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Abstract

We consider incremental problem arising in elasto-plastic models with isotropic hardening. Our goal is to derive computable and guaranteed bounds of the difference between the exact solution and any function in the admissible (energy) class of the problem considered. Such estimates are obtained by an advanced version of the variational approach earlier used for linear boundary-value problems and nonlinear variational problems with convex functionals [19, 23]. They do no contain mesh-dependent constants and are valid for any conforming approximations regardless of the method used for their derivation. It is shown that the structure of error majorant reflects properties of the exact solution so that the majorant vanishes only if an approximate solution coincides with the exact one. Moreover, it possesses necessary continuity properties, so that any sequence of approximations converging to the exact solution in the energy space generates a sequence of positive numbers (explicitly computable by the majorant functional) that tends to zero.

1 Introduction

Incremental models in the theory of elasto-plasticity are among the most widely used in the numerical analysis of processes that include plasticity phenomenon. These typically include memory effect and exhibit hysteresis [6, 17] behavior which are described by time-dependent variational inequalities. If an implicit Euler scheme is used, then the evolutionary variational inequality is approximated by a sequence of stationary variational inequalities of the second kind [16, 3] in which the unknown functions are displacement u and plastic strain p. Each of these inequalities is equivalent to a minimization problem with a convex but non-smooth energy functional, $J(u, p) \to \min$. The minimization problem is solved by iterative methods like a classical return mapping algorithm [26], inexact Newton methods [10] or SQP method [27] among many others.

The main focus here is not to develop new methods for solving the minimization problems, but to deduce a guaranteed a posteriori estimate of the difference between exact and numerical solutions. A posteriori estimates are intended to (a) give an indication of the overall accuracy of an approximate solution and (b) serve as an error indicator that show regions with excessively high errors (typically a new finite dimensional space constructed on the basis of this information has extra trial functions in each regions). There exist various approaches to the construction of a posteriori error estimates (a discussion of them can be found e. g., in the monographs [5, 1, 7, 19] or in the recent overview [23], whereas for applications to elasto-plasticity, we refer to the works [20, 12].

In this paper, we apply the framework introduced in the book [19], where the estimates are derived by the analysis of the variational problem and its dual counterpart. A computable upper bound of the error is obtained on a purely functional level without exploitation of specific properties of the approximation or the method used for its computation. Estimates of such a type are often called "functional a posteriori estimates". One of the first publications presenting this method was [25] where a posteriori estimates were derived for a deformation plasticity model with hardening. Recently, the method was applied to the Ramberg-Osgood model (sometimes also called Norton-Hoff) in the theory of nonlinear solid media [9], to nonlinear viscous flow problems [8, 14] and to problems with nonlinear boundary conditions [24].

2 Minimization problem and variational inequality

We consider the first time-step problem for the elasto-plasticity model with isotropic hardening and von Mises yield criterion. It can be represented in a variational form as an energy minimization problem (see [3], Definition 3.3)

$$J(v,q) := \frac{1}{2}\int\limits_{\Omega} \left(\mathbb{C}(\boldsymbol{\varepsilon}(v)-q) : (\boldsymbol{\varepsilon}(v)-q) + \sigma_y^2 H^2 |q|^2\right)\,dx + \int\limits_{\Omega} \left(\sigma_y |q| - fv\right)\,dx \to \min, \quad (2.1)$$

for an unknown displacement v and a plastic strain q. Here, $\Omega \subset \mathbb{R}^d$ is a bounded connected domain with Lipschitz boundary Γ. In (2.1), $\boldsymbol{\varepsilon}(\cdot)$ denotes the linearized Green-St. Venant strain tensor defined as

$$\boldsymbol{\varepsilon}(v) := \frac{1}{2}\left(\nabla v + (\nabla v)^T\right), \quad (2.2)$$

where ∇ denotes a vector gradient operator. $J(v,q)$ is minimized over

$$v \in V_0 + u_0 := \left\{ w + u_0 \;\middle|\; w \in H_0^1(\Omega;\mathbb{R}^d) \right\},$$

and

$$q \in Q_0 := \{q \in Q := L^2(\Omega; \mathbb{R}^{d\times d}_{sym}) : \operatorname{tr} q = 0 \ \ a.\ e. \text{ in } \Omega\},$$

where the function $u_0 \in H^1(\Omega; \mathbb{R}^d)$ defines a nonhomogeneous Dirichlet boundary condition on Γ (which is understood in the sense of traces) and tr denotes the trace operator defined by the relation $\operatorname{tr} A = A : \mathbb{I}$ for all $A \in \mathbb{R}^{d\times d}$ with $\mathbb{I} \in \mathbb{R}^{d\times d}$ meaning the identity matrix. The positive scalar constants H and σ_y represent the modulus of hardening and yield stress, respectively, and

$$\mathbb{C} \in \mathcal{L}(\mathbb{R}^{d\times d}_{sym}, \mathbb{R}^{d\times d}_{sym})$$

denotes the fourth-order elastic stiffness tensor which satisfies the relation (for known positive constants $c_1 \leq c_2$)

$$c_1|q|^2 \leq \mathbb{C}q : q \leq c_2|q|^2 \tag{2.3}$$

for all $q \in \mathbb{R}^{d\times d}_{sym}$. Finally, the vector

$$f \in L^2(\Omega; \mathbb{R}^d)$$

expresses external forces acting on an elastoplastic continuum located in the domain Ω. For more details on mechanical aspects of this mathematical model and its possible generalization please refer to [16, 11].

Theorem 1 *There exists a pair $(u, p) \in (V_0 + u_0) \times Q_0$ that solves* (2.1)*. It satisfies the variational inequality*

$$a(u, p; v - u, q - p) + \Psi(q) - \Psi(p) - l(v - u) \geq 0, \tag{2.4}$$

where

$$\begin{aligned}
a(u, p; v, q) &:= \int_\Omega \left(\mathbb{C}(\boldsymbol{\varepsilon}(u) - p) : (\boldsymbol{\varepsilon}(v) - q) + \sigma_y^2 H^2 p : q\right) dx, \\
\Psi(q) &:= \int_\Omega \sigma_y |q| \, dx, \\
l(v) &:= \int_\Omega f v \, dx
\end{aligned}$$

for all $(v, q) \in V_0 \times Q_0$.

Proof. Existence of $(u, p) \in (V_0 + u_0) \times Q_0$ follows from known results in the calculus of variations. Indeed, the functional $J(v, q)$ is convex and coercive on

$(V_0+u_0)\times Q_0$, which is a convex closed subset of the product space $V\times Q$, where $V=H^1(\Omega,\mathbb{R}^d)$ and Q are reflexive spaces. Due to the assumption (2.3), the ellipticity and boundedness of the bilinear form $a(u,p;v,q)$ are easily proved. Then, the equivalence of the variational inequality (2.4) and (2.1) follows from the Lions-Stampacchia Theorem [18]. □

Remark 1 *By the variation of the energy functional $J(v,q)$, it is simple to show that the minimizer (u,p) must satisfy the following relations almost everywhere in Ω:*

$$\sigma := \mathbb{C}(\boldsymbol{\varepsilon}(u)-p), \tag{2.5}$$

$$\operatorname{div}\sigma + f = 0, \tag{2.6}$$

$$\sigma^D = \sigma_y^2 H^2 p + \sigma_y \lambda, \tag{2.7}$$

$$\lambda \in \begin{cases} \Lambda & \text{if } p=0, \\ \frac{p}{|p|} & \text{otherwise,} \end{cases} \tag{2.8}$$

where σ is the stress tensor associated with the exact solution. These relations have a clear physical meaning: (2.5) *expresses an exact stress tensor σ as an additive decomposition of a linearized elastic strain $\boldsymbol{\varepsilon}(v)$ and a plastic strain p combined with a Hook's law.* (2.6) *formulates an equilibrium of internal and external forces in the quasistatic case.* (2.7) *and* (2.8) *formulate a plasticity flow law in the case of von Mises yield function. See [3] for more details on the mechanical model and its mathematical aspects.*

3 Basic estimate of the deviation from exact solution

Theorem 2 *For any $(v,q)\in(V_0+u_0)\times Q_0$, the estimate*

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \le J(v,q)-J(u,p) \tag{3.1}$$

holds, where the norm in the left hand side is defined by the relation

$$\begin{aligned} |||(u-v),(p-q)|||^2 &:= a(u-v,p-q;u-v,p-q) \\ &= \|\mathbb{C}(\boldsymbol{\varepsilon}(u-v)-p+q)\|^2_{\mathbb{C}^{-1};\Omega} + \sigma_y^2 H^2\,\|p-q\|^2_\Omega \end{aligned} \tag{3.2}$$

and

$$\|\kappa\|^2_{\mathbb{C}^{-1};\Omega} := \int_\Omega \mathbb{C}^{-1}\kappa : \kappa\,dx.$$

Proof. The direct calculation shows

$$J(v,q) - J(u,p) = \tfrac{1}{2}a(v,q;v,q) - \tfrac{1}{2}a(u,p;u,p) + \Psi(q) - \Psi(p) - l(v) + l(u)$$
$$= \tfrac{1}{2}a(u-v,p-q;u-v,p-q) + a(u,p;v-u,q-p) + \Psi(q) - \Psi(p) - l(v-u).$$

In view of (2.4), we obtain (3.1). □

Remark 2 *Since $\sigma_y^2 H^2 > 0$, the term $\sigma_y^2 H^2 \|p-q\|_\Omega^2$ can be dropped to obtain a weaker estimate (formulated as (28) in [12])*

$$\frac{1}{2}\|\sigma - \tau\|^2_{\mathbb{C}^{-1};\Omega} \le J(v,q) - J(u,p),$$

where $\sigma := \mathbb{C}(\boldsymbol{\varepsilon}(u) - p)$ and $\tau := \mathbb{C}(\boldsymbol{\varepsilon}(v) - q)$ have meaning of an exact stress σ and an approximate stress τ.

In the following, we will bound the difference $J(v,q) - J(u,p)$ in (3.1) from above by a directly computable and physically meaningful term, which does not involve the exact solution (u,p).

4 Perturbed problem and Lagrangian

The value of $J(u,p)$ is unknown in the estimate (3.1). To use this estimate, we need to find a computable lower bound of this quantity. For this purpose, we introduce a "perturbed" functional

$$J_\lambda(v,q) := \frac{1}{2}a(v,q;v,q) + \int_\Omega \sigma_y \lambda : q\,dx - l(v),$$

where the multiplier λ belongs to the set

$$\Lambda := \{\lambda \in L^\infty(\Omega, \mathbb{R}^{d\times d}) : |\lambda| \le 1, \operatorname{tr}\lambda = 0 \;\; a.\ e. \text{ in } \Omega\}.$$

The relation of the original and the perturbed problem is given by

$$\sup_{\lambda\in\Lambda} J_\lambda(v,q) = J(v,q) \tag{4.1}$$

for all $(v,q) \in (V_0 + u_0) \times Q_0$. Further we define the respective "perturbed" Lagrangian

$$\begin{aligned} L_\lambda(v,q;\tau,\xi) : &= \int_\Omega \left(\tau : (\boldsymbol{\varepsilon}(v) - q) - \frac{\mathbb{C}^{-1}\tau : \tau}{2} + \xi : q - \frac{|\xi|^2}{2\sigma_y^2 H^2} \right) dx \\ &+ \int_\Omega \left(\sigma_y \lambda : q - f v\right)\, dx, \end{aligned} \tag{4.2}$$

where new tensor-valued functions $\tau \in Q, \xi \in Q_0$. Since

$$\begin{aligned}
\sup_{\tau \in Q} \int\limits_\Omega \left(\tau : (\boldsymbol{\varepsilon}(v) - q) - \frac{\mathbb{C}^{-1}\tau : \tau}{2} \right) dx &= \frac{1}{2} \int\limits_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q) : (\boldsymbol{\varepsilon}(v) - q)\, dx, \\
\sup_{\xi \in Q_0} \int\limits_\Omega \left(\xi : q - \frac{|\xi|^2}{2\sigma_y^2 H^2} \right) dx &= \frac{1}{2} \int\limits_\Omega \sigma_y^2 H^2 |q|^2\, dx,
\end{aligned}$$

it is easy to see

$$\sup_{\substack{\tau \in Q \\ \xi \in Q_0}} L_\lambda(v, q; \tau, \xi) = J_\lambda(v, q) \tag{4.3}$$

for all $(v, q) \in (V_0 + u_0) \times Q_0$. Thus, the combination of (4.1) and (4.3) provides an estimate

$$\begin{aligned}
J(u, p) = \inf_{\substack{v \in V_0 + u_0 \\ q \in Q_0}} J(v, q) &\geq \inf_{\substack{v \in V_0 + u_0 \\ q \in Q_0}} J_\lambda(v, q) = \inf_{\substack{v \in V_0 + u_0 \\ q \in Q_0}} \sup_{\substack{\tau \in Q \\ \xi \in Q_0}} L_\lambda(v, q; \tau, \xi) \\
&\geq \sup_{\substack{\tau \in Q \\ \xi \in Q_0}} \inf_{\substack{v \in V_0 + u_0 \\ q \in Q_0}} L_\lambda(v, q; \tau, \xi) \\
&\geq \inf_{\substack{v \in V_0 + u_0 \\ q \in Q_0}} L_\lambda(v, q; \tau, \xi)
\end{aligned}$$

whose substitution in (3.1) yields the inequality

$$\frac{1}{2} |||(u - v), (p - q)|||^2 \leq J(v, q) - \inf_{\substack{v \in V_0 + u_0 \\ q \in Q_0}} L_\lambda(v, q; \tau, \xi) =: \mathcal{M}(v, q; \tau, \xi, \lambda)$$

valid for all $\tau \in Q, \xi \in Q_0$. The right hand side of the last estimate defines an error majorant $\mathcal{M}(v, q; \tau, \xi, \lambda)$. Its explicit form of is derived in the next section.

5 Derivation of the error majorant

The infimum of Lagrangian (4.2) can be rewritten in the form

$$\begin{aligned}
&\inf_{\substack{v \in V_0 + u_0 \\ q \in Q_0}} L_\lambda(v, q; \tau, \xi) = \inf_{\substack{w \in V_0 \\ q \in Q_0}} L_\lambda(w + u_0, q; \tau, \xi) \\
= \quad & -\frac{1}{2} \int\limits_\Omega \left(\mathbb{C}^{-1}\tau : \tau + \frac{|\xi|^2}{\sigma_y^2 H^2} \right) dx + \int\limits_\Omega \left(\tau : \boldsymbol{\varepsilon}(u_0) - f u_0 \right) dx \\
&+ \inf_{w \in V_0} \int\limits_\Omega (\tau : \boldsymbol{\varepsilon}(w) - f w)\, dx + \inf_{q \in Q_0} \int\limits_\Omega (\xi + \sigma_y \lambda - \tau) : q\, dx.
\end{aligned} \tag{5.1}$$

Note it holds

$$\inf_{w\in V_0}\int_\Omega (\tau : \boldsymbol{\varepsilon}(w) - fw)\,dx = \begin{cases} 0 & \text{if } \operatorname{div}\tau + f = 0 \ \ a.\ e. \text{ in } \Omega, \\ -\infty & \text{otherwise}, \end{cases}$$

$$\inf_{q\in Q_0}\int_\Omega (\xi + \sigma_y\lambda - \tau) : q\,dx = \begin{cases} 0 & \text{if } \tau^D = \xi + \sigma_y\lambda \ \ a.\ e. \text{ in } \Omega, \\ -\infty & \text{otherwise}, \end{cases}$$

where $\cdot^D$ denotes a deviatoric operator, i.e., $A^D = A - \frac{\operatorname{tr} A}{d}\mathbb{I}$, for all $A \in \mathbb{R}^{d\times d}$. Hence, we arrive at the following result.

$$\inf_{\substack{v\in V_0+u_0\\ q\in Q_0}} L_\lambda(v,q;\tau,\xi) = \begin{cases} -\int_\Omega \left(\frac{\mathbb{C}^{-1}\tau:\tau}{2} + \frac{|\xi|^2}{2\sigma_y^2H^2} - \tau : \boldsymbol{\varepsilon}(u_0) + fu_0\right)dx \\ \quad \text{if } (\tau,\xi) \in Q_{f_\lambda}, \\ -\infty \quad \text{otherwise}, \end{cases} \tag{5.2}$$

where

$$Q_{f_\lambda} := \{(\tau,\xi) \in Q \times Q_0 : \operatorname{div}\tau + f = 0, \tau^D = \xi + \sigma_y\lambda \ \ a.\ e. \text{ in } \Omega\}. \tag{5.3}$$

The combination of (2.1) and (5.2) yields an explicit form of the error majorant estimate under the assumption $(\tau,\xi) \in Q_{f_\lambda}$,

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left(\mathbb{C}(\boldsymbol{\varepsilon}(v) - q) : (\boldsymbol{\varepsilon}(v) - q) + \sigma_y^2H^2|q|^2\right)dx + \int_\Omega (\sigma_y|q| - fv)\,dx \\ &+\frac{1}{2}\int_\Omega \left(\mathbb{C}^{-1}\tau : \tau + \frac{|\xi|^2}{\sigma_y^2H^2}\right)dx - \int_\Omega (\tau : \boldsymbol{\varepsilon}(u_0) - fu_0)\,dx \\ = \ &\frac{1}{2}\int_\Omega \left(\mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) + \sigma_y^2H^2(q - \frac{\xi}{\sigma_y^2H^2})^2\right)dx \\ &+ \int_\Omega \sigma_y|q|\,dx - \int_\Omega (q : \tau - \xi : q)\,dx + \int_\Omega (\tau : \boldsymbol{\varepsilon}(v - u_0) - f(v - u_0))\,dx. \end{aligned}$$

After the simplification of the last integral terms due to the constrain $(\tau,\xi) \in Q_{f_\lambda}$,

$$\int_\Omega q : \tau - \xi : q\,dx = \int_\Omega \sigma_y\lambda : q\,dx \qquad \text{for all } q \in Q_0,$$

$$\int_\Omega \tau : \boldsymbol{\varepsilon}(v - u_0) - f(v - u_0)\,dx = 0,$$

we deduce an explicit form of the error majorant

$$\mathcal{M}(v,q;\tau,\xi,\lambda) = \frac{1}{2}\int\limits_{\Omega} \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau)\,dx$$

$$+\frac{1}{2}\int\limits_{\Omega} \sigma_y^2 H^2 \left(q - \frac{1}{\sigma_y^2 H^2}\xi\right)^2 dx + \int\limits_{\Omega} (\sigma_y|q| - \sigma_y\lambda : q)\,dx. \quad (5.4)$$

Summarizing the above considerations, we formulate the following result.

Theorem 3 *The majorant (5.4) represents a guaranteed upper bound of the combined error norm*

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \;\leq\; \mathcal{M}(v,q;\tau,\xi,\lambda). \quad (5.5)$$

It is valid for any $(v,q) \in (V_0 + u_0) \times Q_0$, $\lambda \in \Lambda$, *and* $(\tau,\xi) \in Q_{f_\lambda}$.

Remark 3 *The majorant (5.4) was derived by purely functional analysis of the problem in question. It does not involve mesh-dependent constants and is valid for any admissible (conforming) approximations from the respective functional classes associated with the primal variational problem. For this reason, error majorants (or a posteriori error estimates) of this type are called functional.*

Remark 4 *In order to get the upper bound as sharp as possible, we should minimize the right hand side with respect to free functions and use the estimate*

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \;\leq\; \inf_{(\tau,\xi)\in Q_{f_\lambda}} \mathcal{M}(v,q;\tau,\xi,\lambda) \quad (5.6)$$

This estimate is also valid for any $(v,q) \in (V_0 + u_0) \times Q_0$, $\lambda \in \Lambda$.

Remark 5 *It is easy to see that the functional error majorant* $\mathcal{M}(v,q;\tau,\xi,\lambda)$ *defined in* (5.4) *reflects natural conditions (2.5)–(2.8). Indeed, it attains the zero value if and only if the following conditions hold almost everywhere in* Ω*:*

$$\tau = \mathbb{C}(\boldsymbol{\varepsilon}(v) - q), \quad (5.7)$$

$$\operatorname{div}\tau + f = 0, \quad (5.8)$$

$$\lambda : q = |q|, \qquad \lambda \in \Lambda, \quad (5.9)$$

$$\tau^D = \xi + \sigma_y\lambda, \quad (5.10)$$

$$\xi = \sigma_y^2 H^2 q. \quad (5.11)$$

These conditions mean that (v,q) *and* τ *satisfy (2.5)–(2.8); in other words they must be equal to the solution* (u,p) *of the minimization problem* (2.1) *and the exact stress tensor* σ*, respectively.*

6 Modification of the majorant

For practical computation, an approximated displacement $v \in (V_0 + u_0)$ and $q \in Q_0$ are computed numerically, e. g., by the finite element method. The error of such approximation is bounded in the combined norm from above by a functional majorant $\mathcal{M}(v, q; \tau, \xi, \lambda)$ defined in (5.4). To obtain a finite and therefore meaningful value of the functional majorant, free parameters must satisfy the conditions $(\tau, \xi) \in Q_{f_\lambda}$ and $\lambda \in \Lambda$. It is known that the equilibrium constrain $\operatorname{div} \tau + f = 0$ or its equivalent weak formulation

$$\int_\Omega (-\tau : \boldsymbol{\varepsilon}(w) + f w)\, dx = 0 \quad \text{for all } w \in V_0 \tag{6.1}$$

is difficult to satisfy. Therefore, an upper bound of $\mathcal{M}(v, q; \tau, \xi, \lambda)$ is derived here. It does not contain the parameter ξ as well as it remains independent of the equilibrium constrain, which is transformed into a penalty term. It turns out useful to split the majorant (5.4) in three parts:

$$\mathcal{M}_1(v, q; \tau) := \frac{1}{2} \int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau)\, dx, \tag{6.2}$$

$$\mathcal{M}_2(q; \xi) := \frac{1}{2} \int_\Omega \sigma_y^2 H^2 \left(q - \frac{1}{\sigma_y^2 H^2} \xi \right)^2 dx, \tag{6.3}$$

$$\mathcal{M}_3(q; \lambda) := \int_\Omega (\sigma_y |q| - \sigma_y \lambda : q)\, dx. \tag{6.4}$$

We exclude ξ by setting $\xi = \tau^D - \sigma_y \lambda$ according to (5.3) to rewrite

$$\begin{aligned} \mathcal{M}_2(q; \xi) &= \frac{1}{2} \int_\Omega \sigma_y^2 H^2 \left(q - \frac{1}{\sigma_y^2 H^2} (\tau^D - \sigma_y \lambda) \right)^2 dx \\ &= \frac{1}{2} \int_\Omega \frac{1}{\sigma_y^2 H^2} (\tau^D - \zeta)^2\, dx =: \bar{\mathcal{M}}_2(q; \tau, \lambda), \end{aligned}$$

where $\zeta := \sigma_y^2 H^2 q + \sigma_y \lambda$ and obtain a simplified error majorant independent of ξ

$$\bar{\mathcal{M}}(v, q; \tau, \lambda) := \mathcal{M}_1(v, q; \tau) + \bar{\mathcal{M}}_2(q; \tau, \lambda) + \mathcal{M}_3(q; \lambda),$$

which is defined on

$$\tau \in Q_f := \{ \tau \in Q : \operatorname{div} \tau + f = 0 \ \ a.\ e. \text{ in } \Omega \}.$$

Let us decompose $\tau = \tau - \hat{\tau} + \hat{\tau}$, where

$$\hat{\tau} \in Q_{\rm div} := \{\tau \in Q : \operatorname{div} \tau \in L^2(\Omega, \mathbb{R}^d)\},$$

i.e., $\hat{\tau}$ does not have to satisfy the equilibrium condition (6.1). Then, we obtain

$$\begin{aligned}
\mathcal{M}_1(v, q; \tau) \quad &\leq \quad \frac{1}{2}(1+\beta) \int\limits_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\hat{\tau}) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\hat{\tau})\, dx \\
&\quad + \frac{1}{2}(1 + \frac{1}{\beta}) \int\limits_\Omega \mathbb{C}^{-1}(\tau - \hat{\tau}) : (\tau - \hat{\tau})\, dx, \\
\bar{\mathcal{M}}_2(q; \tau, \lambda) \quad &\leq \quad \frac{1}{2}(1+\delta) \int\limits_\Omega \frac{1}{\sigma_y^2 H^2} (\hat{\tau}^D - \zeta)^2\, dx \\
&\quad + \frac{1}{2}(1 + \frac{1}{\delta}) \int\limits_\Omega \frac{1}{\sigma_y^2 H^2} (\tau^D - \hat{\tau}^D)^2\, dx,
\end{aligned}$$

which is valid for $\hat{\tau} \in Q_{\rm div}, \tau \in Q_f$ and for all $\beta, \delta > 0$. Here, we used the inequality

$$(a+b)^2 \leq (1+\beta)a^2 + (1 + \frac{1}{\beta})b^2$$

valid for all $\beta > 0, a, b \in \mathbb{R}$ and its modification for $\beta = \delta$. Since the last integral in the $\bar{\mathcal{M}}_2(q; \tau, \lambda)$ estimate fulfills

$$\int\limits_\Omega (\tau^D - \hat{\tau}^D)^2\, dx \leq \int\limits_\Omega (\tau - \hat{\tau})^2\, dx \leq c_2 \int\limits_\Omega \mathbb{C}^{-1}(\tau - \hat{\tau}) : (\tau - \hat{\tau})\, dx,$$

we combine available bounds on $\mathcal{M}_1(v, q; \tau)$ and $\bar{\mathcal{M}}_2(q; \tau, \lambda)$ to obtain an ξ-independent estimate

$$\begin{aligned}
& \bar{\mathcal{M}}(v, q; \tau, \lambda, \beta, \delta) && (6.5) \\
\leq \quad & \frac{1}{2}(1+\beta) \int\limits_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\hat{\tau}) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\hat{\tau})\, dx \\
& + \frac{1}{2}(1+\delta) \int\limits_\Omega \frac{1}{\sigma_y^2 H^2} (\hat{\tau}^D - \zeta)^2 dx + \int\limits_\Omega (\sigma_y |q| - \sigma_y \lambda : q)\, dx \\
& + \frac{1}{2} \left[(1 + \frac{1}{\beta}) + \frac{c_2}{\sigma_y^2 H^2}(1 + \frac{1}{\delta}) \right] \int\limits_\Omega \mathbb{C}^{-1}(\tau - \hat{\tau}) : (\tau - \hat{\tau})\, dx && (6.6) \\
=: \quad & \bar{\mathcal{M}}(v, q; \tau, \lambda, \beta, \delta, \hat{\tau}).
\end{aligned}$$

valid for $\hat{\tau} \in Q_{\rm div}$.

Lemma 1 *Let $\hat{\tau} \in Q_{\rm div}$. Then, it holds*

$$\inf_{\tau \in Q_f} \frac{1}{2} \int\limits_\Omega \mathbb{C}^{-1}(\tau - \hat{\tau}) : (\tau - \hat{\tau})\, dx \leq \frac{1}{2} C^2 \left\| \operatorname{div} \hat{\tau} + f \right\|^2,$$

where $C > 0$ satisfies the inequality

$$\|w\| \leq C \left\| \boldsymbol{\varepsilon}(w) \right\|_{\mathbb{C}} \quad \textit{for all } w \in V_0. \tag{6.7}$$

Proof. The direct calculation reveals

$$\begin{aligned}
I(\hat{\tau}) \ &:= \ \inf_{\tau \in \mathcal{Q}_f} \frac{1}{2} \int\limits_\Omega \mathbb{C}^{-1}(\tau - \hat{\tau}) : (\tau - \hat{\tau})\, dx \\
&= \ \inf_{\tau \in Q} \sup_{w \in V_0} \frac{1}{2} \int\limits_\Omega \left(\mathbb{C}^{-1}(\tau - \hat{\tau}) : (\tau - \hat{\tau}) + \tau : \boldsymbol{\varepsilon}(w) - f w \right) dx \\
&\quad \ \text{the interchange of operators follows e. g., from [13], Theorem 4.1.} \\
&= \ \sup_{w \in V_0} \inf_{\tau \in Q} \int\limits_\Omega \left(\frac{1}{2} \mathbb{C}^{-1}(\tau - \hat{\tau}) : (\tau - \hat{\tau}) + \tau : \boldsymbol{\varepsilon}(w) - f w \right) dx \\
&\quad \ \text{the infimum is attained in the argument } \tau = \hat{\tau} - \mathbb{C}\boldsymbol{\varepsilon}(w) \\
&= \ \sup_{w \in V_0} \left(-\frac{1}{2} \left\| \boldsymbol{\varepsilon}(w) \right\|_{\mathbb{C}}^2 - \int\limits_\Omega (\operatorname{div} \hat{\tau} + f) w \, dx \right) \\
&\leq \ \sup_{w \in V_0} \left(-\frac{1}{2} \left\| \boldsymbol{\varepsilon}(w) \right\|_{\mathbb{C}}^2 + \left\| \operatorname{div} \hat{\tau} + f \right\| \|w\| \right) \\
&\leq \ \sup_{w \in V_0} \left(-\frac{1}{2} \left\| \boldsymbol{\varepsilon}(w) \right\|_{\mathbb{C}}^2 + C \left\| \operatorname{div} \hat{\tau} + f \right\| \left\| \boldsymbol{\varepsilon}(w) \right\|_{\mathbb{C}} \right) \\
&\leq \ \sup_{t \geq 0} \left(-\frac{1}{2} t^2 + C \left\| \operatorname{div} \hat{\tau} + f \right\| t \right) = \frac{1}{2} C^2 \left\| \operatorname{div} \hat{\tau} + f \right\|^2,
\end{aligned}$$

where the constant $C > 0$ comes from (6.7). The existence of such constant follows from the Korn's and Friedrichs' inequalities. □
Lemma 1 allows for the reformulation of (6.7) in

$$\bar{\mathcal{M}}(v, q; \tau, \lambda, \beta, \delta) \leq \inf_{\tau \in \mathcal{Q}_f} \bar{\mathcal{M}}(v, q; \tau, \lambda, \beta, \delta, \hat{\tau}) =: \hat{\mathcal{M}}(v, q; \hat{\tau}, \lambda, \beta, \delta),$$

where an non-equilibrated majorant $\hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta)$ is defined as

$$\begin{aligned}\hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta) \;:=\; & \frac{1}{2}(1+\beta)\int\limits_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v)-q-\mathbb{C}^{-1}\hat{\tau}):(\boldsymbol{\varepsilon}(v)-q-\mathbb{C}^{-1}\hat{\tau})\,dx \\ & +\frac{1}{2}(1+\delta)\int\limits_\Omega \frac{1}{\sigma_y^2H^2}(\hat{\tau}^D-\zeta)^2dx+\int\limits_\Omega(\sigma_y|q|-\sigma_y\lambda:q)\,dx \\ & +\frac{1}{2}\left[(1+\frac{1}{\beta})+\frac{c_2}{\sigma_y^2H^2}(1+\frac{1}{\delta})\right]C^2\left\|\operatorname{div}\hat{\tau}+f\right\|^2. \qquad (6.8)\end{aligned}$$

It is clear that the majorant vanishes if and only if (v,q) and $\hat{\tau}$ satisfy (2.5)–(2.8), i.e., if these functions coincide with exact solutions. Hence, we arrive at the following result:

Theorem 4 *The majorant (6.8) represents a guaranteed upper bound of the combined error norm*

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \;\leq\; \hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta). \qquad (6.9)$$

It is valid for any $(v,q)\in(V_0+u_0)\times Q_0$, $\lambda\in\Lambda$, $\hat{\tau}\in Q_{\rm div}$, $\beta,\delta>0$. *The majorant vanishes if and only if* (v,q) *coincides with the solution* (u,p) *of the minimization problem (2.1) and* $\hat{\tau}$ *coincides with the exact stress* $\sigma=\mathbb{C}(\boldsymbol{\varepsilon}(u)-p)$.

Remark 6 *We see that* λ *enters only the second and the third integral terms of (6.8) so that the best* λ *can be found by minimizing them over* $\lambda\in\Lambda$. *If the constrain is not active, then the variation of* $\hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta)$ *with respect to* λ *provides the relation*

$$(1+\delta)\frac{1}{\sigma_y^2H^2}(\zeta-\hat{\tau}^D)\frac{\partial\zeta}{\partial\lambda}+\sigma_yq=0,$$

which implies (we recall $\zeta=\sigma_y^2H^2q+\sigma_y\lambda$*)*

$$\lambda=\lambda_0:=\frac{1}{\sigma_y}\hat{\tau}^D-\frac{\delta\sigma_yH^2q}{1+\delta}.$$

If $|\lambda_0|>1$, *then minimal value is attained for* $|\lambda|=1$, *which yields*

$$\lambda=\lambda_1:=\frac{\lambda_0}{|\lambda_0|}.$$

Remark 7 *From Theorem 4 it follows that*

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \quad \leq \quad \inf_{\substack{\hat{\tau}\in Q_{\rm div}\\ \lambda\in\Lambda}} \hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta) =: \varrho(v,q). \quad (6.10)$$

By the construction, the quantity $\varrho(v,q)$ gives the best upper bound of the combined error norm. For practical applications, λ and $\hat{\tau}$ should be selected such that the value of $\hat{\mathcal{M}}(v,q,\hat{\tau},\lambda)$ does not differ much from $\varrho(v,q)$. Finding suitable λ is not a complicated task. A nature choice is to take λ as in (2.8), *i.e.,*

$$\lambda = \lambda(q) \in \begin{cases} \Lambda & \text{if } q = 0, \\ \frac{q}{|q|} & \text{otherwise.} \end{cases} \qquad (6.11)$$

Hence, the third integral in (6.8) vanishes. The practical choice of $\hat{\tau}$ is not studied here, but should be a topic of the forthcoming paper about implementation of the estimates derived here. The natural choice to consider is the exact stress

$$\hat{\tau} = \mathbb{C}(\boldsymbol{\varepsilon}(u) - p), \qquad (6.12)$$

which is only theoretically interesting, since the exact solution (u,p) is not known. Then the last term in (6.8) *also vanishes and we can take $\delta = \beta = 0$ to simplify the majorant in*

$$\hat{\mathcal{M}}(v,q;\sigma,\lambda(q)) = \frac{1}{2}\left\|\mathbb{C}(\boldsymbol{\varepsilon}(u-v) - p + q)\right\|^2_{\mathbb{C}^{-1}} + \frac{1}{2}\int_\Omega \frac{1}{\sigma_y^2 H^2}(\sigma^D - \sigma_y^2 H^2 q - \sigma_y \lambda(q))^2 dx.$$

The last integral is rewritten using (2.7) *(i.e., $\sigma^D = \sigma_y^2 H^2 p + \sigma_y \lambda(p)$) and bounded as*

$$\frac{1}{2}\int_\Omega \frac{\left(\sigma_y^2 H^2 (p-q) + \sigma_y(\lambda(p) - \lambda(q))\right)^2}{\sigma_y^2 H^2}\, dx \leq \int_\Omega \sigma_y^2 H^2 (p-q)^2 dx + \int_\Omega \frac{\left(\lambda(p) - \lambda(q)\right)^2}{H^2}\, dx.$$

By introduction of

$$\varrho_1(p-q) := \frac{1}{2}\int_\Omega \sigma_y^2 H^2 (p-q)^2 dx + \int_\Omega \frac{\left(\lambda(p) - \lambda(q)\right)^2}{H^2}\, dx,$$

we can write the estimate in the combined norm (3.2),

$$\hat{\mathcal{M}}(v,q;\sigma,\lambda(q)) \le \frac{1}{2}|||(u-v),(p-q)|||^2 + \varrho_1(p-q). \qquad (6.13)$$

The value of $\varrho_1(p-q)$ *measures, how the majorant value* $\hat{\mathcal{M}}(v,q;\sigma,\lambda(q))$ *overestimates the value of error in the combined norm. Since* $\varrho_1(p-q)$ *depends on the exact plain strain* p*, it can not be practically computed, but can be at least simplified. To do it, let us decompose the domain* Ω *in its elastic and plastic parts in dependence of* p *or* q *as*

$$\Omega := \Omega^p_{ela} \cup \Omega^p_{pla}, \quad \Omega := \Omega^q_{ela} \cup \Omega^q_{pla},$$

where

$$\Omega^p_{ela} := \{x \in \Omega : p(x) = 0\}, \quad \Omega^p_{pla} := \Omega \setminus \Omega^p_{ela},$$
$$\Omega^q_{ela} := \{x \in \Omega : q(x) = 0\}, \quad \Omega^q_{pla} := \Omega \setminus \Omega^q_{ela},$$

and let us define

$$\omega_1 := \Omega^q_{ela} \cap \Omega^p_{ela}, \quad \omega_2 := \Omega^q_{pla} \cap \Omega^p_{pla}, \quad \omega_3 := \Omega \setminus \{\omega_1 \cup \omega_2\}.$$

Then, ω_1 *represents a part of domain* Ω*, where both exact and approximate plastic strains show elasticity behavior. It this subdomain, we can choose any* $\lambda(p) = \lambda(q) \in \Lambda$ *to obtain*

$$\int_{\omega_1} (\lambda(p) - \lambda(q))^2 \, dx = 0.$$

In the subdomain ω_3*, where the exact and approximate plastic strains indicate different behaviors (one indicates elastic behavior, whereas the other one plastic behavior), we can choose one parameter (either* $\lambda(p)$ *from* (2.8) *or* $\lambda(q)$ *from* (6.11)*) to ensure*

$$\int_{\omega_3} (\lambda(p) - \lambda(q))^2 \, dx = 0.$$

In ω_2*, both exact and approximate plastic strains show plasticity behavior and*

$$\begin{aligned}
\int_{\omega_2} (\lambda(p) - \lambda(q))^2 \, dx &= \int_{\omega_2} \left(\frac{p}{|p|} - \frac{q}{|q|} \right)^2 dx \\
&= \int_{\omega_2} \left(\frac{p}{|p|}\frac{|q|-|p|}{|q|} - \frac{q-p}{|q|} \right)^2 dx \\
&\le 2\int_{\omega_2} \left(\frac{(|q|-|p|)^2}{|q|^2} + \frac{(q-p)^2}{|q|^2} \right) dx \\
&\le 4\int_{\omega_2} \frac{(q-p)^2}{|q|^2} \, dx \le 4\int_{\Omega^q_{pla}} \frac{(q-p)^2}{|q|^2} \, dx.
\end{aligned}$$

Together, we have formulated an estimate

$$\varrho_1(p-q) \leq \varrho_2(p-q) := \frac{1}{2}\int\limits_{\Omega} \sigma_y^2 H^2 (p-q)^2 dx + 4\int_{\Omega_{pla}^q} \frac{(q-p)^2}{|q|^2}\, dx, \quad (6.14)$$

which results in

Theorem 5 *The majorant value* $\hat{\mathcal{M}}(v, q; \sigma, \lambda(q))$ *in the case of the exact stress* (6.12) *and* $\lambda(q)$ *from* (6.11) *satisfies*

$$\hat{\mathcal{M}}(v, q; \sigma, \lambda(q)) \leq \frac{1}{2}|||(u-v), (p-q)|||^2 + \varrho_2(p-q), \quad (6.15)$$

where the overestimation functional $\varrho_2(p-q)$ *is defined in* (6.14).

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