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A note on splitting-type variational problems with subquadratic growth

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Abstract

We consider variational problems of splitting-type, i.e. we want to minimize

$$\int_\Omega [f(\widetilde{\nabla} w) + g(\partial_n w)]\, dx$$

where $\widetilde{\nabla} = (\partial_1, ..., \partial_{n-1})$. Thereby f and g are two C^2-functions which satisfy power growth conditions with exponents $1 < p \le q < \infty$. In case $p \ge 2$ there is a regularity theory for minimizers $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ without further restrictions on p and q if $n = 2$ or $N = 1$. In the subquadratic case the results are much weaker: we get $C^{1,\alpha}$-regularity, if we require $q \le 2p + 2$ for $n = 2$ or $q < p + 2$ for $N = 1$. In this paper we show $C^{1,\alpha}$-regularity under the bounds $q < \frac{2p+4}{2-p}$ resp. $q < \infty$.

1 Introduction

In this paper we discuss regularity results for local minimizers $u : \Omega \to \mathbb{R}^N$ of variational integrals

$$I[u, \Omega] := \int_\Omega F(\nabla u)\, dx \tag{1.1}$$

where Ω denotes an open set in $\mathbb{R}^n$ and where $F : \mathbb{R}^{nN} \to [0, \infty)$ satisfies an anisotropic growth condition, i.e.

$$C_1|Z|^p - c_1 \le F(Z) \le C_2|Z|^q + c_2, \qquad Z \in \mathbb{R}^{nN} \tag{1.2}$$

with constants $C_1, C_2 > 0$, $c_1, c_2 \ge 0$ and exponents $1 < p \le q < \infty$. The study of such problems was pushed by Marcellini (see [Ma1] and [Ma2]) and today it is a well known fact that there is no hope for regularity of minimizers if p and q differ too much (compare [Gi] and [Ho] for counter examples). Under mild smoothness conditions on F (the case of (p, q)-elliptic integrands) the best known statement is the bound

$$q < p + 2 \tag{1.3}$$

for regularity proved by Bildhauer and Fuchs [BF1], where one has to suppose local boundedness of minimizers. To get better results additional assumptions are necessary. Therefore we consider decomposable integrands, which means we have

$$F(Z) = f(\widetilde{Z}) + g(Z_n) \tag{A1}$$

for $Z = (Z_1, ..., Z_n)$ with $Z_i \in \mathbb{R}^N$ and $\widetilde{Z} = (Z_1, ..., Z_{n-1})$. Thereby f and g are functions of class C^2 and we assume power growth conditions:

$$\begin{aligned} \lambda(1+|\widetilde{Z}|^2)^{\frac{p-2}{2}}|\widetilde{X}|^2 \leq \ & D^2 f(\widetilde{Z})(\widetilde{X},\widetilde{X}) \quad \leq \Lambda(1+|\widetilde{Z}|^2)^{\frac{p-2}{2}}|\widetilde{X}|^2, \\ \lambda\left(1+|Z_n|^2\right)^{\frac{q-2}{2}}|X_n|^2 \leq \ & D^2 g(Z_n)(X_n,X_n) \leq \Lambda\left(1+|Z_n|^2\right)^{\frac{q-2}{2}}|X_n|^2 \end{aligned} \tag{A2}$$

for all $Z = (\widetilde{Z}, Z_n), X = (\widetilde{X}, X_n) \in \mathbb{R}^{nN}$ with positive constants λ, Λ and exponents $1 < p \leq q < \infty$. Assuming (A2) it is easy to see, that we have a condition of the form (1.2) for F.

In case $p \geq 2$ Bildhauer, Fuchs and Zhong show, that local minimizers $u \in W^{1,p}_{loc} \cap L^\infty_{loc}(\Omega, \mathbb{R}^N)$ of (1.1) are of class $C^{1,\alpha}$ without further assumptions on p and q, if $n = 2$ or $N = 1$ (see [BF2] and [BFZ]). Additionally to (A1) and (A2) in case $n = 2$ they have to suppose

$$f(Z_1) = \widehat{f}(|Z_1|) \text{ and } g(Z_n) = \widehat{g}(|Z_2|), \tag{A3}$$

with two functions $\widehat{f}$ and $\widehat{g}$ which are strictly increasing. This is for using the maximum principle of [DLM]. In [BF3] one can find partial regularity results in this topic, but they are much weaker and not independent of dimension.

If we have a look at the subquadratic situation, we find strong restrictions on p and q for receiving regular solutions:

- $q < p + 2$ for $N = 1$, see [BF1], and
- $q \leq 2p + 2$ for $n = 2$, see [BF2], Remark 5.

Thereby in both cases the assumption $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$ is necessary which we can get rid of if $n = 2$. The aim of this paper is to improve the above statements for local minimizers of (1.1).

Definition 1.1 *We call a function $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ a local minimizer of (1.1), if we have for all $\Omega' \Subset \Omega$*

- $\int_{\Omega'} F(\nabla u)\, dx < \infty$ *and*
- $\int_{\Omega'} F(\nabla u)\, dx \leq \int_{\Omega'} F(\nabla v)\, dx$ *for all* $v \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$, $\operatorname{spt}(u - v) \Subset \Omega$.

Our main Theorem reads as follows:

THEOREM 1.1 *For any local minimizer $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ of (1.1) with $1 < p < 2$ we have under the assumptions (A1) and (A2):*

(a) *If we have* $n = 2$, *(A3) and*

$$q < \frac{2p+4}{2-p}, \tag{A4}$$

then $u \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$ *for all* $\alpha < 1$.

(b) *If* $N = 1$ *and* $u \in L^{\infty}_{loc}(\Omega)$, *so one gets* $u \in C^{1,\alpha}(\Omega)$ *for all* $\alpha < 1$.

Remark 1.2

- *If we have* $N = 1$ *Theorem 1.1 b) gives (together with the results from [BFZ1])* $C^{1,\alpha}$*-regularity for all choices of* $1 < p \le q < \infty$. *In the 2D-case we additionally have (A4). This hypothesis is needed for calculating the term* ($\Gamma_i = 1 + |\partial_i u|^2$, $i = 1, 2$)

$$\int \Gamma_2^{\frac{q-2}{2}} \Gamma_1 \, dx.$$

Note that we have an arbitrary wide range of anisotropy for $p \to 2$. *But for* $p \to 1$ *the bound is also much better than the bound* $q \le 2p + 2$ *from [BF2].*

- *In the situation* $n = 2$ *we can get rid of the assumption* $u \in L^{\infty}_{loc}(\Omega, \mathbb{R}^N)$, *see [Bi2] (section 4) for details.*
- *Under suitable conditions on* $D_x D_{\widetilde{P}} f$ *and* $D_x D_{P_n} g$ *it is possible to extend our result to the non-autonomous situation, which means densities* $F = F(x, Z)$ *and "splitting-type" integrands (compare [BF2], Remark 3 and [BFZ1], Remark 1.4).*

2 $\mathbf{C^{1,\alpha}}$-regularity for $\mathbf{n = 2}$

From now on we assume the conditions of Theorem 1.1 a). Let $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of (1.1) and fix $x_0 \in \Omega$. Now it is possible to find a radius $R > 0$ such that $u \in L^{\infty}_{loc}(\partial B_R(x_0), \mathbb{R}^N)$ (compare [Bi2], section 4, for details). From (A3) and the maximum-principle of [DLM] we get $u \in L^{\infty}_{loc}(B_R(x_0), \mathbb{R}^N)$. For $0 < \epsilon \ll 1$ $(u)_\epsilon$ denotes the mollification of u with radius ϵ (see [Ad]). Now we choose $R_0 < R$ and get $\sup_{\epsilon > 0} \|(u)_\epsilon\|_\infty < \infty$. For a fixed $\widetilde{q} > \max\{q, 2\}$ let

$$\delta := \delta(\epsilon) := \frac{1}{1 + \epsilon^{-1} + \|(\nabla u)_\epsilon\|^{2\widetilde{q}}_{L^{\widetilde{q}}(B)}}$$

$$\text{and } F_\delta(Z) := \delta \left(1 + |Z|^2\right)^{\frac{\widetilde{q}}{2}} + F(Z)$$

for $Z \in \mathbb{R}^{nN}$. With $B := B_{R_0}(x_0)$ we define u_δ as the unique minimizer of

$$I_\delta\left[w, B\right] := \int_B F_\delta(\nabla w) dx \tag{2.1}$$

in $(u)_\epsilon + W_0^{1,\widetilde{q}}(B, \mathbb{R}^N)$. Some elementary properties of u_δ are summarized in the following Lemma (see [BF2], Lemma 1, for further references):

Lemma 2.1 • *We have as* $\epsilon \to 0$*:* $u_\delta \rightharpoondown u$ *in* $W^{1,p}(B, \mathbb{R}^N)$ *,*

$$\delta \int_B \left(1 + |\nabla u_\delta|^2\right)^{\frac{\widetilde{q}}{2}} dx \to 0 \quad \textit{and} \quad \int_B F(\nabla u_\delta) dx \to \int_B F(\nabla u) dx.$$

- $\sup_{\delta > 0} \|u_\delta\|_{L^\infty(B)} < \infty.$
- $\nabla u_\delta \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega, \mathbb{R}^N).$

We need the following Caccioppoli-type inequality which is standard to proof:

Lemma 2.2 *For* $\eta \in C_0^\infty(B)$*, arbitrary* $\gamma \in \{1, ..., n\}$ *and* $Q \in \mathbb{R}^{nN}$ *we have*

$$\int_B \eta^2 D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) dx$$
$$\leq c \int_B D^2 F_\delta(\nabla u_\delta)([\partial_\gamma u_\delta - Q_\gamma] \otimes \nabla \eta, [\partial_\gamma u_\delta - Q_\gamma] \otimes \nabla \eta) dx$$

for a constant $c > 0$ *independent of* δ*.*

Analogous to [BF2] we must prove the following statement for H_δ which is defined by

$$H_\delta^2 := D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)$$

with sum over $\gamma \in \{1, 2\}$:

Lemma 2.3 • *We have* $H_\delta \in L^2_{loc}(B)$ *uniform in* ϵ *and*

- $u_\delta \in W^{1,t}_{loc}(B)$ *uniform in* ϵ *for all* $t < \infty$*.*

Proof: We consider for $\Gamma_{i,\delta} := 1 + |\partial_i u_\delta|^2$, $i \in \{1, 2\}$,

$$f_1(\rho) := \int_{B_\rho} \Gamma_{1,\delta}^{\frac{p+2}{2}} dx \quad \text{and} \quad f_2(\rho) := \int_{B_\rho} \Gamma_{2,\delta}^q dx$$

separately. Let $\eta \in C_0^\infty(B_r)$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on B_ρ and $|\nabla\eta| \le c/(r-\rho)^{-1}$. Following [BF2] we see

$$f_1(\rho) \le c\left[1 + \int_{B_r} |\nabla\eta|\eta\Gamma_{1,\delta}^{\frac{p+1}{2}}\,dx + \int_{B_r} \eta^2\Gamma_{1,\delta}^{\frac{p}{2}}|\partial_1\partial_1 u_\delta|\,dx\right]$$

for a constant c independent of ρ, r and δ using uniform bounds on u_δ. By Young's inequality we get for a suitable $\beta > 0$ the upper bound

$$c(\tau)(r-\rho)^{-\beta} + \tau\int_{B_r} \Gamma_{1,\delta}^{\frac{p+2}{2}}\,dx$$

for the first term on the r.h.s. ($\tau > 0$ is arbitrary). For the second one we obtain by (A2)

$$\begin{aligned}\int_{B_r} \eta^2\Gamma_{1,\delta}^{\frac{p}{2}}|\partial_1\partial_1 u_\delta|\,dx &\le c(\tau)\int_{B_r} \eta^2\Gamma_{1,\delta}^{\frac{p-2}{2}}|\partial_1\partial_1 u_\delta|^2\,dx + \tau\int_{B_r} \eta^2\Gamma_{1,\delta}^{\frac{p+2}{2}}\,dx\\ &\le c(\tau)\int_{B_r} \eta^2 H_\delta^2\,dx + \tau\int_{B_r} \eta^2\Gamma_{1,\delta}^{\frac{p+2}{2}}\,dx.\end{aligned}$$

As a consequence

$$f_1(\rho) \le c(\tau)\int_{B_r} \eta^2 H_\delta^2\,dx + c(\tau)(r-\rho)^{-\beta} + \tau\int_{B_r} \Gamma_{1,\delta}^{\frac{p+2}{2}}\,dx. \tag{2.2}$$

For $f_2(\rho)$ we receive (following ideas of [BF5]) by Sobolev's inequality

$$\begin{aligned}f_2(\rho) = \int_{B_\rho} \Gamma_{2,\delta}^q\,dx &\le \int_{B_r} \left(\eta\Gamma_{2,\delta}^{\frac{q}{2}}\right)^2\,dx\\ &\le c\left[\int_{B_r} |\nabla\eta|\Gamma_{2,\delta}^{\frac{q}{2}}\,dx + \int_{B_r} \eta\Gamma_{2,\delta}^{\frac{q-1}{2}}|\partial_2\nabla u_\delta|\,dx\right]^2.\end{aligned}$$

Using Lemma 2.1, we get

$$f_2(\rho)\,dx \le c(r-\rho)^{-1} + c\left[\int_{B_r} \eta\Gamma_{2,\delta}^{\frac{q-1}{2}}|\partial_2\nabla u_\delta|\,dx\right]^2.$$

From Hölder's inequality we deduce

$$\begin{aligned}[...]^2 &\le c\int_{B_r} \Gamma_{2,\delta}^{\frac{q}{2}}\,dx\int_{B_r} \eta^2\Gamma_{2,\delta}^{\frac{q-2}{2}}|\partial_2\nabla u_\delta|^2\,dx\\ &\le c\int_{B_r} \eta^2 H_\delta^2\,dx\end{aligned}$$

by Lemma 2.1, part 1. Combining this with (2.2) and choosing τ small enough we receive

$$\int_{B_\rho}\left(\Gamma_{1,\delta}^{\frac{p+2}{2}}+\Gamma_{2,\delta}^{q}\right)dx\le c(r-\rho)^{-\beta}+c\int_{B_r}\eta^2H_\delta^2\,dx+\frac{1}{4}\int_{B_r}\Gamma_{1,\delta}^{\frac{p+2}{2}}\,dx. \quad (2.3)$$

From Lemma 2.2 we deduce for $Q=0$

$$\begin{aligned}\int_{B_r}\eta^2H_\delta^2\,dx &\le c\int_{B_r}D^2F_\delta(\nabla u_\delta)(\partial_1u_\delta\otimes\nabla\eta,\partial_1u_\delta\otimes\nabla\eta)\,dx\\ &\quad +c\int_{B_r}D^2F_\delta(\nabla u_\delta)(\partial_2u_\delta\otimes\nabla\eta,\partial_2u_\delta\otimes\nabla\eta)\,dx\\ &=: c\,[J_1+J_2]\,.\end{aligned}$$

Thus we have by (A2)

$$\begin{aligned}J_2 &\le c\int_{B_r}|\nabla\eta|^2\Gamma_{2,\delta}^{\frac{q-2}{2}}\Gamma_{2,\delta}\,dx+c\int_{B_r}|\nabla\eta|^2\Gamma_{1,\delta}^{\frac{p-2}{2}}\Gamma_{2,\delta}\,dx\\ &\quad +c\delta\int_{B_r}|\nabla\eta|^2\Gamma_{\delta}^{\frac{\tilde q-2}{2}}\Gamma_{2,\delta}\,dx\\ &\le c\int_{B_r}|\nabla\eta|^2\Gamma_{2,\delta}^{\frac{q}{2}}\,dx+c\int_{B_r}|\nabla\eta|^2\Gamma_{2,\delta}\,dx\\ &\quad +c\delta\int_{B_r}|\nabla\eta|^2\Gamma_{\delta}^{\frac{\tilde q}{2}}\,dx\le c(r-\rho)^{-2},\end{aligned}$$

if we note Lemma 2.1, part 1, and $p\le 2\le q$. Examining J_1 one sees

$$\begin{aligned}J_1 &\le c\int_{B_r}|\nabla\eta|^2\Gamma_{2,\delta}^{\frac{q-2}{2}}\Gamma_{1,\delta}\,dx+c\int_{B_r}|\nabla\eta|^2\Gamma_{1,\delta}^{\frac{p-2}{2}}\Gamma_{1,\delta}\,dx\\ &\quad +c\delta\int_{B_r}|\nabla\eta|^2\Gamma_{\delta}^{\frac{\tilde q-2}{2}}\Gamma_{1,\delta}\,dx\\ &\le c(r-\rho)^{-2}+c\int_{B_r}|\nabla\eta|^2\Gamma_{2,\delta}^{\frac{q-2}{2}}\Gamma_{1,\delta}\,dx. \quad (2.4)\end{aligned}$$

Considerating the last critical term, one can follow by Young's inequality ($\tau'>0$ is arbitrary)

$$c\int_{B_r}|\nabla\eta|^2\Gamma_{2,\delta}^{\frac{q-2}{2}}\Gamma_{1,\delta}\,dx\le\tau'\int_{B_r}\Gamma_{1,\delta}^{\frac{p+2}{2}}\,dx+c(\tau')\int_{B_r}|\nabla\eta|^{2\frac{p+2}{p}}\Gamma_{2,\delta}^{\frac{q-2}{2}\frac{p+2}{p}}\,dx.$$

(A4) gives

$$\frac{q-2}{2}\frac{p+2}{p}<q.$$

We deduce from Young's inequality

$$\int_{B_r} |\nabla\eta|^2 \Gamma_{2,\delta}^{\frac{q-2}{2}} \Gamma_{1,\delta}\, dx \le c(\tau')(r-\rho)^{-\beta} + \tau' \int_{B_r} \Gamma_{1,\delta}^{\frac{p+2}{2}}\, dx + \tau' \int_{B_r} \Gamma_{2,\delta}^{q}\, dx. \tag{2.5}$$

Now we combine (2.4) and (2.5) and get by a suitable choice of τ'

$$c\int_{B_r} \eta^2 H_\delta^2\, dx \le c(r-\rho)^{-\beta} + \frac{1}{4}\int_{B_r} \Gamma_{1,\delta}^{\frac{p+2}{2}}\, dx + \frac{1}{4}\int_{B_r} \Gamma_{2,\delta}^{q}\, dx. \tag{2.6}$$

Inserting this into (2.3) we get

$$\int_{B_\rho} \left(\Gamma_{1,\delta}^{\frac{p+2}{2}} + \Gamma_{2,\delta}^{q}\right) dx \le c(r-\rho)^{-\beta} + \frac{1}{2}\int_{B_r} \left(\Gamma_{1,\delta}^{\frac{p+2}{2}} + \Gamma_{2,\delta}^{q}\right) dx.$$

for all $\rho < r \le R' < R_0$ with $c = c(R')$. From [Gi2] (Lemma 5.1, S. 81) we deduce uniform boundedness of $\partial_1 u_\delta$ in $L^{p+2}_{loc}(B,\mathbb{R}^N)$ and $\partial_2 u_\delta$ in $L^{2q}_{loc}(B,\mathbb{R}^N)$, as well as weak convergence of subsequences in these spaces. So we get $\partial_1 u \in L^{p+2}_{loc}(\Omega,\mathbb{R}^N)$ and $\partial_2 u \in L^{2q}_{loc}(\Omega,\mathbb{R}^N)$. By this result we can infer from (2.6) uniform boundedness of H_δ in $L^2_{loc}(B)$. Since

$$|\nabla\Gamma_{1,\delta}^{\frac{p}{4}}| \le \Gamma_{1,\delta}^{\frac{p-2}{4}} |\partial_1\nabla u_\delta| \le cH_\delta,$$
$$|\nabla\Gamma_{2,\delta}^{\frac{q}{4}}| \le \Gamma_{2,\delta}^{\frac{q-2}{4}} |\partial_2\nabla u_\delta| \le cH_\delta$$

we obtain (by Lemma 2.1) uniform boundedness of $\Gamma_{1,\delta}^{\frac{p}{4}}$ and $\Gamma_{2,\delta}^{\frac{q}{4}}$ in $W^{1,2}_{loc}(B)$ and so we have arbitrary high uniform integrability of $\partial_1 u_\delta$ and $\partial_2 u_\delta$. □

Now we define

$$h_{1,\delta} := \Gamma_{1,\delta}^{\frac{2-p}{4}},\ \ h_{2,\delta} := \Gamma_{2,\delta}^{\frac{q-2}{4}},\ \ h_{3,\delta} := \sqrt{\delta}\Gamma_\delta^{\frac{\tilde{q}-2}{4}}$$
$$\text{and } h_\delta := \left(h_{1,\delta}^2 + h_{2,\delta}^2 + h_{3,\delta}^2\right)^{\frac{1}{2}}.$$

We see following [BF2] for $B_{2r}(z_0) \Subset B$

$$\fint_{B_r(z_0)} H_\delta^2 dx \le c\left[\fint_{B_{2r}(z_0)} (H_\delta h_\delta)^s\, dx\right]^{\frac{1}{s}} \left[\fint_{B_{2r}(z_0)} \left|\nabla^2 u_\delta\right|^s dx\right]^{\frac{1}{s}}, \tag{2.7}$$

where $\fint \ldots$ denotes the mean value. Note $h_{1,\delta} := \Gamma_{1,\delta}^{\frac{2-p}{4}} \ge \Gamma_{1,\delta}^{\frac{p-2}{4}}$ on account of $p<2$. By definition of h_δ, H_δ and (A2) we receive

$$|\nabla^2 u_\delta|^2 \le cH_\delta^2 h_\delta^2$$

and thereby

$$\left[\ \fint_{B_r(z_0)} H_\delta^2 dx\right]^{\frac{1}{2}} \le c \left[\ \fint_{B_{2r}(z_0)} (H_\delta h_\delta)^s\, dx\right]^{\frac{1}{s}}, \tag{2.8}$$

which is exactly (30) in [BF2]. To use further arguments of [BF2], let

$$\widetilde{h}_{1,\delta} := \Gamma_{1,\delta}^{\frac{p}{4}},\ \ \widetilde{h}_{2,\delta} := \Gamma_{2,\delta}^{\frac{q}{4}},\ \ \widetilde{h}_{3,\delta} := \sqrt{\delta}\Gamma_{\delta}^{\frac{\widetilde{q}}{4}}$$
$$\text{and } \widetilde{h}_\delta := \left(\widetilde{h}_{1,\delta}^2 + \widetilde{h}_{2,\delta}^2 + \widetilde{h}_{3,\delta}^2\right)^{\frac{1}{2}}.$$

For $\kappa := \min\{p/(2-p), q/(q-2), \widetilde{q}/(\widetilde{q}-2)\} > 1$ (note $p > 1$ and $q > 2$, if $q \le 2$ we have a range between p and q, small enough to quote the results of [BF4]) we have

$$h_\delta^\kappa \le c\widetilde{h}_\delta \text{ and thereby}$$
$$h_\delta^2 \le \mu \widetilde{h}_\delta^2 + \frac{c}{\mu} \text{ for all } \mu > 0.$$

Now one can end up the proof as in [BF2].

3 $\mathbf{C^{1,\alpha}}$-regularity for $\mathbf{N = 1}$

In this section we work with the Hilbert Haar-regularization (see [BFZ]): Let $B := B_R(x_0) \Subset \Omega$ fixed, then we define u_ϵ as the unique minimizer of $I[\cdot, B]$ in the space of Lipschitz-functions $\overline{B} \to \mathbb{R}$ on boundary data $(u)_\epsilon$ (see [MM], Thm. 4, p. 162), which denotes the mollification of u. So we can quote (compare [BFZ], p. 4, and [MM], Thm. 5, p. 16)

Lemma 3.1
- *We have as $\epsilon \to 0$: $u_\epsilon \rightharpoondown u$ in $W^{1,p}(B)$,*

$$\int_B F(\nabla u_\epsilon) dx \to \int_B F(\nabla u) dx;$$

- $\sup_{\epsilon > 0} \|u_\epsilon\|_{L^\infty(B)} < \infty$*;*
- *$u_\epsilon \in C^{1,\mu}(B) \cap W^{2,2}_{loc}(B)$ for all $\mu < 1$.*

With these preparations, Bildhauer, Fuchs und Zhong show

$$\sup_{\epsilon>0} \|\nabla u_\epsilon\|_{L^t(B_\rho(x_0))} < \infty \tag{3.1}$$

for all $t < \infty$ and all $\rho < R$ (see [BFZ]). W.l.o.g. we assume $p \le 2 \le q$. In this case we have (compare (A2))

$$\lambda(1+|Z|^2)^{\frac{p-2}{2}}|X|^2 \le D^2F(Z)(X,X) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|X|^2. \tag{3.2}$$

Now we can reproduce the proof of [Bi], Thm 5.22. Let $\Gamma_\epsilon := 1 + |\nabla u_\epsilon|^2$ and

$$\tau(k,r) := \int\limits_{A(k,r)} \Gamma_\epsilon^{\frac{q-2}{2}} (\Gamma_\epsilon - k)^2 \, dx$$

with $A(k,r) := B_r \cap [\Gamma_\epsilon > k]$. By arguments from [Bi] one can show

$$\tau(h,r) \le \frac{c}{(\widehat{r}-r)^{\frac{n}{n-1}\frac{1}{s}}(h-k)^{\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}} \tau(k,\widehat{r})^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[1+\frac{1}{t}\right]}$$

for $0 < k < h$ and $0 < r < \widehat{r} < R$. Here $s, t > 1$ are chosen such that

$$\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[1+\frac{1}{t}\right] > 1$$

and c is independent of $h, k, r, \widehat{r}$ and ϵ. If we use [St], Lemma 5.1, we get ∇u in $L^\infty_{loc}(B, \mathbb{R}^n)$ (see [Bi], p. 66, for details). According to the standard theory for elliptic equations or variational problems with standard growth conditions (compare [Gi2]) we can follow the claim of Theorem 1.1. □

References

[Ad] R. A. Adams (1975): Sobolev spaces. Academic Press, New York-San Francisco-London.

[Bi] M. Bildhauer (2003): Convex variational problems: linear, nearly linear and anisotropic growth conditions. Lecture Notes in Mathematics 1818, Springer, Berlin-Heidelberg-New York.

[Bi2] M. Bildhauer (2003): Two-dimensional variational problems with linear growth. Manus. Math. 110, 325-342.

[Br] D. Breit (2009): Regularitätssätze für Variationsprobleme mit anisotropen Wachstumsbedingungen. PhD thesis, Saarland University.

[BF1] M. Bildhauer, M. Fuchs (2002): Elliptic variational problems with nonstandard growth. International Mathematical Series, Vol. I, Nonlinear problems in mathematical physics and related topcs I, in honor of Prof. O.A. Ladyzhenskaya. By Tamara Rozhkovska, Novosibirsk, Russia (in Russian), 49-62. By Kluwer/Plenum Publishers (in English), 53-66.

[BF2] M. Bildhauer, M. Fuchs (2007): Higher integrability of the gradient for vectorial minimizers of decomposable variational integrals. Manus. Math. 123, 269-283.

[BF3] M. Bildhauer, M. Fuchs (2007): Partial regularity for minimizers of splitting-type variational integrals. Preprint 195, Universität des Saarlandes.

[BF4] M. Bildhauer, M. Fuchs (2003): Twodimensional anisotropic variational problems. Calcules of Variations Variations 16, 177-186.

[BF5] M. Bildhauer, M. Fuchs (2009): Differentiability and higher integrability results for local minimizers of splitting-type variational integrals in 2D with applications to nonlinear Hencky-materials. Preprint 223, Universität des Saarlandes.

[BFZ] M. Bildhauer, M. Fuchs, X. Zhong (2007): A regularity theory for scalar local minimizers of splitting-type variational integrals. Ann. SNS Pisa VI(5), 385-404.

[DLM] A. D'Ottavio, F. Leonetti, C. Mussciano (1998): Maximum prinziple for vector valued mappings minimizing variational integrals. Atti Sem. Mat. Fis. Uni. Modena XLVI, 677-683.

[Gi] M. Giaquinta (1987): Growth conditons and regularity, a counterexample. Manus. Math. 59, 245-248.

[Gi2] M. Giaquinta (1983): Mulitiple integrals in the calcules of variations an nonlinear elliptic systems. Ann. Math. Studies 105, Princeton University Press, Princeton.

[Ho] M. C. Hong (1992): Some remarks on the minimizers of variational integrals with non standard growth conditions. Boll. U.M.I. (7) 6-A, 91-101.

[Ma1] P. Marcellini (1989): Regularity of minimizers of integrals in the calcules of variations with non standard growth condtions. Arch. Rat. Mech. Anal. 105, 267-284.

[Ma2] P. Marcellini (1991): Regularity and existence of elliptic equations with (p,q)-growth conditions. Journal of Differential Equations 90, 1-30.

[MM] U. Massari, M. Miranda (1984): Minimal surfaces of codimension one. North-Holland Math. Stud. 91, North-Holland, Amsterdam.

[St] G. Stampacchia (1965): Le problème de Dirichlet pour les equations elliptiques du second ordre à coefficients dicontinus. Ann Inst. Fourier Grenoble 15.1, 189-258.