\,\mathrm{d}|\nabla^s u| \\ &\quad + \int_{\partial\Omega} f_\infty((u_0 - u)\otimes\mathcal{N})\,\mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

for the extension of I to the space $BV(\Omega;\mathbb{R}^N)$. We recall the following facts established in [BF2] (compare also [Bi], Appendix A1):

i) $I[\cdot,\Omega] = K[\cdot,\Omega]$ on $u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$;

ii) $K[\cdot,\Omega] \to \min$ admits at least one solution in $BV(\Omega;\mathbb{R}^N)$;

iii) these minimizers are exactly the elements of $\mathcal{M}$;

iv) $\inf\limits_{u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} I[\cdot,\Omega] = \inf\limits_{BV(\Omega;\mathbb{R}^N)} K[\cdot,\Omega]$.

Based on these facts it is reasonable to address the elements of the set $\mathcal{M}$ as generalized solutions of problem $(\mathcal{P})$.

Now we can state our main result:

Theorem 1. *Suppose that u_0 satisfies (1) for a closed and convex set $K \subset \mathbb{R}^N$. Assume further that we have (2) and (3) for the density h. Then it holds $u(x) \in K$ a.e. for any generalized solution of problem $(\mathcal{P})$.*

Corollary 1. *(Maximum-principle) Suppose that h satisfies (2) and (3). Assume further that $u_0 \in W^1_1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$. Then any generalized minimizer $u \in BV(\Omega;\mathbb{R}^N)$ of problem $(\mathcal{P})$ satisfies $\|u\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$.*

Remark 1. *The proof of Theorem 1 given below immediately extends to integrands of the form*

$$f(Z) = \sum_{i=1}^n h_i(|Z_i|)\,, \qquad Z = (Z_1,\dots,Z_n) \in \mathbb{R}^{nN}\,,\;\; Z_i \in \mathbb{R}^N\,,$$

with functions h_1, ..., h_n satisfying (2) and having the property that

$$\bar{c}_i := \lim_{t\to\infty} \frac{h_i(t)}{t}$$

exists in $(0,\infty)$. In this case it holds

$$f_\infty(Z) = \sum_{i=1}^n \bar{c}_i |Z_i|\,.$$

Of course any other additive decomposition of f depending on the moduli of the Z_i can be considered, e.g.

$$f(Z) = h_1\big(\sqrt{|Z_1|^2 + |Z_2|^2}\big) + h_2(|Z_3|) \quad \text{or} \quad f(Z) = h_1(|Z_1|) + h_2\big(\sqrt{|Z_2|^2 + |Z_3|^2}\big)$$

are admissible in the case $n = 3$. In fact, a careful inspection of the proof of the chain rule shows the validity of

$$|\partial_i(\Phi \circ u)| \le \mathrm{Lip}(\Phi)|\partial_i u|\,, \qquad i = 1,\dots,n\,,$$

so that $|\partial_i(\Phi \circ u)| \le |\partial_i u|$.

Proof. We fix a Lipschitz domain $\hat{\Omega} \Supset \Omega$, extend u_0 to an element of $W^1_1(\hat{\Omega};\mathbb{R}^N)$ with values in K and let

$$BV_{u_0}(\Omega;\mathbb{R}^N) := \{w \in BV(\hat{\Omega};\mathbb{R}^N) :\, w = u_0 \ \text{ on } \ \hat{\Omega} - \Omega\}\,.$$

Following [GMS] we define

$$\begin{aligned} \hat{I}[w,\hat{\Omega}] \ &:= \ \int_{\hat{\Omega}} f(\nabla^a w)\,\mathrm{d}x + \int_{\hat{\Omega}} f_\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)\,\mathrm{d}|\nabla^s w| \\ &= \ \int_{\hat{\Omega}} h(|\nabla^a w|)\,\mathrm{d}x + \bar{c}|\nabla^s w|(\hat{\Omega}) \end{aligned}$$

for $w \in BV_{u_0}(\Omega;\mathbb{R}^N)$, and as outlined in [BF2] we have

$$\hat{I}[w,\hat{\Omega}] = K[w_{|\Omega},\Omega] + const\,.$$

Conversely, if $v \in BV(\Omega;\mathbb{R}^N)$ and if we put

$$\hat{v} := \left\{ \begin{array}{lll} v & \text{on} & \Omega \\ u_0 & \text{on} & \hat{\Omega} - \Omega \end{array} \right\} \in BV_{u_0}(\Omega;\mathbb{R}^N)\,,$$

then

$$\hat{I}[\hat{v},\hat{\Omega}] = K[v,\Omega] + const\,,$$

where $const = \int_{\hat{\Omega}-\Omega} h(|\nabla u_0|)\,\mathrm{d}x$. Due to this observation it is sufficient to consider a solution $u \in BV_{u_0}(\Omega;\mathbb{R}^N)$ of

$$\hat{I}[\cdot,\hat{\Omega}] \to \min \qquad \text{in } \ BV_{u_0}(\Omega;\mathbb{R}^N)$$

and to prove that $u(x) \in K$ a.e.

To this purpose we consider the retraction Φ: $\mathbb{R}^N \to K$ and let as before $v := \Phi \circ u$. According to the comments given at the beginning of the proof of Theorem 3.96 in [AFP] v is in $BV(\hat{\Omega};\mathbb{R}^N)$ and (recall $\mathrm{Lip}(\Phi) = 1$)

$$|\nabla v| \le \mathrm{Lip}(\Phi)|\nabla u| = |\nabla u|\,, \tag{5}$$

where $|\nabla v|$ and $|\nabla u|$ denote the total variations of the vector measures ∇v and ∇u. Here we like to emphasize again that a general chain rule formula as stated for example in Theorem 3.101 of [AFP] is due to Ambrosio and Dal Maso [ADM1], and that (5) is a simple consequence of this important formula. Clearly $v \in BV_{u_0}(\Omega;\mathbb{R}^N)$ so that

$$\hat{I}[u,\hat{\Omega}] \le \hat{I}[v,\hat{\Omega}]\,. \tag{6}$$

Now we use (4) for u and v which implies in combination with (5) for $\mathcal{L}^n$-a.a. $x \in \hat{\Omega}$

$$|\nabla^a v(x)| = \lim_{\rho\downarrow 0} \frac{|\nabla v|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} \le \lim_{\rho\downarrow 0} \frac{|\nabla u|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} = |\nabla^a u(x)|\,,$$

and the monotonicity of h gives

$$\int_{\hat{\Omega}} h(|\nabla^a v|)\,\mathrm{d}x \le \int_{\hat{\Omega}} h(|\nabla^a u|)\,\mathrm{d}x\,. \tag{7}$$

Quoting [AFP], Proposition 3.92 (a), we may write for functions $w \in BV(\hat{\Omega};\mathbb{R}^N)$

$$\nabla^s w = \nabla w \llcorner S_w\,, \qquad S_w := \left\{ x \in \hat{\Omega} : \lim_{\rho \downarrow 0} \frac{|\nabla w|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} = \infty \right\}, \tag{8}$$

and deduce from (5) that

$$S_v \subset S_u\,, \tag{9}$$

since

$$|\nabla v|(B_\rho(x)) \le |\nabla u|(B_\rho(x))\,.$$

Next we use (5), (8) and (9) and get

$$|\nabla^s v|(\hat{\Omega}) = |\nabla v|(S_v) \le |\nabla u|(S_u) = |\nabla^s u|(\hat{\Omega}) \tag{10}$$

which in combination with (7) leads to

$$\hat{I}[v,\hat{\Omega}] \le \hat{I}[u,\hat{\Omega}]\,.$$

By (6) we must have

$$\hat{I}[v,\hat{\Omega}] = \hat{I}[u,\hat{\Omega}]\,,$$

and by (7) and (10) this is only possible if

$$\int_{\hat{\Omega}} h(|\nabla^a u|)\,\mathrm{d}x = \int_{\hat{\Omega}} h(|\nabla^a v|)\,\mathrm{d}x\,, \tag{11}$$

$$|\nabla^s u|(\hat{\Omega}) = |\nabla^s v|(\hat{\Omega})\,. \tag{12}$$

¿From (11), from $|\nabla^a v| \le |\nabla^a u|$ and from the requirement (2) it is immediate that

$$|\nabla^a u| = |\nabla^a v| \quad \mathcal{L}^n\text{-a.e. on } \hat{\Omega}\,. \tag{13}$$

If $E \subset \hat{\Omega}$ is a Borel set, then analogous to (10) we get from (5) and (9)

$$|\nabla^s v|(E) = |\nabla v|(S_v \cap E) \le |\nabla u|(S_u \cap E) = |\nabla^s u|(E)\,. \tag{14}$$

At the same time – using (14) with E replaced by $\hat{\Omega} - E$ – it holds on account of (12)

$$\begin{aligned} |\nabla^s v|(E) &= |\nabla^s v|(\hat{\Omega}) - |\nabla^s v|(\hat{\Omega} - E) \ \ge\ |\nabla^s v|(\hat{\Omega}) - |\nabla^s u|(\hat{\Omega} - E) \\ &= |\nabla^s u|(\hat{\Omega}) - |\nabla^s u|(\hat{\Omega} - E) \ =\ |\nabla^s u|(E)\,, \end{aligned}$$

and with (14) it is shown that

$$|\nabla^s u| = |\nabla^s v|\,. \tag{15}$$

Suppose that

$$\mathcal{L}^n\Big(\big\{x \in \hat{\Omega} : \nabla^a u(x) \neq \nabla^a v(x)\big\}\Big) > 0 . \tag{16}$$

We have

$$\int_{[\nabla^a u \neq \nabla^a v]} (|\nabla^a u| + |\nabla^a v| - |\nabla^a u + \nabla^a v|)\, \mathrm{d}x > 0 , \tag{17}$$

since otherwise

$$|\nabla^a u + \nabla^a v| = |\nabla^a u| + |\nabla^a v|$$

a.e. on $[\nabla^a u \neq \nabla^a v]$ and therefore

$$\nabla^a u = \lambda \nabla^a v$$

on this set with a non-negative function λ. But (13) then gives the contradiction $\lambda = 1$. ¿From (17) we get recalling (2)

$$\begin{aligned} \int_{\hat{\Omega}} h\bigg(\Big|\nabla^a\Big(\frac{u+v}{2}\Big)\Big|\bigg)\, \mathrm{d}x \;&<\; \int_{\hat{\Omega}} h\Big(\frac{1}{2}|\nabla^a u| + \frac{1}{2}|\nabla^a v|\Big)\, \mathrm{d}x \\ &\leq\; \frac{1}{2}\int_{\hat{\Omega}} h(|\nabla^a u|)\, \mathrm{d}x + \frac{1}{2}\int_{\hat{\Omega}} h(|\nabla^a v|)\, \mathrm{d}x , \end{aligned}$$

and since $|\nabla^s(u+v)| \leq |\nabla^s u| + |\nabla^s v|$ it follows from (13) and (15) that

$$\hat{I}\Big[\frac{u+v}{2}, \hat{\Omega}\Big] < \hat{I}[u, \hat{\Omega}] . \tag{18}$$

But $(u+v)/2$ belongs to $BV_{u_0}(\Omega; \mathbb{R}^N)$, thus the strict inequality (18) contradicts the minimizing property of u, and assumption (16) is wrong which means

$$\nabla^a u = \nabla^a v \quad \mathcal{L}^n\text{-a.e. on } \hat{\Omega} . \tag{19}$$

Consider the measure $\mu := |\nabla^s u|$. Using (15) we find μ-measurable functions Θ_u, Θ_v: $\hat{\Omega} \to \mathbb{R}^{nN}$ s.t. $|\Theta_u| = 1 = |\Theta_v|$ μ-a.e. and

$$\nabla^s u = \Theta_u \llcorner \mu , \qquad \nabla^s v = \Theta_v \llcorner \mu . \tag{20}$$

Let us assume that

$$\Big|\nabla^s\Big(\frac{u+v}{2}\Big)\Big|(\hat{\Omega}) < |\nabla^s u|(\hat{\Omega}) . \tag{21}$$

This implies on account of (19)

$$\hat{I}\Big[\frac{u+v}{2}, \hat{\Omega}\Big] = \int_{\hat{\Omega}} h(|\nabla^a u|)\, \mathrm{d}x + \bar{c}\Big|\nabla^s\Big(\frac{u+v}{2}\Big)\Big|(\hat{\Omega}) < \hat{I}[u, \hat{\Omega}]$$

which is in contradiction to the minimality of u. We therefore have in place of (21)

$$\bigg|\int_{\hat{\Omega}} \frac{1}{2}(\Theta_u + \Theta_v)\, \mathrm{d}\mu\bigg| = \mu(\hat{\Omega}) ,$$

hence
$$\mu(\hat{\Omega}) \leq \frac{1}{2}\int_{\hat{\Omega}} |\Theta_u + \Theta_v| \,\mathrm{d}\mu \leq \frac{1}{2}\int_{\hat{\Omega}} \big(|\Theta_u| + |\Theta_v|\big) \,\mathrm{d}\mu = \mu(\hat{\Omega})$$
and in conclusion
$$|\Theta_u + \Theta_v| = |\Theta_u| + |\Theta_v| \quad \mu\text{-a.e.}$$
For this reason we can write
$$\Theta_u = \bar{\lambda}\Theta_v$$
with $\bar{\lambda}$ non-negative and μ-measurable, but $|\Theta_u| = 1 = |\Theta_v|$ gives $\bar{\lambda} \equiv 1$, i.e. $\Theta_u = \Theta_v$ μ-a.e. From (20) it follows $\nabla^s u = \nabla^s v$ which together with (19) shows that $\nabla u = \nabla v$. Quoting Proposition 3.2 of [AFP] we see $u - v \equiv \mathit{const}$ and $u = u_0 = v$ on $\hat{\Omega} - \Omega$ yields $u = v$ and in conclusion $u(x) \in K$ a.e. The proof of Theorem 1 is complete. $\square$

For the sake of completeness we have a look at the scalar case for which it is possible to give up the special structure of the integrand and to obtain a maximum principle close to the classical one. To be precise, let us assume that $F\colon \mathbb{R}^n \to [0,\infty)$ is strictly convex together with $F(0) = 0$. For $u_0 \in W^1_1(\Omega)$ we consider again the variational problem
$$I[u,\Omega] = \int_\Omega F(\nabla u)\,\mathrm{d}x \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}^1_1(\Omega)\,, \tag{$\mathcal{P}$}$$
and observe
$$\inf_{\partial\Omega} u_0 \leq u \leq \sup_{\partial\Omega} u_0 \tag{22}$$
provided we can find a soluton $u \in W^1_1(\Omega)$ of $(\mathcal{P})$. In fact, if we assume $M := \sup_{\partial\Omega} u_0 < \infty$, then we deduce from
$$I[u,\Omega] \leq I\big[\min(u,M),\Omega\big]$$
that
$$\int_{[u>M]} F(\nabla u)\,\mathrm{d}x = 0\,,$$
and $0 \leq F(\nabla u/2) < F(\nabla u)/2$ on $[\nabla u \neq 0]$ implies $\nabla u = 0$ on $[u > M]$, hence $\nabla \max(u,M) = 0$, which shows $u \leq M$.

Let us now assume that F is of linear growth, i.e. with constants a, $A > 0$, b, $B \in \mathbb{R}$ it holds
$$a|\xi| + b \leq F(\xi) \leq A|\xi| + B \tag{23}$$
for all $\xi \in \mathbb{R}^n$. Moreover, we require
$$F(-\eta) = F(\eta) \quad \text{for all } \eta \in \mathbb{R}^n\,. \tag{24}$$

Then we have

Theorem 2. *Let the strictly convex function F satisfy (23) and (24) together with $F(0) = 0$. If $u \in \mathcal{M}$ denotes a generalized minimizer of problem ($\mathcal{P}$), then (the slightly weaker variant of (22))*

$$\inf_{\Omega} u_0 \le u(x) \le \sup_{\Omega} u_0 \tag{25}$$

is satisfied for a.a. $x \in \Omega$.

Proof. It is sufficient to consider the case $M := \sup_\Omega u_0 < \infty$ and to prove the second inequality stated in (25). We extend u_0 to a function of class $W^1_1(\hat\Omega)$ on a bounded Lipschitz domain $\hat\Omega \Supset \Omega$ assuming that this extension – again denoted by u_0 – still satisfies $u_0 \le M$ a.e. (now on $\hat\Omega$), since otherwise we may compose it with the function $\psi(t) := \min(M, t)$, $t \in \mathbb{R}$. As outlined in the proof of Theorem 1 the claim of Theorem 2 will follow if we can show that any solution $u \in BV_{u_0}(\Omega)$ of

$$\hat I[w, \hat\Omega] := \int_{\hat\Omega} F(\nabla^a w)\,\mathrm{d}x + \int_{\hat\Omega} F_\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)\,\mathrm{d}|\nabla^s w| \to \min \quad \text{in } BV_{u_0}(\Omega)$$

satisfies $u \le M$ a.e. Quoting the chain rule for real valued functions as stated in Theorem 3.99 of [AFP] we have $v := \psi \circ u \in BV_{u_0}(\Omega)$ together with

$$\nabla v = \psi'(u)\nabla^a u \llcorner \mathcal{L}^n + \big(\psi(u^+) - \psi(u^-)\big)\nu_u \mathcal{H}^{n-1} \llcorner J_u + \psi'(\tilde u)\nabla^c u\,,$$

where our notation follows the terminology of [AFP]. Let us look at the part $\psi'(u)\nabla^a u \llcorner \mathcal{L}^n$ of the vector measure ∇v being absolutely continuous w.r.t. $\mathcal{L}^n$. It holds $\psi'(u) = 0$ a.e. on the set $[u > M]$, wheras $\psi'(u) = 1$ a.e. on $[u < M]$. Since the density $\nabla^a u$ equals the approximative differential of u (see [AFP], Theorem 3.83), and since the approximative differential of u vanishes a.e. on $[u = M]$ (see [AFP], Proposition 3.73 (c)), we get

$$\int_{\hat\Omega} F(\nabla^a v)\,\mathrm{d}x = \int_{[u<M]} F(\nabla^a u)\,\mathrm{d}x\,. \tag{26}$$

Notice that the measures $\nabla^j v$ and $\nabla^c v$ are mutually orthogonal, hence we can write

$$\int_{\hat\Omega} F_\infty\Big(\frac{\nabla^s v}{|\nabla^s v|}\Big)\,\mathrm{d}|\nabla^s v| = \int_{J_u} F_\infty\big(\psi(u^+) - \psi(u^-)\big)\nu_u\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\hat\Omega} F_\infty\Big(\psi'(\tilde u)\frac{\nabla^c u}{|\nabla^c u|}\Big)\,\mathrm{d}|\nabla^c u|\,. \tag{27}$$

The function $\psi'(\tilde u)$ has values in $\{0, 1\}$, which means

$$F_\infty\Big(\psi'(\tilde u)\frac{\nabla^c u}{|\nabla^c u|}\Big) \le F_\infty\Big(\frac{\nabla^c u}{|\nabla^c u|}\Big)$$

$|\nabla^c u|$-a.e. At the same time we have $\mathcal{H}^{n-1}$-a.e. on J_u

$$\begin{aligned}
F_\infty\big(\big(\psi(u^+) - \psi(u^-)\big)\nu_u\big) &= |\psi(u^+) - \psi(u^-)| F_\infty\big(\mathrm{sign}\big[\psi(u^+) - \psi(u^-)\big]\nu_u\big) \\
&= |\psi(u^+) - \psi(u^-)| F_\infty(\nu_u) \\
&\le |u^+ - u^-| F_\infty(\nu_u) \\
&= F_\infty\big((u^+ - u^-)\nu_u\big)\,.
\end{aligned}$$

Here the first equality sign follows from the fact that the recession function is positively homogeneous of degree one, the second is a consequence of (24) and the last equation is established in the same way. Combing the inequalities from above with (26) and (27) and using the minimality of u we find

$$\int_{[u\geq M]} F(\nabla^a u)\,\mathrm{d}x = 0 \tag{28}$$

together with

$$\int_{J_u} F_\infty\big((\psi(u^+)-\psi(u^-))\nu_u\big)\,\mathrm{d}\mathcal{H}^{n-1} = \int_{J_u} F_\infty\big((u^+-u^-)\nu_u\big)\,\mathrm{d}\mathcal{H}^{n-1} \tag{29}$$

and

$$\int_{\hat\Omega} F_\infty\Big(\psi'(\tilde u)\frac{\nabla^c u}{|\nabla^c u|}\Big)\,\mathrm{d}|\nabla^c u| = \int_{\hat\Omega} F_\infty\Big(\frac{\nabla^c u}{|\nabla^c u|}\Big)\,\mathrm{d}|\nabla^c u|\,. \tag{30}$$

¿From (28) we deduce using the strict convexity of F together with $F(0)=0$ that

$$\nabla^a u = 0 \quad \mathcal{L}^n\text{-a.e. on } [u\geq M]\,. \tag{31}$$

¿From (29) and

$$F_\infty\big((\psi(u^+)-\psi(u^-))\nu_u\big) \leq F_\infty\big((u^+-u^-)\nu_u\big)$$

$\mathcal{H}^{n-1}$-a.e. on J_u it follows that

$$F_\infty\big((\psi(u^+)-\psi(u^-))\nu_u\big) = F_\infty\big((u^+-u^-)\nu_u\big) \tag{32}$$

$\mathcal{H}^{n-1}$-a.e. on J_u, since otherwise we would have a contradiction to the minimality of u. (32) gives

$$|\psi(u^+)-\psi(u^-)| = |u^+-u^-| \tag{33}$$

$\mathcal{H}^{n-1}$-a.e. on J_u (recall $F_\infty(t\xi)=|t|F_\infty(\xi)$) but by definition of ψ this means

$$\psi(u^+)-\psi(u^-) = u^+-u^- \tag{34}$$

$\mathcal{H}^{n-1}$-a.e. on J_u. In the same way we obtain from (30), from

$$F_\infty\Big(\psi'(\tilde u)\frac{\nabla^c u}{|\nabla^c u|}\Big) \leq F_\infty\Big(\frac{\nabla^c u}{|\nabla^c u|}\Big)$$

and from the minimality of u that

$$\psi'(\tilde u) = 1 \quad |\nabla^c u| - \text{a.e.} \tag{35}$$

Recalling the formula for ∇v and using (31), (34) and (35) we arrive at $\nabla v = \nabla u$, hence $v=u$ and in conclusion $u\leq M$ a.e. on $\hat\Omega$. □

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