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Abstract. We consider the problem of minimizing

$$J(u,E) \;=\; \int_E f_h^+\big(\cdot,\varepsilon(u)\big)\,dx + \int_{\Omega-E} f^-\big(\cdot,\varepsilon(u)\big)\,dx + \sigma\,|\partial E\cap\Omega|$$

among functions $u:\ \mathbb{R}^d\supset\Omega\to\mathbb{R}^d$, $u_{|\partial\Omega}=0$, and measurable subsets E of Ω. Here f_h^+, f^- denote quadratic potentials defined on $\overline{\Omega}\times\{\text{symmetric } d\times d \text{ matrices}\}$, h is the minimum energy of f_h^+ and $\varepsilon(u)$ is the symmetric gradient of the displacement field u. An equilibrium state $\hat{u}$, $\hat{E}$ of $J(u,E)$ is called one–phase if $E=\emptyset$ or $E=\Omega$, two–phase otherwise. For two–phase states $\sigma|\partial E\cap\Omega|$ measures the effect of the separating surface, and we investigate in which way the distribution of phases is affected by the choice of the parameters $h\in\mathbb{R}$, $\sigma>0$. Additional results concern the smoothness of two–phase equilibrium states and the behaviour of $\inf J(u,E)$ in the limit $\sigma\downarrow 0$. Moreover, we discuss the case of additional volume force potentials, and extend the previous results to non–zero boundary values.

AMS Subject Classification: 74 B 05, 74 G 65, 74 N 99.
Key words: elastic materials, phase transition, equilibrium states, functions of bounded variation.

Contents

1 Introduction

We consider an elastic medium occupying a bounded Lipschitz region $\Omega \subset \mathbb{R}^d$ and assume that the medium can exist in two different phases. Each phase is characterized by its deformation energy density

$$f^{\pm}\big(x, \varepsilon(u)(x)\big) \;=\; \big\langle A^{\pm}(x)\big(\varepsilon(u)(x) - \xi^{\pm}(x)\big), \varepsilon(u)(x) - \xi^{\pm}(x)\big\rangle + a^{\pm}$$

and its location in the non–deformed state, i.e. by sets $\Omega^{\pm} \subset \Omega$, where $\Omega^+ \cap \Omega^- = \emptyset$ and $\overline{\Omega^+ \cup \Omega^-} = \overline{\Omega}$. The plus and minus superscripts correspond to the first and second phase, respectively, $u(x) = (u^1(x), \ldots, u^d(x))$, $x \in \overline{\Omega}$, is the field of displacements with corresponding strain tensor $\varepsilon(u)$, and we assume that $u(x)$ vanishes on $\partial\Omega$. According to the definition the energy density $f^{\pm}$ of each phase is a quadratic function of the linear strain $\varepsilon(u)$. $\xi^{\pm}$ denotes the stress–free strain of the i^{th} phase; $A^{\pm}(x)$ is the tensor of the elastic moduli viewed as a positve definite, symmetric linear map on the space of symmetric tensors. We do not assume that A^+ and A^- coincide but their difference measured in L^∞-norm should be small (see Section 2 for precise statements). Finally, $a^{\pm}$ is the associated minimum energy, w.l.o.g. we will assume that $a^+ = h \in \mathbb{R}$ and $a^- = 0$. In order to indicate the dependence of f^+ on the parameter h, we will write $f_h^+(x, \varepsilon)$ in place of $f^+(x, \varepsilon)$. If χ denotes the characteristic function of the set Ω^+ occupied by the first phase, then it is natural to take the functional (neglecting for the moment volume force potentials)

$$J[u, \chi] \;=\; \int_\Omega \chi f_h^+\big(\cdot, \varepsilon(u)\big) + (1-\chi) f^-\big(\cdot, \varepsilon(u)\big)\, dx \tag{1.1}$$

as the total deformation energy and to investigate the existence and behaviour of equilibrium states, i.e. of pairs $\hat{u}$, $\hat{\chi}$ such that

$$J\big[\hat{u}, \hat{\chi}\big] = \inf J\,,$$

where the infimum has to be taken w.r.t. to all deformations $u : \overline{\Omega} \to \mathbb{R}^d$, $u_{|\partial\Omega} = 0$, and all measurable characteristic functions $\chi : \Omega \to \mathbb{R}$. A state of

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equilibrium is termed one–phase, if $\hat{\chi} \equiv 0$ or $\hat{\chi} \equiv 1$, and two–phase otherwise. Unfortunately, the variational problem $J \to \min$ may fail to have solutions as it is shown by an example in [MO]. One way to overcome this difficulty is the observation that

$$\begin{aligned} \chi f_h^+\big(\cdot, \varepsilon(u)\big) + (1-\chi) f^-\big(\cdot, \varepsilon(u)\big) \;&\geq\; \min\Big\{ f_h^+\big(\cdot, \varepsilon(u)\big), f^-\big(\cdot, \varepsilon(u)\big)\Big\} \\ &=: \; f\big(\cdot, \varepsilon(u)\big), \end{aligned}$$

hence we may introduce the functional

$$I[u] \;=\; \int_\Omega f\big(\cdot, \varepsilon(u)\big)\, dx \tag{1.2}$$

whose energy density is the non–convex double well potential $f(\cdot, \varepsilon(u))$ and whose infimum agrees with $\inf J$ (compare Theorem 7.1). Again, the existence of I–minimizing displacement fields can not be guaranteed but the quasiconvex envelope $\tilde{f}(\cdot, \varepsilon(u))$ of $f(\cdot, \varepsilon(u))$ provides a natural regularisation $\tilde{I}[u] = \int_\Omega \tilde{f}(\cdot, \varepsilon(u))\, dx$ of the functional I which means that $\tilde{I}$ attains its minimum among all admissible displacements. Moreover, the $\tilde{I}$–minimizing displacement fields are exactly the weak cluster points of I–minimizing sequences. There are many papers devoted to the study of the relaxed variational problem

$$\tilde{I}[u] \;\to\; \min \tag{1.3}$$

on suitable classes of displacements $u\colon \overline{\Omega} \to \mathbb{R}^d$. Without being complete we mention [DM], [BJ] and the references quoted therein.

There is another way to obtain a regularization of the functional (1.1): following [GR] it is natural to introduce an additional term in (1.1) similar to the Griffith surface energy and proportional to the area of the surface S separating the regions $\Omega^+ = \{x \in \Omega : \chi(x) = 1\}$ and $\Omega^- = \{x \in \Omega : \chi(x) = 0\}$, more precisely we let

$$I[u, \chi] = \int_\Omega \Big[\chi f_h^+\big(\cdot, \varepsilon(u)\big) + (1-\chi) f^-\big(\cdot, \varepsilon(u)\big)\Big]\, dx + \sigma\, |S|\,, \tag{1.4}$$

where $|S|$ denotes the area of the separating surface and σ is a positive constant. Reformulating the variational problem for (1.4) in suitable spaces like $X = \overset{\circ}{W}{}_2^1(\Omega; \mathbb{R}^d) \times \{\chi \in BV(\Omega) : \chi(x) = 0 \;\text{ or }\; 1\}$ by observing that

$$|S| = \int_\Omega |\nabla \chi|\,,$$

if $S := \partial(\operatorname{spt}\chi \cap \Omega)$ is smooth, it is easy to show the existence of equilibrium states $\hat{u}$, $\hat{\chi}$. Moreover, we have $\inf I[u,\chi] \to \inf J[u,\chi]$ as $\sigma \downarrow 0$ (see Theorem 7.1).

The main purpose of our paper is to investigate in which way equilibrium states depend on the parameters $h \in \mathbb{R}$ and $\sigma > 0$, in particular, we describe the range of parameters for which only two–phase (one–phase) equilibria exist and under which conditions bifurcation occurs. A precise formulation is given in Theorem 2.1, and in Theorem 8.1 we include additional volume force terms, Theorem 9.1 addresses the case of non–zero boundary values. In this paper we will make use of various methods introduced by the third author in [OS1] for the investigation of phase transition problems in elastic media with residual stress operators. The reader who wants to learn more about the mathematical and physical background should consult the monograph [OS2].

Finally, we wish to mention that there exists a third way of regularizing (1.1) where the surface energy term $\sigma|S|$ from (1.4) is replaced by a quantity involving higher order weak derivatives like $\sigma \int_\Omega |\Delta u|^p \, dx$ of the deformation field u. This model was proposed in [KM] and [MU], and in the particular case $h = 0$ there is an approach to investigate the minimization problem for the functional (1.2) without using any regularization. This approach is based on the construction of a deformation u s.t. $\varepsilon(u)(x) \in \{\xi^+(x), \xi^-(x)\}$ holds a.e. (see again [MU]).

REMARK 1.1 *The reader should note that variational problems with a perimeter penalization naturally occur in the setting of optimal design theory, we refer e.g. to [AB].*

2 Notation and results

Let $\mathbb{S}^d$ denote the space of all symmetric $d \times d$ matrices. We define for $u = (u_i)$, $v = (v_i) \in \mathbb{R}^d$ and for $\varkappa = (\varkappa_{ij})$, $\kappa = (\kappa_{ij}) \in \mathbb{S}^d$, $u \cdot v := u_i v_i$, $|u| = \sqrt{u \cdot u}$, $\langle \varkappa, \kappa \rangle := \operatorname{tr}(\varkappa\, \kappa) = \varkappa_{ij}\, \kappa_{ij}$, $|\varkappa| := \sqrt{\langle \varkappa, \varkappa \rangle}$, $\varkappa\, u := (\varkappa_{ij} u_j) \in \mathbb{R}^d$, where we always take the sum over repeated Latin indices from 1 to d. If A: $\mathbb{S}^d \to \mathbb{S}^d$ denotes a symmetric linear operator, i.e.

$$\langle A\alpha, \beta \rangle \;=\; \langle \alpha, A\beta \rangle \quad \text{for all } \alpha,\, \beta \in \mathbb{S}^d,$$

we will use a coordinate representation in the form

$$(A\alpha)_{ij} \;=\; a_{ij,kl}\, \alpha_{kl}\,, \qquad i,\, j = 1, \ldots, d\,.$$

In terms of the coefficients $a_{ij,kl} \in \mathbb{R}$ symmetry of A means

$$a_{ij,kl} \;=\; a_{kl,ij}\,,\;\; a_{ij,kl} \;=\; a_{ji,kl}\,,\;\; a_{ij,kl} \;=\; a_{ij,lk}\,. \tag{2.1}$$

In the following $\Omega \subset \mathbb{R}^d$ is assumed to be a bounded Lipschitz domain. For functions u: $\Omega \to \mathbb{R}^d$ from the Sobolev space $\overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^d)$ (see [AD]) we define the strain tensor

$$\big(\varepsilon(u)\big)_{ij} \;=\; \frac{1}{2}\big(\partial_i u^j + \partial_j u^i\big)\,, \qquad i,\, j = 1,\dots,d\,, \tag{2.2}$$

and observe that $\varepsilon(u)(x) \in \mathbb{S}^d$ for a.a. $x \in \Omega$. Note also that by Korn's inequality (compare, e.g. [ZE] for a list of references) there is a constant c independent of u such that

$$\big\|\nabla u\big\|_{L^2(\Omega)} \;\le\; c\,\big\|\varepsilon(u)\big\|_{L^2(\Omega)}$$

holds for any u from the space $\overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^d)$. Suppose now that for each $x \in \overline{\Omega}$ two symmetric, linear operators $A^{\pm}(x)$: $\mathbb{S}^d \to \mathbb{S}^d$ are given with coordinates of the form

$$a^{\pm}_{ij,kl}(x) \;=\; a_{ij,kl}(x) + \alpha^{\pm}_{ij,kl}(x)\,, \tag{2.3^1}$$

$a_{ij,kl}$ and $\alpha^{\pm}_{ij,kl}$ being symmetric and satisfying

$$\left.\begin{aligned} & a_{ij,kl} \in C^0(\overline{\Omega})\,,\;\; \alpha^{\pm}_{ij,kl} \in L^\infty(\Omega)\,, \\ & \big\|\alpha^{\pm}_{ij,kl}\big\|_{L^\infty(\Omega)} < \varepsilon\,. \end{aligned}\right\} \tag{2.3^2}$$

Here ε is a sufficiently small positive real number being specified in Lemma 3.5 below. In addition to (2.3) we assume the operators $A^{\pm}$ to be positive definite, i.e. for some $\nu > 0$ we have

$$\nu\,|\alpha|^2 \;\le\; \big\langle A^{\pm}(x)\,\alpha,\alpha\big\rangle \;\le\; \nu^{-1}\,|\alpha|^2 \tag{2.4}$$

being valid for all $x \in \overline{\Omega}$ and $\alpha \in \mathbb{S}^d$. Next, let us state our hypotheses concerning the stress–free strains $\xi^{\pm}$: for some finite $q > d$ we have

$$\xi^{\pm} \in L^{2q}(\Omega;\mathbb{S}^d)\,, \tag{2.5}$$

moreover, $\xi^{\pm}$ are generalized solutions of the equilibrium equations, i.e.

$$\int_\Omega \big\langle A^{\pm}\,\xi^{\pm},\varepsilon(v)\big\rangle\,dx \;=\; 0 \quad \text{for all } \; v \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^d)\,. \tag{2.6}$$

Note that (2.6) holds in the case that $A^{\pm}$ as well as $\xi^{\pm}$ do not depend on $x \in \Omega$. Besides of this $A^{\pm}$ and $\xi^{\pm}$ should satisfy <u>one of</u> the following additional conditions:

$$\left.\begin{array}{l}\text{there is a subset } E \text{ of } \Omega \text{ with positive measure such that}\\ \big\langle A^+(x)\,\xi^+(x), \xi^+(x)\big\rangle - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \big\langle A^+(y)\,\xi^+(y), \xi^+(y)\big\rangle\,dy \\ \qquad < \;\big\langle A^-(x)\,\xi^-(x), \xi^-(x)\big\rangle - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \big\langle A^-(y)\,\xi^-(y), \xi^-(y)\big\rangle\,dy \\ \text{is true for a.a. } x \in E.\end{array}\right\} \qquad (2.7)$$

$$\left.\begin{array}{l}\big\langle A^+(x)\,\xi^+(x), \xi^+(x)\big\rangle - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \big\langle A^+(y)\,\xi^+(y), \xi^+(y)\big\rangle\,dy \\ \qquad \le \;\big\langle A^-(x)\,\xi^-(x), \xi^-(x)\big\rangle - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \big\langle A^-(y)\,\xi^-(y), \xi^-(y)\big\rangle\,dy \\ \text{a.e. on } \Omega \text{ and } \; A^+\xi^+ \ne A^-\xi^- \,.\end{array}\right\} \qquad (2.7^*)$$

In Section 4 the hypotheses (2.7) and (2.7*) will imply the existence ot two–phase equilibria. (2.7) should be viewed as a kind of sufficient condition for this fact in the case of variable data $A^{\pm}(x)$, $\xi^{\pm}(x)$. Clearly (2.7) is violated in the case when $A^{\pm} \equiv A_0^{\pm}$, $\xi^{\pm} \equiv \xi_0^{\pm}$ with constant operators $A_0^{\pm}$ and constant symmetric matrices $\xi_0^{\pm}$ but then (2.7*) reduces to the natural requirement that $A_0^+\,\xi_0^+ \ne A_0^-\,\xi_0^-$. An example satisfying (2.7) will be given at the end of Section 6.

Let us now recall our definitions of $f^{\pm}(\cdot,\varepsilon)$, $f_h^+(\cdot,\varepsilon)$, $h \in \mathbb{R}$, $\varepsilon \in \mathbb{S}^d$, from Section 1 and define for $\sigma \ge 0$

$$I[u,\chi,h,\sigma] \;:=\; \int_\Omega \Big(\chi\, f_h^+\big(\cdot,\varepsilon(u)\big) + (1-\chi)\, f^-\big(\cdot,\varepsilon(u)\big)\Big)\,dx + \sigma\int_\Omega |\nabla\chi| \quad (2.8)$$

where the pair (u,χ) is taken from the space

$$X \;:=\; \overset{\circ}{W}{}_2^1\big(\Omega;\mathbb{R}^d\big) \times \Big\{\chi \in BV(\Omega) \,:\, \chi(x) \in \{0,1\} \;\text{ a.e.}\Big\}\,, \qquad (2.9)$$

i.e. χ is a measurable characteristic function of finite total variation

$$\int_\Omega |\nabla\chi| \;:=\; \sup\Big\{\int_\Omega \chi \operatorname{div}\varphi\,dx \,:\, \varphi \in C_0^1\big(\Omega;\mathbb{R}^d\big),\; |\varphi| \le 1 \,\text{a.e.}\Big\} \;<\; +\infty\,.$$

For a definition of the space $BV(\Omega)$ we refer to [GI] or [AFP] where the reader will also find the proofs of the following facts:

a) lower semicontinuity: if $\{\chi_n\}$ is a sequence of measurable characteristic functions $\chi_n \in BV(\Omega)$ s.t. $\chi_n \to \chi$ a.e., then

$$\int_\Omega |\nabla \chi| \;\leq\; \liminf_{n\to\infty} \int_\Omega |\nabla \chi_n| \,.$$

b) compactness: if for a sequence $\{\chi_n\}$ as above we have $\sup_n \int_\Omega |\nabla \chi_n| < \infty$, then there is a subsequence $\{\tilde{\chi}_n\}$ and a measurable characteristic function $\chi \in BV(\Omega)$ s.t. $\chi_n \to \chi$ a.e.

c) isoperimetric inequality: suppose that the measurable characteristic function $\chi \in BV(\Omega)$ satisfies $\frac{1}{|\Omega|}\int_\Omega \chi\, dx \leq \frac{1}{2}$. Then there is a number $\kappa = \kappa(d, \Omega)$ s.t.

$$\left(\int_\Omega \chi\, dx\right)^{\frac{d-1}{d}} \;\leq\; \kappa \int_\Omega |\nabla \chi| \,.$$

d) density: for any measurable characteristic function χ there exists a sequence $\{\chi_n\}$ of measurable characteristic functions in $BV(\Omega)$ s.t. $\chi_n \to \chi$ a.e.

(Property d) is proved in [OS2], for convenience we sketch the proof in the Appendix.)
Now we state the main result of our paper in which we describe the dependence of equilibrium states $\hat{u}, \hat{\chi}$ of the functional (2.8) on the parameters h and σ.

THEOREM 2.1 *Let all the hypotheses stated before be satisfied. Then, for any $h \in \mathbb{R}$ and $\sigma > 0$, the functional from (2.8) attains its minimum on the set X defined in (2.9). The half–plane of parameters $\sigma > 0$ and $h \in \mathbb{R}$ is divided into three open regions A, B, C (see the figure below) such that the following holds:*

a) for $(\sigma, h) \in A$ we only have the one–phase equilibrium $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$;

b) for $(\sigma, h) \in C$ only the one–phase equilibrium state $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ exists;

c) within the region B only two–phase states of equilibria exist.

Region A (C) is separated from region B by the graph of a continuous function $h^+(\sigma)$ ($h^-(\sigma)$), $0 < \sigma < \sigma^*$ for some $\sigma^* > 0$; the functions $h^\pm$ are defined on $(0, +\infty)$ and have the following properties: there exists a number $\hat{h}$ (an expression for this quantity is given in (3.7)) such that:

on $(0,\sigma^*)$ h^+ is strictly decreasing and $h^+ > \hat{h}$;
on $(0,\sigma^*)$ h^- is strictly increasing and $h^- < \hat{h}$;
for $\sigma \in [\sigma^*, \infty)$ we have $h^+(\sigma) = h^-(\sigma) = \hat{h}$.

On the graphs of $h^\pm$ we have the following description of equilibrium states:

d) for $h = h^+(\sigma)$, $\sigma \in (0,\sigma^*)$, we have the one–phase equilibrium state $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ and at least one additional two–phase equilibrium state;

e) for $h = h^-(\sigma)$, $\sigma \in (0,\sigma^*)$, there is at least one two–phase equilibrium state together with the one–phase equilibrium $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$;

f) for $h = \hat{h}$, $\sigma \in (\sigma^*, \infty)$, the equilibrium states consist of the pairs $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ and $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$;

g) for $h = \hat{h}$, $\sigma = \sigma^*$ we have the equilibrium states $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$, $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ plus at least one additional two–phase equilibrium.

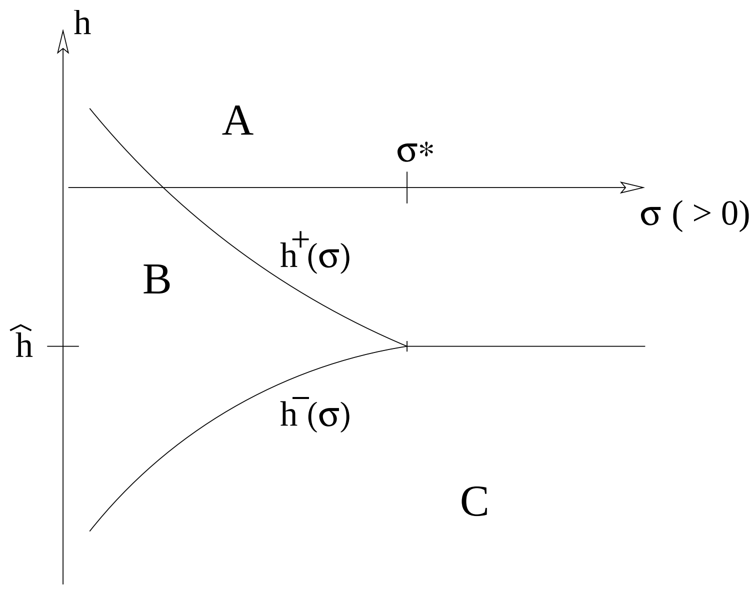

Figure 1: The σ, h half–plane

As to the regularity for two–phase state equilibria we have the following result (assuming the same hypotheses as for Theorem 2.1).

THEOREM 2.2 *Consider a two–phase equilibrium state $(\hat{u}, \hat{\chi}) \in X$ of the functional from (2.8) with $\sigma > 0$ and let $E = \{x \in \Omega : \hat{\chi}(x) = 1\}$. Then, if*

$d \le 7$, $\Omega \cap \partial E$ is a hypersurface of class C^1 separating Ω into two open sets on which $\hat{u}$ is smooth provided that the coefficients $a^{\pm}_{ij,kl}$ are regular.

REMARK 2.3 *Regarding the definition of ∂E we adopt the standard convention (see, e.g. [GI], Proposition 3.1 and Remark 3.2) that*

$$0 \ < \ |E \cap B_r(x)| \ < \ \omega_d r^d$$

holds for any $x \in \partial E$, ω_d denoting the volume of the d–dimensional unit ball. The latter condition can always be achieved by replacing E through a set $\tilde{E}$ such that $|E - \tilde{E}| = |\tilde{E} - E| = 0$.

REMARK 2.4 *In Theorem 8.1 (and 8.2) we prove that Theorem 2.1 (and also Theorem 2.2) extend to the case when we add a potential like $\int_\Omega p \cdot u \, dx$ to the energy $I[u, \chi, h, \sigma]$. Moreover, we can include the case of non–vanishing boundary values u_0 (see Theorem 9.1). In both cases the data have to be sufficiently small.*

REMARK 2.5 *By further decreasing the quantity ε from (2.3^2) (if necessary) the functions $h^{\pm}(\sigma)$ are seen to be bounded if so are the stress–free strains $\xi^{\pm}(x)$ (see Lemma 7.3).*

The proof of Theorem 2.1 is organized in a series of lemmas presented in Section 3 to Section 5. In Section 6 we put together these auxiliary results by the way completing the proof of Theorem 2.1. Moreover, Section 6 contains the proof of Theorem 2.2. We finish Section 6 by adding an example for which condition (2.7) is satisfied. In Section 7 we give some further comments on our results, in particular we show that $\{\hat{u}_n\}$ is a minimizing sequence for the functional from (1.2) whenever $(\hat{u}_n, \hat{\chi}_n) \in X$ is an equilibrium state of $I[u, \chi, h, \sigma_n]$ for a sequence $\{\sigma_n\}$ such that $\sigma_n \ge \sigma_{n+1}$, $\sigma_n > 0$, $\lim_{n\to\infty} \sigma_n = 0$. Finally, we discuss in Section 8 the case involving an additional volume force term, in Section 9 we add some remarks on non–zero boundary values.

3 Some existence results

From now on we assume that all the conditions stated before Theorem 2.1 are valid but let us remark explicitly that we neither need (2.7) nor (2.7*) throughout this section. We start with the following simple observation: consider a one–phase equilibrium state $(\hat{u}, \hat{\chi}) \in X$ of the functional $I[u, \chi, h, \sigma]$, $h \in \mathbb{R}$, $\sigma \ge 0$, i.e. $\hat{\chi} \equiv 0$ or $\hat{\chi} \equiv 1$. Then we have $\hat{u} \equiv 0$. For the proof let us consider the case $\hat{\chi} \equiv 1$. Then

$$I[\hat{u}, 1, h, \sigma] \ \le \ I[0, 1, h, \sigma]$$

implies (compare (2.8) and write $f_h^+ = f_0^+ + h$, $f_0^+(\cdot,\varepsilon) := \langle A^+(\varepsilon-\xi^+), \varepsilon-\xi^+\rangle$)

$$\int_\Omega f_0^+\big(\cdot,\varepsilon(\hat u)\big)\,dx \;\le\; \int_\Omega f_0^+(\cdot,0)\,dx\,,$$

and by (2.6) (recall $\hat u \in \overset{\circ}{W}{}_2^1(\Omega;\mathbb{R}^d)$) this reduces to

$$\int_\Omega \big\langle A^+\,\varepsilon(\hat u),\varepsilon(\hat u)\big\rangle\,dx \;\le\; 0\,,$$

hence the ellipticity condition (2.4) together with Korn's inequality gives the claim. The case $\hat\chi \equiv 0$ is treated in the same way.
The next result is a trivial application of (2.4) combined with Young's inequality.

LEMMA 3.1 *Let $h \in \mathbb{R}$, $\sigma \ge 0$ be given. Then for any $(u,\chi) \in X$ we have the estimate*

$$\begin{aligned}\frac{\nu}{2}\int_\Omega |\varepsilon(u)|^2\,dx + \sigma\int_\Omega |\nabla\chi| \\ \le\; I[u,\chi,h,\sigma] + |h|\,|\Omega| + \frac{\nu^2+4}{\nu^3}\int_\Omega \big(|\xi^+|^2+|\xi^-|^2\big)\,dx\,.\end{aligned} \tag{3.1}$$

Proof. We have on account of (2.4)

$$\begin{aligned} I[u,\chi,h,\sigma] \;\ge\; & \nu\int_\Omega |\varepsilon(u)|^2\,dx - |h|\,|\Omega| + \sigma\int_\Omega|\nabla\chi| \\ & -\frac{1}{\nu}\int_\Omega\big(|\xi^+|^2+|\xi^-|^2\big)\,dx \\ & -2\int_\Omega\Big(\big|\big\langle A^+\,\varepsilon(u),\xi^+\big\rangle\big| + \big|\big\langle A^-\,\varepsilon(u),\xi^-\big\rangle\big|\Big)\,dx\,.\end{aligned}$$

Observing $|\langle A^\pm\varepsilon,\tilde\varepsilon\rangle| \le \sqrt{\langle A^\pm\varepsilon,\varepsilon\rangle}\,\sqrt{\langle A^\pm\tilde\varepsilon,\tilde\varepsilon\rangle}$, inequality (3.1) is immediate. ∎

The functional $I[u,\chi,h,\sigma]$ has nice lower semicontinuity properties.

LEMMA 3.2 *Consider a sequence (u_n,χ_n) from the space X and sequences $h_n \in \mathbb{R}$, $\sigma_n \ge 0$ such that*

$$h_n \to: h\,,\quad \sigma_n \to: \sigma\,,\quad u_n \rightharpoonup: u \;\; \textit{in} \;\; \overset{\circ}{W}{}_2^1\big(\Omega;\mathbb{R}^d\big) \;\; \textit{and} \;\; \chi_n \to: \chi \;\; \textit{a.e.} \tag{3.2}$$

with $(u,\chi) \in X$. Then

$$I[u,\chi,h,\sigma] \;\le\; \liminf_{n\to\infty} I[u_n,\chi_n,h_n,\sigma_n]\,. \tag{3.3}$$

REMARK 3.3 *Clearly, the assumption that (u,χ) is in the space X is equivalent to $\chi \in BV(\Omega)$. On account of*

$$\int_\Omega |\nabla \chi| \;\leq\; \liminf_{n\to\infty} \int_\Omega |\nabla \chi_n|$$

this would follow if the total variations of the χ_n stay bounded which can not be deduced from the convergences stated in (3.2).

Proof. Let us first show that

$$I[u,\chi,h,0] \;\leq\; \liminf_{n\to\infty} I[u_n,\chi_n,h_n,0]\,. \tag{3.4}$$

Assuming (3.4) we get

$$\begin{aligned} I[u,\chi,h,\sigma] &= I[u,\chi,h,0] + \sigma \int_\Omega |\nabla \chi| \\ &\leq \liminf_{n\to\infty} I[u_n,\chi_n,h_n,0] + \liminf_{n\to\infty} \int_\Omega |\nabla(\sigma_n \chi_n)| \end{aligned}$$

where $\sigma \int_\Omega |\nabla\chi| \leq \liminf_{n\to\infty} \int_\Omega |\nabla(\sigma_n\chi_n)|$ follows from $\sup_n \|\sigma_n\chi_n\|_{L^\infty(\Omega)} < \infty$ together with $\sigma_n\chi_n \to \sigma\chi$ by a simple application of Lebesgues's theorem on dominated convergence. This shows (3.3). For (3.4) we observe

$$\begin{aligned} I[u_n,\chi_n,h_n,0] &= \int_\Omega \Big(\chi_n f^+_{h_n}\big(\cdot,\varepsilon(u_n)\big) + (1-\chi_n)\, f^-\big(\cdot,\varepsilon(u_n)\big)\Big)\, dx \\ &= \int_\Omega \Big(\chi_n f^+_{h_n}\big(\cdot,\varepsilon(u)\big) + (1-\chi_n)\, f^-\big(\cdot,\varepsilon(u)\big)\Big)\, dx \\ &\quad + \int_\Omega \Big[\chi_n \Big(f^+_{h_n}\big(\cdot,\varepsilon(u_n)\big) - f^+_{h_n}\big(\cdot,\varepsilon(u)\big)\Big) \\ &\qquad\quad +(1-\chi_n)\Big(f^-\big(\cdot,\varepsilon(u_n)\big) - f^-\big(\cdot,\varepsilon(u)\big)\Big)\Big]\, dx \\ &:= (1) + (2)\,, \end{aligned}$$

where

$$(1) \;=\; I[u,\chi_n,h_n,0] \;\overset{n\to\infty}{\to}\; I[u,\chi,h,0]\,.$$

Let $\varepsilon_n := \varepsilon(u_n)$, $\varepsilon := \varepsilon(u)$. Then

$$\begin{aligned} \big\langle A^\pm(\varepsilon_n-\xi^\pm),\varepsilon_n-\xi^\pm\big\rangle &= \big\langle A^\pm(\varepsilon_n-\varepsilon),\varepsilon_n-\varepsilon\big\rangle + \big\langle A^\pm(\varepsilon-\xi^\pm),\varepsilon-\xi^\pm\big\rangle \\ &\quad +2\big\langle A^\pm(\varepsilon_n-\varepsilon),\varepsilon-\xi^\pm\big\rangle\,, \end{aligned}$$

and by ellipticity this implies the lower bound

$$
\begin{aligned}
(2) \quad \geq \quad & 2\int_\Omega \chi_n \big\langle A^+\big(\varepsilon(u_n)-\varepsilon(u)\big), \varepsilon(u)-\xi^+\big\rangle\, dx \\
+ \quad & 2\int_\Omega (1-\chi_n)\big\langle A^-\big(\varepsilon(u_n)-\varepsilon(u)\big), \varepsilon(u)-\xi^-\big\rangle\, dx \quad \overset{n\to\infty}{\to} \quad 0\,,
\end{aligned}
$$

where the limit behaviour of the right–hand side follows from assumption (3.2). This proves (3.4), Lemma 3.2 is established. ■

Putting together Lemma 3.1 and Lemma 3.2 we get

THEOREM 3.4 *Given arbitrary parameters $h \in \mathbb{R}$ and $\sigma > 0$, there exists an equilibrium state $(\hat{u}, \hat{\chi}) \in X$ of the functional $I[u, \chi, h, \sigma]$.*

Proof. Let $(u_n, \chi_n) \in X$ denote a $I[\cdot, \cdot, h, \sigma]$–minimizing sequence, i.e.

$$
I[u_n, \chi_n, h, \sigma] \quad \overset{n\to\infty}{\to} \quad \inf_X I[\cdot, \cdot, h, \sigma]\,.
$$

Lemma 3.1 together with Korn's inequality and the compactness of the embedding $BV(\Omega) \to L^1(\Omega)$ implies the existence of a subsequence and the existence of a pair $(u, \chi) \in X$ such that the convergences (3.2) hold. The minimizing property of (u, χ) is a consequence of (3.3) (with the choice $h_n = h$, $\sigma_n = \sigma$). ■

In the next lemma we give an upper bound for the value of the quantity ε occurring in condition (2.3^2). It should be noted that ε does neither depend on h nor on $\sigma > 0$.

LEMMA 3.5 *There exist constants $\varepsilon > 0$ and $R > 0$ just depending on Ω, d, q, $\|a_{ij,kl}\|_{L^\infty(\Omega)}$ and the ellipticity constant ν (R is also depending on $\|\xi^\pm\|_{L^{2q}(\Omega)}$) such that if ($2.3^2$) is satisfied for this choice of ε, we have $\hat{u} \in \overset{\circ}{W}{}^1_{2q}(\Omega; \mathbb{R}^d)$ together with $\|\hat{u}\|_{W^1_{2q}(\Omega;\mathbb{R}^d)} \leq R$, whenever $(\hat{u}, \hat{\chi}) \in X$ is an equilibrium state of $I[u, \chi, h, \sigma]$ with $h \in \mathbb{R}$, $\sigma > 0$.*

Proof. Let $(\hat{u}, \hat{\chi})$ denote an equilibrium state of $I[u, \chi, h, \sigma]$. Then $I[\hat{u}, \hat{\chi}, h, \sigma] \leq I[\hat{u} + tv, \hat{\chi}, h, \sigma]$ for any $t \in \mathbb{R}$, $v \in \overset{\circ}{W}{}^1_2(\Omega; \mathbb{R}^d)$, therefore

$$
\int_\Omega \Big\{ \hat{\chi}\big\langle A^+\big(\varepsilon(\hat{u})-\xi^+\big), \varepsilon(v)\big\rangle + (1-\hat{\chi})\big\langle A^-\big(\varepsilon(\hat{u})-\xi^-\big), \varepsilon(v)\big\rangle \Big\}\, dx \;=\; 0\,.
$$

Let $A := (a_{ij,kl})$, $\tilde{A} := (\hat{\chi}\,\alpha^+_{ij,kl} + [1-\hat{\chi}]\,\alpha^-_{ij,kl})$, hence

$$\begin{aligned}\int_\Omega \big\langle A\,\varepsilon(\hat{u}), \varepsilon(v)\big\rangle\,dx + \int_\Omega \big\langle \tilde{A}\,\varepsilon(\hat{u}), \varepsilon(v)\big\rangle\,dx \\ = \int_\Omega \big\langle \hat{\chi}\,A^+\,\xi^+ + (1-\hat{\chi})\,A^-\,\xi^-, \varepsilon(v)\big\rangle\,dx\end{aligned}$$

being valid for any $v \in \overset{\circ}{W}{}_2^{\,1}(\Omega;\mathbb{R}^d)$. For ε small enough (depending on $\|A\|_{L^\infty(\Omega)}$ and ν) the operator $A(x)$: $\mathbb{S}^d \to \mathbb{S}^d$ satisfies (2.4) with ν replaced by $\frac{\nu}{2}$ for any $x \in \overline{\Omega}$, therefore the unique weak solution $u \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^d)$ of

$$\int_\Omega \big\langle A\,\varepsilon(u), \varepsilon(v)\big\rangle\,dx \;=\; \langle T, v\rangle$$

belongs to the space $\overset{\circ}{W}{}^1_{2q}(\Omega;\mathbb{R}^d)$ provided

$$T \in W^{-1}_{2q}(\Omega;\mathbb{R}^d) := \big(\overset{\circ}{W}{}^1_{(2q)'}(\Omega;\mathbb{R}^d)\big)^*$$

(compare, e.g. [MOR], Chap. 6.4, or [RS]), more precisely, the mapping

$$J:\;\; W^{-1}_{2q}\big(\Omega;\mathbb{R}\big) \ni T \;\longmapsto\; u \in \overset{\circ}{W}{}^1_{2q}\big(\Omega;\mathbb{R}^d\big)$$

is an isomorphism. Clearly

$$\begin{aligned} L:\; \overset{\circ}{W}{}^1_{2q}\big(\Omega;\mathbb{R}^d\big) \ni u \;&\mapsto\; l_u \in W^{-1}_{2q}\big(\Omega;\mathbb{R}^d\big)\,,\\ \langle l_u, \varphi\rangle \;&:=\; \int_\Omega \big\langle \tilde{A}\,\varepsilon(u), \varepsilon(\varphi)\big\rangle\,dx\,,\;\; \varphi \in \overset{\circ}{W}{}^1_{(2q)'}\big(\Omega;\mathbb{R}^d\big)\,,\end{aligned}$$

is a continuous linear mapping whose norm can be bounded independent of $\hat{\chi}$, and from the definition of $\tilde{A}$ is follows that

$$J^{-1} + L:\; \overset{\circ}{W}{}^1_{2q}\big(\Omega;\mathbb{R}^d\big) \;\to\; W^{-1}_{2q}\big(\Omega;\mathbb{R}^d\big)$$

is an isomorphism provided that $\|L\|$ is small enough. The last requirement can be fulfilled (independent of $\hat{\chi}$) if we choose

$$\|\alpha^\pm_{ij,kl}\|_{L^\infty(\Omega)} \;<\; \varepsilon$$

with ε as in the lemma. Let $b \in W^{-1}_{2q}(\Omega;\mathbb{R}^d)$ be given by

$$\langle b, v\rangle \;=\; \int_\Omega \langle \hat{\chi}\,A^+\,\xi^+ + (1-\hat{\chi})\,A^-\,\xi^-, \varepsilon(v)\big\rangle\,dx\,.$$

By the above considerations there exists a unique function $u' \in \overset{\circ}{W}{}^1_{2q}(\Omega;\mathbb{R}^d)$ such that

$$\big(J^{-1}+L\big)(u') \;=\; b\,,$$

and from the equation satisfied by $\hat u$ we immediately deduce $\hat u = u'$. This shows

$$\hat u \in \overset{\circ}{W}{}^1_{2q}\big(\Omega;\mathbb{R}^d\big) \quad \text{together with} \quad \|\hat u\|_{W^1_{2q}(\Omega)} \;\le\; C\,\|b\|_{W^{-1}_{2q}(\Omega)}\,.$$

By definition (recall (2.5)) the norm of b can be bounded uniformly w.r.t. to $\hat\chi$, the lemma is established. ■

To finish this section let us consider

$$\begin{aligned} I^+[u,h] &:= I[u,1,h,0] &=& \int_\Omega f_h^+\big(\cdot,\varepsilon(u)\big)\,dx\,, \\ I^-[u] &:= I[u,0,h,0] &=& \int_\Omega f^-\big(\cdot,\varepsilon(u)\big)\,dx\,, \quad u \in \overset{\circ}{W}{}^1_2\big(\Omega;\mathbb{R}^d\big)\,. \end{aligned} \tag{3.5}$$

$I^\pm$ represent the total deformation energy of one–phase elastic media with energy density $f_h^+(\cdot,\varepsilon)$ and $f^-(\cdot,\varepsilon)$, respectively.

LEMMA 3.6 *On $\overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^d)$ the functionals $I^\pm$ attain their unique minima at $\hat u^\pm \equiv 0$.*

Proof. Existence and uniqueness of minimizers $\hat u^\pm$ for $I^\pm$ follows from condition (2.4). The corresponding Euler equation reads

$$\int_\Omega \big\langle A^\pm\big(\varepsilon(\hat u^\pm)-\xi^\pm\big),\varepsilon(v)\big\rangle\,dx \;=\; 0 \quad \text{for all } v \in \overset{\circ}{W}{}^1_2\big(\Omega;\mathbb{R}^d\big)\,,$$

and by (2.6) this reduces to

$$\int_\Omega \big\langle A^\pm\,\varepsilon(\hat u^\pm),\varepsilon(v)\big\rangle\,dx \;=\; 0\,.$$

Taking $v = \hat u^\pm$ the claim follows. ■

Let us consider the case that the variational problem

$$I[u,\chi,h,\sigma] \;\to\; \min \quad \text{in } X \quad (h\in\mathbb{R},\;\; \sigma>0)$$

admits only one–phase equilibria $(\hat u,\hat\chi)$, i.e. either $\hat\chi\equiv 0$ or $\hat\chi\equiv 1$ together with $\hat u\equiv 0$ (see the beginning of this section). Then the quantity

$$\hat I_0[h] \;:=\; \min\Big\{I^+[\hat u^+,h],\,I^-[\hat u^-]\Big\} \tag{3.6}$$

determines the dependence of the energy of an equilibrium state on the quantity h. It is easy to check that

$$\hat{I}_0[h] \;=\; \begin{cases} \displaystyle\int_\Omega \langle A^+\xi^+,\xi^+\rangle\,dx + h\,|\Omega|\,, & h \le \hat{h}\,, \\ \displaystyle\int_\Omega \langle A^-\xi^-,\xi^-\rangle\,dx\,, & h \ge \hat{h}\,, \end{cases} \tag{3.7}$$

$$\hat{h} \;=\; \frac{1}{|\Omega|}\int_\Omega \big(-\langle A^+\xi^+,\xi^+\rangle + \langle A^-\xi^-,\xi^-\rangle\big)\,dx\,.$$

4 The behaviour of the volume of the phases of equilibrium states as a function of the parameter h

Roughly speaking, the next lemma implies that under certain hypotheses sequences of two–phase equilibria will converge weakly to a two–phase equilibrium.

LEMMA 4.1 *For any $k > 1$ there exists a number $\delta = \delta(k) \in (0, 1/2)$ (depending also on d, Ω, q, $\|\xi^\pm\|_{L^{2q}(\Omega)}$) such that the following is true: suppose that $(\hat{u},\hat{\chi}) \in X$ is a two–phase equilibrium of the energy $I[u,\chi,h,\sigma]$ with $|h| \le k$ and $k^{-1} \le \sigma \le k$. Then we have*

$$\delta \;\le\; \frac{1}{|\Omega|}\int_\Omega \hat{\chi}\,dx \;\le\; 1-\delta.$$

Proof. Let $(\hat{u},\hat{\chi})$ denote a two–phase equilibrium state of $I[u,\chi,h,\sigma]$. Then

$$\begin{aligned} I[\hat{u},\hat{\chi},h,\sigma] &\;\le\; I[0,0,h,\sigma]\,, \\ I[\hat{u},\hat{\chi},h,\sigma] &\;\le\; I[0,1,h,\sigma]\,. \end{aligned} \tag{4.1}$$

The first inequality in (4.1) implies

$$\begin{aligned} &\int_\Omega \Big\{ f^-\big(\cdot,\varepsilon(\hat{u})\big) + \hat{\chi}\Big(f_h^+\big(\cdot,\varepsilon(\hat{u})\big) - f^-\big(\cdot,\varepsilon(\hat{u})\big)\Big)\Big\}\,dx + \sigma\int_\Omega |\nabla\hat{\chi}| \\ &\qquad \le\; \int_\Omega f^-(\cdot,0)\,dx \;\le\; \int_\Omega f^-\big(\cdot,\varepsilon(\hat{u})\big)\,dx\,, \end{aligned}$$

where the last inequality follows from Lemma 3.6. Thus we obtain

$$\sigma \int_\Omega |\nabla \hat{\chi}| \;\leq\; \int_\Omega \hat{\chi} \Big\{ f^-\big(\cdot, \varepsilon(\hat{u})\big) - f_h^+\big(\cdot, \varepsilon(\hat{u})\big) \Big\} dx \,. \tag{4.2}$$

In a similar way we may use the second inequality in (4.1) to get

$$\begin{aligned} \int_\Omega \Big\{ f_h^+\big(\cdot, \varepsilon(\hat{u})\big) + (1-\hat{\chi}) \Big(f^-\big(\cdot, \varepsilon(\hat{u})\big) - f_h^+\big(\cdot, \varepsilon(\hat{u})\big) \Big) \Big\}\, dx & \\ +\sigma \int_\Omega \big|\nabla(1-\hat{\chi})\big| \;\leq\; \int_\Omega f_h^+(\cdot, 0)\, dx \;\leq\; \int_\Omega f_h^+\big(\cdot, \varepsilon(\hat{u})\big)\, dx \,, & \end{aligned}$$

in conclusion

$$\sigma \int_\Omega |\nabla(1-\hat{\chi})| \;\leq\; \int_\Omega (1-\hat{\chi}) \Big\{ f_h^+\big(\cdot, \varepsilon(\hat{u})\big) - f^-\big(\cdot, \varepsilon(\hat{u})\big) \Big\}\, dx \,. \tag{4.3}$$

Let $G := |f_0^+(\cdot, \varepsilon(\hat{u})) - f^-(\cdot, \varepsilon(\hat{u}))|$. Recalling Lemma 3.5 and assumption (2.5) we get

$$\|G\|_{L^q(\Omega)} \;\leq\; G_0$$

for a finite constant G_0 independent of h and σ (but depending on the same quantities as R from Lemma 3.5). Let us denote by $\tilde{\chi}$ one of the functions $\hat{\chi}$ or $1-\hat{\chi}$ for which

$$\frac{1}{|\Omega|} \int_\Omega \tilde{\chi}\, dx \;\leq\; \frac{1}{2}$$

is true. (4.2), (4.3) and Hölder's inequality imply

$$\begin{aligned} \sigma \int_\Omega |\nabla \tilde{\chi}| \;&\leq\; \Big[G_0 + |h|\, \|\tilde{\chi}\|_{L^1(\Omega)}^{1/q} \Big]\, \|\tilde{\chi}\|_{L^1(\Omega)}^{\frac{q-1}{q}} \\ &=\; \Big[G_0 + |h|\, \|\tilde{\chi}\|_{L^1(\Omega)}^{1/q} \Big]\, \|\tilde{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}}\, \|\tilde{\chi}\|_{L^1(\Omega)}^{\frac{d-1}{d}} \\ &\leq\; \Big[G_0 + |h|\, \|\tilde{\chi}\|_{L^1(\Omega)}^{1/q} \Big]\, \|\tilde{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}}\, \kappa \int_\Omega |\nabla \tilde{\chi}| \end{aligned}$$

where we used the isoperimetric inequality (with constant κ) to bound the quantity $\|\tilde{\chi}\|_{L^1(\Omega)}^{\frac{d-1}{d}}$. Recall that $(\hat{u}, \hat{\chi})$ is a two–phase equilibrium state, hence $\int_\Omega |\nabla \tilde{\chi}| \not\equiv 0$, and we deduce from the above inequality

$$\frac{1}{|\Omega|} \left\{ \frac{\sigma}{\kappa\big[G_0 + |h|\, (|\Omega|/2)^{1/q}\big]} \right\}^{\frac{qd}{q-d}} \;\leq\; \frac{1}{|\Omega|}\, \|\tilde{\chi}\|_{L^1(\Omega)} \,. \tag{4.4}$$

From (4.4) the claim of the lemma follows if we define δ as the minimum of the left–hand side for all choices of $|h| \leq k$ and $\sigma \in [\frac{1}{k}, k]$. ∎

An application of Lemma 4.1 is

LEMMA 4.2 *Consider sequences $h_n \in \mathbb{R}$, $\sigma_n > 0$ such that $h_n \to h$ and $\sigma_n \to \sigma > 0$ as $n \to \infty$. For each n let $(\hat{u}_n, \hat{\chi}_n) \in X$ denote an equilibrium state of the functional $I[u, \chi, h_n, \sigma_n]$. Suppose that a subsequence of two–phase equilibria (or one–phase equilibria with $\hat{\chi}_n \equiv 0$ or one–phase equilibria with $\hat{\chi}_n \equiv 1$) exists. Then, for $I[u, \chi, h, \sigma]$ there also exists a two–phase equilibrium state (or a one–phase equilibrium state with $\hat{\chi} = 0$ or a one–phase equilibrium state with $\hat{\chi} \equiv 1$).*

Proof. Passing to a subsequence let us first assume that $(\hat{u}_n, \hat{\chi}_n)$ is a sequence of two–phase equilibria. By Lemma 3.1 we can extract a subsequence having the convergence properties stated in (3.2), in particular $\hat{u}_n \rightharpoonup \hat{u}$ in $\overset{\circ}{W}{}_2^1(\Omega; \mathbb{R}^d)$, $\hat{\chi}_n \to \chi$ a.e. Lemma 3.2 implies

$$I[\hat{u}, \hat{\chi}, h, \sigma] \;\leq\; \liminf_{n\to\infty} I[\hat{u}_n, \hat{\chi}_n, h_n, \sigma_n] \,.$$

On the other hand, for any $(u, \chi) \in X$, we have by minimality

$$I[\hat{u}_n, \hat{\chi}_n, h_n, \sigma_n] \;\leq\; I[u, \chi, h_n, \sigma_n] \,,$$

and since the right–hand side converges to $I[u, \chi, h, \sigma]$, we see that $(\hat{u}, \hat{\chi})$ is an equilibrium state of $I[u, \chi, h, \sigma]$. Let us fix $k > 1$ such that $|h_n| \leq k$, $\sigma_n \in [1/k, k]$. Then, according to Lemma 4.1, we have

$$\delta \;\leq\; \frac{1}{|\Omega|} \int_\Omega \hat{\chi}_n \, dx \;\leq\; 1 - \delta$$

for all n with δ independent of n, therefore

$$\delta \;\leq\; \frac{1}{|\Omega|} \int_\Omega \hat{\chi} \, dx \;\leq\; 1 - \delta \,,$$

and $(\hat{u}, \hat{\chi})$ is a two–phase equilibrium state. The corresponding result for single–phase equilibria is trivial, since the property $\hat{\chi}_n \equiv 0$ ($\hat{\chi}_n \equiv 1$) is stable in the limit. ∎

LEMMA 4.3 *For $i = 1, 2$ let $(\hat{u}_i, \hat{\chi}_i) \in X$ denote an equilibrium state of the functional $I[u, \chi, h_i, \sigma]$. Then we have*

$$(h_1 - h_2) \left(\|\hat{\chi}_1\|_{L^1(\Omega)} - \|\hat{\chi}_2\|_{L^1(\Omega)} \right) \;\leq\; 0 \,.$$

Proof. The proof is a simple calculation using

$$I[\hat{u}_1, \hat{\chi}_1, h_1, \sigma] \;\leq\; I[\hat{u}_2, \hat{\chi}_2, h_1, \sigma]$$

and

$$I[\hat{u}_2, \hat{\chi}_2, h_2, \sigma] \;\leq\; I[\hat{u}_1, \hat{\chi}_1, h_2, \sigma]\,.$$

■

REMARK 4.4 *Suppose that for some h_0 there exists an equilibrium state $(\hat{u}_0, \hat{\chi}_0)$ of $I[u, \chi, h_0, \sigma]$ such that $\hat{\chi}_0 \equiv 0$ ($\hat{\chi}_0 \equiv 1$). Then, according to Lemma 4.3, all equilibrium states $(\hat{u}, \hat{\chi})$ of $I[u, \chi, h, \sigma]$ with $h > h_0$ ($h < h_0$) satisfy $\hat{\chi} \equiv 0$ ($\hat{\chi} \equiv 1$).*

The next lemma shows that for $|h|$ large enough only one–phase equilibrium states can exist.

LEMMA 4.5 *There are numbers $h^+ > h^-$ depending in particular on σ with the following property: if $h > h^+$ ($h < h^-$) and if $(\hat{u}, \hat{\chi})$ denotes an equilibrium state of $I[u, \chi, h, \sigma]$, then we have $\hat{\chi} \equiv 0$ ($\hat{\chi} \equiv 1$).*

Proof. Let us first suppose that for all numbers $H > 0$ there exists $h > H$ and an equilibrium state $(\hat{u}_h, \hat{\chi}_h)$ of $I[u, \chi, h, \sigma]$ such that

$$\frac{1}{|\Omega|} \int_\Omega \hat{\chi}_h \, dx \;\geq\; \frac{1}{2}\,. \tag{4.5}$$

From estimate (4.2) we get ($\hat{\chi} := \hat{\chi}_h, \hat{u} := \hat{u}_h$)

$$\sigma \int_\Omega |\nabla \hat{\chi}| + h \int_\Omega \hat{\chi} \, dx \;\leq\; \int_\Omega \hat{\chi} \Big(f^-\big(\cdot, \varepsilon(\hat{u})\big) - f_0^+\big(\cdot, \varepsilon(\hat{u})\big) \Big) \, dx\,. \tag{4.6}$$

(4.6) implies

$$\sigma \int_\Omega |\nabla \hat{\chi}| + h \int_\Omega \hat{\chi} \, dx \;\leq\; G_0 \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-1}{q}} \;\leq\; G_0 \, |\Omega|^{\frac{q-1}{q}}\,. \tag{4.7}$$

If we replace h by a sequence $h_n \uparrow \infty$ and use (4.5), then (4.7) gives a contradiction. Hence there exists $\tilde{h}^+ > 0$ such that

$$\frac{1}{|\Omega|} \int_\Omega \hat{\chi} \, dx \;\leq\; \frac{1}{2} \quad \text{for all } \; h > \tilde{h}^+\,. \tag{4.8}$$

Returning to (4.7) and quoting (4.8) we may use the isoperimetric inequality to get ($h > \tilde{h}^+$)

$$\begin{aligned} \sigma \int_\Omega |\nabla \hat{\chi}| &\le G_0 \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-1}{q}} = G_0 \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{d-1}{d}} \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \\ &\le \kappa G_0 \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \int_\Omega |\nabla \hat{\chi}| \end{aligned}$$

being valid for all equilibrium states $(\hat{u}, \hat{\chi})$ of $I[u, \chi, h, \sigma]$. By (4.8) $\hat{\chi} \equiv 1$ is not possible. If $(\hat{u}, \hat{\chi})$ is a two–phase equilibrium, then $\int_\Omega |\nabla \hat{\chi}| \neq 0$. But the same argument leading to (4.8) also shows the existence of h^+ (w.l.o.g. $\ge \tilde{h}^+$) such that

$$\sigma^{-1} \kappa G_0 \, \|\hat{\chi}\|^{\frac{q-d}{qd}} < 1 \quad \text{for all } h > h^+ . \tag{4.9}$$

Inserting this into the estimate for $\sigma \int_\Omega |\nabla \hat{\chi}|$, we see that $(\hat{u}, \hat{\chi})$ must be a one–phase equilibrium, in conclusion $\hat{\chi} \equiv 0$ follows. The existence of h^- is proved in a similar way: we recall (4.3)

$$\begin{aligned} &\sigma \int_\Omega |\nabla (1 - \hat{\chi})| - h \int_\Omega (1 - \hat{\chi}) \, dx \\ &\qquad \le \int_\Omega (1 - \hat{\chi}) \Big(f_0^+\big(\cdot, \varepsilon(\hat{u})\big) - f^-\big(\cdot, \varepsilon(\hat{u})\big) \Big) \, dx \end{aligned} \tag{4.10}$$

valid for an arbitrary equilibrium state $(\hat{u}, \hat{\chi})$ of $I[u, \chi, h, \sigma]$ and use (4.10) to get the estimate

$$\begin{aligned} \sigma \int_\Omega |\nabla (1 - \hat{\chi})| - h \int_\Omega (1 - \hat{\chi}) \, dx &\le G_0 \, \|1 - \hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \, \|1 - \hat{\chi}\|_{L^1(\Omega)}^{\frac{d-1}{d}} \\ &\le G_0 \, |\Omega|^{\frac{d-1}{d}} . \end{aligned} \tag{4.11}$$

As before (4.11) gives the existence of a number $\tilde{h}^- < 0$ s.t.

$$\frac{1}{|\Omega|} \int_\Omega (1 - \hat{\chi}) \, dx \le \frac{1}{2} \quad \text{for all } h < \tilde{h}^- . \tag{4.12}$$

(4.12) enables us to use the isoperimetric inequality with the result

$$\sigma \int_\Omega |\nabla (1 - \hat{\chi})| \le \kappa G_0 \, \|1 - \hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \int_\Omega \big|\nabla (1 - \hat{\chi})\big| . \tag{4.13}$$

Next we determine h^- (w.l.o.g. $< \tilde{h}^-$) according to

$$\sigma^{-1} \kappa G_0 \, \|1 - \hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} < 1 \quad \text{for all } h < h^- \tag{4.14}$$

to get from (4.13) $\int_\Omega |\nabla(1-\hat{\chi})| = 0$. Finally, (4.12) implies $\hat{\chi} \equiv 1$. ∎

To proceed further we now present some necessary conditions for the parameter h under which two–phase equilibria for $I[u,\chi,h,\sigma]$ can exist.

LEMMA 4.6 *Suppose that for the functional $I[u,\chi,h,\sigma]$ at least one two–phase equilibrium state exists. Then we have*

$$|h| \; < \; h_0(\sigma) \; := \; \max\left\{(2/|\Omega|)^{1/q} G_0\,,\, (\kappa/\sigma)^{d/(q-d)}\, G_0^{q/(q-d)}\right\}.$$

Here G_0 is defined after (4.3) and κ denotes the constant from the isoperimetric inequality.

Proof. Let $(\hat{u},\hat{\chi})$ denote a two–phase equilibrium state.
Case 1: $h \geq 0$
Going through the proof of Lemma 4.5, we see that (4.8) or (4.9) must be violated, i.e. we have

$$\frac{1}{|\Omega|}\,\|\hat{\chi}\|_{L^1(\Omega)} \; > \; \frac{1}{2} \quad \text{or} \quad \sigma^{-1}\,\kappa G_0\,\|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \; \geq \; 1 \tag{4.15}$$

since in the opposite case $(\hat{u},\hat{\chi})$ is a one–phase equilibrium state. Quoting (4.7) in the form

$$h\int_\Omega \hat{\chi}\,dx \; \leq \; G_0\,\|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-1}{q}}\,, \qquad \text{i.e. } h \; \leq \; G_0\,\|\hat{\chi}\|_{L^1(\Omega)}^{-\frac{1}{q}}$$

and using (4.15) to estimate the right–hand side we get

$$0 \; \leq \; h \; < \; \max\left\{(2/|\Omega|)^{\frac{1}{q}}\, G_0\,,\, (\kappa/\sigma)^{d/(q-d)}\, G_0^{q/(q-d)}\right\}.$$

Case 2: $h \leq 0$
Going back to the proof of Lemma 4.5 we see that now (4.12) or (4.14) must be wrong. Then we may argue as in Case 1 with the result

$$0 \; \leq \; -h \; < \; \max\left\{(2/|\Omega|)^{1/q} G_0\,,\, (\kappa/\sigma)^{d/(q-d)}\, G_0^{q/(q-d)}\right\}.$$

∎

LEMMA 4.7 *There exists a number σ^* such that for $\sigma > \sigma^*$ and all $h \in \mathbb{R}$ all equilibrium states of $I[u,\chi,h,\sigma]$ are one–phase equilibria.*

Proof.: Consider an arbitrary equilibrium state $(\hat{u}, \hat{\chi}) \in X$ of the functional $I[u, \chi, h, \sigma]$. According to (4.7) we have

$$\sigma \int_\Omega |\nabla \hat{\chi}| + h \int_\Omega \hat{\chi}\, dx \ \leq \ G_0 \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{d-1}{d}} ,$$

hence

$$\sigma \int_\Omega |\nabla \hat{\chi}| \ \leq \ \left\{ |h| \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{1}{d}} + G_0 \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \right\} \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{d-1}{d}} .$$

If $(\hat{u}, \hat{\chi})$ is a two–phase equilibrium state, then, according to Lemma 4.6, $|h|$ can be replaced by $h_0(\sigma)$, in the one–phase case this is obvious, therefore

$$\sigma \int_\Omega |\nabla \hat{\chi}| \ \leq \ \left\{ h_0(\sigma) \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{1}{d}} + G_0 \, \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \right\} \|\hat{\chi}\|_{L^1(\Omega)}^{\frac{d-1}{d}} .$$

Using (4.11) we get in the same manner

$$\sigma \int_\Omega |\nabla(1 - \hat{\chi})| \ \leq \ \left\{ h_0(\sigma) \, \|1 - \hat{\chi}\|_{L^1(\Omega)}^{\frac{1}{d}} + G_0 \, \|1 - \hat{\chi}\|_{L^1(\Omega)}^{\frac{q-d}{qd}} \right\} \|1 - \hat{\chi}\|_{L^1(\Omega)}^{\frac{d-1}{d}} .$$

Let $\tilde{\chi}$ denote the function $\hat{\chi}$ or $1 - \hat{\chi}$ for which

$$\frac{1}{|\Omega|} \int_\Omega \tilde{\chi}\, dx \ \leq \ \frac{1}{2} .$$

Using the isoperimetric inequality and the estimates for $\hat{\chi}$ and $1 - \hat{\chi}$, we deduce

$$\begin{aligned} \int_\Omega |\nabla \tilde{\chi}| \ &\leq \ h_1(\sigma) \int_\Omega |\nabla \tilde{\chi}| , \\ h_1(\sigma) \ &:= \ \frac{\kappa}{\sigma} \left[h_0(\sigma) \, (|\Omega|/2)^{\frac{1}{d}} + G_0 \, (|\Omega|/2)^{\frac{q-d}{qd}} \right] . \end{aligned}$$

Since $h_0(\sigma)$ stays bounded as $\sigma \to \infty$, it is clear that there exists a number $\sigma^* > 0$ such that $h_1(\sigma) < 1$ for all $\sigma > \sigma^*$. But then $(\hat{u}, \hat{\chi})$ is a one–phase equilibrium state. ■

LEMMA 4.8 *Let $\hat{h}$ denote the number defined in formula (3.7). Then, for all $\sigma > 0$ small enough, the energy $I[u, \chi, \hat{h}, \sigma]$ has only two–phase equilibrium states.*

Proof. By contradiction we assume that there exists a sequence $\sigma_n > 0$, $\sigma_n \to 0$, such that $I[u, \chi, \hat{h}, \sigma_n]$ admits a one–phase equilibria $(\hat{u}_n, \hat{\chi}_n)$, i.e. $\hat u_n \equiv 0$, $\hat\chi_n \equiv 0$ or $\hat u_n \equiv 0$, $\hat\chi_n \equiv 1$. For any $(u,\chi) \in X$ we get

$$\begin{aligned} I[u,\chi,\hat h,\sigma_n] \;&\ge\; I[\hat u_n,\hat\chi_n,\hat h,\sigma_n] \;=\; I[0,\hat\chi_n,\hat h,0] \\ &=\; \begin{cases} \displaystyle\int_\Omega f^+_{\hat h}(\cdot,0)\,dx\,, & \hat\chi_n \equiv 1\,, \\ \displaystyle\int_\Omega f^-(\cdot,0)\,dx\,, & \hat\chi_n \equiv 0\,, \end{cases} \end{aligned}$$

thus (compare (3.7))

$$I[u,\chi,\hat h,\sigma_n] \;\ge\; \hat I_0[\hat h] \quad \text{for all } (u,\chi) \in X\,.$$

Choosing $u \equiv 0$ we get for any characteristic function $\chi \in BV(\Omega)$

$$\begin{aligned} &\int_\Omega \Big(\chi\,\langle A^+\xi^+,\xi^+\rangle + (1-\chi)\,\langle A^-\xi^-,\xi^-\rangle + \hat h\,\chi\Big)\,dx + \sigma_n\int_\Omega |\nabla\chi| \\ &\qquad \ge\; \hat I_0[\hat h] \;=\; \int_\Omega \langle A^-\xi^-,\xi^-\rangle\,dx\,, \end{aligned}$$

hence

$$\int_\Omega \chi\Big[\langle A^+\xi^+,\xi^+\rangle - \langle A^-\xi^-,\xi^-\rangle + \hat h\Big]\,dx + \sigma_n\int_\Omega |\nabla\chi| \;\ge\; 0\,.$$

Passing to the limit $n \to \infty$ and using the definition of $\hat h$, we obtain

$$\begin{aligned} &\int_\Omega \chi\Big[\langle A^+\xi^+,\xi^+\rangle - \frac{1}{|\Omega|}\int_\Omega \langle A^+\xi^+,\xi^+\rangle\,dy \\ &\qquad -\Big(\langle A^-\xi^-,\xi^-\rangle - \frac{1}{|\Omega|}\int_\Omega \langle A^-\xi^-,\xi^-\rangle\,dy\Big)\Big]\,dx \;\ge\; 0 \end{aligned}$$

valid for all χ as above. Therefore

$$\langle A^+\xi^+,\xi^+\rangle - \frac{1}{|\Omega|}\int_\Omega \langle A^+\xi^+,\xi^+\rangle\,dx \;\ge\; \langle A^-\xi^-,\xi^-\rangle - \frac{1}{|\Omega|}\int_\Omega \langle A^-\xi^-,\xi^-\rangle\,dx$$

which is in contradiction to (2.7). Let us now assume condition (2.7*) in place of (2.7). With the same notation as before we get

$$I[u,\chi,\hat h,\sigma_n] \;\ge\; \hat I_0[\hat h] \quad \text{for all } (u,\chi) \in X\,, \qquad \text{i.e.}$$

$$\begin{aligned} &\int_\Omega \Big[\chi\Big(\big\langle A^+\big(\varepsilon(u)-\xi^+\big),\varepsilon(u)-\xi^+\big\rangle + \hat h\Big) \\ &\qquad +(1-\chi)\big\langle A^-\big(\varepsilon(u)-\xi^-\big),\varepsilon(u)-\xi^-\big\rangle\Big]\,dx + \sigma_n\int_\Omega |\nabla\chi| \\ &\qquad \ge\; \hat I_0[\hat h] \;=\; \int_\Omega \langle A^-\xi^-,\xi^-\rangle\,dx\,. \end{aligned}$$

Observing $\int_\Omega \langle A^-\xi^-, \varepsilon(u)\rangle \, dx = \int_\Omega \langle \xi^-, A\varepsilon(u)\rangle \, dx = 0$ (see (2.6)) and letting $n \to \infty$, we arrive at

$$\begin{aligned}
&\int_\Omega \big\langle A^-\varepsilon(u), \varepsilon(u)\big\rangle + \chi\Big[\big\langle A^+\varepsilon(u), \varepsilon(u)\big\rangle + 2\big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+\big\rangle \\
&\qquad -\big\langle A^-\varepsilon(u), \varepsilon(u)\big\rangle\Big] \, dx \; + \int_\Omega \chi\Big[\langle A^+\xi^+, \xi^+\rangle - \frac{1}{|\Omega|}\int_\Omega \langle A^+\xi^+, \xi^+\rangle \, dy \\
&\qquad -\Big(\langle A^-\xi^-, \xi^-\rangle - \frac{1}{|\Omega|}\int_\Omega \langle A^-\xi^-, \xi^-\rangle \, dy\Big)\Big] \, dx \\
&\quad \geq \; 0\,,
\end{aligned}$$

and (2.7*) implies

$$\begin{aligned}
&\int_\Omega \big\langle A^-\varepsilon(u), \varepsilon(u)\big\rangle + \chi\Big[\big\langle A^+\varepsilon(u), \varepsilon(u)\big\rangle + 2\big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+\big\rangle \\
&\qquad -\big\langle A^-\varepsilon(u), \varepsilon(u)\big\rangle\Big] \, dx \;\; \geq \;\; 0 \quad \text{for all } \; (u,\chi) \in X\,.
\end{aligned}$$

In a next step we replace u by λu, $\lambda > 0$, divide through λ and pass to the limit $\lambda \downarrow 0$ with the result

$$\int_\Omega \chi \big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+\big\rangle \, dx \;\; \geq \;\; 0 \quad \text{for all } \; (u,\chi) \in X\,. \tag{4.16}$$

We claim that (2.7*) (i.e. $A^+\xi^+ \not\equiv A^-\xi^-$) implies the existence of $u \in C_0^\infty(\Omega;\mathbb{R}^d)$ s.t.

$$\big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+\big\rangle \;\; \neq \;\; 0 \tag{4.17}$$

holds on a set $E \subset \Omega$ with positive measure. If not, then

$$\big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+\big\rangle \;\; = \;\; 0 \quad \text{a.e. and for all } \; u \in C_0^\infty\big(\Omega;\mathbb{R}^d\big)\,,$$

hence

$$\int_\Omega \varphi \big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+\big\rangle \, dx \;\; = \;\; 0 \quad \text{for all } \; \varphi \in C^\infty(\overline{\Omega})\,, \;\; u \in C_0^\infty\big(\Omega;\mathbb{R}^d\big)\,.$$

Observing

$$\varepsilon(\varphi u) \;\; = \;\; \frac{1}{2}(\nabla\varphi \otimes u + u \otimes \nabla\varphi) + \varepsilon(u)\,\varphi$$

and (compare (2.6))

$$\int_\Omega \big\langle \varepsilon(\varphi u), A^-\xi^- - A^+\xi^+\big\rangle \, dx \;\; = \;\; 0$$

we get $((\gamma_{ij}) := A^-\xi^- - A^+\xi^+)$

$$\int_\Omega \partial_j \varphi \, u^i \gamma_{ij} \, dx \;=\; 0\,,$$

and since u is arbitrary, this implies

$$\partial_j \varphi \, \gamma_{ij} \;=\; 0\,, \qquad i = 1, \ldots, d\,, \;\; \text{for all} \;\; \varphi \in C^\infty(\overline{\Omega})\,.$$

Letting $\varphi(x) = x_k$, $k = 1, \ldots, d$, we obtain the contradiction

$$\gamma_{ik} \;=\; 0\,, \qquad i, k = 1, \ldots, d\,.$$

Thus we have (4.17). W.l.o.g. we may assume that

$$E^- \;:=\; \Big\{ x \in \Omega : \big\langle \varepsilon(u)(x), A^-(x)\,\xi^-(x) - A^+(x)\,\xi^+(x) \big\rangle < 0 \Big\}$$

has positive measure (otherwise replace u by $-u$). Let χ denote the characteristic function of E^-. We do not know that χ is in $BV(\Omega)$ but according to the density property we find measurable characteristic functions $\chi_n \in BV(\Omega)$ such that $\chi_n \to \chi$ a.e. We get

$$0 \;>\; \int_\Omega \chi \big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+ \big\rangle \, dx \;=\; \lim_{n\to\infty} \int_\Omega \chi_n \big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+ \big\rangle \, dx\,,$$

hence

$$\int_\Omega \chi_n \big\langle \varepsilon(u), A^-\xi^- - A^+\xi^+ \big\rangle \, dx \;<\; 0$$

for large enough n. But $(u, \chi_n) \in X$ and so the last inequality contradicts (4.16). ■

5 The behaviour of the energy of equilibrium states as a function of the parameter h

If $h \in \mathbb{R}$ and $\sigma > 0$, we set

$$\hat{I}[h, \sigma] \;:=\; \inf_{(u,\chi)\in X} I[u, \chi, h, \sigma]\,. \tag{5.1}$$

According to Theorem 3.4 the value $\hat{I}[h, \sigma]$ is attained by at least one equilibrium state $(\hat{u}, \hat{\chi})$.

LEMMA 5.1 *We have the estimates*

$$\begin{aligned} \hat{I}[h,\sigma] &\le \hat{I}_0[h]\,, \\ |\hat{I}[h_2,\sigma]-\hat{I}[h_1,\sigma]| &\le |\Omega|\,|h_2-h_1| \end{aligned} \tag{5.2}$$

valid for all $\sigma > 0$, h, h_1, $h_2 \in \mathbb{R}$, in particular, the function $h \mapsto \hat{I}[h,\sigma]$ is continuous for all $\sigma > 0$.

Proof. For (5.2) we just observe

$$\hat{I}[h,\sigma] \le I[0,1,h,\sigma] \quad \text{and} \quad \hat{I}[h,\sigma] \le I[0,0,h,\sigma]\,.$$

Consider h_1, $h_2 \in \mathbb{R}$ and let $(\hat{u}_i, \hat{\chi}_i)$ denote equilibrium states of $I[u,\chi,h_i,\sigma]$, $i = 1,2$. Then

$$\begin{aligned} \hat{I}[h_2,\sigma]-\hat{I}[h_1,\sigma] &\le I[\hat{u}_1,\hat{\chi}_1,h_2,\sigma]-I[\hat{u}_1,\hat{\chi}_1,h_1,\sigma] \\ &= (h_2-h_1)\int_\Omega \hat{\chi}_1\,dx \le |\Omega|\,|h_2-h_1| \end{aligned}$$

and in the same way

$$\hat{I}[h_1,\sigma]-\hat{I}[h_2,\sigma] \le \int_\Omega \hat{\chi}_2\,dx\,(h_1-h_2)\,,$$

from which the claim follows. ∎

Let us define the number

$$\sigma^* := \inf\Big\{\sigma_1 > 0 : \text{for } \sigma > \sigma_1 \text{ and all } h \in \mathbb{R} \text{ the energy } I[u,\chi,h,\sigma] \text{ has only one–phase state equilibria}\Big\}\,.$$

By Lemma 4.7 the set of numbers $\sigma_1 > 0$ is non-empty, hence $\inf\{\ldots\} \ge 0$, but on account of Lemma 4.8 we know $\sigma^* > 0$.

LEMMA 5.2 *For any $\sigma > 0$ there exist unique numbers $h^+(\sigma) \ge h^-(\sigma)$ as follows:*

$$h^-(\sigma) = h^+(\sigma) = \hat{h} \quad \textit{for all } \sigma \ge \sigma^*\,; \tag{5.3}$$

$$h^-(\sigma) < \hat{h} < h^+(\sigma) \quad \textit{for all } \sigma \in (0,\sigma^*)\,; \tag{5.4}$$

$$\left.\begin{array}{l} \textit{for } h \in \big(h^-(\sigma), h^+(\sigma)\big),\ 0 < \sigma < \sigma^*, \textit{ all equilibrium states} \\ \textit{are two–phase and } \hat{I}[h,\sigma] \ < \ \hat{I}_0[h]\,; \end{array}\right\} \tag{5.5}$$

$$\left.\begin{array}{l} \textit{for } h < h^-(\sigma),\ \sigma > 0, \textit{ only the one–phase equilibrium state} \\ \hat{u} \equiv 0,\ \hat{\chi} \equiv 1 \textit{ exists, we have } \hat{I}[h,\sigma] \ = \ \hat{I}_0[h]\,; \end{array}\right\} \tag{5.6}$$

$$\left.\begin{array}{l} \textit{for } h > h^+(\sigma),\ \sigma > 0, \textit{ only the one–phase equilibrium state} \\ \hat{u} \equiv 0,\ \hat{\chi} \equiv 0 \textit{ exists, we have } \hat{I}[h,\sigma] \ = \ \hat{I}_0[h]\,. \end{array}\right\} \tag{5.7}$$

Proof. Fix $\sigma > 0$ and quote Lemma 4.5 to see that

$$\begin{aligned} h^+(\sigma) \ &:= \ \inf\Big\{H \in \mathbb{R} :\ \hat{u} \equiv 0,\ \hat{\chi} \equiv 0 \\ &\qquad\qquad \text{is the only equilibrium state of } I[u,\chi,H,\sigma]\Big\}\,, \\ h^-(\sigma) \ &:= \ \sup\Big\{H \in \mathbb{R} :\ \hat{u} \equiv 0,\ \hat{\chi} \equiv 1 \\ &\qquad\qquad \text{is the only equilibrium state of } I[u,\chi,H,\sigma]\Big\} \end{aligned}$$

$h^+(\sigma) \geq h^-(\sigma)$, are well–defined and that for $h > h^+(\sigma)$ ($h < h^-(\sigma)$) we must have $\hat{\chi} \equiv 0$ ($\hat{\chi} \equiv 1$) for any equilibrium state $(\hat{u}, \hat{\chi})$ of $I[u,\chi,h,\sigma]$. Let us start with the proof of (5.3) assuming first that $\sigma > \sigma^*$ is fixed. Recalling the definition of σ^* we see

$$\hat{I}[h,\sigma] \ = \ \left\{\begin{array}{ll} \displaystyle\int_\Omega \big\langle A^-\xi^-, \xi^-\big\rangle\, dx \ := \ E & (\text{“}\hat{\chi} \equiv 0\text{”}) \\ \quad\text{or} & \\ \displaystyle\int_\Omega \big\langle A^+\xi^+, \xi^+\big\rangle\, dx + h\,|\Omega| & (\text{“}\hat{\chi} \equiv 1\text{”}) \end{array}\right. \tag{5.8}$$

with $\hat{I}[h,\sigma] \equiv E$ at least for $h > h^+(\sigma)$ whereas the second line of (5.8) is valid for $h < h^-(\sigma)$. We claim that $h^+(\sigma) = h^-(\sigma)$ ($= \hat{h}$). If this is not the case, then $h^-(\sigma) < h^+(\sigma)$. Suppose that for some $h_1 \in (h^-(\sigma), h^+(\sigma))$ we have

$$\hat{I}[h_1,\sigma] \ = \ \hat{I}[h^+(\sigma),\sigma]\,.$$

By continuity of $\hat{I}[h,\sigma]$ we get $\hat{I}[h^+(\sigma),\sigma] = E$, hence $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ is an equilibrium state of $I[u,\chi,h_1,\sigma]$. But then Remark 4.4 shows that for $h > h_1$

all equilibria of $I[u,\chi,h,\sigma]$ are of this form contradicting the definition of $h^+(\sigma)$ and the choice $h_1 < h^+(\sigma)$. For this reason we must have

$$\hat{I}[h,\sigma] \;=\; \int_\Omega \langle A^+\xi^+,\xi^+\rangle\,dx + h\,|\Omega|$$

also on $(h^-(\sigma),h^+(\sigma))$, in particular $\hat{u}\equiv 0$, $\hat{\chi}\equiv 1$ is the only equilibrium state for h in this range (compare Remark 4.4) contradicting the definition of $h^-(\sigma)$. Thus we have $h^+(\sigma)=h^-(\sigma)$ and by continuity of $\hat{I}[h,\sigma]$ this can only happen for the common value $\hat{h}$.
Let us extend (5.3) to the limit case $\sigma=\sigma^*$: consider a sequence $\sigma_n > \sigma^*$ such that $\sigma_n \to \sigma^*$. For $h > \hat{h} = h^+(\sigma_n)$ we have the only equilibrium state $\hat{u}_n \equiv 0$, $\hat{\chi}_n \equiv 0$, by Lemma 4.2 $\hat{u}\equiv 0$, $\hat{\chi}\equiv 0$ is an equilibrium state of $I[u,\chi,h,\sigma^*]$, and with the same reasoning $\hat{u}\equiv 0$, $\hat{\chi}\equiv 1$ is an equilibrium state of $I[u,\chi,h,\sigma^*]$, $h<\hat{h}$. According to Remark 4.4 we see that for $h>\hat{h}$ the only equilibrium state of $I[u,\chi,h,\sigma^*]$ is given by $\hat{u}\equiv 0$, $\hat{\chi}\equiv 0$, whereas for $h<\hat{h}$ we only have $\hat{u}\equiv 0$, $\hat{\chi}\equiv 1$. This implies $h^+(\sigma^*)=h^-(\sigma^*)=\hat{h}$ by definition of $h^\pm(\sigma^*)$. Altogether we have shown (5.3), (5.6) and (5.7). For example, $\hat{I}[h,\sigma]=\hat{I}_0[h]$ in case $h>h^+(\sigma)$ follows from

$$I^+[\hat{u}^+,h] \;=\; I[0,1,h,\sigma] \;\geq\; \hat{I}[h,\sigma] \;=\; I[0,0,h,\sigma] \;=\; I^-[\hat{u}^-]\,,$$

hence according to the definition of $\hat{I}_0[h]$ it is seen $I[0,0,h,\sigma] = \hat{I}_0[h] = \hat{I}[h,\sigma]$.

To proceed further we claim that at $\hat{h}$, σ^* there exist the equilibrium states $\hat{u}\equiv 0$, $\hat{\chi}\equiv 0$ or $\hat{\chi}\equiv 1$ and at least one two–phase equilibrium state $\check{u}$, $\check{\chi}$.

In fact, the existence of the one–phase equilibria follows from Lemma 4.2 together with (5.3), (5.6) and (5.7). By definition of σ^* there exists a sequence $\sigma_n < \sigma^*$, $\sigma_n \to \sigma^*$, such that at least for one $h=h_n$ a two–phase equilibrium state $(\check{u}_n,\check{\chi}_n)$ of $I[u,\chi,h_n,\sigma_n]$ must exist. Lemma 4.6 implies $\sup_n |h_n| < +\infty$, hence $h_n \to: \bar{h}$ at least for a subsequence, and Lemma 4.2 shows that at $\bar{h}$, σ^* a two–phase equilibrium state exists. In case $\bar{h}>\hat{h}$ we get a contradiction: since $\hat{u}\equiv 0$, $\hat{\chi}\equiv 0$ is an equilibrium state at $\hat{h},\sigma^*$, we would get again by Remark 4.4 that for $h>\hat{h}$ all equilibrium states are of this kind. The same argument excludes the case $\bar{h}<\hat{h}$, hence $\bar{h}=\hat{h}$.

Let us fix $\sigma<\sigma^*$. If $(\check{u},\check{\chi})$ is the two–phase equilibrium state at $\hat{h},\sigma^*$, we get (use the existence of one–phase equilibria for the first equation)

$$\hat{I}_0[\hat{h}] \;=\; \hat{I}[\hat{h},\sigma^*] \;=\; I[\check{u},\check{\chi},\hat{h},\sigma^*] \;>\; I[\check{u},\check{\chi},\hat{h},\sigma] \;\geq\; \hat{I}[\hat{h},\sigma]\,.$$

Since $h \mapsto \hat{I}_0[h]$ and $h \mapsto I[h,\sigma]$ are continuous, we see $\hat{I}_0[h] > \hat{I}[h,\sigma]$ valid for h on an open interval $(H^-(\sigma), H^+(\sigma))$ containing $\hat{h}$. Clearly $H^\pm(\sigma) = h^\pm(\sigma)$: by definition of $h^\pm(\sigma)$ we have $h^-(\sigma) \le H^-(\sigma)$, $H^+(\sigma) \le h^+(\sigma)$, since for example $H^-(\sigma) < h^-(\sigma)$ would imply the existence of $h \in \big(H^-(\sigma), h^-(\sigma)\big)$ with equilibrium state only $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$, but then $H^-(\sigma)$ has also $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ as the only equilibrium state (see Remark 4.4) contradicting $\hat{I}_0[h] > \hat{I}[h,\sigma]$. On the other hand, by definition of $h^\pm(\sigma)$, the open interval (h^-, h^+) contains only numbers h for which only two–phase state equilibria of $I[u,\chi,h,\sigma]$ can exist (again use Remark 4.4), hence $\hat{I}_0[h] > \hat{I}[h,\sigma]$. This proves $H^\pm(\sigma) = h^\pm(\sigma)$, i.e. (5.4) and (5.5).■

Next we discuss the behaviour of the phases for the cases $h = h^\pm(\sigma)$.

LEMMA 5.3 *i) For $\sigma \in (0,\sigma^*)$ and $h = h^+(\sigma)$ the states of equilibrium of the energy $I[u,\chi,h,\sigma]$ consist of $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ and at least one additional two–phase equilibrium.*

ii) For $\sigma \in (0,\sigma^*)$ and $h = h^-(\sigma)$ the states of equilibrium of the energy $I[u,\chi,h,\sigma]$ consist of $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ and at least one additional two-phase equilibrium.

iii) In case $h = \hat{h}$, $\sigma > \sigma^*$ only the one–phase state equilibria $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ and $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ (both) occur.

iv) At $h = \hat{h}$, $\sigma = \sigma^*$ we have one–phase equilibrium states $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$, $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$, and at least one two–phase equilibrium.

Proof. i), ii), iii) follow directly from Lemma 5.2 and Lemma 4.2 (with $\sigma_n = \sigma$), iv) is contained in the proof of Lemma 5.2. ■

Finally, we discuss some analytic aspects concerning the functions $h^\pm(\sigma)$.

LEMMA 5.4 *The functions $\sigma \mapsto h^\pm(\sigma)$ are continuous on $(0,\infty)$. h^+ is strictly decreasing on $(0,\sigma^*)$, whereas h^- is strictly increasing on this set.*

Proof. It is sufficient to discuss h^+, the results for h^- follow with obvious modifications. So let $0 < \sigma_2 < \sigma_1 \le \sigma^*$, $h_i := h^+(\sigma_i)$, $i = 1,2$, and consider a two–phase equilibrium state $\hat{u}_i$, $\hat{\chi}_i$ of $I[u,\chi,h_i,\sigma_i]$, $i = 1,2$, whose existence follows from Lemma 5.3. Since there also exists the one–phase equilibria $\tilde{u}_i \equiv 0$, $\tilde{\chi}_i \equiv 0$, we have

$$\hat{I}[h_i,\sigma_i] \;=\; \hat{I}_0[h_i]\,,$$

hence

$$\hat{I}_0[h_1] \;=\; I[\hat{u}_1,\hat{\chi}_1,h_1,\sigma_1] \;>\; I[\hat{u}_1,\hat{\chi}_1,h_1,\sigma_2] \;\ge\; \hat{I}[h_1,\sigma_2]\,.$$

But $\hat{I}_0[h_1] > \hat{I}[h_1, \sigma_2]$ implies that $I[u, \chi, h_1, \sigma_2]$ admits only two–phase equilibria which means $h_1 \in (h^-(\sigma_2), h^+(\sigma_2))$ (see Lemma 5.2), i.e. $h^+(\sigma_1) < h^+(\sigma_2)$.
It is enough to discuss the continuity of h^+ on $(0, \sigma^*]$, since $h^+ \equiv \hat{h}$ on $[\sigma^*, \infty)$. Assume by contradiction that h^+ is discontinuous at some point $\sigma_0 \in (0, \sigma^*]$. The monotonicity of h^+ implies

$$\lim_{\sigma \uparrow \sigma_0} h^+(\sigma) \;=:\; \alpha \;>\; \beta \;:=\; \lim_{\sigma \downarrow \sigma_0} h^+(\sigma)\,.$$

Let us fix $h \in [\beta, \alpha]$ and consider a sequence $\sigma_n \uparrow \sigma_0$. We have $h^+(\sigma_n) > \lim_{\sigma \uparrow \sigma_0} h^+(\sigma) \geq h$. By Lemma 5.2, (5.5), there exists a two–phase equilibrium state for $I[u, \chi, h, \sigma_n]$, Lemma 4.2 implies the same for $I[u, \chi, h, \sigma_0]$. Next, let $\sigma_n \downarrow \sigma_0$. Then $h^+(\sigma_n) < \lim_{\sigma \downarrow \sigma_0} h^+(\sigma) \leq h$, on account of Lemma 5.2, (5.7), we find the one–phase equilibrium state $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ for the energy $I[u, \chi, h, \sigma_n]$, hence the same is true for $I[u, \chi, h, \sigma_0]$. Thus, for any $h \in [\beta, \alpha]$, there exists the one–phase equilibrium $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ and also a two–phase equilibrium of $I[u, \chi, h, \sigma_0]$. By Lemma 5.2 and 5.3 this implies $h = h^+(\sigma_0)$ for any $h \in [\beta, \alpha]$ which contradicts $\beta < \alpha$. ■

6 Proofs of Theorem 2.1 and Theorem 2.2

The existence of equilibrium states $(\hat{u}, \hat{\chi}) \in X$ for the energy $I[u, \chi, h, \sigma]$, $(u, \chi) \in X$, $h \in \mathbb{R}$, $\sigma > 0$, is established in Theorem 3.4. The subdivision of the parameter half–plane $\sigma > 0$, $h \in \mathbb{R}$ into the open regions A, B, C together with a description of the corresponding phase states is given in Lemma 5.2. In Lemma 5.3 the behaviour of the distribution of the phases on the boundaries of the regions A, B and C is analyzed, Lemma 5.4 contains the information concerning the functions $\sigma \mapsto h^{\pm}(\sigma)$ whose graphs generate the subdivision of the parameter half-plane. Thus we have a complete proof of Theorem 2.1. ■

Next consider a two–phase equilibrium state $(\hat{u}, \hat{\chi})$ of $I[u, \chi, h, \sigma]$. Let $H = \sigma^{-1}\big(f_h^+(\cdot, \varepsilon(\hat{u})) - f^-(\cdot, \varepsilon(\hat{u}))\big)$ and observe

$$\int_\Omega |\nabla \hat{\chi}| + \int_\Omega \hat{\chi}\, H\, dx \;\leq\; \int_\Omega |\nabla \chi| + \int_\Omega \chi\, H\, dx$$

for any characteristic function $\chi \in BV(\Omega)$. This implies

$$\int_\Omega |\nabla \mathbf{1}_E| + \int_{E \cap \Omega} H\, dx \;\leq\; \int_\Omega |\nabla \mathbf{1}_F| + \int_{F \cap \Omega} H\, dx$$

for any set F of finite perimeter in Ω, and following [TA] we see that E is a set of generalized mean curvature H in Ω. From (2.3), (2.5) and Lemma 3.5 we deduce $H \in L^q(\Omega)$e and since $q > d$, the regularity of $\partial E \cap \Omega$ follows from [TA], 1.9 and 1.14. It is wellknown that $\nabla \hat{\chi}$ is supported on the reduced boundary $\partial^* E$ which on account of the above result coincides with ∂E if $d \leq 7$. Let $x_0 \in \Omega - \partial E$. Due to the smoothness of $\partial E \cap \Omega$ we find a ball $B_\rho(x_0)$ such that $B_\rho(x_0) \cap \partial E = \emptyset$, hence $|\nabla \hat{\chi}|(B_\rho(x_0)) = 0$, thus either $\hat{\chi} \equiv 1$ on $B_\rho(x_0)$ or $\hat{\chi} \equiv 0$ on this ball. Let us consider the case $\hat{\chi} \equiv 0$. Then, for any $v \in C_0^1(B_\rho(x_0), \mathbb{R}^d)$ we have

$$
\begin{aligned}
0 &= \frac{d}{dt}_{|t=0} \int_{B_\rho(x_0)} f^-\big(\cdot, \varepsilon(\hat{u}) + t\,\varepsilon(v)\big)\, dx \\
&= 2 \textstyle\int_{B_\rho(x_0)} \big\langle A^- \big(\varepsilon(\hat{u}) - \xi^-\big), \varepsilon(v)\big\rangle\, dx \\
&\overset{(2.6)}{=} 2 \int_{B_\rho(x_0)} \big\langle A^- \varepsilon(\hat{u}), \varepsilon(v)\big\rangle\, dx\,,
\end{aligned}
$$

hence $\hat{u}$ is a solution of the equilibrium equations of linear elasticity and therefore smooth in case of regular coefficients. The other case is treated in the same way which gives the proof of Theorem 2.2 . ■

As stated in Section 2 we require the tensors of elastic moduli $A^\pm(x)$ and the stress–free strains $\xi^\pm(x)$ to satisfy one of the conditions (2.7) or (2.7*) which in turn are used to prove the existence of two–phase equilibrium states for $I[u, \chi, \hat{h}, \sigma]$ for small positive σ. In the case of constant data (2.7*) seems to be quite natural, now we would like to add an example for which 2.7 is true. Let

$$
\langle A^\pm \xi, \xi\rangle \;:=\; a^\pm(x)\,\mathrm{tr}\,(\xi^2) + b^\pm(x)\,(\mathrm{tr}\,\xi)^2\,, \qquad \xi \in \mathbb{S}^d,\;\; x \in \Omega\,,
$$

with functions $a^\pm$, $b^\pm \in L^\infty(\Omega)$ such that

$$
a^\pm(x) \;\geq\; \nu \;>\; 0\,,\;\; b^\pm(x) \;\geq\; \nu \;>\; 0\,.
$$

The equilibrium equations (2.6) to be satisfied by the stress–free strains $\xi^\pm$ now read

$$
\frac{\partial}{\partial x_j}\big(a^\pm \xi_{ij}^\pm\big) + \frac{\partial}{\partial x_i}\big(b^\pm \mathrm{tr}\,\xi^\pm\big) \;=\; 0\,, \qquad i = 1, \ldots, d\,. \tag{6.1}
$$

Let us assume that

$$
a^+ \;\equiv\; a\,,\;\; b^+ \;\equiv\; b \text{ and } \xi^+ \;\equiv\; \xi_0^+
$$

with constants a, $b \geq \nu$, $\xi_0^+ \in \mathbb{S}^d$. Then (6.1) holds in the $+$–case, and (2.7) reduces to

$$\frac{1}{|\Omega|} \int_\Omega \langle A^- \xi^-, \xi^- \rangle \, dx \;<\; \langle A^- \xi^-, \xi^- \rangle \tag{6.2}$$

valid on a set $E \subset \Omega$ with $|E| > 0$. Obviously, (6.2) holds if we assume that $\langle A^- \xi^-, \xi^- \rangle \neq const.$ Let us write

$$a^-(x) \;=\; a + \alpha^-(x)\,, \qquad b^-(x) \;=\; b + \beta^-(x)$$

with functions α^-, $\beta^- \in L^\infty(\Omega)$ whose norm is sufficiently small (see (2.3)). Moreover, let

$$\xi_{ij}^-(x) \;=\; c(x)\,\delta_{ij}\,.$$

Returning to (6.1) we find for some constant γ

$$c(x) \;=\; \gamma/\big(a + \alpha^- + d\,[b + \beta^-]\big)\,. \tag{6.3}$$

Conversely, if we define c through (6.3) and let ξ^-, a^-, b^- be defined as above, then we see that $A^\pm$, $\xi^\pm$ satisfy 2.7 together with the other requirements from Section 2.

7 The case $\sigma_n \downarrow 0$

In this section we first investigate the behaviour of

$$\alpha_n \;:=\; \inf_{(u,\chi)\in X} I[u, \chi, h, \sigma_n]$$

for a sequence $\{\sigma_n\}$, $\sigma_n > 0$, $\sigma_{n+1} \leq \sigma_n$, such that $\lim_{n\to\infty} \sigma_n = 0$. To this purpose define $J[u, \chi]$ and $I[u]$ according to (1.1) and (1.2), respectively, and let

$$Y \;:=\; \overset{\circ}{W}{}_2^1\big(\Omega; \mathbb{R}^d\big) \times \Big\{\chi : \Omega \to \mathbb{R},\ \chi \text{ is measurable},\ \chi(x) \in \{0, 1\} \text{ a.e.}\Big\}\,.$$

We further define

$$\alpha \;:=\; \inf_X J\,, \quad \beta \;:=\; \inf_Y J\,, \quad \gamma \;:=\; \inf_{\overset{\circ}{W}{}_2^1(\Omega;\mathbb{R}^d)} I$$

and select for each $n \in \mathbb{N}$ an equilibrium state $(\hat{u}_n, \hat{\chi}_n) \in X$ of $I[u, \chi, h, \sigma_n]$, i.e. $I[\hat{u}_n, \hat{\chi}_n, h, \sigma_n] = \alpha_n$. Then we have

THEOREM 7.1 *Let the above assumptions hold, in particular, we assume that all the hypothese needed for Theorem 2.1 are satisfied.*

a) *We have* $\alpha = \beta = \gamma$, *and the common value is given by* $\lim_{n\to\infty} \alpha_n$.

b) $(\hat{u}_n, \hat{\chi}_n)$ *provides a minimizing sequence for the functional* J *considered either on the space* X *or on the space* Y.

c) $\{\hat{u}_n\}$ *is a minimizing sequence for the energy* I *on the space* $\overset{\circ}{W}{}_2^1(\Omega;\mathbb{R}^d)$.

REMARK 7.2 *According to Lemma 3.1 there exits a subsequence* $\{\hat{u}_n\}$ *(not relabeled) and a function* $\hat{u} \in W_2^1(\Omega;\mathbb{R}^d)$ *such that* $\hat{u}_n \rightharpoonup \hat{u}$ *in* $\overset{\circ}{W}{}_2^1(\Omega;\mathbb{R}^d)$. *Since* $\hat{u}$ *is a weak cluster point of an* I*–minimizing sequence, we see that* $\hat{u}$ *is a minimizer of the relaxed energy* $\tilde{I}[u] = \int_\Omega \tilde{f}(\cdot, \varepsilon(u))\, dx$, $\tilde{f}$ *denoting the quasiconvex envelope of* f. *On account of Theorem 2.2 the functions* $\hat{u}_n$ *have good smoothness properties, and it is an interesting question if these properties are preserved to some extend in the limit* $n \to \infty$.

Proof of Theorem 7.1. From

$$I[\hat{u}_{n+1}, \hat{\chi}_{n+1}, h, \sigma_{n+1}] \;\leq\; I[\hat{u}_n, \hat{\chi}_n, h, \sigma_{n+1}] \;\leq\; I[\hat{u}_n, \hat{\chi}_n, h, \sigma_n]$$

we see $\alpha_{n+1} \leq \alpha_n$. Moreover

$$\alpha \;=\; \inf_X J \;=\; \inf_X I[u, \chi, h, 0] \;\leq\; I[\hat{u}_n, \hat{\chi}_n, h, 0] \;\leq\; I[\hat{u}_n, \hat{\chi}_n, h, \sigma_n] \;=\; \alpha_n\,,$$

thus $\alpha \leq \lim_{n\to\infty} \alpha_n$. To prove the reverse inequality, choose $\varepsilon > 0$ and $(\tilde{u}, \tilde{\chi}) \in X$ such that

$$\alpha \;\geq\; J[\tilde{u}, \tilde{\chi}] - \varepsilon\,.$$

Since $\tilde{\chi} \in BV(\Omega)$ we have $\sigma_n \int_\Omega |\nabla \tilde{\chi}| < \varepsilon$ for $n \gg 1$, in conclusion

$$\alpha \;\geq\; I[\tilde{u}, \tilde{\chi}, h, \sigma_n] - 2\varepsilon \;\geq\; \alpha_n - 2\varepsilon$$

valid for large enough n. By definition we have $\beta \leq \alpha$. Again, for given $\varepsilon > 0$, select $(u, \chi) \in Y$ such that

$$\beta \;\geq\; J[u, \chi] - \varepsilon\,.$$

The density property d) stated before Theorem 2.1 implies the existence of a sequence $\{\chi_m\}$ of characteristic functions in $BV(\Omega)$ such that $\chi_m \to \chi$ a.e. For large enough m we get (by dominated convergence) $J[u, \chi] \geq J[u, \chi_m] - \varepsilon$, and having fixed such a m we see as before $J[u, \chi_m] \geq I[u, \chi_m, h, \sigma_n] - \varepsilon$ for $n \gg 1$, thus

$$\beta \;\geq\; I[u, \chi_m, h, \sigma_n] - 3\varepsilon \;\geq\; \alpha_n - 3\varepsilon\,, \quad \text{i.e.} \quad \beta \;\geq\; \lim_{n\to\infty} \alpha_n\,.$$

From $\beta \le \alpha$ and $\alpha = \lim_{n\to\infty} \alpha_n$ together with the foregoing inequality we get

$$\alpha \; = \; \beta \; = \; \lim_{n\to\infty} \alpha_n \, .$$

Obviously

$$I[\hat{u}_n, \hat{\chi}_n, h, \sigma_n] \; \ge \; J[\hat{u}_n, \hat{\chi}_n] \; \ge \; \alpha \, ,$$

thus $(\hat{u}_n, \hat{\chi}_n)$ is a J–minimizing sequence w.r.t. both spaces X and Y. By definition we have

$$f\big(\cdot, \varepsilon(u)\big) \;\; \le \;\; \chi \, f_h^+\big(\cdot, \varepsilon(u)\big) + (1-\chi)\, f^-\big(\cdot, \varepsilon(u)\big)$$

for any $(u, \chi) \in Y$ which shows $\gamma \le \beta$. Consider now an arbitrary function $u \in \overset{\circ}{W}{}_2^1(\Omega; \mathbb{R}^d)$ and let

$$\Omega_1 \;\; := \;\; \Big\{ x \in \Omega : \; f_h^+\big(\cdot, \varepsilon(u)\big) \le f^-\big(\cdot, \varepsilon(u)\big) \Big\} \, ,$$

$$\Omega_2 \;\; := \;\; \Big\{ x \in \Omega : \; f_h^+\big(\cdot, \varepsilon(u)\big) > f^-\big(\cdot, \varepsilon(u)\big) \Big\} \, .$$

Then we have $(\chi := \mathbf{1}_{\Omega_1})$

$$\int_\Omega f\big(\cdot, \varepsilon(u)\big)\, dx \;\; = \;\; J[u, \chi] \;\; \ge \;\; \inf_Y J \;\; = \;\; \beta \, ,$$

hence $\gamma \ge \beta$, in conclusion $\beta = \gamma$. Finally, we observe

$$\gamma \;\; \le \;\; I[\hat{u}_n] \;\; \le \;\; J[\hat{u}_n, \hat{\chi}_n] \;\; \overset{n\to\infty}{\to} \;\; \beta \;\; = \;\; \gamma \, ,$$

and therefore $\{\hat{u}_n\}$ is an I–minimizing sequence. This completes the proof of Theorem 7.1. ■

We finish this section with the following

LEMMA 7.3 *If we assume in addition to the hypotheses of Theorem 2.1 that (2.5) is replaced by the requirement*

$$\xi^\pm \in L^\infty\big(\Omega; \mathbb{S}^d\big)$$

and if we further impose the bound (see (2.3) and (2.4)) $\varepsilon < \nu/2$*, then the functions* $h^\pm(\sigma)$ *have finite limits as* $\sigma \downarrow 0$.

Proof. Let us suppose that we can find a finite number h_0^+ such that

$$I[u, \chi, h_0^+, \sigma] \;\; \ge \;\; I[0, 0, h_0^+, \sigma] \tag{7.1}$$

holds for any $\sigma > 0$ and all pairs $(u, \chi) \in X$. From Remark 4.4 we deduce that $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ is the only equilibrium state of $I[\cdot,\cdot,h,\sigma]$ for $h > h_0^+$ which means (recall the definition of $h^+(\sigma)$) that $h^+(\sigma) \leq h_0^+$ for any $\sigma > 0$. Obviously (7.1) is equivalent to

$$
\begin{aligned}
&\int_\Omega \chi \Big[\big\langle (A^+ - A^-)\,\varepsilon(u), \varepsilon(u) \big\rangle - 2 \big\langle A^+ \xi^+ - A^- \xi^-, \varepsilon(u) \big\rangle \\
&\quad + \big\langle A^+ \xi^+, \xi^+ \big\rangle - \big\langle A^- \xi^-, \xi^- \big\rangle + h_0^+ \Big]\, dx + \int_\Omega \big\langle A^- \varepsilon(u), \varepsilon(u) \big\rangle\, dx \qquad (7.2) \\
&\quad + \sigma \int_\Omega |\nabla \chi| \;\geq\; 0\,.
\end{aligned}
$$

Using the estimates $(0 < \mu < 1)$

$$
\begin{aligned}
&\big\langle A^- \varepsilon(u), \varepsilon(u) \big\rangle + \chi \big\langle (A^+ - A^-)\,\varepsilon(u), \varepsilon(u) \big\rangle \\
&\qquad\qquad \geq\; \big\langle A^- \varepsilon(u), \varepsilon(u) \big\rangle - \big| \big\langle (A^+ - A^-)\,\varepsilon(u), \varepsilon(u) \big\rangle \big|\,, \\
&2 \big| \big\langle A^\pm \xi^\pm, \varepsilon(u) \big\rangle \big| \;\leq\; \mu \big\langle A^\pm \varepsilon(u), \varepsilon(u) \big\rangle + \frac{1}{\mu} \big\langle A^\pm \xi^\pm, \xi^\pm \big\rangle
\end{aligned}
$$

we see that (7.2) follows from

$$
\begin{aligned}
&\int_\Omega \Big[\big\langle A^- \varepsilon(u), \varepsilon(u) \big\rangle - \big| \big\langle (A^+ - A^-)\,\varepsilon(u), \varepsilon(u) \big\rangle \big| \\
&\quad - \mu \big\langle (A^+ + A^-)\,\varepsilon(u), \varepsilon(u) \big\rangle \Big]\, dx \\
&\quad + \int_\Omega \chi \Big[h_0^+ + \Big(1 - \frac{1}{\mu}\Big) \big\langle A^+ \xi^+, \xi^+ \big\rangle - \Big(\frac{1}{\mu} + 1\Big) \big\langle A^- \xi^-, \xi^- \big\rangle \Big]\, dx \qquad (7.3) \\
&\geq\; 0\,.
\end{aligned}
$$

By (2.4) and the bound for ε the first term in (7.3) is greater than or equal to

$$
\int_\Omega \Big(\nu - \frac{\nu}{2} - 2\,\mu\,\nu^{-1} \Big)\, |\varepsilon(u)|^2\, dx\,.
$$

Thus we fix $\mu < 1$ such that $\frac{\nu}{2} - 2\mu\nu^{-1} > 0$ and define

$$
h_0^+ \;=\; \Big\| \Big(1 - \frac{1}{\mu}\Big) \big\langle A^+ \xi^+, \xi^+ \big\rangle - \Big(\frac{1}{\mu} + 1\Big) \big\langle A^- \xi^-, \xi^- \big\rangle \Big\|_{L^\infty(\Omega)}\,.
$$

This proves (7.3) and by the way (7.1).

In a similar way we prove the existence of a number $h_0^- > 0$ such that

$$I[u, \chi, -h_0^-, \sigma] \;\geq\; I[0, 1, -h_0^-, \sigma]$$

is true for all $\sigma > 0$ and $(u, \chi) \in X$, i.e. $h^-(\sigma) \geq -h_0^-$. ■

8 The case of non–zero volume forces

From now on suppose that (2.1), (2.3), (2.4), (2.5), (2.6) and either (2.7) or (2.7*) are satisfied. Suppose further that a function

$$p \in L^s\big(\Omega; \mathbb{R}^d\big)\,, \qquad s \;\geq\; \frac{2\,q\,d}{2\,q + d}\,, \tag{8.1}$$

is given. (Note that $s > 2d/(d+2)$, hence $\overset{\circ}{W}{}_2^1(\Omega; \mathbb{R}^d) \ni u \mapsto \int_\Omega p \cdot u \, dx$ is compact.) With $I[u, \chi, h, \sigma]$ from (2.8) we now let

$$I_p[u, \chi, h, \sigma] \;=\; I[u, \chi, h, \sigma] + \int_\Omega p \cdot u \, dx\,, \tag{8.2}$$

$h \in \mathbb{R}$, $\sigma \geq 0$, $(u, \chi) \in X$. As before we say that a minimizer $(\hat{u}, \hat{\chi})$ of I_p is a one–phase equilibrium state if either $\hat{\chi} \equiv 1$ or $\hat{\chi} \equiv 0$, otherwise $(\hat{u}, \hat{\chi})$ is termed a two–phase equilibrium state. Clearly, Lemma 3.2 and Theorem 3.4 hold for the functional from (8.2). From (8.1) we see that Lemma 3.5 is true for equilibrium states $(\hat{u}, \hat{\chi})$ of $I_p[u, \chi, h, \sigma]$ with the quantity ε as before but R now also depending on $\|p\|_{L^s(\Omega)}$. Let

$$\begin{aligned} I^+[u, h] \;&:=\; I[u, 1, h, 0] + \int_\Omega p \cdot u \, dx \;=\; \int_\Omega f_h^+\big(\cdot, \varepsilon(u)\big) + \int_\Omega p \cdot u \, dx\,, \\ I^-[u] \;&:=\; I[u, 0, h, 0] + \int_\Omega p \cdot u \, dx \;=\; \int_\Omega f^-\big(\cdot, \varepsilon(u)\big) + \int_\Omega p \cdot u \, dx\,, \end{aligned}$$

$u \in \overset{\circ}{W}{}_2^1(\Omega; \mathbb{R}^d)$. Let $\hat{u}^+$ and $\hat{u}^-$ denote the unique minimizer of $I^+[u, h]$ and $I^-[u]$, respectively. Obviously $(\hat{u}, \hat{\chi})$ is a one–phase equilibrium state if and only if $(\hat{u}, \hat{\chi}) = (\hat{u}^+, 1)$ or $= (\hat{u}^-, 0)$. Using (2.6) we find for any $v \in \overset{\circ}{W}{}_2^1(\Omega; \mathbb{R}^d)$

$$2 \int_\Omega \big\langle A^\pm \varepsilon(\hat{u}^\pm), \varepsilon(v) \big\rangle \, dx + \int_\Omega p \cdot v \, dx \;=\; 0\,. \tag{8.3}$$

This implies

$$\begin{aligned} I^+[\hat{u}^+, h] \quad &= \quad \int_\Omega \left\langle A^+\left(\varepsilon(\hat{u}^+) - \xi^+\right), \varepsilon(\hat{u}^+) - \xi^+ \right\rangle dx \\ &\qquad\qquad +h\,|\Omega| + \int_\Omega p \cdot \hat{u}^+\, dx \\ &\overset{(2.6),(8.3)}{=} \quad h\,|\Omega| + \int_\Omega \langle A^+\xi^+, \xi^+\rangle\, dx + \frac{1}{2}\int_\Omega p\cdot \hat{u}^+\, dx \end{aligned} \tag{8.4}$$

and in the same way

$$I^-[\hat{u}^-] \;=\; \int_\Omega \langle A^-\xi^-, \xi^-\rangle\, dx + \frac{1}{2}\int_\Omega p\cdot \hat{u}^-\, dx\,. \tag{8.5}$$

Let us define $\hat{I}_0[h]$ as in (3.6). Then

$$\hat{I}_0[h] \;=\; \begin{cases} \displaystyle\int_\Omega \langle A^+\xi^+, \xi^+\rangle\, dx + h\,|\Omega| + \frac{1}{2}\int_\Omega p\cdot\hat{u}^+\, dx\,, & h \le \hat{h}\,, \\ \displaystyle\int_\Omega \langle A^-\xi^-, \xi^-\rangle\, dx + \frac{1}{2}\int_\Omega p\cdot\hat{u}^-\, dx\,, & h \ge \hat{h}\,, \end{cases}$$

where now

$$\hat{h} \;=\; \frac{1}{|\Omega|}\int_\Omega \big(-\langle A^+\xi^+, \xi^+\rangle + \langle A^-\xi^-, \xi^-\rangle \big)\, dx + \frac{1}{2}\frac{1}{|\Omega|}\int_\Omega p\cdot(\hat{u}^- - \hat{u}^+)\, dx\,.$$

Consider next an equilibrium state $(\hat{u}, \hat{\chi})$ of $I_p[u, \chi, h, \sigma]$. Then

$$I_p[\hat{u}, \hat{\chi}, h, \sigma] \;\le\; I_p[\hat{u}^+, 1, h, \sigma]\,,$$

i.e.

$$\begin{aligned} &\int_\Omega \Big[\hat{\chi}\Big(f_h^+\big(\cdot, \varepsilon(\hat{u})\big) - f^-\big(\cdot, \varepsilon(\hat{u})\big)\Big) - f^-\big(\cdot, \varepsilon(\hat{u})\big)\Big]\, dx \\ &\qquad +\sigma\int_\Omega |\nabla\hat{\chi}| + \int_\Omega p\cdot\hat{u}\, dx \;\le\; I^+[\hat{u}^+, h] \;\le\; I^+[\hat{u}, h]\,, \end{aligned}$$

so that

$$\sigma\int_\Omega |\nabla(1-\hat{\chi})| \;\le\; \int_\Omega (1-\hat{\chi})\Big(f_h^+\big(\cdot, \varepsilon(\hat{u})\big) - f^-\big(\cdot, \varepsilon(\hat{u})\big)\Big)\, dx\,.$$

Thus we have again inequality (4.3), inequality (4.2) follows from

$$I_p[\hat{u}, \hat{\chi}, h, \sigma] \;\le\; I_p[\hat{u}^-, 0, h, \sigma] \;=\; I^-[\hat{u}^-] \;\le\; I^-[\hat{u}]\,.$$

With (4.2) and (4.3) the proof of Lemma 4.1 can be finished as before, hence we have Lemma 4.1 for the functional I_p with δ also depending on p. Since Lemma 4.2 is reduced to Lemma 3.2 and Lemma 4.1, the conclusion of Lemma 4.2 holds for the p–case, too; the validity of Lemma 4.3 for equilibrium states of $I_p[u,\chi,h_i,\sigma]$ is immediate. From Lemma 4.3 we get Remark 4.4 as before, and since the proof of Lemma 4.5 just uses inequality (4.2) and (4.3), the result of Lemma 4.5 is not affected by the presence of the p–term. Finally, Lemma 4.6 and Lemma 4.7 remain unchanged: their proofs again rely on estimates presented in the proof of Lemma 4.5. Let us pass to the proof of Lemma 4.8: again, assume by contradiction, that for a sequence $\sigma_n \downarrow 0$ the functional $I_p[u,\chi,\hat{h},\sigma_n]$ admits only one–phase equilibria $(\hat{u}_n,\hat{\chi}_n)$, i.e. $(\hat{u}_n,\hat{\chi}_n) = (\hat{u}^+,1)$ or $= (\hat{u}^-,0)$. Exactly the same calculations as before — using (8.4) and (8.5) — imply

$$I_p[u,\chi,\hat{h},\sigma_n] \;\geq\; \hat{I}_0[\hat{h}] \quad \text{for all } (u,\chi)\in X\,, \tag{8.6}$$

and if we choose $u\equiv 0$, (8.6) implies after passing to the limit $n\to\infty$

$$\begin{aligned}&\int_\Omega \Big(\chi\Big[\big\langle A^+\xi^+,\xi^+\big\rangle + \hat{h}\Big] + (1-\chi)\big\langle A^-\xi^-,\xi^-\big\rangle\Big)\,dx\\ &\qquad\geq\; \int_\Omega \big\langle A^-\xi^-,\xi^-\big\rangle\,dx + \frac{1}{2}\int_\Omega p\cdot\hat{u}^-\,dx\end{aligned}$$

being valid for any characteristic function $\chi\in BV(\Omega)$. Inserting the value of $\hat{h}$ we arrive at

$$\begin{aligned}&\int_\Omega \Big\{\chi\Big[\big\langle A^+\xi^+,\xi^+\big\rangle - \frac{1}{|\Omega|}\int_\Omega \big\langle A^+\xi^+,\xi^+\big\rangle\,dy - \Big(\big\langle A^-\xi^-,\xi^-\big\rangle\\ &\qquad -\frac{1}{|\Omega|}\int_\Omega \big\langle A^-\xi^-,\xi^-\big\rangle\,dy\Big)\Big] + \chi\frac{1}{|\Omega|}\frac{1}{2}\int_\Omega p\cdot(\hat{u}^- - \hat{u}^+)\,dy\Big\}\,dx\\ &\qquad\geq\; \frac{1}{2}\int_\Omega p\cdot\hat{u}^-\,dx\,.\end{aligned} \tag{8.7}$$

Suppose now that (2.7) is valid. Approximating $\mathbf{1}_E$ with characteristic functions $\chi_k\in BV(\Omega)$ we see that

$$\begin{aligned}&\int_\Omega \chi_k\Big[\big\langle A^+\xi^+,\xi^+\big\rangle - \frac{1}{|\Omega|}\int_\Omega \big\langle A^+\xi^+,\xi^+\big\rangle\,dy\\ &\qquad -\Big(\big\langle A^-\xi^-,\xi^-\big\rangle - \frac{1}{|\Omega|}\int_\Omega \big\langle A^-\xi^-,\xi^-\big\rangle\,dy\Big)\Big]\,dx\end{aligned}$$

converges to the negative number

$$b(E) \;:=\; \int_E [\ldots]\,dx\,,$$

and (8.7) implies

$$
\begin{aligned}
b(E) &\geq -\int_\Omega |p| \left(|\hat{u}^+| + |\hat{u}^-|\right) dx \\
&\geq -\|p\|_{L^2(\Omega;\mathbb{R}^d)} \left(\|\hat{u}^+\|_{L^2(\Omega;\mathbb{R}^d)} + \|\hat{u}^-\|_{L^2(\Omega;\mathbb{R}^d)}\right).
\end{aligned}
\tag{8.8}
$$

Let us estimate the norms of $\hat{u}^\pm$: minimality of $\hat{u}^+$ implies

$$
\begin{aligned}
&\int_\Omega \left\langle A^+ \left(\varepsilon(\hat{u}^+) - \xi^+\right), \varepsilon(\hat{u}^+) - \xi^+ \right\rangle dx + \hat{h}\,|\Omega| + \int_\Omega p \cdot \hat{u}^+ \, dx \\
&\qquad \leq \int_\Omega \left\langle A^+ \xi^+, \xi^+ \right\rangle dx + \hat{h}\,|\Omega|,
\end{aligned}
$$

in conclusion (recall (2.6))

$$
\int_\Omega \left\langle A^+ \varepsilon(\hat{u}^+), \varepsilon(\hat{u}^-) \right\rangle dx + \int_\Omega p \cdot \hat{u}^+ \, dx \;\leq\; 0,
$$

and from Korn's inequality we get

$$
\|\nabla \hat{u}^+\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \;\leq\; c_1 \|\hat{u}^+\|_{L^2(\Omega;\mathbb{R}^d)} \|p\|_{L^2(\Omega;\mathbb{R}^d)}.
$$

By Poincaré's inequality this turns into the estimate

$$
\|\nabla \hat{u}^+\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \;\leq\; c_2 \|p\|_{L^2(\Omega;\mathbb{R}^d)},
$$

and the same result holds for $\|\nabla \hat{u}^-\|_{L^2(\Omega,\mathbb{R}^{d\times d})}$. Inserting this into (8.8) we end up with

$$
b \;\geq\; -c_3 \|p\|^2_{L^2(\Omega;\mathbb{R}^d)},
$$

which is a contradiction if we assume

$$
\|p\|^2_{L^2(\Omega;\mathbb{R}^d)} \;\leq\; \Theta \tag{8.9}
$$

for some sufficiently small positive number Θ depending on the data.

Suppose next that (2.7*) holds. Then we get for any $(u,\chi) \in X$, using (8.2) and passing to the limit $n \to \infty$

$$
\begin{aligned}
&\int_\Omega \Big\{ \left\langle A^- \varepsilon(u), \varepsilon(u) \right\rangle + \chi \Big[\left\langle A^+ \varepsilon(u), \varepsilon(u) \right\rangle + 2 \left\langle A^- \xi^- - A^+ \xi^+, \varepsilon(u) \right\rangle \\
&\quad - \left\langle A^- \varepsilon(u), \varepsilon(u) \right\rangle \Big] \Big\} dx + \int_\Omega \chi \Big[\left\langle A^+ \xi^+, \xi^+ \right\rangle - \frac{1}{|\Omega|} \int_\Omega \left\langle A^+ \xi^+, \xi^+ \right\rangle dy \\
&\quad - \Big(\left\langle A^- \xi^-, \xi^- \right\rangle - \frac{1}{|\Omega|} \int_\Omega \left\langle A^- \xi^-, \xi^- \right\rangle dy \Big) \Big] \, dx \\
&\geq \frac{1}{2} \int_\Omega p \cdot \hat{u}^- \, dx - \int_\Omega p \cdot u \, dx - \frac{1}{2} \frac{1}{|\Omega|} \int_\Omega \chi \, dx \int_\Omega p \cdot \left(\hat{u}^- - \hat{u}^+\right) dx.
\end{aligned}
$$

By (2.7*) we may drop the second integral on the left–hand side, thus

$$\begin{aligned}
&\int_\Omega \Big[\chi \big\langle A^+ \varepsilon(u), \varepsilon(u)\big\rangle + (1-\chi)\big\langle A^- \varepsilon(u), \varepsilon(u)\big\rangle\Big]\, dx \\
&\quad +2\int_\Omega \chi \big\langle A^-\xi^- - A^+\xi^+, \varepsilon(u)\big\rangle\, dx + \int_\Omega p\cdot u\, dx \\
&\quad \geq\; -c_4\, \|p\|^2_{L^2(\Omega;\mathbb{R}^d)}\,.
\end{aligned} \tag{8.10}$$

We recall (see (4.17)) that (2.7*) implies the existence of a measurable set E with positive measure and of a function $\varphi \in C_0^\infty(\Omega;\mathbb{R}^d)$ with the property $\langle \varepsilon(\varphi), A^-\xi^- - A^+\xi^+\rangle \neq 0$ a.e. on E. W.l.o.g. we may assume that

$$\int_E \langle \varepsilon(\varphi), A^-\xi^- - A^+\xi^+\rangle\, dx \;=:\; \tilde{b} \;>\; 0\,.$$

Let $\chi_k \in BV(\Omega)$ denote a sequence of characteristic functions such that $\chi_k \to \mathbf{1}_E$ a.e. We then use (8.10) with $\chi = \chi_k$ and $u = \lambda\varphi$, $\lambda \in \mathbb{R}$, and get after passing to the limit $k \to \infty$

$$\begin{aligned}
&\lambda^2 \int_\Omega \Big[\mathbf{1}_E \big\langle A^+ \varepsilon(\varphi), \varepsilon(\varphi)\big\rangle + (1-\mathbf{1}_E)\big\langle A^- \varepsilon(\varphi), \varepsilon(\varphi)\big\rangle\Big]\, dx \\
&\quad +2\lambda\left[\tilde{b} + \frac{1}{2}\int_\Omega p\cdot\varphi\, dx\right] \;\geq\; -c_4\, \|p\|_{L^2(\Omega;\mathbb{R}^d)}\,,
\end{aligned}$$

i.e.

$$P(\lambda) \;:=\; \lambda^2 A + 2\lambda B + C \;\geq\; 0\,.$$

For $\|p\|_{L^2(\Omega;\mathbb{R}^d)}$ small enough we can arrange $B \geq \tilde{b}/2 > 0$ so that for $B^2 - C > 0$ $P(\lambda)$ has two different negative zeros. Thus we can find $\lambda < 0$ such that $P(\lambda) < 0$. By definition of C the required condition $B^2 > C$ holds under suitable smallness assumptions for $\|p\|_{L^2(\Omega;\mathbb{R}^d)}$. Summing up we have shown that a condition of the form (8.9) implies a contradiction also in the case (2.7*) which finally proves Lemma 4.8 to be valid also in the presence of a volume force term p whose L^2–norm is small enough.

In accordance with (5.1) we let for $h \in \mathbb{R}$, $\sigma > 0$

$$\hat{I}[h,\sigma] \;:=\; \inf_{(u,\chi)\in X} I_p[u,\chi,h,\sigma]$$

and obtain (5.2) from

$$
\begin{aligned}
\hat{I}[h,\sigma] \quad &\leq \quad I_p[\hat{u}^+,1,h,\sigma] \quad = \quad I^+[\hat{u}^+,h] \\
&\overset{(8.4)}{=} \quad h\,|\Omega| + \int_\Omega \langle A^+\xi^+,\xi^+\rangle\,dx + \frac{1}{2}\int_\Omega p\cdot\hat{u}^+\,dx\,, \\
\hat{I}[h,\sigma] \quad &\leq \quad I_p[\hat{u}^-,0,h,\sigma] \quad = \quad I^-[\hat{u}^-] \\
&\overset{(8.5)}{=} \quad \int_\Omega \langle A^-\xi^-,\xi^-\rangle\,dx + \frac{1}{2}\int_\Omega p\cdot\hat{u}^-\,dx\,.
\end{aligned}
$$

The second inequality in Lemma 5.1 can be proved as before if $I[u,\chi,h_i,\sigma]$ is replaced by $I_p[u,\chi,h_i,\sigma]$.
Lemma 5.2 up to Lemma 5.4 were established just by combining the previous results. So they remain valid with obvious changes: $I[u,\chi,h,\sigma]$ has to be replaced by $I_p[u,\chi,h,\sigma]$, the one–phase equilibrium states $(0,0)$ and $(0,1)$ have to be interpreted as $(\hat{u}^+,1)$ and $(\hat{u}^-,0)$, respectively. Summing up we have proved the first part of Remark 2.4, precisely, the validity of the following result is shown:

THEOREM 8.1 *Assume (2.1), (2.3)–(2.6), (2.7) or* (2.7^*)*, (8.1) and (8.9). Then, with the notational changes just stated above, Theorem 2.1 remains valid if the functional from (2.8) is replaced by the energy defined in (8.2).*

It is immediate that the volume force term $\int_\Omega p\cdot u\,dx$ does not affect the proof of Theorem 2.2, hence we get

THEOREM 8.2 *Under the assumptions of Theorem 8.1 the result of Theorem 2.2 is true for two–phase equilibrium states of the functional from (8.2).*

In the same spirit we can prove Theorem 7.1 if the potential $\int_\Omega p\cdot u\,dx$ is added to any energy under consideration.

9 Remarks on the case of non–zero boundary values

Given $u_0 \in W^1_2(\Omega;\mathbb{R}^d)$ we now denote by $\hat{u}^+$ $(\hat{u}^-)$ the unique minimizer of $I^+[u,h]$ $(I^-[u])$ in $u_0 + \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^d)$ (compare Lemma 3.6). We further replace X (see (2.9)) by

$$
\tilde{X} \;:=\; \left\{u_0 + \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^d)\right\} \times \left\{\chi\in BV(\Omega)\,:\,\chi(x)\in\{0,1\}\ \text{a.e.}\right\},
$$

and with I defined according to (2.8) we consider the variational problem

$$I[u, \chi, h, \sigma] \;\to\; \min \;\text{ in }\; \tilde{X} \quad (h \in \mathbb{R}, \;\; \sigma > 0)\,. \tag{9.1}$$

The existence of equilibrium states for (9.1) is ensured exactly as in Theorem 3.4. Let $(\hat{u}, \hat{\chi})$ denote a one–phase equilibrium of $I[u, \chi, h, \sigma]$ in $\tilde{X}$, i.e. $\hat{\chi} \equiv 1$ or $\hat{\chi} \equiv 0$. If $\hat{\chi} \equiv 1$ we have for all $u \in u_0 + \overset{\circ}{W}{}^1_2(\Omega; \mathbb{R}^d)$

$$\int_\Omega f_h^+\big(\cdot, \varepsilon(\hat{u})\big)\, dx \;=\; I[\hat{u}, 1, h, \sigma] \;\le\; I[u, 1, h, \sigma] \;=\; I^+[u, h]\,,$$

hence (together with the analogous inequality for I^-) the one–phase equilibria are seen to be given by $(\hat{u}^+, 1)$ and by $(\hat{u}^-, 0)$. Before going through the arguments of Chapter 4, we first note that Lemma 3.5 remains valid if we assume in addition that $u_0 \in W^1_{2q}(\Omega; \mathbb{R}^d)$ (of course we now claim $\hat{u} \in u_0 + \overset{\circ}{W}{}^1_{2q}(\Omega; \mathbb{R}^d)$ with corresponding apriori bound for $\|\hat{u}\|_{W^1_{2q}(\Omega;\mathbb{R}^d)}$, where the quantity R also depends on $\|u_0\|_{W^1_{2q}(\Omega;\mathbb{R}^d)}$). Moreover, the quantity ε is seen to be independent of u_0. Further, observe that now (3.7) reads as

$$\begin{aligned}
\hat{I}_0[h] \;&=\; \begin{cases} c^+ + \displaystyle\int_\Omega \langle A^+\xi^+, \xi^+\rangle\, dx + h\,|\Omega|\,, & h \le \hat{h}\,, \\[2ex] c^- + \displaystyle\int_\Omega \langle A^-\xi^-, \xi^-\rangle\, dx\,, & h \ge \hat{h}\,, \end{cases} \\
\hat{h} \;&=\; \frac{c^- - c^+}{|\Omega|} + \frac{1}{|\Omega|}\int_\Omega \big(- \langle A^+\xi^+, \xi^+\rangle + \langle A^-\xi^-, \xi^-\rangle\big)\, dx\,.
\end{aligned} \tag{9.2}$$

Here we have set

$$c^\pm \;:=\; \int_\Omega \Big(\big\langle A^\pm\varepsilon(\hat{u}^\pm), \varepsilon(\hat{u}^\pm)\big\rangle - 2\big\langle A^\pm\varepsilon(\hat{u}^\pm), \xi^\pm\big\rangle \Big)\, dx\,.$$

Writing $\hat{u}^\pm = u_0 + \hat{\varphi}^\pm$, $\hat{\varphi}^\pm \in \overset{\circ}{W}{}^1_2(\Omega; \mathbb{R}^d)$, the inequalities $I^+[\hat{u}^+, h] \le I^+[u_0, h]$ $(I^-[\hat{u}^-] \le I^-[u_0])$ imply

$$\int_\Omega \Big(\big\langle A^\pm\varepsilon(\hat{u}^\pm), \varepsilon(\hat{u}^\pm)\big\rangle - 2\big\langle A^\pm\varepsilon(\hat{\varphi}^\pm), \xi^\pm\big\rangle \Big)\, dx \;\le\; \int_\Omega \big\langle A^\pm\varepsilon(u_0), \varepsilon(u_0)\big\rangle\, dx\,,$$

and, as a consequence of (2.4) and (2.6),

$$|c^\pm| \;=\; |c^\pm(u_0)| \;\to\; 0 \quad \text{as } \|\varepsilon(u_0)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \to 0\,. \tag{9.3}$$

To check the arguments of Chapter 4 in the case of non–zero boundary values, we replace the one–phase equilibria $(0,1)$ and $(0,0)$ by $(\hat{u}^+,1)$ and $(\hat{u}^-,0)$, respectively, for example instead of (4.1) we obtain

$$I[\hat{u},\hat{\chi},h,\sigma] \;\le\; I[\hat{u}^-,0,h,\sigma]\,, \qquad I[\hat{u},\hat{\chi},h,\sigma] \;\le\; I[\hat{u}^+,1,h,\sigma]\,.$$

The minimality of $\hat{u}^\pm$ yields the fundamental inequalities (4.2) and (4.3) as before. This proves Lemma 4.1 with G_0 and δ also depending on $\|u_0\|_{W^1_{2q}(\Omega;\mathbb{R}^d)}$. Lemma 4.2, Lemma 4.3 and Remark 4.4 remain valid without any changes, going through the proofs of Lemma 4.5, Lemma 4.6 and Lemma 4.7 we just have to observe that G_0, in particular the quantities $h^\pm$, $h_0(\sigma)$ and σ^*, now depend on $\|u_0\|_{W^1_{2q}(\Omega;\mathbb{R}^d)}$. As in Section 8 again Lemma 4.8 has to be studied in detail: with $\hat{h}$ given in (9.2) again assume that there exists a sequence $\sigma_n > 0$, $\sigma_n \to 0$ such that $I[u,\chi,\hat{h},\sigma_n]$ admits only one–phase equilibria $(\hat{u}_n,\hat{\chi}_n)$ in $\tilde{X}$, i.e. $\hat{u}_n \equiv \hat{u}^+$, $\hat{\chi}_n \equiv 1$ or $\hat{u}_n \equiv \hat{u}^-$, $\hat{\chi}_n \equiv 0$. Again we obtain (for $\hat{I}_0[\hat{h}]$ defined in (9.2))

$$I[u,\chi,\hat{h},\sigma_n] \;\ge\; \hat{I}_0[\hat{h}] \quad \text{for all } (u,\chi)\in\tilde{X}\,. \tag{9.4}$$

Case 1.) Assume that a set $E \subset \Omega$ satisfies (2.7) and choose $u = u_0$ as well as the approximation of $\mathbf{1}_E$ with characteristic functions $\chi_k \in BV(\Omega)$ (compare the density property d)). (9.4) implies (passing to the limit $n\to\infty$)

$$\begin{aligned}
&\int_\Omega \chi_k \Big\{\big\langle A^+\varepsilon(u_0),\varepsilon(u_0)\big\rangle - 2\big\langle A^+\varepsilon(u_0),\xi^+\big\rangle\Big\}\,dx \\
&\quad + \int_\Omega (1-\chi_k)\Big\{\big\langle A^-\varepsilon(u_0),\varepsilon(u_0)\big\rangle - 2\big\langle A^-\varepsilon(u_0),\xi^-\big\rangle\Big\}\,dx \\
&\quad + \int_\Omega \chi_k\Big[\big\langle A^+\xi^+,\xi^+\big\rangle - \big\langle A^-\xi^-,\xi^-\big\rangle + \hat{h}\Big]\,dx \;\ge\; c^-\,.
\end{aligned}$$

The absolute value of the first and the second integral on the left–hand side is bounded independent of χ_k by a quantity $c_0 \ge 0$ satisfying $c_0 \to 0$ as $\|\varepsilon(u)\|_{L^2} \to 0$. The definition of $\hat{h}$ gives

$$\begin{aligned}
&\int_\Omega \chi_k\Big[\big\langle A^+\xi^+,\xi^+\big\rangle - \frac{1}{|\Omega|}\int_\Omega \big\langle A^+\xi^+,\xi^+\big\rangle\,dy \\
&\qquad -\Big(\big\langle A^-\xi^-,\xi^-\big\rangle - \frac{1}{|\Omega|}\int_\Omega \big\langle A^-\xi^-,\xi^-\big\rangle\,dy\Big)\Big]\,dx \\
&\qquad \ge\; c^-\Big(1 - \fint_\Omega \chi_k\,dx\Big) + c^+ \fint_\Omega \chi_k\,dx - c_0\,.
\end{aligned}$$

Passing to the limit $k \to \infty$ the left–hand side converges to a negative number $\tilde{b}(E) = b(A^\pm, \xi^\pm)$ which gives a contradiction and the lemma is established if we assume $\|\varepsilon(u_0)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}$ to be small enough such that

$$c_0 + |c^-| + |c^+| \ < \ |b(A^\pm, \xi^\pm)|\,. \tag{9.5}$$

Case 2.) Assume now (2.7*) to be true. According to (4.17) we find $\varphi \in C_0^\infty(\Omega; \mathbb{R}^d)$ and a sequence of characteristic functions $\chi_k \in BV(\Omega)$ such that

$$\int_\Omega \chi_k \big\langle \varepsilon(\varphi), A^- \xi^- - A^+ \xi^+ \big\rangle \, dx \ \to: \ b(A^\pm, \xi^\pm) \neq 0 \tag{9.6}$$

as $k \to \infty$. Again we may assume w.l.o.g. that $b > 0$. Choosing $u = u_0 + \lambda\varphi$, $\lambda \in \mathbb{R}$, and $\chi = \chi_k$ in (9.4) we obtain in this case by a direct calculation (using (2.7*))

$$\begin{aligned}
& a\lambda^2 + \tilde{b}\lambda + c \\
& \quad := \ \lambda^2 \int_\Omega \Big[\chi_k \big\langle A^+ \varepsilon(\varphi), \varepsilon(\varphi)\big\rangle + (1-\chi_k)\big\langle A^- \varepsilon(\varphi), \varepsilon(\varphi)\big\rangle\Big]\, dx \\
& \qquad + \lambda 2 \int_\Omega \Big[\chi_k \big\langle A^+ \varepsilon(u_0), \varepsilon(\varphi)\big\rangle + (1-\chi_k)\big\langle A^- \varepsilon(u_0), \varepsilon(\varphi)\big\rangle \\
& \qquad\qquad + \chi_k \big\langle \varepsilon(\varphi), A^-\xi^- - A^+\xi^+ \big\rangle\Big]\, dx \\
& \qquad + \big\{c_0 + |c^-| + |c^+|\big\} \ \geq \ 0
\end{aligned}$$

being valid for all $\lambda \in \mathbb{R}$. Note that φ was chosen only depending on the data $A^\pm$, $\xi^\pm$ and we assume now $\|\varepsilon(u_0)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}$ to be sufficiently small such that

$$\int_\Omega \Big[\chi_k \big\langle A^+ \varepsilon(u_0), \varepsilon(\varphi)\big\rangle + (1-\chi_k)\big\langle A^- \varepsilon(u_0), \varepsilon(\varphi)\big\rangle\Big]\, dx \ > \ -\frac{b}{2}\,. \tag{9.7}$$

Passing to the limit $k \to \infty$ and considering the case $\lambda \leq 0$ we arrive at (see (9.6))

$$a(A^\pm, \xi^\pm)\,\lambda^2 + b(A^\pm, \xi^\pm)\,\lambda + c \ \geq \ 0 \quad \text{for all } \lambda \leq 0\,. \tag{9.8}$$

Note that a, b, $c > 0$, i.e. the possible zeros of (9.8) are negative and a contradiction is obtained if

$$b^2(A^\pm, \xi^\pm) - 4\,a(A^\pm, \xi^\pm)\,c \ > \ 0\,. \tag{9.9}$$

In conclusion, the Lemma is proved for boundary value problems with $\|\varepsilon(u_0)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}$ sufficiently small such that in addition to (9.7) inequality (9.9) holds.

Chapter 5 and Chapter 6 are carried over with the obvious changes and we have proved

THEOREM 9.1 *Assume that $u_0 \in W^1_{2q}(\Omega; \mathbb{R}^d)$ is given such that*

$$\|\varepsilon(u_0)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \;<\; \gamma\,,$$

where γ is sufficiently small depending on the data $A^\pm$, $\xi^\pm$ (compare (9.5), (9.7), (9.9)). Then Theorem 2.1 and Theorem 2.2 remain valid if we replace X by $\tilde{X}$.

REMARK 9.2 *i) In the same way small pertubations of solutions of the equilibrium equations (2.6) can be treated as boundary values.*

ii) Small pertubations of the equilibrium equations (2.6) are by the above arguments also seen to be an admissible choice for $\xi^\pm$.

iii) Of course we can combine the case of non–zero boundary values with the presence of an additional volume force term p. The calculations are in principal the same as needed for establishing the results of Theorem 8.1, 8.2 and 9.1.

Appendix

For completeness we give a proof of the density result d) from Section 2 following the lines of [OS2]. Consider a measurable characteristic function χ: $\Omega \to \{0,1\}$. In order to construct a sequence $\{\chi_n\}$ of measurable characteristic functions $\chi_n \in BV(\Omega)$ such that $\chi_n(x) \to \chi(x)$ for a.a. $x \in \Omega$, we introduce a subdivision of $\mathbb{R}^d$ into cubes $K^{(n)}$ with disjoint interior and side length equal to $1/n$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we select those cubes $K^{(n)}_j$, $j = 1, \dots, N(n)$, with the properties

$$\bigcup_{j=1}^{N(n)} K_j^{(n)} \supset \Omega\,, \qquad K_j^{(n)} \cap \Omega \neq \emptyset\,, \qquad \Big|\bigcup_{j=1}^{N(n)} K_j^{(n)} - \Omega\Big| \;\le\; 1/n\,.$$

Clearly, the last condition holds for n large enough. Let

$$c_j^{(n)} \;=\; \fint_{K_j^{(n)}\cap\Omega} \chi\,dx$$

and choose a smooth domain $\omega_j^{(n)} \subset \overline{\omega_j^{(n)}} \subset K_j^{(n)} \cap \Omega$ such that

$$|\omega_j^{(n)}| \;=\; (1 - 1/n)\, c_j^{(n)} \left|K_j^{(n)} \cap \Omega\right| . \tag{A.1}$$

Next we define $\chi_j^{(n)} := \mathbf{1}_{\omega_j^{(n)}}$ and let

$$\chi^{(n)} \;:=\; \sum_{j=1}^{N(n)} \chi_j^{(n)} \;\in\; BV(\Omega)$$

which clearly is a measurable characteristic function. We have

$$\begin{aligned}
\int_\Omega \left(\chi - \chi^{(n)}\right) dx \;&=\; \sum_{j=1}^{N(n)} \int_{K_j^{(n)} \cap \Omega} \left(\chi - \chi_j^{(n)}\right) dx \\
&=\; \sum_{j=1}^{N(n)} \left[c_j^{(n)} \left|K_j^{(n)} \cap \Omega\right| - |\omega_j^{(n)}| \right] \\
&\overset{(A.1)}{=}\; \sum_{j=1}^{N(n)} \frac{1}{n} c_j^{(n)} \left|K_j^{(n)} \cap \Omega\right| ,
\end{aligned}$$

hence, using $0 \le c_j^{(n)} \le 1$,

$$\begin{aligned}
\left| \int_\Omega \chi\, dx - \int_\Omega \chi^{(n)}\, dx \right| \;&=\; \frac{1}{n} \sum_{j=1}^{N(n)} c_j^{(n)} \left|K_j^{(n)} \cap \Omega\right| \\
&\le\; \frac{1}{n} \sum_{j=1}^{N(n)} \left|K_j^{(n)} \cap \Omega\right| \;=\; \frac{1}{n} |\Omega|
\end{aligned}$$

which means

$$\lim_{n \to \infty} \int_\Omega \chi^{(n)}\, dx \;=\; \int_\Omega \chi\, dx . \tag{A.2}$$

Consider $g \in C_0^1(\Omega)$ (to be extended by zero to the whole space $\mathbb{R}^d$) and let $x_j^{(n)}$ denote the center of $K_j^{(n)}$. Then

$$\int_\Omega g \left(\chi - \chi^{(n)}\right) dx \;=\; \sum_{j=1}^{N(n)} \int_{K_j^{(n)} \cap \Omega} g \left(\chi - \chi_j^{(n)}\right) dx \;=$$

$$= \sum_{j=1}^{N(n)} g\big(x_j^{(n)}\big) \int_{K_j^{(n)}\cap\Omega} \big(\chi(x) - \chi_j^{(n)}(x)\big)\, dx$$

$$+ \sum_{j=1}^{N(n)} \int_{K_j^{(n)}\cap\Omega} \big(g(x) - g\big(x_j^{(n)}\big)\big)\big(\chi(x) - \chi_j^{(n)}(x)\big)\, dx \ =: \ I + II\,.$$

For I we observe as before (see (A.1))

$$\Big| \int_{K_j^{(n)}\cap\Omega} \big(\chi - \chi_j^{(n)}\big)\, dx \Big| \ = \ \frac{1}{n}\, c_j^{(n)} \, \big|K_j^{(n)} \cap \Omega\big|\,,$$

hence

$$|I| \ \leq \ \sup_{\Omega} |g| \sum_{j=1}^{N(n)} \frac{1}{n} c_j^{(n)} \big|K_j^{(n)} \cap \Omega\big| \ \leq \ \frac{1}{n}\, |\Omega| \sup_{\Omega} |g|\,.$$

Since $\big|g(x) - g\big(x_j^{(n)}\big)\big| \leq (1/n)\, const\, \|\nabla g\|_\infty$ on $K_j^{(n)} \cap \Omega$, we get

$$|II| \leq const\, \frac{1}{n}\, |\Omega|\, \|\nabla g\|_\infty\,,$$

therefore

$$\lim_{n\to\infty} \int_{\Omega} g\,\big(\chi - \chi^{(n)}\big)\, dx \ = \ 0 \quad \text{for all } \ g \in C_0^1(\Omega)\,. \tag{A.3}$$

Suppose now that $1 < p < \infty$ and $g \in L^{p'}(\Omega)$, $p' := p/(p-1)$, are given. If we choose $\tilde{g} \in C_0^1(\Omega)$ such that

$$\|g - \tilde{g}\|_{L^{p'}(\Omega)} \ < \ \varepsilon$$

for some given $\varepsilon > 0$, we deduce

$$\Big| \int_{\Omega} \big(\chi - \chi^{(n)}\big)\, g\, dx \Big| \ \leq \ \Big| \int_{\Omega} \big(\chi - \chi^{(n)}\big)\, \tilde{g}\, dx \Big| + \Big| \int_{\Omega} \big(\chi - \chi^{(n)}\big)\, (g - \tilde{g})\, dx \Big|\,,$$

and the first integral on the right–hand side vanishes as $n \to \infty$ according to (A.3). For the second one we observe (using $|\chi - \chi^{(n)}| \leq 1$)

$$\Big| \int_{\Omega} \big(\chi - \chi^{(n)}\big)\, (g - \tilde{g})\, dx \Big| \ \leq \ \|g - \tilde{g}\|_{L^{p'}(\Omega)}\, \big\|\chi - \chi^{(n)}\big\|_{L^p(\Omega)} \ \leq \ \varepsilon\, |\Omega|^{\frac{1}{p}}\,.$$

This shows

$$\chi^{(n)} \rightharpoonup \chi \quad \text{in } L^p(\Omega) \quad \text{for all } 1 < p < \infty\,, \tag{A.4}$$

in particular this is true for $p = 2$. On the other hand we have by (A.2)

$$\int_\Omega \left(\chi^{(n)}\right)^2 dx \;=\; \int_\Omega \chi^{(n)}\, dx \;\overset{n\to\infty}{\to}\; \int_\Omega \chi\, dx \;=\; \int_\Omega \chi^2\, dx\,,$$

hence $\|\chi^{(n)}\|_{L^2(\Omega)} \to \|\chi\|_{L^2(\Omega)}$. This together with (A.4) implies $\|\chi^n - \chi\|_{L^2(\Omega)} \to 0$ (in fact L^p–convergence holds, $p < \infty$) and we may extract a subsequence $\{\chi^{(n')}\}$ s.t. $\chi^{(n')} \to \chi$ a.e. on Ω. ■

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