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Heinz König

Dedicated to the Memory of Igor Kluvánek

Abstract. The work of the author in measure and integration is based on parallel extension theories from inner and outer premeasures to their maximal extensions, both times in three different columns (finite, sequential, nonsequential). The present paper characterizes those contents and measures which occur as these maximal extensions.

Keywords. Inner and outer premeasures and their maximal extensions, complete, saturated, and SC contents and measures, tame inner and outer premeasures, quasi-Radon measures.

The present article is devoted to the foundational part of the theory of measure and integration developed in the author's book [4] and in a series of subsequent papers, summarized in [6] and [7]. It consists of parallel *inner* and *outer* extension theories which proceed

from the inner $\bullet$ *premeasures* $\varphi : \mathfrak{S} \to [0, \infty[$
to their *maximal* inner $\bullet$ *extensions* $\Phi = \varphi_\bullet | \mathfrak{C}(\varphi_\bullet)$, and
from the outer $\bullet$ *premeasures* $\varphi : \mathfrak{S} \to [0, \infty]$
to their *maximal* outer $\bullet$ *extensions* $\Phi = \varphi^\bullet | \mathfrak{C}(\varphi^\bullet)$,

both times in the three parallel procedures

$\bullet = \star$: the *finite* one, based on finite formations,
$\bullet = \sigma$: the *sequential* one, based on countable formations,
$\bullet = \tau$: the *nonsequential* one, based on arbitrary formations.

The set functions Φ thus produced are contents on algebras in case $\bullet = \star$ and measures on σ algebras for $\bullet = \sigma\tau$.

As a rule the main theorems in the new theory start from assumptions on certain initial inner or outer $\bullet$ premeasures φ, and the assertions are for their maximal $\bullet$ extensions Φ, or for related entities. The essential point in this set-up is the both decisive and flexible position of the basic data φ. It so happened that another issue did not come to the surface so far: the problem to *characterize* those contents and measures which occur as the maximal $\bullet$ extensions Φ of the different kinds.

The present article wants to obtain such a characterization. It is of course of interest, in particular for the comparison with the more traditional extension procedures in measure and integration. In both the inner and the outer situation the characterization has a common form in the three cases $\bullet = \star\sigma\tau$. The main characteristic properties are the familiar notions *complete* and *saturated*, but the top one is a new notion named SC, which is close to, but not equivalent to the combination of complete and saturated. These properties will be discussed in section 1. Section 2 will then be devoted to the characterizations in question. In section 3 we shall add another characterization theorem, which is under the régime of *local finiteness*: On the side of the new theory there are the inner $\bullet$ premeasures φ which are of local finiteness type with respect to certain $\bullet$ *complemental pairs* of lattices in the sense of [6] section 4, while on the traditional side there are the quasi-Radon measures of Fremlin [2][3] in case $\bullet = \tau$ and their relatives for $\bullet = \star\sigma$. Section 4 will then be devoted to the comparison quoted above.

1. The Relevant Properties of Contents and Measures

We start to recall the basic concepts and facts. Our main reference will be the survey article [6]. Let X be a nonvoid set, which carries the set systems under consideration. For an isotone set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ the inner and outer $\bullet$ envelopes $\varphi_\bullet, \varphi^\bullet : \mathfrak{P}(X) \to [0,\infty]$ for $\bullet = \star\sigma\tau$ are in the usual terms

$$\varphi_\bullet(A) = \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \downarrow\subset A\},$$

$$\varphi^\bullet(A) = \inf\{\sup_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \uparrow\supset A\},$$

with $\inf\varnothing := \infty$. It follows that $\varphi_\star \leqq \varphi^\star$. If moreover φ is submodular, then $[\varphi < \infty] := \{S \in \mathfrak{S} : \varphi(S) < \infty\} \subset \mathfrak{S}$ is a lattice as well, and we can define $\varphi_\circ : \mathfrak{P}(X) \to [0,\infty]$ to be $\varphi_\circ = (\varphi|[\varphi < \infty])_\star$. Thus $\varphi_\circ \leqq \varphi_\star$, and $\varphi_\circ(A) = \varphi_\star(A)$ when $\varphi_\star(A) < \infty$. We start with a basic fact [8] section 2.

1.1 LEMMA. *Let $\varphi : \mathfrak{S} \to [0,\infty]$ be a content on a ring $\mathfrak{S}$. Then*
$\varphi(S) = \varphi_\star(S\cap E) + \varphi^\star(S\cap E')$ for $S \in \mathfrak{S}$ and $E \subset X$, and hence
$\varphi(S) = \varphi_\circ(S\cap E) + \varphi^\star(S\cap E')$ for $S \in [\varphi < \infty]$ and $E \subset X$.

For the sake of completeness we include a proof. $\leqq$) For $A \in \mathfrak{S}$ with $S\cap E' \subset A$ we have $S \cap A' \subset S \cap E$ and hence

$$\varphi(S) = \varphi(S\cap A') + \varphi(S\cap A) \leqq \varphi_\star(S\cap E) + \varphi(A).$$

It follows that $\varphi(S) \leqq \varphi_\star(S\cap E) + \varphi^\star(S\cap E')$. $\geqq$) For $A \in \mathfrak{S}$ with $A \subset S\cap E$ we have $S\cap E' \subset S\cap A'$ and hence

$$\varphi(S) = \varphi(A) + \varphi(S\cap A') \geqq \varphi(A) + \varphi^\star(S\cap E').$$

It follows that $\varphi(S) \geqq \varphi_\star(S \cap E) + \varphi^\star(S \cap E')$. $\square$

Next we recall for a set function $\Theta : \mathfrak{P}(X) \to [0,\infty]$ with $\Theta(\varnothing) = 0$ the Carathéodory class

$$\mathfrak{C}(\Theta) := \{A \subset X : \Theta(M) = \Theta(M \cap A) + \Theta(M \cap A')\ \forall M \subset X\},$$

the members of which are called *measurable* Θ. One verifies that $\Theta|\mathfrak{C}(\Theta)$ is a content on the algebra $\mathfrak{C}(\Theta)$.

1.2 Remark. *Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a content on an algebra $\mathfrak{A}$. For $A \subset X$ then*

$\star$) $A \in \mathfrak{C}(\alpha^\star) \iff \alpha(S) \geqq \alpha^\star(S \cap A) + \alpha^\star(S \cap A')$ *for all $S \in [\alpha < \infty]$,*
$\circ$) $A \in \mathfrak{C}(\alpha_\circ) \iff \alpha(S) \leqq \alpha_\circ(S \cap A) + \alpha_\circ(S \cap A')$ *for all $S \in [\alpha < \infty]$.*

Proof. To be shown is $\Longleftarrow$. $\star$) The right side holds true for all $S \in \mathfrak{A}$. We fix $M \subset X$ and use this fact for the $S \in \mathfrak{A}$ with $S \supset M$, which furnishes $\alpha^\star(M) \geqq \alpha^\star(M \cap A) + \alpha^\star(M \cap A')$. It follows that $\alpha^\star(M) = \alpha^\star(M \cap A) + \alpha^\star(M \cap A')$ since $\alpha^\star$ is submodular. $\circ$) is obtained as before. $\square$

1.3 .Proposition. *Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a content on an algebra $\mathfrak{A}$. Then* 1) $\mathfrak{C}(\alpha^\star) = \mathfrak{C}(\alpha_\circ) \supset \mathfrak{A}$. 2) *If $E \in \mathfrak{C}(\alpha^\star) = \mathfrak{C}(\alpha_\circ)$ is upward enclosable $[\alpha < \infty]$ then $\alpha^\star(E) = \alpha_\circ(E)$.*

Proof. i) The inclusions $\mathfrak{A} \subset \mathfrak{C}(\alpha^\star)$ and $\mathfrak{A} \subset \mathfrak{C}(\alpha_\circ)$ are obvious from 1.2. ii) Assume that $E \in \mathfrak{C}(\alpha^\star)$. For $S \in [\alpha < \infty]$ then 1.1 and 1.2.$\star$) furnish

$$\alpha(S) = \alpha_\circ(S \cap E) + \alpha^\star(S \cap E') \leqq \alpha^\star(S \cap E) + \alpha^\star(S \cap E') \leqq \alpha(S) < \infty,$$

and hence $\alpha_\circ(S \cap E) = \alpha^\star(S \cap E)$. Likewise of course $\alpha_\circ(S \cap E') = \alpha^\star(S \cap E')$. Thus 1.1 and 1.2.$\circ$) show that $E \in \mathfrak{C}(\alpha_\circ)$. Moreover $\alpha_\circ(E) = \alpha^\star(E)$ when there exists an $S \in [\alpha < \infty]$ with $S \supset E$. iii) Assume that $E \in \mathfrak{C}(\alpha_\circ)$. For $S \in [\alpha < \infty]$ as above 1.1 and 1.2.$\circ$) furnish $\alpha^\star(S \cap E') = \alpha_\circ(S \cap E')$ and $\alpha^\star(S \cap E) = \alpha_\circ(S \cap E)$, and thus 1.1 and 1.2.$\star$) show that $E \in \mathfrak{C}(\alpha^\star)$. $\square$

After this we define a content $\alpha : \mathfrak{A} \to [0,\infty]$ on an algebra $\mathfrak{A}$ to be SC iff $\mathfrak{C}(\alpha^\star) = \mathfrak{C}(\alpha_\circ) = \mathfrak{A}$. We recall that α is called *complete* iff $Q \in \mathfrak{A}$ with $\alpha(Q) = 0$ implies that all $P \subset Q$ are in $\mathfrak{A}$, and is called *saturated* iff $[\alpha < \infty]\top\mathfrak{A} \subset \mathfrak{A}$, with $\top$ the *transporter* as in [4][6][7]. We mention that Fremlin [2] 64G and [3] 211H defines a measure α on a σ algebra $\mathfrak{A}$ to be *locally determined* iff it is both saturated and *semifinite* (which means that α is inner regular $[\alpha < \infty]$).

1.4 Theorem. *Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a content on an algebra $\mathfrak{A}$. Then* 1) *α is* SC $\Longrightarrow$ *α complete and saturated.* 2) *The converse need not be true.* 3) *Assume that $\mathfrak{A}$ is a σ algebra. Then α is* SC $\Longleftarrow$ *α complete and saturated.*

Proof of 1). Assume that α is SC. i) To see that α is complete let $P \subset Q \in \mathfrak{A}$ with $\alpha(Q) = 0$. Then $\alpha^\star(P) \leqq \alpha^\star(Q) = 0$ and hence $\alpha^\star(P) = 0$. Therefore for $S \in [\alpha < \infty]$ we have

$$\alpha^\star(S \cap P) + \alpha^\star(S \cap P') = \alpha^\star(S \cap P') \leqq \alpha(S),$$

so that 1.2.$\star$) implies that $P \in \mathfrak{C}(\alpha^\star) = \mathfrak{A}$. ii) To see that α is saturated let $A \in [\alpha < \infty]\top\mathfrak{A}$. For $S \in [\alpha < \infty]$ thus $S \cap A \in \mathfrak{A}$ and hence $S \cap A' = S \setminus$

$(S \cap A) \in \mathfrak{A}$. It follows that $\alpha(S) = \alpha(S \cap A) + \alpha(S \cap A')$. Thus 1.2 furnishes $A \in \mathfrak{C}(\alpha^\star) = \mathfrak{C}(\alpha_\circ) = \mathfrak{A}$. $\square$

Assertion 2) will be proved with the counterexample below.

Proof of 3). Assume that α is complete and saturated, and that $\mathfrak{A}$ is a σ algebra. We shall prove that $\mathfrak{C}(\alpha_\circ) \subset \mathfrak{A}$ and fix $E \in \mathfrak{C}(\alpha_\circ)$. It suffices to prove that $E \in [\alpha < \infty]\top\mathfrak{A}$. Thus we fix $S \in [\alpha < \infty]$ and claim that $S \cap E \in \mathfrak{A}$.

i) We know that $\alpha(S) = \alpha_\circ(S \cap E) + \alpha_\circ(S \cap E')$. From the definition of $\alpha_\circ$ we obtain sequences of $P_n, Q_n \in [\alpha < \infty]$ with

$$\begin{aligned} &P_n \subset S \cap E \quad \text{and} \quad \alpha(P_n) \to \alpha_\circ(S \cap E),\\ &Q_n \subset S \cap E' \quad \text{and} \quad \alpha(Q_n) \to \alpha_\circ(S \cap E'). \end{aligned}$$

We can achieve that $P_n \uparrow$ some $P \subset S \cap E$ and $Q_n \uparrow$ some $Q \subset S \cap E'$. Then $P, Q \in \mathfrak{A}$ since $\mathfrak{A}$ is a σ algebra, and

$$\begin{aligned} \alpha(P_n) \leqq \alpha(P) \leqq \alpha_\circ(S \cap E) \quad &\text{implies that} \quad \alpha(P) = \alpha_\circ(S \cap E),\\ \alpha(Q_n) \leqq \alpha(Q) \leqq \alpha_\circ(S \cap E') \quad &\text{implies that} \quad \alpha(Q) = \alpha_\circ(S \cap E'), \end{aligned}$$

in particular $P, Q \in [\alpha < \infty]$. ii) From $P \cap Q = \varnothing$ we obtain

$$\alpha(P \cup Q) = \alpha(P) + \alpha(Q) = \alpha_\circ(S \cap E) + \alpha_\circ(S \cap E') = \alpha(S),$$

so that $P \cup Q \in [\alpha < \infty]$ fulfils $P \cup Q \subset S$ and $\alpha\big(S \setminus (P \cup Q)\big) = 0$. But

$$S \setminus (P \cup Q) = \big((S \cap E) \setminus P\big) \cup \big((S \cap E') \setminus Q\big),$$

so that $(S \cap E) \setminus P \subset S \setminus (P \cup Q)$, and hence $(S \cap E) \setminus P \in \mathfrak{A}$ since α is complete. It follows that $S \cap E \in \mathfrak{A}$. $\square$

1.5 Example. Let $X = [0, 1]$ and $\lambda : \mathfrak{L} \to [0, \infty[$ be the Lebesgue measure on X (which is known to be complete). We define $\mathfrak{A} \subset \mathfrak{L}$ to consist of those $A \in \mathfrak{L}$ which fulfil either $[0, \delta] \subset A$ or $[0, \delta] \cap A = \varnothing$ for some $0 < \delta < 1$. Then $\mathfrak{A}$ is an algebra, but not a σ algebra. And $\alpha := \lambda|\mathfrak{A}$ is a finite content on $\mathfrak{A}$ and hence saturated, and moreover upward σ continuous. The completeness of λ combined with the definition of $\mathfrak{A}$ implies at once that α is complete. But α is *not* SC: To see this consider $A = \{0\}$. We have $A \in \mathfrak{L}$ but $A \notin \mathfrak{A}$. However, for $S \in \mathfrak{A}$ we have $\alpha^\star(S \cap A) \leqq \alpha^\star(A) = 0$ and hence

$$\alpha^\star(S \cap A) + \alpha^\star(S \cap A') = \alpha^\star(S \cap A') \leqq \alpha(S),$$

so that 1.2.$\star$) implies that $A \in \mathfrak{C}(\alpha^\star)$. Thus we have indeed $\mathfrak{C}(\alpha^\star) \neq \mathfrak{A}$. $\square$

2. The Characterization Theorems

We continue to assume a nonvoid set X and $\bullet = \star\sigma\tau$.

2.1 Inner Remark. Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ and $\varphi : \mathfrak{S} \to [0, \infty[$ be isotone with $\varphi(\varnothing) = 0$. By definition and the inner $\bullet$ extension theorem [6] 3.5 then φ is an inner $\bullet$ premeasure iff there exist contents $\alpha : \mathfrak{A} \to [0, \infty]$ on algebras $\mathfrak{A} \supset \mathfrak{S}$ which are inner $\bullet$ extensions of φ. *Of these* α *a unique one is* SC*: this is the maximal* $\alpha = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$.

We recall that an *inner* $\bullet$ *extension* of φ in the sense of [6] section 3 is defined to be a content $\alpha:\mathfrak{A}\to[0,\infty]$ on a ring $\mathfrak{A}$ which is an extension of φ and satisfies $\mathfrak{S}\subset\mathfrak{S}_\bullet\subset\mathfrak{A}$ with

α is inner regular $\mathfrak{S}_\bullet$,
$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (note that $\alpha|\mathfrak{S}_\bullet<\infty$).

Proof. The $\alpha:\mathfrak{A}\to[0,\infty]$ in question are restrictions of $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. Thus $\mathfrak{S}\subset\mathfrak{S}_\bullet\subset[\alpha<\infty]\subset\mathfrak{A}\subset\mathfrak{C}(\varphi_\bullet)$ and $\alpha=\varphi_\bullet|\mathfrak{A}$. On $[\alpha<\infty]$ we have $\varphi_\bullet=\alpha=\alpha|[\alpha<\infty]=\alpha_\circ$, and hence $\varphi_\bullet=\alpha_\circ$ *partout* since both sides are inner regular $[\alpha<\infty]$. It follows that α is SC $\Leftrightarrow\mathfrak{A}=\mathfrak{C}(\alpha_\circ)=\mathfrak{C}(\varphi_\bullet)\Leftrightarrow\alpha=\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. $\square$

2.2 Inner Characterization Theorem. *Let $\alpha:\mathfrak{A}\to[0,\infty]$ be a content on an algebra $\mathfrak{A}$. Then*

$$\begin{aligned}&\alpha=\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)\ \textit{for some inner}\ \bullet\ \textit{premeasure}\ \varphi:\mathfrak{S}\to[0,\infty[\\ \Longleftrightarrow\ &\textit{there exists a lattice}\ \mathfrak{S}\ \textit{with}\ \varnothing\subset\mathfrak{S}\subset\mathfrak{A}\ \textit{and}\ \alpha|\mathfrak{S}<\infty\ \textit{such that}\ \alpha\ \textit{is}\\ &\textit{an inner}\ \bullet\ \textit{extension of}\ \alpha|\mathfrak{S},\ \textit{and}\ \alpha\ \textit{is}\ \mathrm{SC}\ .\end{aligned}$$

This is an immediate consequence of 2.1. We continue with an additional equivalence in the cases $\bullet=\star\sigma$.

2.2 Continuation. *Moreover*

$$\begin{aligned}&\textit{in case}\ \bullet=\star\colon\Longleftrightarrow\ \alpha\ \textit{is semifinite and}\ \mathrm{SC};\\ &\textit{in case}\ \bullet=\sigma\colon\Longleftrightarrow\ \alpha\ \textit{is a measure on the}\ \sigma\ \textit{algebra}\ \mathfrak{A},\ \textit{and is semifinite}\\ &\qquad\qquad\textit{and}\ \mathrm{SC}\ \text{(that is complete and saturated).}\end{aligned}$$

Proof. Both times $\Longrightarrow$ is clear, and $\Longleftarrow$ results for $\mathfrak{S}:=[\alpha<\infty]$. $\square$

We turn to the outer counterpart.

2.3 Outer Remark. Let $\mathfrak{S}$ be a lattice with $\varnothing\in\mathfrak{S}$ and $\varphi:\mathfrak{S}\to[0,\infty]$ be isotone with $\varphi(\varnothing)=0$. By definition and the outer $\bullet$ extension theorem [6] 3.1 then φ is an outer $\bullet$ premeasure iff there exist contents $\alpha:\mathfrak{A}\to[0,\infty]$ on algebras $\mathfrak{A}\supset\mathfrak{S}$ which are outer $\bullet$ extensions of φ. *Of these α a unique one is* SC*: this is the maximal $\alpha=\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$.*

We recall that an *outer* $\bullet$ *extension* of φ in the sense of [6] section 3 is defined to be a content $\alpha:\mathfrak{A}\to[0,\infty]$ on a ring $\mathfrak{A}$ which is an extension of φ and satisfies $\mathfrak{S}\subset\mathfrak{S}^\bullet\subset\mathfrak{A}$ with

α is outer regular $\mathfrak{S}^\bullet$,
$\alpha|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous.

Proof. The $\alpha:\mathfrak{A}\to[0,\infty]$ in question are restrictions of $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$. Thus $\mathfrak{S}\subset\mathfrak{S}^\bullet\subset\mathfrak{A}\subset\mathfrak{C}(\varphi^\bullet)$ and $\alpha=\varphi^\bullet|\mathfrak{A}$. On $\mathfrak{A}$ we have $\varphi^\bullet=\alpha=\alpha^\star$, and hence $\varphi^\bullet=\alpha^\star$ *partout* since both sides are outer regular $\mathfrak{A}$. It follows that α is SC $\Leftrightarrow$ $\mathfrak{A}=\mathfrak{C}(\alpha^\star)=\mathfrak{C}(\varphi^\bullet)\Leftrightarrow\alpha=\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$. $\square$

2.4 Outer Characterization Theorem. *Let* $\alpha : \mathfrak{A} \to [0,\infty]$ *be a content on an algebra* $\mathfrak{A}$. *Then*

$$\alpha = \varphi^\bullet|\mathfrak{C}(\varphi^\bullet) \text{ for some outer } \bullet \text{ premeasure } \varphi : \mathfrak{S} \to [0,\infty]$$
$$\Longleftrightarrow \text{there exists a lattice } \mathfrak{S} \text{ with } \varnothing \subset \mathfrak{S} \subset \mathfrak{A} \text{ such that } \alpha \text{ is an outer } \bullet$$
$$\text{extension of } \alpha|\mathfrak{S}, \text{ and } \alpha \text{ is SC}.$$

This is an immediate consequence of 2.3 as before. We continue with an additional equivalence in the cases $\bullet = \star\sigma$.

2.4 Continuation. *Moreover*

in case $\bullet = \star$: $\Longleftrightarrow$ α *is* SC;

in case $\bullet = \sigma$: $\Longleftrightarrow$ α *is a measure on the* σ *algebra* $\mathfrak{A}$, *and is* SC (that is complete and saturated).

Proof. Both times $\Longrightarrow$ is clear, and $\Longleftarrow$ results for $\mathfrak{S} := \mathfrak{A}$. $\square$

2.5 Remark. Assume that

$\alpha = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ for an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$, or
$\alpha = \varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ for an outer $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty]$.

Then α need not be the completion of the restriction $\alpha|\mathrm{A}\sigma(\mathfrak{S})$ of α to the generated σ algebra $\mathrm{A}\sigma(\mathfrak{S})$ when $\bullet = \sigma\tau$, but can be much more comprehensive, and the like for $\bullet = \star$. As an example let $\mathfrak{S}$ consist of the finite subsets of an uncountable set X and $\varphi : \mathfrak{S} \to [0,\infty[$ be the *cardinality* restricted to $\mathfrak{S}$. In all cases then $\varphi_\bullet = \varphi^\bullet = \text{card}$, so that $\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\varphi^\bullet) = \mathfrak{P}(X)$ and $\alpha = \text{card}$. Now $\mathrm{A}\sigma(\mathfrak{S})$ consists of the countable and the cocountable subsets of X. Thus if $E \subset X$ is neither countable nor cocountable, then for each $A \in \mathrm{A}\sigma(\mathfrak{S})$ the difference set $A\Delta E = (A' \cap E) \cup (A \cap E')$ is uncountable and hence has $\alpha(A\Delta E) = \infty$.

3. Another Inner Characterization Theorem

The topic of the present section came up in the frame of Radon measures. Let X be a Hausdorff topological space with the obvious set systems $\mathrm{Op}(X)$ and $\mathrm{Comp}(X)$ and the Borel σ algebra $\mathrm{Bor}(X)$. A measure $\alpha : \mathfrak{A} \to [0,\infty]$ on a σ algebra $\mathfrak{A} \supset \mathrm{Bor}(X)$ is called *Radon* iff $\alpha|\mathrm{Comp}(X) < \infty$ and α is inner regular $\mathrm{Comp}(X)$. This is the *actual* definition initiated - as far as the author is aware - in Berg-Christensen-Ressel [1] chapter 2, while the *traditional* definition fortified $\alpha|\mathrm{Comp}(X) < \infty$ to *local finiteness*: Each point of X and hence each $A \in \mathrm{Comp}(X)$ is contained in some $U \in \mathrm{Op}(X)$ with $\alpha(U) < \infty$. The traditional definition is still in frequent use, for example in Fremlin [3].

In the present new development of measure and integration, as before on a nonvoid set X and for $\bullet = \star\sigma\tau$, the counterpart of the Radon measures in the *actual* sense can be viewed to be the inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$ (for $\bullet = \tau$ or for $\bullet = \star\sigma\tau$). A counterpart of the Radon measures in the *traditional* sense can then be obtained in form of certain particular inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$

in the context of the $\bullet$ *complemental pairs* in the sense of [6] section 4. These particular inner $\bullet$ premeasures φ will be the heroes of the present section.

We start to recall the relevant concepts and facts. We define a pair of lattices $\mathfrak{S}$ and $\mathfrak{T}$ with $\varnothing$ to be $\bullet$ *complemental* iff $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp$ and $\mathfrak{S} \subset (\mathfrak{T}\top\mathfrak{T}^\bullet)\perp$, with $\mathfrak{M}\perp := \{M' : M \in \mathfrak{M}\}$ for $\mathfrak{M}$ a nonvoid set system. In this situation an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ is called $\bullet$ *tame for* $\mathfrak{S}$ *and* $\mathfrak{T}$ iff $\varphi_\bullet$ is outer regular $\mathfrak{T}^\bullet$ at $\mathfrak{S}$; note that $\mathfrak{T}^\bullet \subset (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp \subset \mathfrak{C}(\varphi_\bullet)$. Equivalent is the much simpler condition that each $S \in \mathfrak{S}$ is contained in some $T \in \mathfrak{T}^\bullet$ with $\varphi_\bullet(T) < \infty$, which is a certain *local finiteness* condition. Likewise an outer $\bullet$ premeasure $\psi : \mathfrak{T} \to [0, \infty]$ is called $\bullet$ *tame for* $\mathfrak{S}$ *and* $\mathfrak{T}$ iff $\psi^\bullet|\mathfrak{S} < \infty$ and $\psi^\bullet$ is inner regular $\mathfrak{S}_\bullet$ at $\mathfrak{T}$; as above note that $\mathfrak{S}_\bullet \subset (\mathfrak{T}\top\mathfrak{T}^\bullet)\perp \subset \mathfrak{C}(\psi^\bullet)$. After these definitions we recall [6] 4.6, which asserts that the two kinds of set functions $\varphi : \mathfrak{S} \to [0, \infty[$ and $\psi : \mathfrak{T} \to [0, \infty]$ are in one-to-one correspondence via $\psi = \varphi_\bullet|\mathfrak{T}$ and $\varphi = \psi^\bullet|\mathfrak{S}$, and henceforth are called $\bullet$ *complemental pairs for* $\mathfrak{S}$ *and* $\mathfrak{T}$. For these pairs one has $\varphi_\bullet \leqq \psi^\bullet$, with $\varphi_\bullet = \psi^\bullet$ on $\mathfrak{S}_\bullet$ and $\mathfrak{T}^\bullet$ and $[\psi^\bullet|\mathfrak{C}(\psi^\bullet) < \infty]$, and $\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\psi^\bullet)$. In the concrete situation of Radon measures this correspondence is due to Laurent Schwartz [9].

In the context of the present paper we shall specialize the lattices $\mathfrak{T}$ with $\varnothing \in \mathfrak{T}$ to those with $\varnothing, X \in \mathfrak{T}$ and $\mathfrak{T} = \mathfrak{T}^\bullet$, for short called the $\bullet$ *topologies*, because in case $\bullet = \tau$ these are the familiar topologies. Then the relation that $\mathfrak{S}$ and $\mathfrak{T}$ be $\bullet$ *complemental* reads $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{T}\perp \subset \mathfrak{S}\top\mathfrak{S}_\bullet$.

After this we define a content $\alpha : \mathfrak{A} \to [0, \infty]$ on an algebra $\mathfrak{A}$ to be $\bullet$ *quasi-Radon for a* $\bullet$ *topology* $\mathfrak{T}$ iff it is SC and satisfies $\mathfrak{T} \subset \mathfrak{A}$ (and hence $\mathfrak{T}\perp \subset \mathfrak{A}$) with

α is inner regular $\mathfrak{H} := \{H \in \mathfrak{T}\perp : H \text{ is enclosable } [\alpha|\mathfrak{T} < \infty]\}$,
$\alpha|\mathfrak{T}$ is upward $\bullet$ continuous;

note that $\mathfrak{H}$ is a lattice with $\varnothing \in \mathfrak{H} = \mathfrak{H}_\bullet \subset \mathfrak{A}$. We shall see next that for $\bullet = \sigma\tau$ these α are measures on σ algebras, and then conclude from the respective definition in [6] section 4 that in case $\bullet = \tau$ we obtain the quasi-Radon measures in the sense of Fremlin [2][3].

3.1 Theorem. *Let* $\alpha : \mathfrak{A} \to [0, \infty]$ *be a content on an algebra* $\mathfrak{A}$ *which is* $\bullet$ *quasi-Radon for the* $\bullet$ *topology* $\mathfrak{T}$. *Then the above* $\mathfrak{H}$ *and* $\mathfrak{T}$ *form a* $\bullet$ *complemental pair* $\subset \mathfrak{A}$. *Moreover*

$\xi := \alpha|\mathfrak{H} < \infty$ *is an inner* $\bullet$ *premeasure,*
$\eta := \alpha|\mathfrak{T}$ *is an outer* $\bullet$ *premeasure,*

and the two are $\bullet$ *tame and form a* $\bullet$ *complemental pair for* $\mathfrak{H}$ *and* $\mathfrak{T}$. *We have* $\alpha = \xi_\bullet|\mathfrak{C}(\xi_\bullet)$ (but α need not be $= \eta^\bullet|\mathfrak{C}(\eta^\bullet)$).

We note that $\alpha = \xi_\bullet|\mathfrak{C}(\xi_\bullet)$ combined with $\xi = \eta^\bullet|\mathfrak{H}$ shows that the restriction $\eta = \alpha|\mathfrak{T}$ determines α. Also note that $\xi_\bullet = \xi_\star$ and $\eta^\bullet = \eta^\star$ from [6] 2.2.4) since $\mathfrak{H} = \mathfrak{H}_\bullet$ and $\mathfrak{T} = \mathfrak{T}^\bullet$. An example for the final assertion can be obtained from [5] example 4.8 due to Dowker.

Proof. 1) We claim that $\mathfrak{H} \subset \mathfrak{T}\perp \subset \mathfrak{H}\top\mathfrak{H}$, so that $\mathfrak{H}$ and $\mathfrak{T}$ are $\bullet$ complemental. To see the second inclusion let $M \in \mathfrak{T}\perp$. For $H \in \mathfrak{H} \subset \mathfrak{T}\perp$ then $M \cap H \in \mathfrak{T}\perp$, and hence $M \cap H \in \mathfrak{H}$ since $M \cap H \subset H$ is enclosable $[\alpha|\mathfrak{T} < \infty]$. Thus $M \in \mathfrak{H}\top\mathfrak{H}$.

2) We claim that $\xi = \alpha|\mathfrak{H}$ is downward $\bullet$ continuous. To see this (for $\bullet = \sigma\tau$) let $\mathfrak{M} \subset \mathfrak{H}$ be nonvoid $\bullet$ with $\mathfrak{M} \downarrow D \in \mathfrak{H}$. To be shown is $\inf\{\alpha(M) : M \in M\} = \alpha(D)$. In view of directedness we can assume that all $M \in \mathfrak{M}$ are contained in some fixed $T \in \mathfrak{T}$ with $\alpha(T) < \infty$. Then $\{T \setminus M : M \in \mathfrak{M}\} \subset \mathfrak{T}$ is nonvoid $\bullet$ with $\uparrow T \setminus D \in \mathfrak{T}$, and the claim reads $\sup\{\alpha(T \setminus M) : M \in \mathfrak{M}\} = \alpha(T \setminus D)$ and hence is true.

3) The definition of $\bullet$ quasi-Radon and 2) assert that α is an inner $\bullet$ extension of $\xi = \alpha|\mathfrak{H}$. Thus ξ is an inner $\bullet$ premeasure. Since α is SC we obtain from 2.1 that $\alpha = \xi_\bullet|\mathfrak{C}(\xi_\bullet)$. Moreover ξ is $\bullet$ tame for $\mathfrak{H}$ and $\mathfrak{T}$, since each $H \in \mathfrak{H}$ is contained in some $T \in \mathfrak{T}$ with $\alpha(T) = \xi_\bullet(T) < \infty$.

4) Now $\eta := \xi_\bullet|\mathfrak{T} = \alpha|\mathfrak{T}$ is the unique outer $\bullet$ premeasure $\eta : \mathfrak{T} \to [0, \infty]$ which is $\bullet$ tame for $\mathfrak{H}$ and $\mathfrak{T}$ and such that ξ and η form a $\bullet$ complemental pair for $\mathfrak{H}$ and $\mathfrak{T}$. This completes the proof. □

3.2 Theorem. *Let $\varphi : \mathfrak{S} \to [0, \infty[$ be an inner $\bullet$ premeasure and $\mathfrak{T}$ be a $\bullet$ topology, and assume that $\mathfrak{S}$ and $\mathfrak{T}$ are $\bullet$ complemental and φ is $\bullet$ tame for $\mathfrak{S}$ and $\mathfrak{T}$. Then $\alpha := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is $\bullet$ quasi-Radon for $\mathfrak{T}$.*

Proof. We have $\alpha = \varphi_\bullet|\mathfrak{A}$ on $\mathfrak{A} := \mathfrak{C}(\varphi_\bullet)$. 1) α is SC in view of 2.1. 2) We have $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp \subset \mathfrak{C}(\varphi_\bullet) = \mathfrak{A}$ from [6] 3.5 and hence $\mathfrak{T}\perp \subset \mathfrak{A}$, so that the above $\mathfrak{H}$ is well-defined. Moreover $\mathfrak{S}_\bullet \subset \mathfrak{H}$, since by assumption on the one hand $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{T}\perp$, and on the other hand each $S \in \mathfrak{S}_\bullet$ is contained in some $T \in \mathfrak{T}$ with $\alpha(T) = \varphi_\bullet(T) < \infty$. It follows that $\alpha = \varphi_\bullet|\mathfrak{A}$ is inner regular $\mathfrak{H}$.

3) $\alpha = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ has $\alpha|(\mathfrak{S}\top\mathfrak{S}_\bullet)\perp$ upward $\bullet$ continuous. This is clear for $\bullet = \star\sigma$ and in [6] 3.6.ii) for $\bullet = \tau$. Thus $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp$ implies that $\alpha|\mathfrak{T}$ is upward $\bullet$ continuous. It follows that α is $\bullet$ quasi-Radon for $\mathfrak{T}$. □

3.3 Inner Characterization Theorem. *Let $\alpha : \mathfrak{A} \to [0, \infty]$ be a content on an algebra $\mathfrak{A}$. For a $\bullet$ topology $\mathfrak{T}$ then*

$$\begin{aligned} & \alpha = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet) \textit{ for some inner } \bullet \textit{ premeasure } \varphi : \mathfrak{S} \to [0, \infty[\textit{ such that} \\ & \mathfrak{S} \textit{ and } \mathfrak{T} \textit{ are } \bullet \textit{ complemental and } \varphi \textit{ is } \bullet \textit{ tame for } \mathfrak{S} \textit{ and } \mathfrak{T} \\ \Longleftrightarrow\ & \alpha \textit{ is } \bullet \textit{ quasi-Radon for } \mathfrak{T}. \end{aligned}$$

This is an immediate consequence of 3.1 and 3.2. We remark that the result in the case $\bullet = \tau$ restricted to measures on σ algebras has been formulated without proof earlier in [6] 4.9.

4. Application to the Inner Measure Constructions

The present inner characterization theorems 2.2 and 3.3 illuminate the connection between the inner $\bullet$ extension theorem [4] 6.31 = [6] 3.5 = [7] section 4 of the present author and the so-called *inner measure constructions* of Fremlin [3], the basic results of which are

lemma 413H for the case $\bullet = \star$,
theorem 413J for the case $\bullet = \sigma$,
theorem 415K for the case $\bullet = \tau$,

where the last one is restricted to the topological situation and to local finiteness. Both times these are the basic results for the fundamental task of extension of basic set functions: One assumed an isotone set function $\varphi : \mathfrak{S} \to [0,\infty[$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ and formulated certain conditions on φ in order that it possesses a (unique) extension which is (at least) a content $\alpha : \mathfrak{A} \to [0,\infty]$ on an algebra $\mathfrak{A}$ and has certain desired properties (in the interest of a common set-up we pass over certain technical deviations). In this context then the present inner characterization theorems 2.2 and 3.3 can be read so as to assert that (at least in the common cases $\bullet = \sigma\tau$) the desired properties in the two theories are equivalent - which is not at all visible at first sight.

After this we turn to the two collections of conditions imposed upon φ. In Fremlin [3] the conditions are (α) the *crude tightness*

$(\star)$ $\qquad \varphi(B) = \varphi(A) + \varphi_\star(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$,

and (β) φ to be (downward) $\bullet$ continuous at $\varnothing$, and in addition in case $\bullet = \tau$, which is restricted to the topological situation and to local finiteness, an appropriate local finiteness condition (γ) with the topology $\mathfrak{T}$ in question. In the new context the conditions can be formulated as $(\alpha\beta)$ the $\bullet$ *tightness*

$(\bullet)$ $\qquad \varphi(B) = \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$,

and under local finiteness with the $\bullet$ topology $\mathfrak{T}$ as above (γ) φ to be $\bullet$ *tame* for $\mathfrak{S}$ and $\mathfrak{T}$.

Then the results in the two theories show a remarkable difference: In Fremlin [3] the conditions $(\alpha)(\beta)$ and $(\alpha)(\beta)(\gamma)$ are *sufficient* conditions for the aims in question, but in cases $\bullet = \sigma\tau$ they are *not equivalent* ones, at least without the additional requirement $\mathfrak{S} = \mathfrak{S}_\bullet$, whereas in the new situation of the author the conditions $(\alpha\beta)$ and $(\alpha\beta)(\gamma)$ are *equivalent* conditions in all cases $\bullet = \star\sigma\tau$ and for all $\mathfrak{S}$. The reason is that the *crude* envelope $\varphi_\star$ is the appropriate one for the case $\bullet = \star$ but not for $\bullet = \sigma\tau$, where the respective $\bullet$ envelope $\varphi_\bullet$ attains its place: see the subsequent simple example extracted from [4] 6.32, and for the entire context [7] sections 3-5. One notes that condition (α) of Fremlin [3] is identical with that in Topsøe [10][11] from 1970. In the meantime the new development due to the present author had made clear that for $\bullet = \sigma\tau$ the new inner and outer $\bullet$ envelopes $\varphi_\bullet$ and $\varphi^\bullet$ are the adequate ones.

4.1 Example. We take $X = \mathbb{R}$ and $\mathfrak{S} = \mathrm{Op}(\mathbb{R})$, and $\varphi = \delta_a|\mathfrak{S}$ for some fixed $a \in \mathbb{R}$. Thus φ is isotone with $\varphi(\varnothing) = 0$, and modular and downward $\bullet$ continuous for $\bullet = \sigma\tau$. In the cases $\bullet = \sigma\tau$ one verifies that $\varphi_\bullet = \delta_a$ partout, so that the $\bullet$ tightness condition $(\bullet)$ is fulfilled. Hence the inner $\bullet$ extension theorem [6] 3.5 asserts that φ is an inner $\bullet$ premeasure, so that the equivalent desired properties in the two theories are fulfilled. But $\varphi_\star(\{a\}) = 0$, which implies that the crude tightness condition $(\star)$ is violated for $B \in \mathfrak{S}$ with $a \in B$ and $A = B \setminus \{a\}$.

References

[1] C.Berg, J.P.R.Christensen and P.Ressel, Harmonic Analysis on Semigroups. Springer 1984.

[2] D.H.Fremlin, Topological Riesz Spaces and Measure Theory. Cambridge Univ.Press 1974.

[3] D.H.Fremlin, Measure Theory Vol.1-4. Torres Fremlin 2000-2003 (in a numbered reference the first digit indicates its volume). http://www.essex.ac.uk/maths/staff/fremlin/mt.htm.

[4] H.König, Measure and Integration: An Advanced Course in Basic Procedures and Applications. Springer 1997.

[5] H.König, Measure and Integration: Mutual generation of outer and inner premeasures. Annales Univ. Saraviensis Ser.Math. 9(1998) No.2, 99-122.

[6] H.König, Measure and Integration: An attempt at unified systematization. Rend. Istit. Mat. Univ. Trieste 34(2002), 155-214. Preprint No.42 under http://www.math.uni-sb.de.

[7] H.König, Measure and Integral: New foundations after one hundred years. In: Functional Analysis and Evolution Equations (The Günter Lumer Volume). Birkhäuser 2007, pp.405-422. Preprint No.175 under http://www.math.uni-sb.de.

[8] J.Łoś and E.Marczewski, Extensions of measure. Fund.Math. 36(1949), 267-276.

[9] L.Schwartz, Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures. Oxford Univ. Press 1973.

[10] F.Topsøe, Compactness in spaces of measures. Studia Math. 36(1970), 195-212.

[11] F.Topsøe, Topology and Measure. Lect.Notes Math. 133, Springer 1970.

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