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Abstract

We discuss partial regularity results concerning local minimizers $u : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^3$ of variational integrals of the form

$$\int_\Omega \{h(|\epsilon(w)|) - f \cdot w\}\, dx$$

defined on appropriate classes of solenoidal fields, where h is a N-function of rather general type. As a byproduct we obtain a theorem on partial C^1-regularity for weak solutions of certain non-uniformly elliptic Stokes-type systems modelling generalized Newtonian fluids.

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1 Introduction

As stated in the monograph of Ladyzhenskaya (see [La], p. 35) the Stokes problem in its classical formulation for the stationary case reads as follows: find a velocity field $v : \Omega \to \mathbb{R}^n$ and a pressure function $\pi : \Omega \to \mathbb{R}$ such that the following system of partial differential equations is satisfied

$$\begin{cases} \Delta v = \nabla\pi - f & \text{on } \Omega, \\ \operatorname{div} v = 0 & \text{on } \Omega, \\ v = v_0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω denotes a domain in $\mathbb{R}^n$ ($n \in \{2,3\}$), $f : \Omega \to \mathbb{R}^n$ is a system of volume forces and $v_0 : \partial\Omega \to \mathbb{R}^n$ represents the given boundary data. For results concerning existence and regularity of solutions of (1.1) we again refer to [La] or to the more recent expositions [Ga1,2] of Galdi. If we let $H(\epsilon) = \frac{1}{2}|\epsilon|^2$, then the solutions of (1.1) are clearly in one-to-one correspondence to the minimizers of

$$J[w] := \int_\Omega \{H(\epsilon(w)) - f \cdot w\}\, dx \tag{1.2}$$

defined on an appropriate class of solenoidal fields $w : \Omega \to \mathbb{R}^n$, $\varepsilon(w)$ denoting the symmetric gradient, i.e. $\varepsilon(w) = \frac{1}{2}\left(\nabla w + \nabla w^T\right)$
A natural extension of this problem also proposed by Ladyzhenskaya (compare [La], p. 193) is to consider minimizers of (1.2) with potential H being of power growth in the sense that

$$\lambda(1 + |\epsilon|^2)^{\frac{p-2}{2}}\, |\sigma|^2 \le D^2H(\epsilon)(\sigma,\sigma) \le \Lambda(1 + |\epsilon|^2)^{\frac{p-2}{2}}\, |\sigma|^2$$

holds for all $\epsilon, \sigma \in \mathbb{S}$ with positive constants λ, Λ and for an exponent $p > 1$, where $\mathbb{S}$ is the space of symmetric $n \times n$-matrices. In this case the first equation in (1.1) is replaced by the nonlinear system

$$\operatorname{div}\{\nabla H(\epsilon(v))\} = \nabla\pi - f \quad \text{on } \Omega,$$

where on the l.h.s. the operator "div" has to be applied linewise. For these power law models full interior $C^{1,\alpha}$-regularity in the 2D case has been proved by Kaplický, Málek and Stará [KMS] and Wolf [Wo], whereas the higher dimensional situation is studied for example in Naumann and Wolf [NW]. For partial regularity results in dimensions $n \geq 3$ we also refer to [FGR] and [Fu4]. The reader should note that the related but much more difficult case of power law models with x-dependent exponents describing the behavior of electrorheological fluids has been investigated by Acerbi and Mingione [AM]. In the paper [BF1] Bildhauer and the second author consider the minimization problem now under so-called anisotropic growth conditions, i.e. they assume the validity of

$$\lambda(1 + |\epsilon|^2)^{\frac{p-2}{2}} |\sigma|^2 \leq D^2 H(\epsilon)(\sigma, \sigma) \leq \Lambda(1 + |\epsilon|^2)^{\frac{q-2}{2}} |\sigma|^2$$

with exponents $1 < p \leq q < \infty$ and constants $\lambda, \Lambda > 0$. It should be remarked that such a behavior of the potential H is suggested for example in Section 5.1 of the monograph [MNRR] of Málek, Necǎs, Rokyta and Růžička. The main result of the paper [BF1] is a partial $C^{1,\alpha}$-regularity theorem in general dimensions n using the hypothesis

$$q < p\,\frac{n+2}{n}$$

limiting the range of anisotropy. This corresponds to the result being valid in the framework of classical variational calculus (see [BF2]), and in general there is no hope for regularity if p and q differ too much (compare the counterexamples of Giaquinta [Gi] and Hong [Ho] in this context). For completeness we like to mention that in the case $n = 2$ the hypothesis $q < \min(2p, p + 2)$ is a sufficient condition for full regularity of stationary and also slow anisotropic flows, see [BFZ].

In this note we will follow the ideas of [Fu1] and [Fu2], where the author proves full regularity in two dimensions including the case of fluids and partial regularity for $n \geq 3$ in the setting of variational calculus for integrands depending on the modulus, i.e. the dissipative potential H is of the special form

$$H(\epsilon) = h(|\epsilon|),\ \epsilon \in \mathbb{S},$$

which seems to be a natural assumption for the study of fluids. Here $h : [0, \infty) \to [0, \infty)$ is a C^2-function satisfying:

$$\begin{aligned}&h \text{ is strictly increasing and convex together with}\\ &h''(0) > 0 \quad \text{and} \quad \lim_{t\downarrow 0} \frac{h(t)}{t} = 0;\end{aligned} \tag{A1}$$

$$\text{there exists a constant } \overline{k} > 0 \text{ such that } h(2t) \le \overline{k} h(t) \text{ for all } t \ge 0; \tag{A2}$$

$$\frac{h'(t)}{t} \le h''(t) \le a(1+t^2)^{\frac{\omega}{2}} \frac{h'(t)}{t} \quad \text{for all } t \ge 0 \tag{A3}$$
with an exponent $\omega \ge 0$ and a constant $a \ge 0$;

Let us give some comments on (A1-3):

i) We have $h(0) = h'(0) = 0$, and by convexity h' is an increasing function with $h'(t) > 0$ for all $t > 0$: otherwise it would follow that $h' = 0$ on some interval $[0, t_0]$, $t_0 > 0$, contradicting the first part of (A1).

ii) The inequality $\frac{h'(t)}{t} \le h''(t)$ implies that the function $t \mapsto \frac{h'(t)}{t}$ is increasing, moreover we deduce the lower bound

$$h(t) \ge \frac{1}{2} h''(0) t^2, \ t \ge 0\,, \tag{1.3}$$

and (A1) combined with (1.3) shows that h is a N-function in the sense of Adams [Ad, Section 8.2].

iii) (A2) states that h satisfies a global ($\Delta 2$)-condition, and it is easy to see that

$$h(t) \le c(t^{\overline{q}} + 1) \tag{1.4}$$

for a suitable exponent $\overline{q} \ge 2$ and a constant c. The convexity of h then implies that $h'(t)$ can be bounded in terms of $t^{\overline{q}-1}$.

iv) From (A2) and from the convexity of h we deduce the inequality

$$\overline{k}^{-1} h'(t)t \le h(t) \le t h'(t), \ t \ge 0\,. \tag{1.5}$$

v) From (A3) we conclude the ellipticity condition

$$\frac{h'(|Z|)}{|Z|}|Y|^2 \le D^2 H(Z)(Y,Y) \le a(1+|Z|^2)^{\frac{\omega}{2}} \frac{h'(|Z|)}{|Z|}|Y|^2\,. \tag{1.6}$$

Recalling iii) and using (see ii)) $\frac{h'(|Z|)}{|Z|} \ge h''(0)$, we get from (1.6) with exponent $q := \overline{q} + \omega$

$$h''(0)|Y|^2 \le D^2 H(Z)(Y,Y) \le C(1+|Z|^2)^{\frac{q-2}{2}} |Y|^2\,, \tag{1.7}$$

and (1.7) means that H is of anisotropic $(2,q)$–growth.

vi) In physical terms our conditions on h imply that the fluid described by the potential H is of shear thickening type.

vii) Let $\Theta : [0,\infty) \to [0,\infty)$ denote a continuous and increasing function such that $\Theta(0) > 0$. If in addition Θ has the (Δ_2)-property, then it is shown in [BF3], that

$$h(t) := \int_0^t \int_0^s \Theta(u)\, du\, ds$$

satisfies (A1)-(A3) with exponent $\omega = 0$.

Suppose now that we are given a function u_0 from the Orlicz-Sobolev class $W^{1,h}(\Omega,\mathbb{R}^n)$ generated by h (see [Ad] for a definition) satisfying $\operatorname{div} u_0 = 0$. We define the class

$$\mathcal{C} := \left\{ w \in u_0 + W_0^{1,h}(\Omega,\mathbb{R}^n) : \operatorname{div} w = 0 \right\},$$

$W_0^{1,h}(\Omega,\mathbb{R}^n)$ denoting the subspace of $W^{1,h}(\Omega,\mathbb{R}^n)$ of fields with zero trace, and deduce from Lemma 4.2 the unique solvability of the variational problem

$$\mathbb{J}[w,\Omega] := \int_\Omega h(|\epsilon(w)|)\, dx \longrightarrow \min \tag{1.8}$$

in $\mathcal{C}$. Of course we could also add a volume force term like $\int_\Omega f \cdot w\, dx$ to (1.8) which under appropriate assumptions on f is of no effect on the argumentation below. Since we will not touch the question of boundary regularity, we restrict ourselves to local minimizers of (1.8).

Definition 1.1. *We call a function u from the local Orlicz-Sobolev space $W^{1,h}_{loc}(\Omega,\mathbb{R}^n)$ satisfying* $\operatorname{div} u = 0$ *a local minimizer of (1.8), if for any subdomain $\Omega' \Subset \Omega$ it holds*

- $\mathbb{J}[u,\Omega'] < \infty$ *and*
- $\mathbb{J}[u,\Omega'] \le \mathbb{J}[v,\Omega']$

for all $v \in W^{1,h}_{loc}(\Omega,\mathbb{R}^n)$ such that $\operatorname{div} v = 0$ *and* $\operatorname{spt}(u-v) \subset \Omega'$.

Abbreviating

$$V^0_{h,loc}(\Omega,\mathbb{R}^n) := \left\{ w \in W^{1,h}_{loc}(\Omega,\mathbb{R}^n) : \operatorname{div} w = 0 \right\}$$

we can now state our main results in case $n = 3$ (full regularity in 2D is proved in [Fu2] even under weaker hypotheses concerning h):

THEOREM 1.1. *Let $u \in V^0_{h,loc}(\Omega)(\Omega,\mathbb{R}^3)$ be a local minimizer of (1.8) under the assumptions (A1)-(A3) with $\omega < 4/3$. Then we have*

a) $\displaystyle\int_0^{|\epsilon(u)|} \sqrt{\frac{h'(t)}{t}}\, dt \in W^{1,2}_{loc}(\Omega)$;

b) $h(|\varepsilon(u)|) \in L^3_{loc}(\Omega)$.

THEOREM 1.2. *Let $u \in V^0_{h,loc}(\Omega, \mathbb{R}^3)$ be a local minimizer of (1.8) under the assumptions (A1)-(A3) with $\omega < 4/3$. Then there is an open subset Ω_0 of Ω with full Lebesgue measure such that $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^3)$ for any $0 < \alpha < 1$.*

An explicit description of the set Ω_0 is given after Lemma 3.1. Unfortunately we could not rule out the occurrence of singular points (for ∇u), but even if they exist, the solution itself is at least continuous. In fact, from Theorem 1.1 b) combined with (1.3) it follows that $|\varepsilon(u)| \in L^6_{loc}(\Omega)$ holds, and we deduce from Korn's inequality (see e.g. [FS] or [AM]) and Sobolev's embedding theorem

Corollary 1.1. *Under the assumptions of Theorem 1.1 and 1.2 any local minimizer of problem (1.8) is locally Hölder continuous with exponent 1/2.*

Remark 1.1. *It is easy to see that the statement of Theorem 1.1 remains valid in higher dimensions, which means that we get $h(|\varepsilon(u)|) \in L^{\frac{n}{n-2}}_{loc}(\Omega)$ provided $\omega < 4/n$. This corresponds to the result obtained in [Fu2], where it is shown that $\omega < 2$ is a sufficient condition for full regularity in the 2D case.*

Remark 1.2. *In the setting of classical variational problems studied in [Fu1] the appropriate variant of Theorem 1.2 requires the bound $\omega \leq 4$, if the case $n = 3$ is considered (compare [Fu1], Remark 1.1). Actually, as it was recently shown in [Fu5], local minimizers satisfy a local Lipschitz condition independent of the value of ω and for any dimension $n \geq 2$.*

Remark 1.3. *It is an open problem if the bound on ω imposed in Theorem 1.1 and Theorem 1.2 can be improved. Clearly, if we drop the side condition* div $= 0$*, then we obtain the result of Theorem 1.2 now for $\omega \leq 4$ by completely adopting the arguments from [Fu1], but this statement seems to be of no physical interest. A very challenging task however is to discuss if in the setting of Theorem 1.2 actually singular points occur and if the value of ω is of any importance for the regular or irregular behaviour of minimizers.*

Our paper is organized as follows:
In section 2 we introduce a suitable sequence of regularized problems whose solutions are smooth enough to carry out the calculations which lead to the proof of Theorem 1.1 after passing to the limit. Section 3 contains the proof of the partial regularity result stated in Theorem 1.2. This step makes essential use of a blow-up argument. In section 4 we present some background material on Korn's inequality in Orlicz spaces.

2 Proof of Theorem 1.1

Let u denote a local minimizer of problem (1.8) under the assumptions of Theorem 1.1. A first step is to approximate (1.8) locally by variational problems with sufficiently regular minimizers. Let

$$\delta := \delta(\rho) := \frac{1}{1 + \rho^{-1} + \|(\varepsilon((u)_\rho)\|^{2q}_{L^q(B)}},$$

$$H_\delta(\epsilon) := \delta \left(1+|\epsilon|^2\right)^{\frac{q}{2}} + H(\epsilon)$$

for $\epsilon \in \mathbb{S}$ and for a small parameter $\rho > 0$. Here the exponent q is defined in (1.7) and $(u)_\rho$ denotes the mollification of u with radius ρ. With $B := B_{R_0}(x_0) \Subset \Omega$ we define u_δ as the unique minimizer of

$$\mathbb{J}_\delta\,[w, B] := \int_B H_\delta(\epsilon(w))dx \tag{2.1}$$

in $(u)_\rho + \overset{\circ}{W}{}^{1,q}_0(B, \mathbb{R}^3)$ subject to the constraint $\operatorname{div} w = 0$. Some elementary properties of u_δ are summarized in the following Lemma (see [BF1], Lemma 3.1, Lemma 4.1 and estimate (4.10), as well as the inequalities (12) and (13) from [Fu2] for part c)):

Lemma 2.1. *Let the hypothesis of Theorem 1.1 hold. Then we have*

a) $u_\delta \in W^{2,2}_{loc}(B, \mathbb{R}^3)$,

b) $\sigma_\delta := DH_\delta(\epsilon(u_\delta)) \in W^{1,q/(q-1)}_{loc}(B, \mathbb{S})$;

c) $(1+|\epsilon(u_\delta)|^2)^{\frac{q}{4}} \in W^{1,2}_{loc}(B)$, *and for all* $\eta \in C^\infty_0(B)$, $Q \in \mathbb{R}^{3\times 3}$ *and* $\gamma \in \{1, ..., n\}$ *we obtain*

$$\int_B \eta^2 D^2 H_\delta(\epsilon(u_\delta))(\partial_\gamma \epsilon(u_\delta), \partial_\gamma \epsilon(u_\delta))\, dx$$
$$\leq -2 \int_B \eta \partial_\gamma \tau_\delta : (\nabla \eta \odot \partial_\gamma [u_\delta - Qx])\, dx.$$

Here we have abbreviated $\tau_\delta := \sigma_\delta - p_\delta I$ *for a suitable pressure function* p_δ, *i.e.* $\nabla p_\delta = \operatorname{div} \sigma_\delta$, *which implies* $\tau_\delta \in W^{1,q/(q-1)}_{loc}(B, \mathbb{S})$ *together with*

$$|\nabla \tau_\delta| \leq c|\nabla \sigma_\delta|.$$

d) *As* $\rho \to 0$ *we have* $u_\delta \rightharpoonup u$ *in* $W^{1,2}(B, \mathbb{R}^3)$ *and*

$$\delta \int_B \left(1+|\epsilon(u_\delta)|^2\right)^{\frac{q}{2}}\, dx \to 0.$$

e) *The integrals* $\int_B h(|\epsilon(u_\delta)|)\, dx$ *are bounded independent of* δ.

Furthermore we need the following statements:

Lemma 2.2. *Under the assumptions of Theorem 1.1 it holds:*

a) u_δ *is uniformly bounded in* $W^{1,h}(B,\mathbb{R}^3)$.

b) The sequence $h(|u_\delta|)$ *is uniformly bounded in any space* $L^\chi(B)$, $\chi < 3$, *so that*

$$h(|u_\delta|)|u_\delta|^\mu \in L^1(B)$$

uniformly, provided $\mu < 4$.

Proof of Lemma 2.2: From Lemma 2.1 e) combined with Korn's inequality formulated in Lemma 4.1 a) we deduce that the L_h-norms of the tensors $\nabla(u_\delta - (u)_\rho)$ can be estimated in terms of the corresponding norms of $\nabla(u)_\rho$, which in turn stay bounded due to Jensen's inequality. The claim of part a) then is a consequence of the Poincaré inequality (applied to $u_\delta - (u)_\rho$) for functions from the space $W_0^{1,h}(B,\mathbb{R}^3)$ (see [FO]). For the higher integrability result stated in part b) we first observe that according to a) the functions $h(|u_\delta|)^{\frac{1}{\beta}}$ stay bounded (uniformly w.r.t. the approximation parameter) in any space $L^\beta(B)$, where $\beta \in (1,2)$. By (1.5) we have

$$\int_B |\nabla h(|u_\delta|)^{\frac{1}{\beta}}|^\beta \, dx \le c \int_B h'(|u_\delta|)|u_\delta|^{1-\beta}|\nabla u_\delta|^\beta \, dx$$

and in order to discuss the integral on the r.h.s. we define the N-function

$$\mathcal{N}(t) := h(t^{\frac{1}{\beta}}).$$

Note that $\mathcal{N}$ actually is a N-function on account of (A3) and the choice of β. From (1.5) we deduce for the conjugate function

$$\begin{aligned}\mathcal{N}^*(t) &= \sup_{s\ge 0}\left[t - \frac{h(s^{\frac{1}{\beta}})}{s}\right] s \le \sup_{s\ge 0}\left[t - \overline{k}^{-1} h'(s^{\frac{1}{\beta}}) s^{\frac{1}{\beta}-1}\right] s \\ &\le \overline{h}^{-1}\left(\overline{k} t\right) t, \quad \overline{h}(t) := h'(t^{\frac{1}{\beta}}) t^{\frac{1}{\beta}-1}.\end{aligned}$$

We remark that the strict monotonicity of $\overline{h}$ also follows from (A3) and our choice $\beta < 2$. We now apply Young's inequality for N-functions and obtain

$$\int_B h'(|u_\delta|)|u_\delta|^{1-\beta}|\nabla u_\delta|^\beta \, dx \le \int_B \mathcal{N}^*\left(\overline{k}^{-1} h'(|u_\delta|)|u_\delta|^{1-\beta}\right) dx + c(\overline{k}) \int_B h(|\nabla u_\delta|).$$

Clearly the last integral is uniformly bounded by part a). For the first one we have the upper bound

$$c \int_B \overline{h}^{-1}\left(h'(|u_\delta|)|u_\delta|^{1-\beta}\right) h'(|u_\delta|)|u_\delta|^{1-\beta} \, dx$$

and the definition of of $\overline{h}$ gives (remember (1.5))

$$\overline{h}^{-1}\left(h'(t)t^{1-\beta}\right) h'(t)t^{1-\beta} = h'(t)t \le \overline{k} h(t),$$

hence we can also control the remaining term independent of δ by part a). Altogether it is shown that for each $\beta \in (1,2)$ the sequence $h(|u_\delta|)^{\frac{1}{\beta}}$ stays bounded in the Sobolev-space $W^{1,\beta}(B)$, and our claim follows from Sobolev's embedding theorem recalling also (1.3).□

After these preparations we come to the proof of Theorem 1.1: in a first step we work with a cut-off function $\eta_1 \in C_0^\infty(B_{\widetilde{r}}(z))$ with $\eta_1 \equiv 1$ on $B_r(z)$, $0 \le \eta_1 \le 1$ and $|\nabla \eta_1| \le c/(\widetilde{r}-r)$, where $0 < r < R$ are such that $B_R(z) \Subset B$ and $\widetilde{r} := \frac{R+r}{2}$. We get by Sobolev's inequality

$$\begin{aligned}&\int_{B_r(z)} h(|\varepsilon(u_\delta)|)^3\,dx \le \int_{B_{\widetilde{r}}(z)} \eta_1^6 h(|\varepsilon(u_\delta)|)^3\,dx\\ &\le c\left\{\int_{B_{\widetilde{r}}(z)} |\nabla\eta_1|^2 h(|\varepsilon(u_\delta)|)\,dx + \int_{B_{\widetilde{r}}(z)} \eta_1^2 \frac{[h'(|\varepsilon(u_\delta)|)]^2}{h(|\varepsilon(u_\delta)|)}|\nabla\varepsilon(u_\delta)|^2\,dx\right\}^3.\end{aligned}$$

Using (1.5), Lemma 2.1 e) and (1.6) we obtain for a suitable positive number β (summation w.r.t. $\gamma \in \{1,2,3\}$)

$$\begin{aligned}&\int_{B_r(z)} h(|\varepsilon(u_\delta)|)^3\,dx \le c(R-r)^{-\beta} + c\left\{\int_{B_{\widetilde{r}}(z)} \eta_1^2 \frac{h'(|\varepsilon(u_\delta)|)}{|\varepsilon(u_\delta)|}|\nabla\varepsilon(u_\delta)|^2\,dx\right\}^3\\ &\le c(R-r)^{-\beta} + c\left\{\int_{B_{\widetilde{r}}(z)} \eta_1^2 D^2H_\delta(\varepsilon(u_\delta))(\partial_\gamma\varepsilon(u_\delta),\partial_\gamma\varepsilon(u_\delta))\,dx\right\}^3. \qquad (2.2)\end{aligned}$$

In order to discuss the integral on the r.h.s. of (2.2) we apply the Caccioppoli-type inequality from Lemma 2.1 c): we have for all $\kappa > 0$

$$\begin{aligned}&\int_{B_{\widetilde{r}}(z)} \eta_1^2 D^2H_\delta(\epsilon(u_\delta))(\partial_\gamma\epsilon(u_\delta),\partial_\gamma\epsilon(u_\delta))\,dx\\ &\qquad\le -2\int_{B_{\widetilde{r}}(z)} \eta_1\partial_\gamma\tau_\delta : (\nabla\eta_1 \odot \partial_\gamma u_\delta)\,dx.\\ &\qquad\le \kappa\int_{B_{\widetilde{r}}(z)} \eta_1^2 D^2H_\delta(\epsilon(u_\delta))(\partial_\gamma\epsilon(u_\delta),\partial_\gamma\epsilon(u_\delta))\,dx\,dx\\ &\qquad+ c(\kappa)\int_{B_{\widetilde{r}}(z)} |\nabla\eta_1|^2\frac{h'(|\epsilon(u_\delta)|)}{|\epsilon(u_\delta)|}\Gamma_\delta^{\frac{\omega}{2}}|\nabla u_\delta|^2\,dx\\ &\qquad+ c(\kappa)\delta\int_{B_{\widetilde{r}}(z)} |\nabla\eta_1|^2\Gamma_\delta^{\frac{q-2}{2}}|\nabla u_\delta|^2 \qquad (2.3)\end{aligned}$$

which follows from

$$|\partial_\gamma\tau_\delta|^2 \le c|\partial_\gamma\sigma_\delta|^2 \le cD^2H_\delta(\epsilon(u_\delta))(\partial_\gamma\epsilon(u_\delta),\partial_\gamma\epsilon(u_\delta))^{\frac{1}{2}}D^2H_\delta(\epsilon(u_\delta))(\partial_\gamma\sigma_\delta,\partial_\gamma\sigma_\delta)^{\frac{1}{2}}$$

in combination with (A3) and Young's inequality. For κ small enough we deduce from (2.3)

$$\begin{aligned}&\int_{B_r(z)} h(|\varepsilon(u_\delta)|)^3\,dx\\ &\le c(R-r)^{-\beta} + \left\{\|\nabla\eta_1\|_\infty^2\int_{B_{\widetilde{r}}(z)} \frac{h'(|\epsilon(u_\delta)|)}{|\epsilon(u_\delta)|}|\epsilon(u_\delta)|^\omega|\nabla u_\delta|^2\,dx\right\}^3. \qquad (2.4)\end{aligned}$$

Here the δ-term from the r.h.s. of (2.3) has been handled as follows: obviously it is enough to control the quantity $\delta \int_{B_{\tilde{r}}(z)} |\nabla u_\delta|^q \, dx$, and according to Korn's inequality an upper bound is given by

$$c\delta \left[\int_{B_{\tilde{r}}(z)} |u_\delta|^q \, dx + \int_{B_{\tilde{r}}(z)} |\varepsilon(u_\delta)|^q \, dx \right].$$

By the interpolation inequality [FS], Lemma 3.0.2, it holds

$$\|u_\delta\|_{L^q(B_{\tilde{r}}(z))} \le c \left[\|u_\delta\|_{L^2(B_{\tilde{r}}(z))} + \|\varepsilon(u_\delta)\|_{L^q(B_{\tilde{r}}(z))} \right]$$

and on account of (1.3) and part a) of Lemma 2.2 the L^2-norms of u_δ are uniformly bounded. Now we quote Lemma 2.1 d) to see that

$$\delta \int_{B_{\tilde{r}}(z)} |\nabla \eta_1|^2 \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta|^2 \, dx \le c(R-r)^{-2}$$

is true.
Let us have a look at the integral on the r.h.s. of (2.4): recalling the monotonicity of $t \mapsto \frac{h'(t)}{t}$ and condition (A3) we find

$$\int_{B_{\tilde{r}}(z)} \frac{h'(|\epsilon(u_\delta)|)}{|\epsilon(u_\delta)|} |\epsilon(u_\delta)|^\omega |\nabla u_\delta|^2 \, dx \le c \int_{B_{\tilde{r}}(z)} \widetilde{h}(|\nabla u_\delta|) \, dx, \tag{2.5}$$

where $\widetilde{h}(t) := h(t)t^\omega$. Consider next a cut-off function $\eta_2 \in C_0^\infty(B_R(z))$ with $\eta_2 \equiv 1$ on $B_{\tilde{r}}(z)$, $0 \le \eta_2 \le 1$ and $|\nabla \eta_2| \le c/(R-\tilde{r})$. Lemma 4.1 in the version for $\widetilde{h}$ implies

$$\begin{aligned}
&\int_{B_{\tilde{r}}(z)} \widetilde{h}(|\nabla u_\delta|) \, dx \le \int_{B_R(z)} \widetilde{h}(|\nabla(\eta_2 u_\delta)|) \, dx \\
&\le c(R-r)^{-\alpha} \left[\int_{B_R(z)} \widetilde{h}(|\varepsilon(u_\delta)|) \, dx + \int_{B_R(z)} \widetilde{h}(|u_\delta|) \, dx \right] \\
&\le c(R-r)^{-\alpha} \left\{ \int_{B_R(z)} \widetilde{h}(|\varepsilon(u_\delta)|) \, dx + 1 \right\}
\end{aligned} \tag{2.6}$$

for an exponent $\alpha > 0$. Note that we have used the (Δ_2)-condition valid also for $\widetilde{h}$ (see [BF3], Lemma A.3) and Lemma 2.2 b) for the derivation of the estimate (2.6). If we combine (2.4)-(2.6) we see (by enlarging β if necessary)

$$\int_{B_r(z)} h(|\varepsilon(u_\delta)|)^3 \, dx \le c(R-r)^{-\beta} \left[1 + \left\{ \int_{B_R(z)} h(|\epsilon(u_\delta)|) |\epsilon(u_\delta)|^\omega \, dx \right\}^3 \right]. \tag{2.7}$$

For $t \in (0,1)$ arbitrary we get

$$\left\{ \int_{B_R(z)} h(|\epsilon(u_\delta)|) |\epsilon(u_\delta)|^\omega \, dx \right\}^3 = \left\{ \int_{B_R(z)} h(|\varepsilon(u_\delta)|)^t h(|\epsilon(u_\delta)|)^{1-t} |\epsilon(u_\delta)|^\omega \, dx \right\}^3$$

$$\leq \left\{ \int_{B_R(z)} h(|\epsilon(u_\delta)|)^{3t}\, dx \right\} \left\{ \int_{B_R(z)} h(|\epsilon(u_\delta)|)^{\frac{3(1-t)}{2}} |\epsilon(u_\delta)|^{\frac{3\omega}{2}}\, dx \right\}^2 .$$

If we split

$$\int_{B_R(z)} h(|\epsilon(u_\delta)|)^{\frac{3(1-t)}{2}} |\epsilon(u_\delta)|^{\frac{3\omega}{2}}\, dx = \int_{B_r(z)\cap[|\varepsilon(u_\delta)|\leq 1]} \dots + \int_{B_r(z)\cap[|\varepsilon(u_\delta)|>1]} \dots \quad ,$$

then clearly the first integral is uniformly bounded. For the second one we choose $t > 0$ sufficiently close to 1 in order to reach

$$h(s)^{\frac{3(1-t)}{2}} s^{\frac{3\omega}{2}} \leq cs^2 \leq ch(s) \quad \text{for } s \geq 1$$

which is possible by (1.3), (1.4) and our assumption $\omega < 4/3$. Hence we can bound the whole integral independent of δ (remember Lemma 2.1 e)), and we have shown

$$\int_{B_r(z)} h(|\varepsilon(u_\delta)|)^3\, dx \leq c(R-r)^{-\beta} \left[1 + \int_{B_R(z)} h(|\epsilon(u_\delta)|)^{3t}\, dx \right] . \tag{2.8}$$

In a final step we use Young's inequality and get for some number $\nu > 0$

$$\int_{B_r(z)} h(|\varepsilon(u_\delta)|)^3\, dx \leq c(R-r)^{-\nu} + \frac{1}{2} \int_{B_R(z)} h(|\varepsilon(u_\delta)|)^3\, dx. \tag{2.9}$$

To inequality (2.9) we may apply Lemma 3.1, p 161, of [Gi3] in order to see that $h(|\varepsilon(u_\delta)|)^3$ is in the space $L^1_{loc}(B)$ uniformly w.r.t. δ. This proves Theorem 1.1 b).
During our calculations we have shown that

$$D^2H(\epsilon(u_\delta))(\partial_\gamma \epsilon(u_\delta), \partial_\gamma \epsilon(u_\delta))\, dx \ \in L^1_{loc}(B) \tag{2.10}$$

holds uniformly w.r.t. the approximation parameter. In fact, if we return to (2.3) and absorb the κ-term in the l.h.s., then (2.10) is an immediate consequence of our integrability result stated after (2.9). From (1.3) and (1.7) in combination with (2.10) we deduce uniform $W^{2,2}_{loc}$-bounds on u_δ, hence $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^3)$ and for suitable subsequences it holds

$$u_\delta \rightharpoonup u \text{ in } W^{2,2}_{loc}(B, \mathbb{R}^3), \quad \nabla u_\delta \to \nabla u \quad \text{a.e. on } B$$

as $\rho \downarrow 0$. Moreover we see that the functions

$$\psi_\delta := \int_0^{|\varepsilon(u_\delta)|} \sqrt{\frac{h'(t)}{t}}\, dt$$

are uniformly bounded in the space $W^{1,2}_{loc}(B)$, thus we have weak $W^{1,2}_{loc}(B)$-convergence of ψ_δ with limit

$$\psi := \int_0^{|\varepsilon(u)|} \sqrt{\frac{h'(t)}{t}}\, dt.,$$

and Theorem 1.1 a) is proved. □

Remark 2.1. *Returning to the Caccioppoli inequality stated in Lemma 2.1 - now with arbitrary matrix $Q \in \mathbb{R}^{3\times 3}$ - it is easy to see that the appropriate variant of (2.3) after absorbing the κ-term and passage to the limit $\rho \to 0$ gives the inequality*

$$\int_B \eta^2 |\nabla \psi|^2 \, dx \leq c \int_B |\nabla \eta|^2 |D^2 H(\varepsilon(u))| |\nabla u - Q|^2 \, dx \tag{2.11}$$

valid for any $\eta \in C_0^\infty(B)$ and all $Q \in \mathbb{R}^{3\times 3}$. Alternatively we may replace $|\nabla\psi|^2$ by $D^2H(\varepsilon(u))(\partial_\gamma \varepsilon(u), \partial_\gamma \varepsilon(u))$ (or just $|\nabla^2 u|^2$) in this inequality. The reader should note that the l.h.s. of the δ-version of (2.11) is treated via lower semicontinuity, whereas on the r.h.s. we use equi-integrability in order to pass to the limit $\rho \to 0$.

3 Proof of Theorem 1.2

Let u denote a local $\mathbb{J}$-minimizer and suppose w.l.o.g. that $\omega \in [1, 4/3)$ in (A3). We further let

$$\widetilde{h}(t) := t^\omega h(t), \ t \geq 0\,,$$

and recall that $\widetilde{h}$ is a N-function. From Lemma 2.2 b) and Theorem 1.1 b) combined with Lemma 4.1 (choosing $\varphi = \widetilde{h}$ there) it follows that u is an element of the space $\in W^1_{\widetilde{h},\mathrm{loc}}(\Omega, \mathbb{R}^3)$, hence the excess-function

$$E(x,r) := \fint_{B_r(x)} |\epsilon(u) - (\epsilon(u))_{x,r}|^2 \, dy + \fint_{B_r(x)} \widetilde{h}(|\epsilon(u) - (\epsilon(u))_{x,r}|) \, dy$$

for balls $B_r(x) \Subset \Omega$ is well-defined. Here and in what follows $\fint f, (f)$ denote the mean value of a function f.

Lemma 3.1. *Fix $L > 0$ and a subdomain $\Omega' \Subset \Omega$. Then there is a constant $C_*(L)$ such that for every $\tau \in (0,1)$ one can find a number $\kappa = \kappa(L, \tau)$ with the following property: if $B_r(x) \subset \Omega'$ and if*

$$|(\epsilon(u))_{x,r}| \leq L, \ E(x,r) \leq \kappa\,, \tag{3.1}$$

then it holds

$$E(x, \tau r) \leq C_*(L) \tau^2 E(x,r)\,. \tag{3.2}$$

Once having established Lemma 3.1, it is standard (see, e.g. Giaquinta's textbook [Gi3]) to prove the desired partial regularity result. It turns out that the regular set Ω_0 is given by

$$\Omega_0 = \left\{ x \in \Omega : \sup_{r>0} |(\epsilon(u))_{x,r}| < \infty \text{ and } \liminf_{r \downarrow 0} E(x,r) = 0 \right\}\,,$$

i.e. Lemma 3.1 shows that the set on the r.h.s. is open and $\nabla u \in C^{0,\alpha}$ there for any $0 < \alpha < 1$. Obviously Ω_0 is a set of full Lebesgue measure.

Proof of Lemma 3.1: We argue by contradiction (compare [Fu1]). Let $L > 0$ and choose $C_* = C_*(L)$ as outlined below. Then, for some $\tau \in (0,1)$, there is a sequence of balls $B_{r_m}(x_m) \Subset \Omega'$ such that

$$\begin{aligned} |(\epsilon(u))_{x_m,r_m}| &\le L,\ E(x_m,r_m) =: \lambda_m^2 \to 0,\ \text{as } m \to \infty\,, & (3.3)\\ E(x_m,\tau r_m) &> C_*\tau^2\lambda_m^2\,. & (3.4) \end{aligned}$$

Letting $A_m := (\epsilon(u))_{x_m,r_m}$ we define for $z \in B_1 := B_1(0)$

$$\begin{aligned} \widetilde{u}_m(z) &:= \frac{1}{\lambda_m r_m}\Big[u(x_m + r_m z) - r_m A_m z\Big], & (3.5)\\ u_m(z) &:= \widetilde{u}_m(z) - R_m(z), & (3.6) \end{aligned}$$

where R_m is the orthogonal projection of $\widetilde{u}_m$ into the space of rigid motions with respect to the $L^2(B_1,\mathbb{R}^3)$ inner product. We get from (3.3) using

$$\epsilon(u_m)(z) = \frac{1}{\lambda_m}\Big[\epsilon(u)(x_m + r_m z) - A_m\Big]$$

the relations

$$|A_m| \le L,\ \fint_{B_1} |\epsilon(u_m)|^2\,dz + \lambda_m^{-2}\fint_{B_1}\widetilde{h}(\lambda_m|\epsilon(u_m)|)\,dz = 1\,. \tag{3.7}$$

On the other hand, (3.4) reads after scaling

$$\fint_{B_\tau} |\epsilon(u_m) - (\epsilon(u_m))_{0,\tau}|^2\,dz + \lambda_m^{-2}\fint_{B_\tau}\widetilde{h}(\lambda_m|\epsilon(u_m) - (\epsilon(u_m))_{0,\tau}|)\,dz > C_*\tau^2\,. \tag{3.8}$$

After passing to suitable subsequences we obtain from (3.7)

$$\begin{aligned} A_m \to: A,\ u_m \rightharpoondown: \overline{u} \quad &\text{in} \quad W_2^1(B_1;\mathbb{R}^3)\,,\\ \lambda_m\epsilon(u_m) \to 0 \quad &\text{in} \quad L^2(B_1;\mathbb{S}) \text{ and a.e.}\,, & (3.9) \end{aligned}$$

where obviously $(\varepsilon(\overline{u}))_{0,1} = 0$. To prove the second convergence we apply Korn's inequality in L^2 (see for example [FS], Lemma 3.0.1 and 3.0.3, and in particular [AM], Proposition 2.6 (g) and Proposition 2.7 (c)) which gives by the choice of R_m

$$\|u_m\|_{W^{1,2}(B)} \le \|\epsilon(u_m)\|_{L^2(B)}\,.$$

If we argue as in [Fu1], (3.8)- (3.15), replacing ∇ by ε and letting $Z_m := Z_m(s,z) := A_m + s\lambda_m\varepsilon(u_m)(z)$, we obtain the limit equation

$$\int_{B_1} D^2H(A)(\epsilon(\overline{u}),\epsilon(\varphi))\,dz = 0$$

valid for any $\varphi \in C_0^\infty(B_1,\mathbb{R}^3)$ such that $\operatorname{div}\varphi = 0$. Quoting standard results on weak solutions of elliptic systems with constant coefficients involving the symmetric gradient as well as the imcompressibility condition (see, e.g., [GM] or [FS], Lemma 3.5) we find that $\overline{u}$ is of class $C^\infty(B_1,\mathbb{R}^3)$ satisfying the Campanato-type estimate

$$\fint_{B_\tau} |\varepsilon(\overline{u}) - (\varepsilon(\overline{u}))_\tau|^2\, dz \le C^*\tau^2 \fint_{B_1} |\varepsilon(\overline{u}) - (\varepsilon(\overline{u}))_1|^2\, dz$$

for a constant $C^* = C^*(L)$. Observing $\fint_{B_1} |\varepsilon(\overline{u})|^2\, dz \le 1$ and $(\varepsilon(u))_1 = 0$, we get

$$\fint_{B_\tau} |\varepsilon(\overline{u}) - (\varepsilon(\overline{u}))_\tau|^2\, dz \le C^*\tau^2.$$

Letting $C_* = 2C^*$ this inequality will give a contradiction to (3.8) as soon as we can show

$$\epsilon(u_m) \to \varepsilon(\overline{u}) \text{ in } L^2_{\mathrm{loc}}(B_1,\mathbb{S})\ , \tag{3.10}$$

$$\lambda_m^{-2} \fint_{B_r} \widetilde{h}\left(\lambda_m|\varepsilon(u_m)|\right)\, dz \to 0,\ r<1. \tag{3.11}$$

For a detailed exposition of how to obtain the desired contradiction we refer to the comments given after (3.18) in [Fu1]. In order to prove (3.10) and (3.11) we return to (2.11) (with $|\nabla u|^2$ in place of $|\nabla\psi|^2$ on the l.h.s.) and get after scaling and with appropriate choice of the testfunction η

$$\int_{B_t} |\nabla^2 u_m|^2\, dz \le C(s-t)^{-2} \int_{B_s} |D^2 H\left(\lambda_m \epsilon(u_m) + A_m\right)| |\nabla u_m|^2\, dz \tag{3.12}$$

valid for $0<t<s<1$. On $[\lambda_m|\epsilon(u_m)| \le K]$ we have

$$\left|D^2 H\left(A_m + \lambda_m\epsilon(u_m)\right)\right| |\nabla u_m|^2 \le c(K)|\nabla u_m|^2\,,$$

whereas on $[\lambda_m|\epsilon(u_m)| \ge K]$ it holds (K large enough)

$$\begin{aligned} &\left|D^2 H\left(\lambda_m\epsilon(u_m) + A_m\right)\right| |\nabla u_m|^2 \\ \le\ & c(K)\left[1 + (\lambda_m|\epsilon(u_m)|)^\omega \frac{h'(\lambda_m|\epsilon(u_m)|)}{\lambda_m|\epsilon(u_m)|}\right] |\nabla u_m|^2 \\ \le\ & c(K)\left[|\nabla u_m|^2 + \lambda_m^{-2}\widetilde{h}\left(\lambda_m|\nabla u_m|\right)\right]. \end{aligned}$$

(3.12) therefore implies (compare (3.20) in [Fu1])

$$\int_{B_t} |\nabla^2 u_m|^2\, dz \le c(s-t)^{-2}\left[\int_{B_s} |\nabla u_m|^2\, dz + \lambda_m^{-2}\int_{B_s} \widetilde{h}\left(\lambda_m|\nabla u_m|\right)\, dz\right]. \tag{3.13}$$

Clearly the first integral on the r.h.s. is uniformly bounded by (3.9). For the second one we deduce from Lemma 4.4 by letting $\widetilde{h}_{\lambda_m}(t) := \lambda_m^{-2}\widetilde{h}(\lambda_m t)$

$$\|\nabla u_m\|_{L_{\widetilde{h}_{\lambda_m}}(B_s)} \le c\, \|\varepsilon(u_m)\|_{L_{\widetilde{h}_{\lambda_m}}(B_1)} + c(s)\, \|u_m\|_{L_{\widetilde{h}_{\lambda_m}}(B_1)}\,.$$

By (3.7) the first term on the r.h.s. is uniformly bounded. In order to get the same result for the second one, we have to estimate the quantity

$$\lambda_m^{-2} \int_{B_1} \widetilde{h}(\lambda_m |u_m|)\, dz.$$

To this purpose we observe that the superquadratic growth of h stated in (1.3) in combination with (3.7) implies the bound

$$\int_{B_1} |\lambda_m^{1-\frac{2}{2+\omega}} \epsilon(u_m)|^{2+\omega}\, dz \leq c\lambda_m^{-2} \int_{B_1} \widetilde{h}(\lambda_m |\epsilon(u_m)|)\, dz \leq c. \tag{3.14}$$

We therefore get by Korn's inequality in spaces L^p (see again [FS] or [AM]) and the choice of R_m

$$||\lambda_m^{1-\frac{2}{2+\omega}} u_m||_{W^{1,2+\omega}(B_1)} \leq c||\lambda_m^{1-\frac{2}{2+\omega}} \epsilon(u_m)||_{L^{2+\omega}(B_1)},$$

hence we find

$$\lambda_m^{1-\frac{2}{2+\omega}} u_m \in L^t(B_1, \mathbb{R}^3) \text{ uniformly in } m \text{ for all } t < \infty \tag{3.15}$$

by quoting Sobolev's theorem for $n = 3$ (recall $\omega \geq 1$). From the inequalities (see (A1) and (1.3))

$$h(t) \leq ct^2 \text{ for } t \leq 1 \text{ and } h(t) \leq ct^q \text{ for } t \geq 1$$

in combination with (3.15) we deduce

$$\lambda_m^{-2} \int_{B_1} \widetilde{h}(\lambda_m |u_m|)\, dz \leq c \int_{B_1} |\lambda_m^{1-\frac{2}{2+\omega}} u_m|^{2+\omega}\, dz + c \int_{B_1} |\lambda_m^{1-\frac{2}{q+\omega}} u_m|^{q+\omega}\, dz \leq c$$

independent of m. Note that we have used the estimate

$$\lambda_m^{1-\frac{2}{q+\omega}} \leq c\lambda_m^{1-\frac{2}{2+\omega}},$$

which follows from $q \geq 2$. Hence we can bound the r.h.s. of (3.13) uniformly in m and therefore we obtain uniform L^2_{loc}-bounds on $\nabla^2 u_m$, which shows (3.10).
For proving our claim (3.11) we return to (2.11) (in the version with $D^2H(\varepsilon(u))(\partial_\gamma \varepsilon(u), \partial_\gamma \varepsilon(u))$ in place of $|\nabla\psi|^2$ on the l.h.s.) and observe that after scaling the r.h.s. of (3.12) provides an upper bound for the quantity

$$\int_{B_t} \frac{h'(|\lambda_m \epsilon(u_m) + A_m|)}{|\lambda_m \epsilon(u_m) + A_m|} |\nabla^2 u_m|^2\, dz := a_m(t)\,.$$

On the other hand, our previous calculations guarantee uniform bounds for the r.h.s. of (3.12) so that we arrive at

$$a_m(t) \leq c(t) \tag{3.16}$$

for finite constants $c(t)$, $0<t<1$. We introduce the auxiliary functions

$$\Psi_m := \frac{1}{\lambda_m}\left\{\int_0^{|\lambda_m\epsilon(u_m)+A_m|}\sqrt{\frac{h'(t)}{t}}\,dt - \int_0^{|A_m|}\sqrt{\frac{h'(t)}{t}}\,dt\right\}$$

and deduce from (3.16)

$$\int_{B_t}|\nabla\Psi_m|^2\,dz \le c(t)\,. \tag{3.17}$$

Following the lines of [Fu1] (after (3.22)) (replacing ∇ by ε) we easily obtain $\int_{B_1}|\Psi_m|^2\,dz \le c$ and therefore together with (3.17) it is shown that

$$\|\Psi_m\|_{W_2^1(B_t)} \le c(t) < \infty\,, 0<t<1. \tag{3.18}$$

With (3.18) we can exactly repeat the arguments presented after (3.23) in the paper [Fu1] ending up with (3.11). Note that the condition

$$t^\omega \le c\left[h(t)^2+1\right] \quad (t\ge 0)$$

required in [Fu1] is clearly satisfied in our context as a consequence of the superquadratic growth of h and the hypothesis $\omega < 4/3$. This completes the proof of Lemma 3.1. □

4 Appendix

In this section we collect some auxiliary material concerning Korn type inequalities, which are a crucial tool for solving the global problem (1.8) and also for proving the strong convergences (3.10) and (3.11).We start with

Lemma 4.1. *a) Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^n$ and let φ denote a N-function of class $(\Delta_2)\cap(\nabla_2)$ (see, e.g., [RR] for a definition). Then there is a constant $C = C(n,\varphi,\Omega)$ such that*

$$\int_\Omega \varphi(|\nabla w|)\,dz \le c\int_\Omega \varphi(|\varepsilon(w)|)\,dz$$

holds for any $w\in W_0^{1,\varphi}(\Omega,\mathbb{R}^n)$.

b) In the case that Ω is a ball $B_R(x_0)$ the constant C has the form

$$C = c(n,\varphi)R^{-\beta}$$

for a positive exponent β.

The proof of Lemma 4.1 a) is presented in [Fu3], part b) can easily be derived from this first inequality by scaling and using the (Δ_2)-property of φ. □

Suppose now that h satisfy (A1)-(A3). Then we have

$$th'(t) = \int_0^t \frac{d}{ds}\left[sh'(s)\right]\,ds = h(t) + \int_0^t sh''(s)\,ds \geq 2h(t),$$

and in conclusion

$$a(h) := \inf_{t>0} \frac{h'(t)t}{h(t)} \geq 2.$$

Therefore h is a N-function of (global) type (∇_2), which follows from Corollary 4 on p. 26 in [RR], and we have

Corollary 4.1. *The Korn type inequalities stated in Lemma 4.1 hold for the N-function h.*

Remark 4.1. *If we consider the N-function $\widetilde{h}(t) = t^\omega h(t)$, then we have $a(\widetilde{h}) \geq 2 + \omega$, hence Lemma 4.1 applies to $\widetilde{h}$ as well.*

Remark 4.2. *Using the interpolation argument outlined in the work of Acerbi and Mingione [AM] we obtain the Korn inequality in terms of the Luxemburg norm*

$$\|\nabla w\|_{L_h(\Omega)} \leq c(n,h,\Omega)\,\|\nabla\varepsilon(w)\|_{L_h(\Omega)}$$

valid for fields $w \in W_0^{1,h}(\Omega,\mathbb{R}^n)$. We refer to Lemma 4.3 and Lemma 4.4, where this interpolation argument is applied to the sequence $\widetilde{h}_{\lambda_m}$ defined in Section 3.

Lemma 4.2. *Let h satisfy (A1)-(A3), consider $u_0 \in W^{1,h}(\Omega,\mathbb{R}^n)$ such that $\operatorname{div} u_0 = 0$ and define the class $\mathcal{C}$ as done in section 1. Then the variational problem (1.8) admits a unique solution u in $\mathcal{C}$.*

Proof: If $u_k \in \mathcal{C}$ denotes a minimizing sequence, then Lemma 4.1 a) (applied to $u_k - u_0$) in combination with the Poincaré inequality from [FO] gives the boundedness of u_k in the space $W^{1,h}(\Omega,\mathbb{R}^n)$. Since h is of type $(\Delta_2) \cap (\nabla_2)$, we see that $W^{1,h}$ is reflexive (compare [RR], Corollary 4 on p. 26, and [Ad], Theorem 8.28), and our claim follows from standard arguments. □

Next we are going to prove that we have uniform Korn type inequalities for the scaled N-functions

$$\widetilde{h}_\lambda(t) := \lambda^{-2}\widetilde{h}(\lambda t),$$

where $\lambda > 0$ denotes a parameter and where $\widetilde{h}(s) := s^\omega h(s)$, h satisfying (A1)-(A3). This will be done along the lines of [AM], proof of Theorem 3.1, using the following auxiliary result:

Lemma 4.3. *We can find some exponents $p_1, p_2 > 1$ such that the function $\widetilde{h}_\lambda(t)/t^{p_1}$ increases and the function $\widetilde{h}_\lambda(t)/t^{p_2}$ decreases. Furthermore there are positive constants k_1 and k_2 independent of λ such that the estimates*

$$\int_0^t \frac{\widetilde{h}_\lambda(s)}{s^{p_1}} \frac{ds}{s} \le k_1 \frac{\widetilde{h}_\lambda(t)}{t^{p_1}}, \tag{4.1}$$

$$\int_t^\infty \frac{\widetilde{h}_\lambda(s)}{s^{p_2}} \frac{ds}{s} \le k_2 \frac{\widetilde{h}_\lambda(t)}{t^{p_2}}, \tag{4.2}$$

hold for all $t > 0$.

Proof: We set $p_1 := 1 + \omega$ and choose $p_2 > \omega + \overline{k}$ with ω and $\overline{k}$ from (A2) and (A3). It follows

$$\frac{\widetilde{h}_\lambda(t)}{t^{p_1}} = \lambda^{\omega-2} \frac{h(\lambda t)}{t}$$

which is increasing on account of $th'(t) - h(t) \ge 0$. Moreover we have by (A3) and $h'(0) = 0$

$$\begin{aligned}\int_0^t \frac{\widetilde{h}_\lambda(s)}{s^{p_1}} \frac{ds}{s} &= \lambda^{\omega-2} \int_0^t \frac{h(\lambda s)}{s} \frac{ds}{s} \le \lambda^{\omega-1} \int_0^t \frac{h'(\lambda s)}{s} \, ds \\ &\le \lambda^{\omega} \int_0^t h''(\lambda s) \, ds = \lambda^{\omega-1} h'(\lambda t).\end{aligned}$$

If we use (1.5), we get from this estimate

$$\int_0^t \frac{\widetilde{h}_\lambda(s)}{s^{p_1}} \frac{ds}{s} \le \overline{k} \lambda^{\omega-1} \frac{h(\lambda t)}{\lambda t} = \overline{k} \frac{\widetilde{h}_\lambda(t)}{t^{p_1}},$$

hence (4.1) holds with $k_1 = \overline{k}$. From (1.5) we obtain

$$\frac{d}{dt}\left[\widetilde{h}_\lambda(t)/t^{p_2}\right] \le \lambda^{\omega-2} t^{\omega-p_2-1}\left[h'(\lambda t)\lambda t - \overline{k} h(\lambda t)\right] \le 0,$$

hence $\widetilde{h}_\lambda(t)/t^{p_2}$ is decreasing. Finally we prove (4.2): since the function $s \mapsto h(s)/s^{\overline{k}}$ is also decreasing, we have

$$\begin{aligned}\int_t^\infty \frac{\widetilde{h}_\lambda(s)}{s^{p_2}} \frac{ds}{s} &= \lambda^{\omega+\overline{k}-2} \int_t^\infty \frac{h(\lambda s)}{(\lambda s)^{\overline{k}}} \frac{1}{s^{p_2-\overline{k}-\omega}} \frac{ds}{s} \\ &\le \lambda^{\omega+\overline{k}-2} \frac{h(\lambda t)}{(\lambda t)^{\overline{k}}} \int_t^\infty s^{-1-p_2+\overline{k}+\omega} \, ds \\ &= \frac{1}{p_2 - \overline{k} - \omega} \lambda^{\omega+\overline{k}-2} \frac{h(\lambda t)}{(\lambda t)^{\overline{k}}} t^{\overline{k}+\omega-p_2} \\ &= \frac{1}{p_2 - \overline{k} - \omega} \frac{\widetilde{h}_\lambda(t)}{t^{p_2}},\end{aligned}$$

which completes the proof of Lemma 4.3. $\square$

Lemma 4.4. *With the notation introduced before Lemma 4.3 we have for all* $w : \Omega \to \mathbb{R}^n$ *with* $|w|, |\varepsilon(w)| \in L_{\widetilde{h}_\lambda}(\Omega)$ *and all* $\Omega^* \Subset \Omega$

$$\|\nabla w\|_{L_{\widetilde{h}_\lambda}(\Omega^*)} \le c_1(h) \|\varepsilon(w)\|_{L_{\widetilde{h}_\lambda}(\Omega)} + c_2(h, \Omega^*) \|w\|_{L_{\widetilde{h}_\lambda}(\Omega)},$$

where the constants c_i *are independent of the parameter* λ. *Moreover,* c_2 *growth like* $\mathrm{dist}(\partial\Omega, \Omega^*)^{-1}$.

Proof: From Lemma 4.3 and [AM], Theorem 3.3, we get for all $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$\|\nabla v\|_{L_{\widetilde{h}_\lambda}(\mathbb{R}^n)} \le c(h) \|\varepsilon(v)\|_{L_{\widetilde{h}_\lambda}(\mathbb{R}^n)} \tag{4.3}$$

with a positive constant $c(h)$ being independent of λ. For w with $|w|, |\varepsilon(w)| \in L_{\widetilde{h}_\lambda}(\Omega)$ and $\Omega^* \Subset \Omega$ we choose $\eta \in C_0^\infty(\Omega)$ such that $\eta \equiv 1$ on Ω^*, $0 \le \eta \le 1$ and $|\nabla\eta| \le c/\,\mathrm{dist}(\Omega^*, \partial\Omega)$. From (4.3) applied to $v := \eta w$ we conclude (using a standard approximation argument)

$$\|\nabla w\|_{L_{\widetilde{h}_\lambda}(\Omega^*)} \le c(h) \|\varepsilon(w)\|_{L_{\widetilde{h}_\lambda}(\Omega)} + c(h) \|\nabla\eta \odot w\|_{L_{\widetilde{h}_\lambda}(\Omega)},$$

and the claim of Lemma 4.4 is a consequence of the choice of η. □

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