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Abstract

In this article we prove regularity results for minimzers $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ of functionals $\int_\Omega \left[(1+|\nabla_1 u|^2)^{\frac{p(x)}{2}} + (1+|\nabla_2 u|^2)^{\frac{q(x)}{2}}\right] dx$, where p and q are Lipschitz-functions and $\nabla u = (\nabla_1 u, \nabla_2 u)$ is an arbitrary decompositon.

1 Introduction

The study of regularity properties for minimizers $u : \Omega \to \mathbb{R}^N$ of energies

$$I[u,\Omega] := \int_\Omega F(\nabla u)\, dx, \tag{1.1}$$

where Ω denotes an open set in $\mathbb{R}^n$ and where $F : \mathbb{R}^{nN} \to [0,\infty)$ satisfies an anisotropic growth condition, i.e.

$$C_1|Z|^{\overline{p}} - c_1 \leq F(Z) \leq C_2|Z|^{\overline{q}} + c_2, \qquad Z \in \mathbb{R}^{nN}$$

with constants $C_1, C_2 > 0$, $c_1, c_2 \geq 0$ and exponents $1 < \overline{p} \leq \overline{q} < \infty$, was pushed by Marcellini (see [Ma1] and [Ma2]). Since the research of Esposito Leonetti and Mingione [ELM] it is known that the statements do not stay true if one allows an additional x-dependence and considers minimizers of functionals

$$J[u,\Omega] := \int_\Omega F(\cdot, \nabla u)\, dx, \tag{1.2}$$

for $F : \Omega \times \mathbb{R}^{nN} \to [0,\infty)$. Already in the autonomous situation it is well-known, that we have no hope for regularity for minimizers of (1.1), if $\overline{p}$ and $\overline{q}$ are too far apart (compare the counterexamples of [Gi] and [Ho]). To get better results additional assumptions are necessary. Therefore Fuchs and Bildhauer consider decomposable integrands, which means we have

$$F(Z) = f(\widetilde{Z}) + g(Z_n)$$

for $Z = (Z_1, ..., Z_n)$ with $Z_i \in \mathbb{R}^N$ and $\widetilde{Z} = (Z_1, ..., Z_{n-1})$ (note that this condition is only an example, we could consider every other decomposition of ∇u into two parts). Bildhauer, Fuchs and Zhong assume power growth conditions for the C^2-functions f and g with exponents $\overline{p} \leq \overline{q}$ and get a very general regularity theory in case $\overline{p} \geq 2$ (see [BF1], [BF2] and [BFZ]). In [Br] we generalize these statements under the assumption

$$f(\widetilde{Z}) = a(|\widetilde{Z}|) \quad \text{and} \quad g(Z_n) = b(|Z_n|)$$

with N-functions a and b. Thereby the main assumptions are (h stands for a or b)

$$\frac{h'(t)}{t} \approx h''(t)$$

and superquadratic growth of h. In [Br2] we extend the results for an x-dependence without severe restrictions. If we want to study the behaviour of minimizers of

$$\mathcal{F}[w] := \int_\Omega \left[(1 + |\widetilde{\nabla} u|^2)^{\frac{p(x)}{2}} + (1 + |\partial_n u|^2)^{\frac{q(x)}{2}} \right] dx \tag{1.3}$$

the functions

$$a(x,t) := \left(1 + t^2\right)^{\frac{p(x)}{2}} - 1 \quad \text{and} \quad b(x,t) := \left(1 + t^2\right)^{\frac{q(x)}{2}} - 1$$

satisfy all conditons assumed in [Br2] (if $p, q \geq 2$) except

$$|\partial_\gamma h^{'}(x,t)| \leq c h^{'}(x,t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+ \tag{1.4}$$

and all $\gamma \in \{1, ..., n\}$ for a constant a $c \geq 0$. Note that (1.4) is the main hypothese to handle the terms involving derivatives with respect to x in [Br2]. Instead of (1.4) we get here

$$|\partial_\gamma h^{'}(x,t)| \leq c(\epsilon)(1 + t^2)^{\frac{\epsilon}{2}} h^{'}(x,t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+ \tag{1.5}$$

and all $\epsilon > 0$. Let us state our new result.

THEOREM 1.1 *Let $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of (1.3) in the class $W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ and $p, q \in W^{1,\infty}_{loc}(\Omega, [2, \infty))$. Then we have*

- *(a) partial $C^{1,\alpha}$-regularity, if $p \leq q < p+2$ on Ω (for $n \geq 5$ we additionally need $p > \|q - p\|_\infty (n-2)/2$);*
- *(b) full $C^{1,\alpha}$-regularity for $n = 2$;*
- *(c) full $C^{1,\alpha}$-regularity for $N = 1$, if $\|p - q\|_\infty < 2$.*

Remark 1.1 • *Results due to minimizers like in 1.1 are not found in literatur. A similar problem is minimizing*

$$\int_\Omega (1 + |\nabla w|^2)^{\frac{p(x)}{2}} dx.$$

Regularity results are stated in [CM].

- *Our result is not restricted to the special integrand in (1.3). We can also consider functions $a, b : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ which satisfy all assumptions from [Br2] except (A5) toghether with (1.5).*

Remark 1.2 • *Let us compare the statements of Theorem 1.1 with the power growth situation: Fuchs and Bildhauer [BF1] proved full regularity for $n = 2$ in the superquadratic situation which we can exactly reproduce. In [BF2] they analyze the general vector case and get partial regularity under the assumptions $p \leq q \leq p + 2$ and $q \leq pn/(n-2)$. The first one is nearly the same as in Theorem 1.1, we can not allow an equality. If we have a look at the second one this corresponds to $p > \|q - p\|_\infty (n-2)/2$ in case of constants p and q but without equality, too. Only the scalar case is a real restriction: In [BFZ] no condition between p and q is needed, but we have to suppose $\|p - q\|_\infty < 2$.*

Remark 1.3 • *If $n = 2$ then we do not have to assume local boundedness of the minimizer. The idea to remove this is outlined in [Bi] (section 4). In 2D it is possible to consider subquadratic problems with restriction between p and q. In this case one can follow the approach of [BF5] and [Br3].*

- *From our proof follows that we do not need superquadratic growth if $N = 1$. We only have to suppose $p > 1$ on Ω. Then the regularized problem (compare Lemma 2.2) has a Lipschitz-solution by [BF4] (Thm. 1.2).*

- *If $n \leq 4$ then we can deduce from $p \geq 2$ and $p \leq q < p+2$ the inequality $p > \|q - p\|_\infty (n-2)/2$.*

2 Proof of Theorem 1.1

Let

$$a(x,t) := \left(1 + t^2\right)^{\frac{p(x)}{2}} - 1 \quad \text{and} \quad b(x,t) := \left(1 + t^2\right)^{\frac{q(x)}{2}} - 1.$$

It is easy to prove that these functions satisfy the assumptions (A1)-(A4) from [Br2] as well as (A9) and (A10). If we define the regularization u_M as there, then we can quote the following results. Thereby h_M stands for the approximation for $h \in \{a, b\}$ and $g(t) := h'(t)/t$.

Lemma 2.1 *For the sequence (h_M) we have:*

(i) $h_M \in C^2(\overline{\Omega} \times [0,\infty))$ *is a* N*-function,* $h_M \geq h_0 > 0$ *uniformly in* M*;*

(ii) $h_M \leq h$ *and* $h''_M \leq c(M)$ *on* $\overline{\Omega} \times \mathbb{R}_0^+$*;*

(iii) we have for positive constants $\overline{\epsilon}, \overline{h}$

$$\overline{\epsilon}\, \frac{h'_M(x,t)}{t} \leq h''_M(x,t) \leq \overline{h}\, \frac{h'_M(x,t)}{t}$$

uniformly in M*;*

(iv) if we have $p \leq q$*, then*

$$a_M(x,t) \leq \overline{c} b_M(x,t) \text{ for all } x \in \overline{\Omega} \text{ and all } t \geq 0;$$

(v) (1.4) extends to h_M *uniformly in* M*:*

$$|\partial_\gamma h'_M(x,t)| \leq c(\epsilon)(1+t^2)^{\frac{\epsilon}{2}} h'_M(x,t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+$$

and all $\gamma \in \{1,...,n\}$*;*

(vi) from $q - p \leq \omega$ *for a positive number* ω *follows*

$$b_M(x,t) \leq c t^\omega a_M(x,t) \text{ uniformly in } M;$$

(vii) h_M *and* h_M^{-1} *satisfy uniform* Δ_2*-conditions, which follows from part (iii);*

(viii) we get from part (iii) and monotonicity of h_M

$$\lambda h'_M(x,t)t \leq h_M(x,t) \leq h'_M(x,t)t \quad \text{uniformly in } M.$$

Only part (v) is not the same as in [Br2], but can be proved similarly to the appropriate version. Now we state the regularity results due to our regularization u_M which minimizes the functional $\int_B F_M(\cdot, \nabla w)\, dx$, where $F_M(x,Z) := a_M(x, |\widetilde{Z}|) + b_M(x, |Z_n|)$ and $B \Subset \Omega$.

Lemma 2.2 *(i)* u_M *belongs to the space* $W^{2,2}_{loc}(B, \mathbb{R}^N)$*;*

(ii) $a_M(\cdot, |\nabla \widetilde{u}_M|)|\widetilde{\nabla} u_M|^2$ *and* $b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2$ *are elements of* $L^1_{loc}(B)$*;*

(iii) if $n = 2$ *or* $N = 1$ *then we have* $u_M \in W^{1,\infty}_{loc}(B, \mathbb{R}^N)$*;*

(iv) for $\gamma \in \{1, ..., n\}$ $\partial_\gamma u_M$ *solves*

$$\int_B D_P^2 F_M(\cdot, \nabla u_M)(\nabla w, \nabla\varphi)\, dx$$
$$+ \int_B \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla\varphi\, dx = 0 \text{ for all } \varphi \in W_0^{1,2}(B, \mathbb{R}^N)$$

with $\operatorname{spt}(\varphi) \Subset B$*;*

(v) u_M *is in* $W^{1,2}(B, \mathbb{R}^N)$ *uniformly bounded and we have*

$$\sup_M \int_B F_M(\cdot, \nabla u_M)\, dx < \infty;$$

(vi) if we have $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$ *then* $\sup_M \|u_M\|_\infty < \infty$*.*

Proof: By construction of F_M we obtain the following growth conditions (compare Lemma 2.2)

$$\lambda|X|^2 \le D_P^2 F_M(x, Z)(X, X) \le \Lambda_M(1 + |Z|^2)^{\frac{\epsilon}{2}}|X|^2,$$
$$|\partial_\gamma D_P F_M(x, Z)| \le \Lambda_M(1 + |Z|^2)^{\frac{1+\epsilon}{2}},$$

for all $X, Z \in \mathbb{R}^{nN}$, all $\gamma \in \{1, .., n\}$ and all $x \in \overline{B}$ for positive constants λ, Λ_M. If we follow the approach of [BF4] (Lemma 2.8 mit $\alpha = 0$) for $p = 2$ and $q = 2 + \epsilon$, we see $\nabla u_M \in L^4_{loc}(B, \mathbb{R}^{nN})$. Note that in case $\alpha = 0$ modulus dependence is not necessary. From the same proof we deduce $u_M \in W^{2,2}_{loc}(B, \mathbb{R}^N)$ and so the first two statements of the Lemma. If we quote [BF4] (Thm. 1.1) then follows $u_M \in W^{1,\infty}_{loc}(B, \mathbb{R}^N)$ for $n = 2$ or $N = 1$ (we can choose ϵ small enough to reach $q < p(n + 1)/n$). By approximation we get part (iv). We can adopt the last two statements from [Br2].

Partial regularity:

Now we have to prove the higher integrability stated in [Br2] (Theorem 1.1) This means we have to show

$$a_M(\cdot, |\nabla\widetilde{u}_M|)|\widetilde{\nabla} u_M|^2, b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2 \in L^1_{loc}(B) \text{ uniformly.} \tag{2.1}$$

If we follow the lines of [Br2] (section2) we get by Young's inequality and Lemma 2.2 (part (v)) on account of (1.5)

$$\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2\, dx \le c(r) + c(r) \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^{2\epsilon}\, dx$$
$$\le c(r, \tau) + \tau \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^2\, dx. \tag{2.2}$$

This is the analogy of inequality (2.5) in [Br2]. Whereas (2.7) of [Br2] now reads as

$$\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^2 \, dx$$
$$\leq c(r) + c(r) \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2 \, dx. \tag{2.3}$$

If we combine (2.2) and (2.3) and choose τ small enough we get (2.1) and can go to the limit.
To modify the blow up-arguments from [Br2] we define on account of (1.5) $\overline{a}(x,t) := a(x,t)t^{\omega+2\epsilon}$. Here we have $\omega := \|p-q\|_\infty < 2$ and we obtain $\omega + 2\epsilon < 2$ for ϵ small enough. This proves the existence of the excess

$$E(x,r) := \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2 \, dy + \fint_{B_r(x)} \overline{a}(\cdot, |\nabla u - (\nabla u)_{x,r}|) \, dy$$

for a small radius r. We have increased $\overline{a}$ in comparison with the version of [Br2] and thereby we can prove the blow up Lemma as in [Br2] in spite of (1.5). In the proof of the strong convergence of the scaled functions we need instead of $a(x,t) \geq \vartheta t^{\frac{\omega}{2}(n-2)}$ the inequality

$$a(x,t) \geq \vartheta t^{\frac{\omega+2\epsilon}{2}(n-2)}.$$

This follows from $p > \|p-q\|_\infty (n-2)/2$ for a suitable choice of ϵ.

Full regularity for $\mathbf{n=2}$:
In [BF5], (2.5), the authors prove an inequality of the form (sum over $\gamma \in \{1,2\}$)

$$\int_{B_r(z)} D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \partial_\gamma \nabla u_M) \, dx$$
$$\leq c(\tau)(R-r)^{-\beta} + \tau \int_{B_R(z)} \left(a_M(\cdot, |\partial_1 u_M|)^2 + b_M(\cdot, |\partial_2 u_M|)^2 \right) \, dx. \tag{2.4}$$

Thereby is $B_r(z) \Subset B_R(z) \Subset B$, $\tau > 0$ arbitrary and $\beta > 0$ a suitable exponent. On account of the x-dependence we have additionally to the terms in [BFt] the integral

$$-\int_{B_R(z)} \eta^2 \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \partial_\gamma \nabla u_M \, dx$$

where η is a suitable cut-off function. Using (1.5) and the splitting-structure we estimate this by

$$c\int_{B_R(z)}\eta^2 a_M'(\cdot,|\partial_1 u_M|)(1+|\partial_1 u_M|^2)^{\frac{\epsilon}{2}}|\partial_\gamma\partial_1 u_M|\,dx$$
$$+c\int_{B_R(z)}\eta^2 b_M'(\cdot,|\partial_2 u_M|)(1+|\partial_2 u_M|^2)^{\frac{\epsilon}{2}}|\partial_\gamma\partial_2 u_M|\,dx.$$

Using Young's inequality we can bound the first term (compare lemma 2.1, part (viii)) through

$$\tau'\int_{B_R(z)}\eta^2\frac{a_M'(\cdot,|\partial_1 u_M|)}{|\partial_1 u_M|}|\partial_\gamma\partial_1 u_M|^2\,dx$$
$$+c(\tau')\int_{B_R(z)}\eta^2 a_M(\cdot,|\partial_1 u_M|)(1+|\partial_1 u_M|^2)^{\epsilon}\,dx.$$

For $\tau'\ll 1$ one can absorb the first integral in the l.h.s. of (2.4). Therefore we use the inequality

$$\frac{a_M'(\cdot,|\widetilde{Z}|)}{|\widetilde{Z}|}|\widetilde{P}|^2\le cD_P^2F_M(x,Z)(P,P)$$

for $Z,P\in\mathbb{R}^{nN}$ (compare Lemma 2.1, part (iii)). For the second one we obtain

$$c(\tau')\int_{B_R(z)}\eta^2 a_M(\cdot,|\partial_1 u_M|)(1+|\partial_1 u_M|^2)^{\epsilon}\,dx$$
$$\le\tau''\int_{B_R(z)}a_M(\cdot,|\partial_1 u_M|)^2\,dx+c(\tau'')\int_{B_R(z)}(1+|\partial_1 u_M|^2)^{2\epsilon}\,dx.$$

We can handle the r.h.s. conveniently, since we can assume $\epsilon\le 1/2$ and receive (compare Lemma 2.2, part (i))

$$\int_{B_R(z)}(1+|\partial_1 u_M|^2)^{2\epsilon}\,dx\le c+\int_{B_R(z)\cap[|\partial_1 u_M|>1]}a_M(\cdot,|\partial_1 u_M|)\,dx\le c.$$

Analogously we can incorporate the term

$$\int_{B_R(z)}\eta^2 b_M'(\cdot,|\partial_2 u_M|)(1+|\partial_2 u_M|^2)^{\frac{\epsilon}{2}}|\partial_\gamma\partial_2 u_M|\,dx$$

in (2.4). In [BF5] we can find the inequality

$$\begin{aligned}&\int_{B_\rho(z)}\left(a_M(\cdot,|\partial_1 u_M|)^2+b_M(\cdot,|\partial_2 u_M|)^2\right)\,dx\\&\quad\le c(R-\rho)^{-2}+c\int_{B_r(z)}D_P^2F_M(\cdot,\nabla u_M)(\partial_\gamma\nabla u_M,\partial_\gamma\nabla u_M)\,dx\end{aligned}\tag{2.5}$$

for $\rho \in (0, R)$ (and $r = (\rho + R)/2$). In our approach we obtain on the r.h.s. of this inequality additionally the term (if we estimate $\nabla_x a_M$ and $\nabla_x b_M$ using (1.5))

$$\left[\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)(1 + |\partial_1 u_M|^2)^{\frac{\epsilon}{2}}\, dx\right]^2 + \left[\int_{B_r(z)} b_M(\cdot, |\partial_2 u_M|)(1 + |\partial_2 u_M|^2)^{\frac{\epsilon}{2}}\, dx\right]^2 .$$

We can handle both terms in a similar way and show the proceeding for the first one. By Hölder's inequality we receive the upper bound

$$Y_M := \left[\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^{s\chi}\, dx\right]^{\frac{2}{\chi}} \times \left[\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^{\frac{\chi - s\chi}{\chi - 1}} (1 + |\partial_1 u_M|^2)^{\frac{\epsilon}{2}\frac{\chi}{\chi-1}}\, dx\right]^{2\frac{\chi-1}{\chi}} .$$

Thereby we have $s \in (0, 1)$ and $\chi \in (1, 2)$ such that $s\chi > 1$. For the second integral Y_M^2 follows by Lemma 2.2 (part (vi))

$$\begin{aligned} Y_M^2 &= \int_{B_r(z)\cap[|\partial_1 u_M| \le 1]} \ldots + \int_{B_r(z)\cap[|\partial_1 u_M| > 1]} \ldots \\ &\le c + \int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)\, dx \le c. \end{aligned}$$

Note thate we have for $t \ge 1$

$$a_M(x, t)^{\frac{\chi - s\chi}{\chi - 1}} (1 + t^2)^{\frac{\epsilon}{2}\frac{\chi}{\chi - 1}} \le c a_M(x, t)$$

for ϵ small enough, since $s\chi > 1$ (remember Lemma 2.1, part (i)). One sees by the inequalities of Jensen and Young

$$\begin{aligned} Y_M &\le c \left[\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^{s\chi}\, dx\right]^{\frac{2}{\chi}} \le c \int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^{2s}\, dx \\ &\le \tau''' \int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^2\, dx + c(\tau'''). \end{aligned}$$

So we have to add

$$\tau''' \int_{B_r(z)} \left(a_M(\cdot, |\partial_1 u_M|)^2 + b_M(\cdot, |\partial_2 u_M|)^2\right)\, dx \qquad (2.6)$$

on the r.h.s. of (2.5). Combining (2.4)-(2.6) we have showed (for a suitable choice of τ and τ''')

$$\int_{B_\rho(z)} \left(a_M(\cdot, |\partial_1 u_M|)^2 + b_M(\cdot, |\partial_2 u_M|)^2\right) \, dx$$
$$\leq c(R-r)^{-\beta} + \frac{1}{2} \int_{B_R(z)} \left(a_M(\cdot, |\partial_1 u_M|)^2 + b_M(\cdot, |\partial_2 u_M|)^2\right) \, dx.$$

Now we get the uniform boundedness of u_M in $W^{2,2}_{loc}(B, \mathbb{R}^N)$ (compare Lemma 2.1, part (i)) and we can reproduce the proof of [BF5] for the rest, whereby the terms which appear additionally on account of (1.5) are uncritical.

Full regularity for N = 1:
In case $N = 1$ it is possible to modify the N-function in [Br2], (4.4). Therefore we need the inequalities

$$b_M(x,t) \leq ct^{2-2\epsilon} a_M(x,t) \quad \text{and} \quad a_M(x,t) \leq ct^{2-2\epsilon} b_M(x,t). \tag{2.7}$$

By Lemma 2.1 (vi) this follows from $\|p - q\|_\infty < 2$ for $\epsilon \ll 1$. So we can separate the mixed integrands of the terms

$$\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) |\Gamma_{n,M}^{\frac{\alpha+2\epsilon}{2}} \, dx \quad \text{and} \quad \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) \widetilde{\Gamma}_M^{\frac{\alpha+2\epsilon}{2}} \, dx,$$

which occur additionally to the integrals in [Br2]. Finally we get instead of [Br2], (4.6),

$$\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\Gamma_{n,M}^{\frac{\alpha+2}{2}} \, dx$$
$$\leq c(\eta) \left[... + \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\Gamma_{n,M}^{\frac{\alpha+2\epsilon}{2}} \, dx + \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \widetilde{\Gamma}_M^{\frac{\alpha+2\epsilon}{2}} \, dx \right]$$

as well as an analogous inequality for $a_M(\cdot, |\widetilde{\nabla} u_M|) \widetilde{\Gamma}_M^{\frac{\alpha+2}{2}}$ instead of [Br2], (4.7). Since we can assume $\epsilon \leq 1/2$ the first integral on the r.h.s. is bounded by (using Young-inequality)

$$\tau \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\Gamma_{n,M}^{\frac{\alpha+2}{2}} \, dx + c(\tau) \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\Gamma_{n,M}^{\frac{\alpha}{2}} \, dx$$

for an arbitrary $\tau > 0$. For the second one we can argue similarly and we obatin

$$\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\Gamma_{n,M}^{\frac{\alpha+2}{2}} \, dx + \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \widetilde{\Gamma}_M^{\frac{\alpha+2}{2}} \, dx$$

$$\leq c(\eta)\left[\int_{\mathrm{spt}(\eta)}\eta^{2k}b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha}{2}}\,dx+\int_{\mathrm{spt}(\eta)}\eta^{2k}a_M(\cdot,|\widetilde{\nabla}u_M|)\widetilde{\Gamma}_M^{\frac{\alpha}{2}}\,dx\right].$$

Now we can iterate as in [Br2] and obtain arbitrary high integrability of ∇u_M uniform in M (the starting point is $\alpha = 0$, see Lemma 2.2, part (v)). This is enough to end up the proof as mentioned there.

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