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THE PRINCIPLES OF THE CALCULUS OF VARIATIONS

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Contents

Glossary

$\overline{\mathbb{R}}$	$= \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$
δ_{ij}	$= \begin{cases} 0 & , \ i \neq j \\ 1 & , \ i = j \end{cases}$ (Kronecker symbol)
$M^{n\times N}$	space of $(n \times N)$–matrices $\cong$ $\mathbb{R}^{nN}$
Du	differential (gradient) of $u : \Omega \to \mathbb{R}^N(\mathbb{R})$
graph u	$= \big\{ \big(x, u(x)\big) : x \in \Omega \big\} \subset \Omega \times \mathbb{R}^N$ for $u : \Omega \to \mathbb{R}^N$
1-graph u	$= \big\{ \big(x, u(x), Du(x)\big) : x \in \Omega \big\} \subset \Omega \times \mathbb{R}^N \times M^{n\times N}$ for $u : \Omega \to \mathbb{R}^N$
a.e.	almost everywhere
l.s.c.	lower semicontinuous

$\mathcal{L}^n$	n–dimensional Lebesgue measure on $\mathbb{R}^n$
$\mathcal{L}^n$ – a.e.	except for some set A with $\mathcal{L}^n(A) = 0$
$L^p(\Omega)$	space of Lebesgue – measurable functions u on Ω with finite L^p–norm $\|u\|_{L^p} = \left(\int_\Omega \|u\|^p d\mathcal{L}^n\right)^{1/p}, \quad 1 \le p < \infty$
$L^\infty(\Omega)$	space of Lebesgue – measurable and essentially bounded functions u on Ω with norm $\|u\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\Omega} \|u(x)\|$
spt u	$= \operatorname{clos}\{x \in \Omega : u(x) \neq 0\}$ (support of a function u on Ω)
$C^1(\Omega)$	space of continuously differentiable functions on Ω
$C_c^\infty(\Omega)$	space of smooth (= arbitrarily often differentiable) functions u on Ω such that spt u is a compact subset of Ω
$\mathrm{BV}(\Omega)$	subspace of $L^1(\Omega)$ – functions of bounded variation
$H^{m,p}(\Omega)$	Sobolev space of functions $u \in L^p(\Omega)$ whose distributional derivatives of order up to m also belong to $L^p(\Omega)$ with norm $\|u\|_{H^{m,p}} = \sum_{k=0}^{m} \sum_{\|\alpha\|=k} \|D^\alpha u\|_{L^p}$
$H_0^{m,p}(\Omega)$	completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^{m,p}}$.

Summary

Assuming the existence of a classical solution for a variational integral one derives a system of second order differential equations, the Euler–Lagrange equations, which necessarily have to be satisfied. To ensure the existence of, for example, a minimizer one uses the direct method of the calculus of variations. This produces a generalized solution and therefore the question of regularity arises. Here, the convexity of the variational integral is an important feature. One method to tackle non–convex problems is the theory of Γ–convergence. Topological conditions enter when investigating unstable critical points.

1. Introduction

As *Giaquinta* and *Hildebrandt* write in the introduction to the first volume of their treatise: "*The Calculus of Variations is the art to find optimal solutions and to describe their essential properties.*" Examples from daily life are: which object has some property to a highest or lowest degree, or what is the optimal strategy to reach some goal. The *Isoperimetric Problem*, already considered in antiquity, is one such question: *Among all possible closed curves of a given length, find those for which the area of the enclosed inner region is maximal.* A property shared by such optimum problems consists in the fact that, usually, they are easy to formulate and to understand, but much less easy to solve.
The principle of economy of means: "*What you can do, you can do simply*" is an idea that dominates many of our everyday actions as well as the most sophisticated inventions or scientific theories. Therefore, it should come as no surprise that this idea was extended to the area of natural phenomena. As *Newton* wrote in his *Principia*: "*Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity and affects not the pomp of superfluous causes.*" Similarly, in the first treatise on the Calculus of Variations, his *Methodus inveniendi* from 1744, *Euler* wrote: "*Because the shape of the whole universe is most perfect and, in fact, designed by the wisest creator, nothing in all of the world will occur in which no maximum or minimum rule is somehow shining forth.*" And, even in the rational world of today's science where apparently no metaphysics is involved, there remains

the fact that many if not all laws of nature can be given the form of an extremal principle.
Apart from this introduction, this article is divided into three sections. The first one, *Classical Theory*, roughly covers the time from *Euler* to the end of 19th century and is concerned with so called *Indirect Methods*. The next section describes the relevant ideas developed during the last 100 years and is entitled *Direct Methods*. An important ingredient here is the introduction of functional analytic techniques. In fact, it was the Calculus of Variations that gave birth to the theory of Functional Analysis. The third and final, extremely short, section bears the title *Unstable Critical Points*, and is concerned with equilibrium solutions which are no longer extrema. Here, an important rôle is played by topological methods.
In this overview of the *Principles of the Calculus of Variations* it was of course neither intended nor possible to cover all the important contributions to the subject. However, in the list of references I have tried to include some of these. In the material presented here I necessarily had to restrict myself to exemplary model cases. For that reason, for example, variational integrals depending on higher derivatives or variational problems with subsidiary conditions are not included.

2. Classical Theory

Compared to the developments in the 20th century which will be the topic of section 3, this part of the calculus of variations could also be called *"Indirect Methods"*. The underlying idea is the following: Suppose you know that a solution to a variational problem (e.g. a minimum) exists. What can you say about such a solution? Which equation(s) does it satisfy? Which properties (e.g. symmetry) of the corresponding variational functional does it inherit?

2.1. The finite dimensional case

First, let us have a look at the finite dimensional situation. Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}$ a smooth function. Suppose f has a *local minimum* at a point $x_0 \in \Omega$, i.e. there is a ball $B_r(x_0) \subset \Omega$, $r > 0$, such that

$$f(x) \geq f(x_0) \text{ for any } x \in B_r(x_0).$$

Then, at such a point $x_0 \in \Omega$ we have

$$Df(x_0) = \text{grad } f(x_0) = 0 \tag{1}$$

where grad $f(x_0) \in \mathbb{R}^n$ is the vector whose components are the partial derivatives of f at x_0.
A point $x_0 \in \Omega$ satisfying (1) is called a *critical point* of f.
Furthermore, using second derivatives, we have:

(a) If x_0 is a *minimal point* of f then $D^2 f(x_0) \geq 0$, i.e. the symmetric matrix of second partial derivatives is *positive semidefinite*.

(b) Suppose, x_0 is a *critical point* of f and furthermore that $D^2 f(x_0) > 0$ (*positive definite*) then x_0 is a *minimal point* of f.

2.2. One–Dimensional Variational Integrals

Let us now turn to the *calculus of variations*. We start with one–dimensional integrals, that is we consider functionals $\mathcal{F}$ of the form

$$\mathcal{F}[u] = \int_I F\big(x, u(x), u'(x)\big)\,dx. \tag{2}$$

Such functionals are called *variational integrals*. Here, $I \subset \mathbb{R}$ is an interval (in general I will be bounded), $F : \overline{I} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is called the *Lagrange function* (we write $F = F(x, z, p)$), and $u : \overline{I} \to \mathbb{R}^N$ is supposed to be smooth. More generally, it suffices to consider the case $F \in C^1(U)$ with $U \subset \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ an open set such that $\{\big(x, u(x), u'(x)\big) : x \in \overline{I}\} \subset U$. In this case, $\mathcal{F}[v]$ is defined for any $v \in C^1(\overline{I}, \mathbb{R}^N)$ provided $\|v - u\|_{C^1(\overline{I})} < \delta$ for $\delta > 0$ sufficiently small.
Thus, for an arbitrary function $\varphi \in C^1(\overline{I}, \mathbb{R}^N)$ we see that

$$\Phi(\varepsilon) := \mathcal{F}[u + \varepsilon\varphi] \tag{3}$$

is defined as soon as $|\varepsilon| < \varepsilon_0 := \delta / \|\varphi\|_{C^1(\overline{I})}$. We get $\Phi \in C^1(-\varepsilon_0, \varepsilon_0)$ and easily compute

$$\Phi'(0) = \int_I \{F_z(x, u, u') \cdot \varphi + F_p(x, u, u') \cdot \varphi'\} dx. \tag{4}$$

In the following we call $\delta\mathcal{F}[u, \varphi] := \Phi'(0)$ the *First Variation of $\mathcal{F}$ at u in direction φ*. From (4) we deduce that $\delta\mathcal{F}[u, \varphi]$ is – with respect to φ – a *linear functional* on $C^1(\overline{I}, \mathbb{R}^N)$.

Definition
A function $u \in C^1(I, \mathbb{R}^N)$ satisfying

$$\int_I \{F_z(x, u, u') \cdot \varphi + F_p(x, u, u') \cdot \varphi'\} dx = 0 \tag{5}$$

for any $\varphi \in C_c^\infty(I, \mathbb{R}^N)$ is called a *weak C^1–extremal* of $\mathcal{F}$. Note, that for $u \in C^1(\overline{I}, \mathbb{R}^N)$ we have that (5) is equivalent to the fact that $\delta\mathcal{F}[u, \cdot] \equiv 0$ on $C_c^\infty(I, \mathbb{R}^N)$.

□

With the above definition in mind we have the following first model result:

Theorem 1
Suppose, that $u \in C^1(\overline{I}, \mathbb{R}^N)$ is a *weak minimizer* of $\mathcal{F}$, that is

$$\mathcal{F}[u] \le \mathcal{F}[u + \varphi] \tag{6}$$

for any $\varphi \in C_c^\infty(I, \mathbb{R}^N)$ such that $\|\varphi\|_{C^1(I)} \le \delta$ for some $\delta \in (0, 1)$.
Then, u is a weak C^1 – extremal of $\mathcal{F}$.

□

For the following considerations we assume that u and F are at least of class C^2.
A partial integration in (5) then implies $\big(I = (a, b)\big)$

$$0 = \int_a^b \{F_z\big(x, u(x), u'(x)\big) - \frac{d}{dx}[F_p\big(x, u(x), u'(x)\big)]\} \cdot \varphi(x) dx \tag{7}$$

for any $\varphi \in C_c^\infty(I, \mathbb{R}^N)$.
We now need the so called

Fundamental Lemma (of the Calculus of Variations)
If $h \in C^0(I, \mathbb{R}^N)$ is such that for any $\varphi \in C_c^\infty(I, \mathbb{R}^N)$ we have

$$\int_a^b h(x) \cdot \varphi(x) dx = 0, \tag{8}$$

then $h \equiv 0$ on $I = (a, b)$.

□

Because of the importance of this result in the calculus of variations we present the simple
Proof
We argue by contradiction, that is we assume there exists $i_0 \in \{1, \ldots, N\}$ and $x_0 \in (a, b)$ such that $h^{i_0}(x_0) \neq 0$. The *continuity* of h then yields the existence of some number $\delta > 0$ with $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ such that

$$|h^{i_0}(x)| > \frac{1}{2}|h^{i_0}(x_0)| \quad \text{for } |x - x_0| < \delta.$$

Now, choose $\eta \in C_c^\infty(I, \mathbb{R}^N)$ in such a way that

$$\begin{aligned} &\eta^{i_0}(x) \equiv 0 \quad \text{for} \quad |x - x_0| \geq \delta, \\ &\eta^{i_0}(x) > 0 \quad \text{for} \quad |x - x_0| < \delta, \\ &\eta^i(x) \equiv 0 \quad \text{for} \quad i \neq i_0. \end{aligned}$$

Finally, define φ by $\varphi(x) := h^{i_0}(x_0)\eta(x)$ so that from (8) we get

$$\begin{aligned} 0 = \int_a^b h(x) \cdot \varphi(x) dx &= \int_{x_0-\delta}^{x_0+\delta} h^{i_0}(x) h^{i_0}(x_0)\eta(x) dx > \\ &> \tfrac{1}{2}|h^{i_0}(x_0)|^2 \int_{x_0-\delta}^{x_0+\delta} \eta(x) dx > 0. \end{aligned}$$

This is a *contradiction* and concludes the proof.

□

Remark
An important *generalization* of the fundamental lemma reads as follows: *Suppose* $h \in L^1(I, \mathbb{R}^N)$ *(instead of* $C^0(I, \mathbb{R}^N)$*) satisfies* (7), *then* $h(x) = 0$ *for* $\mathcal{L}^1$–a.e. $x \in I$. The proof is similar and uses the fact that $C_c^\infty(I, \mathbb{R}^N)$ is dense in $L^2(I, \mathbb{R}^N)$.

□

As a consequence of the fundamental lemma we get

Theorem 2
Suppose $u \in C^2(I, \mathbb{R}^N)$ is a weak extremal of $\mathcal{F}$ and that F is of class $C^2(U)$ where U is an open set containing the 1–graph of u. We then have

$$\frac{d}{dx}[F_p(x, u(x), u'(x))] - F_z(x, u(x), u'(x)) = 0 \quad \text{on } I. \tag{9}$$

□

Remark
Note, that (9) is a *system of ordinary differential equations*, the so called *Euler–Lagrange equations:*

$$\tfrac{d}{dx}[F_{p^i}\big(x,u(x),u'(x)\big)] - F_{z^i}\big(x,u(x),u'(x)\big) = 0, \quad i=1,\dots,N. \tag{9^i}$$

To be more precise, we get a system of N quasilinear ordinary differential equations of second order for the N unknown functions $u^1,\dots,u^N$.

□

At this point let us discuss several
Examples

1. The Lagrange function $F(x,z,p) = \omega(x,z)\sqrt{1+|p|^2}$ with $N=1$ and $\omega > 0$ leads to the variational integral

$$\mathcal{F}[u] = \int\limits_a^b \omega(x,u)\sqrt{1+(u')^2}dx$$

and the Euler–Lagrange equation

$$\frac{d}{dx}\left[\omega(x,u)\frac{u'}{\sqrt{1+(u')^2}}\right] - \omega_z(x,u)\sqrt{1+(u')^2} = 0.$$

This can be written as

$$\kappa\omega\sqrt{1+(u')^2} = \omega_z - u'\omega_x$$

where

$$\kappa := \frac{d}{dx}\left[\frac{u'}{\sqrt{1+(u')^2}}\right]$$

is the *curvature* of the curve graph $u \subset \mathbb{R}^2$.
In case $\omega \equiv 1$ the variational integral $\mathcal{F}$ is just the *length* of graph u and we get $\kappa \equiv 0$, i.e. $u'' \equiv 0$. Thus, the weak extremals of class C^2 of the length functional are the (affine) linear functions $u(x) = \alpha x + \beta$ $(\alpha, \beta \in \mathbb{R})$.

2. The choice $F(x,z,p) = F(p) = \frac{1}{2}|p|^2 = \frac{1}{2}\sum\limits_{i=1}^{N}|p^i|^2$, $N \geq 1$, leads to *Dirichlet's integral:*

$$\mathcal{D}[u] = \frac{1}{2}\int\limits_a^b |u'|^2 dx$$

and the Euler–Lagrange equations

$$\frac{d}{dx}(u^i)' = (u^i)'' = 0 \quad i=1,\dots,N.$$

Again, we identify the extremals as the affine linear functions.

3. A *classical problem* in the calculus of variations is the so called *brachistochrone problem* first formulated by Galileo in 1638:

Find a curve, connecting two given points A and B, on which a point mass moves without friction under the influence of gravity in the least possible time from the initial point A to the end point B below A.

Galileo believed the optimal curve to be a circular arc. However, this is wrong and the correct solution was finally found by Johann Bernoulli in 1697:
Suppose, that in a Cartesian coordinate system with gravity acting in direction of the negative y–axis, $A = (x_1, y_1)$, $B = (x_2, y_2)$, $x_1 < x_2$, $y_1 > y_2$. Then, for a function $u : [x_1, x_2] \to \mathbb{R}$ with $u(x_i) = y_i$ and $u(x) < y_1$ for $x \in (x_1, x_2]$, the time needed by the point mass to slide from A to B along the graph of u, starting at A with zero velocity, is given by the quantity

$$\mathcal{F}[u] = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + |u'(x)|^2}{y_1 - u(x)}}\, dx$$

where g denotes the acceleration due to gravity.
The *solution* turns out to be a *cycloid*, which in parametric form can be given as

$$\left\{ \begin{array}{rcl} x(t) & = & x_1 + k(t - \sin t), \\ u(t) & = & y_1 - k(1 - \cos t). \end{array} \right. \quad t \in [0, T]$$

Here, the constants k and T are determined by the conditions $x(T) = x_2$ and $u(T) = y_2$.
$\square$

In addition to the Euler–Lagrange equations there are further conditions for a minimum (compare the beginning of this chapter).
The *necessary Legendre – condition*:

$$\sum_{i,k=1}^{N} F_{p^i p^k}\big(x, u(x), u'(x)\big)\xi^i \xi^k \geq 0 \tag{10}$$

for any vector $\xi \in \mathbb{R}^N$ and every $x \in \overline{I}$. (Follows from $\Phi''(0) \geq 0$.)
The *sufficient Legendre – condition*:
There is a number $m > 0$ such that

$$\sum_{i,k=1}^{N} F_{p^i p^k}(x, z, p)\xi^i \xi^k \geq m|\xi|^2 \tag{11}$$

for any vector $\xi \in \mathbb{R}^N$ and every $(x, z, p) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

$\square$

2.3. Multiple Integrals

We now turn to multiple integrals in the calculus of variations. Again, we start with necessary conditions.
Suppose that $\Omega \subset \mathbb{R}^n$ is open (and in most cases bounded), $u : \Omega \to \mathbb{R}^N$ is a smooth function, and that

$$f = f(x, z, p) : \Omega \times \mathbb{R}^N \times M^{n \times N} \to \mathbb{R}$$

is a Lagrange function. Thus, the corresponding variational integral is given by

$$I[u] := \int_\Omega f\big(x, u(x), Du(x)\big) d\mathcal{L}^n(x).$$

For $\varphi \in C_c^1(\Omega, \mathbb{R}^N)$ und $\varepsilon > 0$ sufficiently small let

$$\Phi(\varepsilon) := I[u + \varepsilon\varphi]$$

and denote by

$$\delta I[u, \varphi] := \Phi'(0)$$

as before the *First Variation* of I at u in direction φ. A calculation yields

$$\begin{aligned} \Phi'(0) &= \int_\Omega \sum_{i=1}^N [f_{z^i}(x, u, Du)\varphi^i + \sum_{\alpha=1}^n f_{p_\alpha^i}(x, u, Du)\varphi^i_{x^\alpha}] d\mathcal{L}^n(x) \\ &= \int_\Omega \sum_{i=1}^N \{f_{z^i}(x, u, Du) - \sum_{\alpha=1}^n \frac{d}{dx^\alpha}[f_{p_\alpha^i}(x, u, Du)]\}\varphi^i d\mathcal{L}^n(x), \end{aligned} \tag{12}$$

where we used partial integration.
Again, we have a corresponding *fundamental lemma* and are thus led to the *system of Euler–Lagrange equations* for a weak extremal of the variational integral I:
For $i \in \{1, \dots, N\}$ we have

$$\sum_{\alpha=1}^m \frac{d}{dx^\alpha}[f_{p_\alpha^i}\big(x, u(x), Du(x)\big)] - f_{z^i}\big(x, u(x), Du(x)\big) = 0.$$

Differentiating, we arrive at

$$\left\{ \begin{aligned} &\sum_{\alpha=1}^n [f_{p_\alpha^i x^\alpha}(x, u, Du) + \sum_{j=1}^N f_{p_\alpha^i z^j}(x, u, Du) D_\alpha u^j + \\ &+ \sum_{j=1}^N \sum_{\beta=1}^n f_{p_\alpha^i p_\beta^j}(x, u, Du) D_\alpha D_\beta u^j] - f_{z^i}(x, u, Du) = 0. \end{aligned} \right. \tag{13}$$

Remark
The most important term in this expression is $f_{p_\alpha^i p_\beta^j}(x, u, Du)$.

$\square$

Here are two
Examples

1. For $n \geq 1$ and $N = 1$ the Lagrange – function

$$f(x, z, p) = f(p) = \frac{1}{2}|p|^2$$

again leads to *Dirichlet's integral*:

$$\mathcal{D}[u] = \frac{1}{2} \int_\Omega |Du|^2 d\mathcal{L}^n.$$

Because $f_{p_\alpha} = p_\alpha$ we get $f_{p_\alpha p_\beta} = \delta_{\alpha\beta}$ and therefore the Euler – Lagrange equation is

$$\sum_{\alpha=1}^{n} D_\alpha D_\alpha u =: \Delta u = 0. \tag{14}$$

We may conclude that the extremals of Dirichlet's integral are the *harmonic functions.*

2. If we choose

$$f(x, z, p) = f(p) = \sqrt{1 + |p|^2}$$

for $n \geq 1$ and $N = 1$ we are led to the *area integral*

$$\mathcal{A}[u] = \int_\Omega \sqrt{1 + |Du|^2} d\mathcal{L}^n,$$

that is $\mathcal{A}[u]$ = area of graph u. From $f_{p_\alpha} = \frac{p_\alpha}{\sqrt{1+|p|^2}}$ we get the Euler–Lagrange equation

$$\sum_{\alpha=1}^{n} \frac{d}{dx^\alpha}\left(\frac{D_\alpha u}{\sqrt{1+|Du|^2}}\right) = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0. \tag{15}$$

Equation (15) is the famous *minimal surface equation.*

$\square$

The analogue to the necessary Legendre–condition (10) in the one–dimensional case is now called the *Legendre – Hadamard condition*:

$$\sum_{i,k=1}^{N} \sum_{\alpha,\beta=1}^{n} f_{p^i_\alpha p^k_\beta}\big(x, u(x), Du(x)\big)\xi^i \xi^k \eta_\alpha \eta_\beta \geq 0 \tag{16}$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^n$. Again, (16) is necessary for u to be a minimum of the variational integral associated with f. Alternatively, (16) can equivalently be written as

$$\sum_{i,k=1}^{N} \sum_{\alpha,\beta=1}^{n} f_{p^i_\alpha p^k_\beta}\big(x, u(x), Du(x)\big)\ \pi^i_\alpha \pi^k_\beta \geq 0 \tag{16'}$$

for any $x \in \Omega$ and every matrix $\pi = (\pi^i_\alpha)^{i=1,\dots,N}_{\alpha=1,\dots,n}$ of rank one. Note, that π has rank one if and only if $\pi = \xi \otimes \eta$, i.e. $\pi^i_\alpha = \xi^i \eta_\alpha$. The *strict* or *sufficient Legendre – Hadamard condition* reads

$$\frac{1}{2}\sum_{i,k=1}^{N} \sum_{\alpha,\beta=1}^{n} f_{p^i_\alpha p^k_\beta}\big(x, u(x), Du(x)\big)\xi^i \xi^k \eta_\alpha \eta_\beta \geq \lambda |\xi|^2 |\eta|^2 \tag{17}$$

for some number $\lambda > 0$. In this case, the Lagrange function f is said to be *strongly elliptic on the function* $u : \Omega \to \mathbb{R}^N$.
We call $f = f(x, z, p)$ *(uniformly) strongly elliptic* if

$$\frac{1}{2}\sum_{i,k=1}^{N} \sum_{\alpha,\beta=1}^{n} f_{p^i_\alpha p^k_\beta}(x, z, p)\xi^i \xi^k \eta_\alpha \eta_\beta \geq \lambda |\xi|^2 |\eta|^2 \tag{18}$$

holds for some $\lambda > 0$ and for arbitrary x, z, p, ξ, η.
In contrast, $f(x,z,p)$ is called *(uniformly) superelliptic* provided

$$\frac{1}{2}\sum_{i,k=1}^{N}\sum_{\alpha,\beta=1}^{n} f_{p^i_\alpha p^k_\beta}(x,z,p)\pi^i_\alpha\pi^k_\beta \geq \lambda|\pi|^2 \tag{19}$$

holds for *any* matrix $\pi \in M^{n\times N}$. Note, that (19) is equivalent to the fact that $M^{n\times N} \ni p \mapsto f(x,z,p)$ is a *convex* function.

□

3. Direct Methods

Suppose, you want to find a harmonic function u with given boundary values f, that is you want to solve the *Dirichlet problem*, let us say on a ball $B \subset \mathbb{R}^n$:

$$\begin{cases} \Delta u = 0 & \text{in} \quad B, \\ u = f & \text{on} \quad \partial B. \end{cases} \tag{20}$$

Using the fact that $\Delta u = 0$ is the Euler–Lagrange equation for Dirichlet's integral $\mathcal{D}[u]$, you might try to minimize $\mathcal{D}[\cdot]$ in a suitable class of functions. This approach is known as *Dirichlet's principle*. However, as *Weierstrass* pointed out at the end of the 19th century this is not always possible because you first have to ensure that a minimum actually exists.
Furthermore, the fact that you have to work on infinite dimensional spaces in general implies that the functionals you consider are no longer continuous. This was first pointed out by *Lebesgue* at the beginning of the 20th century who observed that the area functional is not continuous but merely semicontinuous with respect to uniform convergence of surfaces. A one–dimensional analogue is a sequence of zig–zag curves $c_k(t)$.
You start with the curve

$$c(t) = \frac{1}{2} - |t - \frac{1}{2}|, \quad t \in [0,1]$$

which is extended periodically to all of $\mathbb{R}$. Now, define c_k by

$$c_k(t) = 2^{-k}c(2^k t) \quad \text{for } t \in [0,1] \text{ and } k \in \mathbb{N}_0.$$

We then have

$$\mathcal{L}(c_k) = \sqrt{2}$$

while their uniform limit is $c(t) = (t, 0)$, $t \in [0,1]$, so that

$$1 = \mathcal{L}(c) < \liminf_{k\to\infty} \mathcal{L}(c_k) = \sqrt{2}.$$

3.1. Tonelli's Program

It was *Tonelli* who noted that the *Arzela–Ascoli compactness* result and *Baire's* concept of *lower semicontinuity* can be generalized from real functions (of one or several variables) to variational integrals. We describe his approach:

Tonelli's method ($\sim$ 1911)

Aim: A functional $\mathcal{F}[u]$ defined on a non–empty class $\mathcal{C}$ (of functions) has an absolute minimum on $\mathcal{C}$.

Step 1: Prove that $\mathcal{F}$ is *bounded from below* on $\mathcal{C}$, that is

$$\mathcal{F}[u] \geq c_0 \quad \text{for any } u \in \mathcal{C}.$$

This implies that

$$\inf\{\mathcal{F}[u] : u \in \mathcal{C}\} \in \mathbb{R}.$$

Step 2: Show that $\mathcal{F}$ is *weakly lower semicontinuous* with respect to a suitable kind of sequential convergence, that is:
If $u_k \to u$ in $\mathcal{C}$ then $\mathcal{F}[u] \leq \liminf\limits_{k\to\infty} \mathcal{F}[u_k]$.

Step 3: Verify the *compactness* of $\mathcal{C}$ with respect to this kind of weak convergence:
For every sequence $\{u_k\} \subset \mathcal{C}$ there exists a subsequence $\{u_{k_\ell}\}$ and $u_0 \in \mathcal{C}$ such that $u_{k_\ell} \to u_0$ in $\mathcal{C}$.
Alternatively:
Show that $\mathcal{C}$ contains at least one *convergent minimizing sequence*, that is:
There is $\{u_k\} \subset \mathcal{C}$ and $u_0 \in \mathcal{C}$ such that $u_k \to u_0$ and $\lim\limits_{k\to\infty} \mathcal{F}[u_k] = \inf\limits_{\mathcal{C}} \mathcal{F}$.

Result: Combining the three steps we get:

$$\inf_{\mathcal{C}} \mathcal{F} \leq \mathcal{F}[u_0] \leq \liminf_{k\to\infty} \mathcal{F}[u_k] = \inf_{\mathcal{C}} \mathcal{F},$$

that is

$$\mathcal{F}[u_0] = \inf_{\mathcal{C}} \mathcal{F}$$

so that u_0 is the desired absolute minimizer of $\mathcal{F}$ on $\mathcal{C}$.

$\square$

To be more precise, Tonelli assumed that the Lagrange function $F(x, z, p)$ of the variational integral

$$\mathcal{F}[u] = \int_a^b F\big(x, u(x), u'(x)\big)\,dx, \quad [a, b] \subset \mathbb{R},$$

satisfies the conditions

$$F_{pp}(x, z, p) \geq 0 \quad (\textit{convexity})$$

and

$$F(x, z, p) \geq c_0|p|^m - c_1,\ m > 1,\ c_0 > 0,\ c_1 \geq 0 \quad (\textit{coercivity}).$$

For the class $\mathcal{C}$ of competing functions he then chose a suitable subset – determined by the boundary conditions – of the space of *absolutely continuous functions* defined on the interval $I = [a, b]$. The notion of convergence he employed was *uniform convergence.*
We get the following

General Existence Theorem
Let $\mathcal{F} : \mathcal{C} \to \mathbb{R} \cup \{+\infty\}$ be a functional defined on a non–empty set $\mathcal{C}$ equipped with such a notion of convergence that $\mathcal{C}$ is sequentially compact and such that $\mathcal{F}$ is lower semicontinuous. In this case there exists a minimizer of $\mathcal{F}$ on $\mathcal{C}$, that is there exists $u_0 \in \mathcal{C}$ such that

$$\mathcal{F}[u_0] = \inf_{\mathcal{C}} \mathcal{F}.$$

$\square$

Examples

1. Minimize *Dirichlet's integral*

$$\mathcal{D}[u] = \frac{1}{2} \int_0^1 |u'(x)|^2 dx$$

within the class

$$K_1 := \{u \in C^0(\overline{I}) \cap C^1(I) : u(0) = \alpha,\ u(1) = \beta\}$$

where $\alpha, \beta \in \mathbb{R}$ and $I = (0,1)$.

2. Minimize the *length* of the *image* or of the *graph* of mappings $u : [0,1] \to \mathbb{R}$ with prescribed values at 0 and 1:

$$\mathcal{L}[u] = \int_0^1 |u'| dx \quad \text{or} \quad \mathcal{A}[u] = \int_0^1 \sqrt{1 + (u')^2}\, dx$$

in K_1 or in $K_2 = \{u \in C^0(\overline{I}) \cap D^1(\overline{I}) : u(0) = \alpha,\ u(1) = \beta\}$. Here, D^1 denotes the class of piecewise C^1–functions.

3. Minimize the *length* of the *image* or of the *graph* of mappings from $[0,1]$ *onto* $S^1 \subset \mathbb{R}^2$ which map 0 and 1 to $(1,0) \in S^1$. Here, $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the unit circle in $\mathbb{R}^2$ centered at the origin. In this case, the class K_3 is given by

$$K_3 = \{u \in C^0(\overline{I}, \mathbb{R}^2) \cap D^1(\overline{I}, \mathbb{R}^2) : u(\overline{I}) = S^1,\ u(0) = u(1) = (1,0)\}.$$

$\square$

Remark
In general $\mathcal{F}$–bounded subset of the classes K_i will not be sequentially compact with respect to C^1–convergence. Thus, one needs a *weaker* notion of convergence τ; but then the K_i will no longer be τ*–complete.*
We therefore have *to complete* the classes $K = K_i$ with respect to τ, that is we are forced to work in a class $K_{(\tau)}$ of *generalized functions* (no longer continuous, not classically differentiable, ...). Furthermore, we have to extend our fundtional $\mathcal{F}$ to a new functional $\mathcal{F}_{(\tau)}$ defined on $K_{(\tau)}$. In general, there will be several ways to achieve this. How should one proceed?
To summarize, we are faced with the following problem: If we want to use direct methods we are forced to work in classes of generalized functions, and thus to accept minimizers which may not be smooth.

3.2. Hilbert's Problems

This was already pointed out by Hilbert in his celebrated lecture at the International Congress of Mathematicians, held in Paris in 1900. Hilbert's 20th problem, stated at this Congress reads as follows:

> *Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also if need be that the notion of solution shall be suitably extended?*

Let us return to the Examples above.

In example 2 for $\alpha = 0$, $\beta = 1$ and

$$u_k(x) := \begin{cases} 0 & , \quad 0 \le x \le \frac{1}{2} - \frac{1}{k} \\ \frac{k}{2}(x + \frac{1}{k} - \frac{1}{2}) & , \quad \frac{1}{2} - \frac{1}{k} \le x \le \frac{1}{2} + \frac{1}{k} \\ 1 & , \quad \frac{1}{2} + \frac{1}{k} \le x \le 1, \end{cases}$$

we have $u_k \in K_2$.

Obviously,

$$\sup_k \mathcal{L}[u_k] < \infty, \qquad \sup_k \mathcal{A}[u_k] < \infty,$$

and

$$\lim_{k\to\infty} u_k(x) =: u(x) = \begin{cases} 0 & , \quad 0 \le x < \frac{1}{2} \\ \frac{1}{2} & , \quad x = \frac{1}{2} \\ 1 & , \quad \frac{1}{2} < x \le 1. \end{cases}$$

For τ = "pointwise convergence" we then get $u \in K_{2(\tau)}$; but for the pointwise derivative $u' = 0$ (which exists on $[0,1] - \{1/2\}$) we have

$$\mathcal{L}[u] = 0.$$

On the other hand, for arbitrary $v \in K_2$ we get $\mathcal{L}[v] \ge 1$.
In example 3 we have $u_k \in K_3$, where

$$u_k(x) = \begin{cases} (1,0) & , \quad 0 \le x \le \frac{1}{2} \\ \big(\cos 2\pi k(x - \frac{1}{2}), \sin 2\pi k(x - \frac{1}{2})\big) & , \quad \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{k} \\ (1,0) & , \quad \frac{1}{2} + \frac{1}{k} \le x \le 1. \end{cases}$$

The *weak limit* of the u_k is the *constant* function $u(x) \equiv (1,0)$ so that $\mathcal{L}[u] = 0$. On the other hand for $v \in K_3$ we have

$$\mathcal{L}[v] \ge 2\pi. \qquad \square$$

We turn to the problem of *extending* the given functional $\mathcal{F}$ to a new functional $\mathcal{F}_{(\tau)}$.
For simplicity, let us assume that $\mathcal{F}$ already is τ–lower semicontinuous on K.
We are not looking for *any* extension of $\mathcal{F}$ but for the *best one*, that is we are interested in

the *largest lower semicontinuous extension* of $\mathcal{F}$ on $K_{(\tau)}$. (That is what Lebesgue did for the area functional.)
The so–called *τ–relaxation* of $\mathcal{F}$ is given by

$$\mathcal{F}_{(\tau)}[u] := \inf\{\liminf_{k\to\infty} \mathcal{F}[u_k] : K \ni u_k \xrightarrow{\tau} u\}.$$

Immediately, we are faced with the following *problems*: Is $\mathcal{F}_{(\tau)}$ a variational integral? If so, how to calculate the corresponding Lagrange function $F_{(\tau)}(x,z,p)$? This is certainly necessary if, for example, we want to work with the Euler–Lagrange equations.
Assuming that the direct method can indeed be applied to $K_{(\tau)}$ and $\mathcal{F}_{(\tau)}$ we are forced to compare the *relaxed problem*: "$\mathcal{F}_{(\tau)} \to$ min in $K_{(\tau)}$" with the *original minimum problem*: "$\mathcal{F} \to$ min in K". This leads to the *regularity problem* for minimizers, a question which also was addressed by Hilbert in 1900. Hilbert's 19th problem reads as follows:

> *Are the solutions of regular problems in the calculus of variations always necessarily regular?*

As an *example* for an existence theorem we have

Theorem 1
Suppose $\Omega \subset \mathbb{R}^n$ is open, and that $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumptions.

(i) $f(\cdot, p)$ is measurable for any $p \in \mathbb{R}^n$;

(ii) $f(x, \cdot)$ is convex for almost every $x \in \Omega$;

(iii) $f(x,p) \geq b|p|^m - a(x)$ for almost all $x \in \Omega$, any $p \in \mathbb{R}^n$, where $a \in L^1(\Omega)$, $b > 0$, and $m > 1$.

Furthermore, let $g \in H^{1,p}(\Omega)$ and $A := g + H_0^{1,p}(\Omega)$.
Then, there exists an absolute minimizer of

$$\mathcal{F}[u] = \int_\Omega f\big(x, Du(x)\big)dx$$

on the class A, that is there is $u_0 \in A$ such that

$$\mathcal{F}[u_0] = \inf_{u\in A} \mathcal{F}[u].$$

$\square$

For Dirichlet's integral ($m = 2$, $b = \frac{1}{2}$, $a \equiv 0$)

$$\mathcal{D}[u] = \frac{1}{2}\int_\Omega |Du|^2 d\mathcal{L}^n$$

Theorem 1 provides us with a minimizer u_0 within the class $g + H_0^{1,2}(\Omega)$ satisying the following Euler–Lagrange equation

$$\int_\Omega Du_0(x) \cdot D\varphi(x)dx = 0 \quad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{21}$$

But we only know $u_0 \in H^{1,2}(\Omega)$, so that $Du_0 \in L^2(\Omega)$ and we are not allowed to perform a partial integration. Thus, we cannot apply the *fundamental lemma* to conclude that $\Delta u_0 = 0$.

We need a *regularity result.*

Theorem 2 (*Weyl's Lemma*, 1940)
Suppose, $u \in L^1(\Omega)$ and that

$$\int\limits_{\Omega} u(x)\Delta\varphi(x)dx = 0 \quad \text{for any} \quad \varphi \in C_c^\infty(\Omega). \tag{22}$$

Then, $u \in C^\infty(\Omega)$, and the fundamental lemma implies $\Delta u \equiv 0$ in Ω. That is, u is harmonic in Ω.

□

Remarks

(i) The *coercivity condition* (iii) in Theorem 1 ensures that minimizing sequences in $H^{1,m}$ stay bounded provided $\mathcal{F} \not\equiv \infty$.
The *convexity condition* (ii) implies the lower semicontinuity of $\mathcal{F}$ on $H^{1,m}$ with respect to both the norm–topology and the weak topology on $H^{1,m}$.
Since bounded sequences in $H^{1,m}$ possess weakly convergent subsequences we see that every minimizing sequence has a weakly convergent subsequence. From the lower semicontinuity it follows that its weak limit is a minimizer of $\mathcal{F}$.

(ii) The proof of Theorem 2 involves smoothing u. That is, one considers for $h > 0$ the function

$$u_h(x) = h^{-n} \int\limits_{\Omega} \rho\left(\frac{|x-y|}{h}\right) u(y)dy,$$

where $\rho = \rho(\cdot) = \rho(|\cdot|) \geq 0$, $\rho(x) \equiv 0$ for $|x| \geq 1$ and

$$\int\limits_{\mathbb{R}^n} \rho(x)dx = \int\limits_{B_1(0)} \rho(x)dx = 1.$$

Such a function is called a *Friedrichs – mollifier.* For given φ in (3) choose $0 < h <$ dist $(\text{spt}\varphi, \partial\Omega)$ and, using Fubini's Theorem we get

$$\begin{aligned} \int\limits_{\Omega} u_h(x)\Delta\varphi(x)dx &= \int\limits_{\Omega} h^{-n}\left[\int\limits_{\Omega} \rho\left(\tfrac{|x-y|}{h}\right)u(y)dy\right]\Delta\varphi(x)dx = \\ &= \int\limits_{\Omega} u(y)\Delta\varphi_h(y)dy. \end{aligned}$$

Note, that we used the fact $(\Delta\varphi)_h = \Delta\varphi_h$ which is easily checked because ρ is rotationally symmetric and the Laplace operator Δ is invariant under rotations. Since $u_h \in C^\infty$ the fundamental lemma may now be applied and yields $\Delta u_h \equiv 0$. Letting $h \downarrow 0$ and using the mean value formula for harmonic functions we get the result of Theorem 2. □

3.3. Regularity Theory

After *Morrey's* fundamental regularity result for minimizers of double integrals, probably the most important contribution to Hilbert's 19th problem was the celebrated regularity theorem by *De Giorgi* and *Nash* around 1957 concerning multiple integrals ($n \geq 3$) depending on *scalar functions*. Their result reads as follows.

Theorem 3 (*De Giorgi* 1957, *Nash* 1958)
Suppose $A = (a_{\alpha\beta})_{\alpha,\beta=1,\dots,n}$, $a_{\alpha\beta} = a_{\beta\alpha} \in L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ is open, such that

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x)\xi^\alpha\xi^\beta \geq |\xi|^2 \quad \text{for a.e. } x \in \Omega. \tag{23}$$

If $u \in H^{1,2}(\Omega)$ is a solution of

$$\int\limits_\Omega \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta} D_\alpha u D_\beta \varphi \, dx = 0 \quad \text{for any } \varphi \in C_c^\infty(\Omega) \tag{24}$$

then u is Hölder–continuous in Ω.

$\square$

Theorem 3 can, for example, be applied in the following situation.
Consider

$$\mathcal{F}[u] := \int\limits_\Omega F(Du)dx.$$

We assume that ($L > 0$)

(i) $F \in C^2(\mathbb{R}^n)$,

(ii) $\sum\limits_{\alpha,\beta=1}^{n} F_{p_\alpha p_\beta}(p)\xi_\alpha\xi_\beta \geq |\xi|^2$ for any $p, \xi \in \mathbb{R}^n$,

(iii) $|F_{pp}| \leq L$.

If u is a minimizer of $\mathcal{F}$ in $H^{1,2}(\Omega)$ then u is a solution of

$$\int\limits_\Omega \sum_{\alpha=1}^{n} F_{p_\alpha}(Du) D_\alpha \varphi \, dx = 0 \quad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{25}$$

Using difference quotients one can then show rather easily that

$$u \in H^{2,2}_{\text{loc}}(\Omega).$$

For arbitrary $\psi \in C_c^\infty(\Omega)$ and $s \in \{1, \dots, n\}$ define $\varphi = D_s\psi \in C_c^\infty(\Omega)$. Inserting φ in (25) and performing a partial integration leads to

$$\int\limits_\Omega \sum_{\alpha,\beta=1}^{n} F_{p_\alpha p_\beta}(Du) D_\beta(D_s u) D_\alpha \psi \, dx = 0.$$

Now, with $a_{\alpha\beta}(x) := F_{p_\alpha p_\beta}\big(Du(x)\big)$ and $\tilde{u} = D_s u$ Theorem 3 is applicable. Therefore, u is Hölder–continuously differentiable.

$\square$

An important contribution to the results by *De Giorgi* and *Nash* is due to *Moser* (1960/61) who introduced *Moser's iteration technique* and who succeeded in proving a *Harnack inequality* for positive solutions.
The corresponding regularity problem for *vector valued solutions* of variational integrals however still remained open. It came as quite a surprise when finally in 1967 *De Giorgi, Giusti* and *M. Miranda* found examples of elliptic variational problems with *irregular weak solutions.* For example, *Giusti* and *Miranda* showed that for the variational integral

$$\mathcal{F}[u] = \int_\Omega \left\{ \sum_{i,j=1}^n |D_i u^j|^2 + \Big[\sum_{i,j=1}^n \big(\delta_{ij} + \frac{4}{n-2} \frac{u^i u^j}{1+|u|^2}\big) D_i u^j \Big]^2 \right\} dx$$

for $n \geq 3$ the vector–valued function $u(x) := \frac{x}{|x|}$ is a solution to the corresponding system of Euler–Lagrange equations which is clearly not a smooth solution in any neighbourhood of the origin; in fact, u is not even *continuous* near 0.
Thus, it became clear that in certain situations one necessarily had to expect *singularities.* Therefore, instead of attempting to prove full regularity one first tries to prove partial regularity. A generic result in that direction could read as follows.

Partial Regularity Result
Suppose, $u : \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^n$ open, is a generalized minimizer of a variational integral. Then, there exists an open and dense set $\Omega' \subset \Omega$ such that $u \in C^0(\Omega')$.

□

Of course, one can then try to improve such a result, for example by estimating the size of the difference set $\Omega \setminus \Omega'$.

3.4. Non–Convex Problems

While convexity is a natural condition to ensure the existence of a minimizer, not every interesting functional in the calculus of variations is necessarily convex. Already *Bolza* pointed out the following

Example (a *non–convex* variational integral)
$\Omega = (0,1) \subset \mathbb{R}$, $u : (0,1) \to \mathbb{R}$, $u(0) = 0 = u(1)$

$$\mathcal{F}[u] := \int_0^1 \Big[u(x)^2 + \big(u'(x)^2 - 1\big)^2 \Big]\, dx.$$

We then have

$$\inf \Big\{ \mathcal{F}[u] : u \in H_0^{1,4}\big((0,1)\big) \Big\} = 0.$$

This can be seen as follows. Consider the sequence $\{u_n\}$ given by

$$u_n(x) = \begin{cases} x - \frac{k}{n} \quad , & 2k \leq 2nx \leq 2k+1 \\ -x + \frac{k+1}{n} \quad , & 2k+1 \leq 2nx \leq 2(k+1). \end{cases}$$

Since u_n is Lipschitz–continuous, i.e. $u_n \in H^{1,\infty}\big((0,1)\big)$, we also have $u_n \in H^{1,4}$. Furthermore, we get (u_n is a zig–zag function)

$$0 \leq u_n \leq \tfrac{1}{2n}, \qquad u_n(0) = 0 = u_n(1),$$

$$|u_n'(x)| = 1 \quad \text{for a.e. } x \in (0,1).$$

We conclude

$$\mathcal{F}[u_n] \leq \frac{1}{4n^2}$$

and thus

$$\lim_{n\to\infty} \mathcal{F}[u_n] = 0.$$

On the other hand there obviously does not exist any $u \in H^{1,4}$ such that $\mathcal{F}[u] = 0$, because this would imply

$$u(x) = 0 \quad \text{a.e. as well as } |u'(x)| = 1 \quad \text{a.e.}$$

□

Remark
Expressions such as $\big(u'(x)^2 - 1\big)^2$ frequently occur in technical problems. An example is cruising with a sailboat.

□

To deal with non–convex problems one has the choice to modify the problem, to generalize the notion of solution, or to do both things simultaneously.
An example is the idea of *relaxation*:
Suppose X is a topological space, and $\mathcal{F} : X \to \overline{\mathbb{R}}$. The *lower semicontinuous envelope* (or *relaxed function*) $\mathrm{sc}^-\mathcal{F}$ of $\mathcal{F}$ is defined for every $x \in X$ by

$$(\mathrm{sc}^-\mathcal{F})(x) := \sup\big\{\Phi(x) : \Phi : X \to \overline{\mathbb{R}} \text{ is l.s.c. and } \Phi \leq \mathcal{F} \text{ on } X\big\}.$$

Thus, $\mathrm{sc}^-\mathcal{F}$ is the largest lower semicontinuous function below $\mathcal{F}$ on X. In particular, we have

$$\mathcal{F} \text{ is l.s.c.} \quad \text{if and only if} \quad \mathrm{sc}^-\mathcal{F} = \mathcal{F}.$$

We have the following fundamental existence result.

Theorem 4
Every accumulation point of a minimizing sequence for $\mathcal{F}$ is a minimizer of $\mathrm{sc}^-\mathcal{F}$. Thus, if $\mathcal{F}$ is coercive, $\mathrm{sc}^-\mathcal{F}$ attains a minimum value, and

$$\min_X \ \mathrm{sc}^-\mathcal{F} = \inf_X \mathcal{F}.$$

□

Remark
For example, in case

$$\mathcal{F}[u] = \int_\Omega f\big(Du(x)\big)dx$$

one can show – if suitable assumptions are satisfied –

$$(\mathrm{sc}^-\mathcal{F})[u] = \int_\Omega (\mathrm{cvx}^- f)\big(Du(x)\big)dx,$$

where $\mathrm{cvx}^- f$ is the largest convex function below f, that is

$$(\mathrm{cvx}^- f)(p) = \sup\big\{g(p) : g \text{ convex}, \quad g \leq f\big\}.$$

□

3.5. Γ–Convergence

A different approach to non–convex problems is the theory of Γ–*convergence* introduced by *De Giorgi* and his school.
Suppose, X is a topological space satisfying the first axiom of countability, and $\mathcal{F}_n : X \to \overline{\mathbb{R}}$ ($n \in \mathbb{N}$) is a sequence of functionals.
The sequence $\{\mathcal{F}_n\}$ is called Γ–*convergent* to $\mathcal{F}$:

$$\mathcal{F} = \Gamma - \lim_{n\to\infty} \mathcal{F}_n,$$

if the following conditions are satisfied:

(i) for every $x \in X$ and *every sequence* $x_n \to x$ we have

$$\mathcal{F}[x] \le \liminf_{n\to\infty} \mathcal{F}_n[x_n];$$

(ii) for every $x \in X$ there *exists* a sequence $x_n \to x$, such that

$$\mathcal{F}[x] = \lim_{n\to\infty} \mathcal{F}_n[x_n].$$

□

Remark
An important application of Γ–convergence is the theory of *homogenisation.*

□

Let us close this section by a famous

Example (*Modica – Mortola*, 1977)
Define $\mathcal{F}_k : L^1(\mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}$ by ($k \in \mathbb{N}$)

$$\mathcal{F}_k[u] := \begin{cases} \int\limits_{\mathbb{R}^n} \left\{\frac{1}{k}|Du|^2 + k\sin^2(\pi k u)\right\} d\mathcal{L}^n & , \quad u \in H^{1,2} \cap L^1 \\ \infty & , \quad \text{otherwise;} \end{cases}$$

and let $\mathcal{F} : L^1(\mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}$ be given by

$$\mathcal{F}[u] := \begin{cases} \frac{4}{\pi} \int\limits_{\mathbb{R}^n} |Du| = \frac{4}{\pi}\|Du\| & , \quad u \in BV(\mathbb{R}^n) \\ \infty & , \quad \text{otherwise.} \end{cases}$$

On $L^1(\mathbb{R}^n)$ we then have

$$\mathcal{F} = \Gamma - \lim_{k\to\infty} \mathcal{F}_k.$$

□

4. Unstable Critical Points

In this last–extremely short–section we are concerned with critical points, that is solutions of the Euler–Lagrange equations, that are neither minimizers nor maximizers. Thus, the direct methods discussed in the previous section are no longer applicable. Here, we want to present

only one method, the so called *Palais–Smale Condition*.
Let V be a Banach space, and suppose $\mathcal{F} \in C^1(V, \mathbb{R})$. Then, $\mathcal{F}$ is said to satisfy the *Palais–Smale condition* (PS), provided the following is true.

Every sequence $\{x_n\}_{n\in\mathbb{N}} \subset V$ such that

(i) $|\mathcal{F}[x_n]| \leq c \quad (\{x_n\}$ is $\mathcal{F}$–bounded),

(ii) $\|D\mathcal{F}[x_n]\| \to 0$ as $n \to \infty$,

contains a convergent subsequence.

$\square$

Remark
If x_0 is the limit of such a sequence the continuity of $D\mathcal{F}$ implies that x_0 is a critical point of $\mathcal{F}$, that is $D\mathcal{F}[x_0] = 0$. An immediate consequence of the definition above is the

Lemma
Suppose, that $\mathcal{F} : V \to \mathbb{R}$ satisfies (PS). Then, for any $\alpha \in \mathbb{R}$ the set

$$K_\alpha := \big\{x \in V : \mathcal{F}[x] = \alpha,\ D\mathcal{F}[x] = 0\big\},$$

that is the set of critical points of $\mathcal{F}$ belonging to the value α, is *compact*.

$\square$

An important application of the Palais–Smale–condition is the

Theorem 1 (*Mountain–Pass Lemma, Ambrosetti–Rabinowitz* 1973)
Suppose, that $\mathcal{F} \in C^1(V, \mathbb{R})$ satisfies (PS), and that $\mathcal{F}[0] = 0$.
Furthermore, assume that

(i) there exists $\rho > 0$, $\beta > 0$ such that $\|u\| = \rho$ implies $\mathcal{F}[u] \geq \beta$;

(ii) there exists $u_1 \in V$ such that $\|u_1\| > \rho$ and $\mathcal{F}[u_1] < \beta$.

If we denote by Γ the set of paths in V connecting 0 and u_1, that is

$$\Gamma = \big\{\gamma \in C^0([0,1], V) : \gamma(0) = 0,\ \gamma(1) = u_1\big\},$$

then we have:

$$\alpha := \inf_{\gamma\in\Gamma} \sup_{\tau\in[0,1]} \mathcal{F}[\gamma(\tau)] \quad (\geq \beta)$$

is a *critical value* of $\mathcal{F}$. Thus, there exists u_0 such that

$$\mathcal{F}[u_0] = \alpha \quad \text{and } D\mathcal{F}[u_0] = 0.$$

$\square$

Let us close this section with a nice application of the *Mountain–Pass Lemma* to a *semilinear elliptic boundary value problem.*

Theorem 2
Suppose, that $\Omega \subset \mathbb{R}^n$ is bounded and that $2 < m < \frac{2n}{n-2}$ (respectively $m < \infty$ for $n = 1, 2$).
Then, the Dirichlet–problem

$$\begin{aligned} \Delta u + |u|^{m-2}u &= 0 \ \text{ in } \Omega \\ u &= 0 \ \text{ on } \partial\Omega \end{aligned}$$

has at least *two non–trivial* (i.e. $\not\equiv 0$) solutions. Note, that if u is a non–trivial solution so is the function $-u$.

□

Remark

(i) For the proof one considers the variational integral

$$\mathcal{F}[u] := \frac{1}{2}\int_\Omega |Du|^2 d\mathcal{L}^n - \frac{1}{m}\int_\Omega |u|^m d\mathcal{L}^n$$

naturally defined on $H_0^{1,2}(\Omega)$ (because of the Sobolev imbedding theorem and the choice of m).

Note, that the direct method is not applicable in this case because

$$\inf_{H_0^{1,2}} \mathcal{F} = -\infty \quad \text{and} \quad \sup_{H_0^{1,2}} \mathcal{F} = +\infty.$$

To see this, first take any $u_1 \in H_0^{1,2}(\Omega)$ such that

$$\int_\Omega |u_1|^m d\mathcal{L}^n > 0.$$

For $u^\lambda = \lambda u_1$ we then have

$$\mathcal{F}[u^\lambda] = \frac{\lambda^2}{2}\int_\Omega |Du_1|^2 d\mathcal{L}^n - \frac{\lambda^m}{m}\int_\Omega |u_1|^m d\mathcal{L}^n,$$

so that – because $\lambda > 2$ – $\lim\limits_{\lambda\to\infty} \mathcal{F}[u^\lambda] = -\infty$.

To see that $\mathcal{F}$ can become arbitrarily large choose a function $u \in H_0^{1,2}(\Omega)$ that oscillates wildly while $\sup\limits_\Omega |u| \le 1$.

(ii) Using the same method one can treat the *eigenvalue problem*

$$\Delta u + |u|^{m-2}u = \lambda u, \quad \lambda \ge 0.$$

(iii) For $m = 2$ we get the *linear equation*

$$\Delta u + u = 0$$

and the non–trivial solutions turn out to be the eigenfunctions of Δ corresponding to the eigenvalue -1.

(iv) The *critical case* is $m = \frac{2n}{n-2}$, the *limit case* of the Sobolev–inequality on $H^{1,2}$. In this case, the Palais–Smale condition is no longer satisfied.

An example is the so called *Yamabe problem* from differential geometry (For a given Riemannian metric g on a manifold is there a conformal one g' with prescribed – for example constant – scalar curvature R'?) which leads to the equation

$$4\frac{n-1}{n-2}\Delta u + Ru = R' u^{\frac{n+2}{n-2}}.$$

Here, one looks for a *positive* solution u, and the new metric is then $g' = u^{\frac{4}{n-2}} g$.

These limit cases where the Palais–Smale–condition fails seem to be typical for the most interesting problems (e.g. minimal surfaces, harmonic mappings, Yang–Mills–fields). □

Apart from the Palais–Smale Condition important other methods are: The Minimax Principle, Index Theory, Ljusternik–Schnirelman Theory, and Morse Theory.

□

Bibliography

Buttazzo G., Giaquinta M., Hildebrandt S. (1998). *One–dimensional Variational Problems*, 262 pp. Oxford, England: Clarendon Press. [A textbook on one–dimensional variational problems with special emphasis on direct methods.]

Dacorogna B. (1989). *Direct Methods in the Calculus of Variations*, 308 pp. Berlin–Heidelberg, Germany: Springer. [A textbook on direct methods, in particular for vector–valued functions and relaxation of non–convex problems.]

Dal Maso G. (1993). *An Introduction to* Γ*–Convergence*, 340 pp. Boston, MA, USA: Birkhäuser. [An introduction to De Giorgi's theory of Γ–convergence].

De Giorgi E. (1957). Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino* **3**, 25–43. [One of the two fundamental papers on the regularity for scalar variational integrals.]

De Giorgi E. (1968). Un esempio di estremali discontinue per un problema variazionali di tipo ellittico. *Boll UMI* **4**, 135–137. [One of the first two counterexamples to the regularity for vector–valued minimizers.]

Euler L. (1744). *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattisimo sensu accepti*, Lausanne et Genevae: Bousquet. [The first treatise on the calculus of variations.]

Federer H. (1969). *Geometric Measure Theory*, 676 pp. Berlin–Heidelberg, Germany: Springer. [The first fundamental monograph on geometric variational problems.]

Giaquinta M. (1983). *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, 296 pp. Princeton, NJ, USA: Princeton University Press. [A monograph on the regularity problem for multiple integrals in the calculus of variations.]

Giaquinta M., Hildebrandt S. (1996). *Calculus of Variations* I and II, 474 and 652 pp. Berlin–Heidelberg, Germany: Springer. [This book describes the classical aspects of the variational calculus which are of interest to analysis, geometers and physicists alike.]

Giaquinta M., Modica G., Souček J. (1998). *Cartesian Currents in the Calculus of Variations* I and II, 711 and 697 pp. Berlin–Heidelberg, Germany: Springer. [This treatise presents a new approach to geometric variational problems.]

Giusti E., Miranda M. (1968). Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni. *Boll. UMI* **2**, 1–8. [One of the first two counterexamples to the regularity for vector–valued minimizers.]

Hildebrandt S., Tromba A. (1996). *The Parsimonious Universe*, 330 pp. New York, NY, USA: Copernicus. [This is a wonderful non–technical book on the calculus of variations and its applications in nature and geometry.]

Kneser A. (1900). Variationsrechnung. *Enzyk. math. Wiss.* II **1**, 571–625. [The first encyclopedia article on the calculus of variations.]

Morrey C.B. Jr. (1966). *Multiple Integrals in the Calculus of Variations*, 506 pp. Berlin–Heidelberg, Germany: Springer. [The first monograph on multiple integrals in the calculus of variations.]

Moser J. (1960). A New Proof of De Giorgi's Theorem Concerning the Regularity Problem for Elliptic Differential Equations. *Comm. Pure Appl. Math.* **13**, 457–468. [This paper introduces Moser's famous iteration technique.]

Moser J. (1961). On Harnack's Theorem for Elliptic Differential Equations. *Comm. Pure

Appl. Math. **14**, 577–591. [The first Harnack inequality for solutions of non–linear equations.]
Nash J. (1958). Continuity of Solutions of Parabolic and Elliptic Equations. *Amer. J. Math.* **8**, 931–954. [The other fundamental paper – besides De Giorgi's one – on the regularity for scalar variational integrals.]
Struwe M. (1996). *Variational Methods*, 272 pp. Berlin–Heidelberg, Germany: Springer. [A recent monograph on the calculus of variations with an emphasis on unstable critical points.]
Zermelo E., Hahn H. (1904). Weiterentwicklung der Variationsrechnung in den letzten Jahren. *Enzyk. math. Wiss.* II **1,1**, 626–641. [The continuation of Kneser's encyclopedia article.]