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# Monotonicity of Implicit Finite Difference Methods for Hyperbolic Conservation Laws

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**Abstract**

Hyperbolic conservation laws (HCLs) are a class of partial differential equations that model transport processes. Many important phenomena in natural sciences are described by them. In this paper we consider finite difference methods for the approximation of HCLs. As HCLs describe an evolution in time, one may distinguish explicit and implicit schemes by the corresponding time integration mechanism employed by them. Explicit numerical schemes are well-analysed. In the explicit setting, the monotonicity property of a method is the key to approximate the physically relevant entropy solution of a HCL. However, there does not exist a rigorous general approach to implicit monotone methods in the literature up to now.

In the current work, this open issue is addressed. We propose monotonicity conditions for fully implicit schemes, and we prove that they are meaningful. The relation between an implicit monotone scheme and a discrete entropy inequality is constructed in a similar fashion as in the classic explicit approach of Crandall and Majda. The convergence of implicit monotone schemes is verified in a framework that does not rely on a compactness property of the underlying function space. All proofs are given for the case of a scalar HCL in multiple space dimensions. They can easily be extended to HCLs with source terms or specialised to the 1-D case.

We apply the developed notion of monotonicity by investigating implicit variants of some well known explicit monotone schemes. As a surprising result, a stability restriction on the time step size may arise. This contradicts the usual intuition that implicit schemes give unconditional stability. Such a restriction is established for the implicit Lax-Friedrichs scheme, and it is illustrated by numerical tests.

# 1 Introduction

Many fundamental physical principles are based on the conservation of certain quantities, like e.g. conservation of mass, momentum or energy. Thus, many important phenomena in natural sciences and engineering are described by conservation laws. By *hyperbolic conservation laws (HCLs)* one denotes a class of partial differential equations (PDEs). They model time-dependent transport processes that obey in addition a conservation principle, cf. [14, 15, 17] for diverse fields of applications where HCLs arise.
In this paper we consider finite difference methods for the approximation of HCLs. As these describe an evolution in time, one may distinguish *explicit* and *implicit schemes* by the time integration mechanism used within the

methods. Explicit schemes can be used under the assumption of finite wave velocities. In this case, the waves can be resolved when they travel from cell to cell on a computational grid. Moreover, given an initial state that is non-zero on a bounded domain, an evolution over a finite time will also stay within a bounded domain. It is well known that the *monotonicity* property of an explicit method is needed to approximate the physically relevant entropy solution of a HCL, cf. [6].

While explicit schemes are well analysed – see e.g. the textbooks [13, 14] and the references therein – there does not exist a mathematically rigorous general approach to monotone implicit methods in the literature up to now. It is just generally *assumed* that an implicit time discretisation gives unconditional stability, i.e. especially no restriction on the time step size, and that the structural properties of an implicit scheme are identical to the properties of its explicit counterpart.

In the current work, we address this open issue. We present a general framework for monotonicity of implicit schemes, and we give proofs of the essential assertions that are the cornerstones of the implicit framework. By applying the new implicit monotonicity conditions, we show that our proceeding gives a meaningful extension of the well known explicit theory, with some surprising results.

**Theoretical background.** As indicated, we consider in this paper finite difference methods for the approximation of scalar HCLs in multiple space dimensions. In this setting, quite general theoretical results are available concerned with existence and uniqueness of entropy solutions, cf. [1, 10, 11] and the references therein. Also, an important structural property is valid, namely a *comparison principle* of solutions: Given two initial states $u_0$ and $v_0$ with $u_0 \geq v_0$ a.e., it also holds $u(\cdot, t) \geq v(\cdot, t)$ when evolving $u_0$ and $v_0$ in time by a HCL.

This comparison principle can be translated into a powerful, non-linear stability notion in the discrete setting of numerical schemes, namely the notion of *monotonicity*. This works in general as follows. Let sets of discrete data $U^n$, $V^n$, $W^n$ and $W^{n+1}$ be given, where $n$ denotes an iteration level. Furthermore, let a numerical method be described by an operator $\mathcal{H}$. Then the method is monotone, if and only if one obtains for the numerical method

$$W^{n+1} = \mathcal{H}\left(W^n, W^{n+1}\right) , \tag{1}$$

the validity of

$$U^n \geq V^n \text{ (componentwise)} \quad \overset{\mathcal{H}}{\Rightarrow} \quad U^{n+1} \geq V^{n+1} \text{ (componentwise)}. \tag{2}$$

The method given by $\mathcal{H}$ is *explicit* if (1) reduces to $W^{n+1} = \mathcal{H}(W^n)$, and *implicit* if the definition (1) is fulfilled with a true dependence of $\mathcal{H}$ on both data sets $(W^n, W^{n+1})$.

It is well known that monotone schemes must be used at non-linear discontinuous solution features, so-called shocks, in order to capture the physically relevant entropy solution. In the discrete setting, this requires the method to obey a discrete entropy inequality. For *explicit* monotone schemes, a meaningful construction of a discrete entropy inequality was given in [6]. In that paper, the mathematical assumptions are that *(i)* the flux functions appearing in HCLs are Lipschitz continuous, and *(ii)* over finite time, the domain of the solution stays bounded.

Let us note that monotone schemes are inevitably of first-order accuracy which somewhat limits their practical use, however, all high-resolution schemes incorporate a monotone method for use at shocks, see e.g. [8, 13, 14] for discussions. Thus, monotone methods are a fundamental building block of numerical schemes in the field of hyperbolic PDEs.

**Our contribution.** The contributions documented in this paper are as follows.

• We propose monotonicity conditions for general implicit schemes, and we verify them rigorously by an unconventional proof in the field.

• In order to ensure that the entropy solution is well approximated, one needs to establish a discrete entropy inequality. We realise this for implicit monotone schemes and in a setting where even just continuous fluxes are allowed in a similar way as in the classic approach of Crandall and Majda [6]. That this can be done is not self-evident; it is based on the fact that even in the original derivation of this relation no technique is used which relies on the Lipschitz property of the flux.

• We prove the convergence of monotone implicit schemes in a framework that does not rely on the compactness of the underlying function space. The idea of this approach has been introduced in [2, 3] and applied with the implicit upwind scheme. By making use of this idea, we do not rely on finite wave speeds or solutions on bounded domains. This distinguishes our procedure from the one of Crandall and Majda [6], from the TVD-approach relying on Helly's theorem – compare the useful discussion in [13] – as well as from the Kuznetsov approach to convergence [12], and from the concept of measure valued solutions [7].

• Intuitively, one may assume that by an implicit formulation one gains the unconditional stability of the numerical schemes, since the characteristics of the true solution are always included in the region of numerical dependence, cf. [14]. This intuition relates to the idea of Courant, Friedrichs and Lewy [5] on numerical stability that gave rise to the celebrated CFL-condition. The implicit upwind scheme may indeed capture solutions even with quite extraordinary waves of infinite speed [3]. However, we show that the use of implicit schemes does in general *not* imply the unconditional stability of the methods. At hand of the example of the implicit Lax-Friedrichs scheme, it becomes evident that a stability restriction on the time step size may nevertheless arise. As this result is quite surprising, it is discussed here in detail.

All our proofs are given for the case of a scalar HCL in multiple space dimensions. They can easily be extended to HCLs with source terms or specialised to the 1-D case. Thus, one may also understand our efforts as a constructive proof of the existence of entropy solutions in multiple dimensions with source terms. The corresponding, general settings are determined by *(i)* smooth fluxes together with non-linear sources, or *(ii)* continuous fluxes plus sources depending on space and time, cf. [1, 10, 11].

**Related work.** While implicit methods for HCLs are often applied in steady state computations, there is no theoretical paper on this topic with the exception of some previous work of the author. By the current article, we significantly extend the work described in [3]. In that paper, only the implicit upwind method is considered in 1-D, and all the proofs given there are also specifically constructed for that relatively simple numerical scheme. The general implicit monotonicity conditions presented here require a new, more unconventional proof. Some other techniques introduced in [2, 3] are generalised within the current work: The approach to prove convergence is extended from the 1-D case without source terms to the more general setting discussed here, and the relationship of monotone implicit schemes to a discrete entropy inequality is put into more general terms similar to the explicit case addressed by Crandall and Majda [6].

**Paper organisation.** The paper is structured in accordance to the points mentioned above. In Section 2, we briefly review the analytical setting of entropy solutions of HCLs in multiple dimensions with source terms, and we briefly recall there the major difficulty of non-compact solution domains. We elaborate on the new monotonicity conditions for numerical schemes in Section 3. We show how to construct a discrete entropy inequality in Section 4. The fifth section is devoted to the convergence proof. In Section 6, we discuss the application of the developed monotonicity conditions to some implicit variations of well known explicit monotone schemes. The paper is finished by a summary with conclusion.

# 2 Analytical results for scalar HCLs revisited

The purpose of this section is to briefly review two basic settings of scalar HDEs in multiple dimensions with source terms that we address in this paper. The crucial point for our work is, that uniqueness results for entropy solutions as well as comparison principles are available in these settings, cf. [1, 9] and the references therein. Also the existence of solutions is shown in that works, however, as indicated the approach in this paper can also be understood as a constructive existence proof. Furthermore, we briefly recall a principle difficulty arising in one of the settings, cf. [3].

## 2.1 HCLs with continuous fluxes and space-dependent source terms

The first type of *initial value problems (IVPs)* under consideration is

$$\frac{\partial}{\partial t} u\left(\mathbf{x},t\right) + \sum_{l=1}^{d} \frac{\partial}{\partial x_l} f_l\left(u\left(\mathbf{x},t\right)\right) \;=\; q \qquad \text{on} \qquad \mathbb{R}^d \times (0,T)\ , \tag{3}$$

$$u\left(\mathbf{x},0\right) \;=\; u_0\left(\mathbf{x}\right) \qquad \text{on} \qquad \mathbb{R}^d\,, \tag{4}$$

where $u(\mathbf{x},t)$ is the sought unknown, $\mathbf{x} := (x_1,\ldots,x_d)^\top$ is to be interpreted as a vector of space variables, $t$ denotes time, and where $T$ is a fixed positive number defining the interval of time integration. Concerning the *flux functions* $f_l$ one assumes

$$f_l(u) \in C\left(\mathbb{R};\,\mathbb{R}\right), \quad l = 1,\,\ldots,\,d\,. \tag{5}$$

In order to apply the uniqueness theorem given in [1], the fluxes are additionally supposed to satisfy the growth conditions

$$|f_l(u) - f_l(\hat{u})| \le \omega_l(u - \hat{u}) \;\text{ a.e. for }\; u \ge \hat{u} \;\text{ and for }\; l = 1,\ldots,d\,, \tag{6}$$

with the moduli of continuity $\omega_l$ featuring

$$\omega_1(0) = \ldots = \omega_d(0) = 0 \quad \text{and} \quad \liminf_{r\to 0} \left[ r^{1-d} \prod_{l=1}^{d} \omega_l(r) \right] < \infty\,. \tag{7}$$

These conditions on the fluxes are less restrictive than the usually assumed Lipschitz continuity. The *initial condition* shall satisfy

$$u_0 \in L^\infty_{loc}\left(\mathbb{R}^d;\,\mathbb{R}\right)\,, \tag{8}$$

and for the source term we consider

$$q \equiv q(\mathbf{x},t) \in L^1_{loc}\left(\mathbb{R}^d \times (0,T);\, \mathbb{R}\right), \tag{9}$$

$$q(\cdot,t) \in L^\infty\left(\mathbb{R}^d;\, \mathbb{R}\right) \quad \text{for a.e.} \quad t \in (0,T) \quad \text{and} \quad \int_0^T ||q(\cdot,t)||_\infty \,\mathrm{d}t < \infty. \tag{10}$$

Under the conditions (5)-(10), Bénilan and Kružkov [1] proved uniqueness of the entropy solution of (3), (4).
Because the solution of the Cauchy problem generally developes discontinuities even if $u_0$ is smooth, it is often considered in its weak form. This means, for all test functions $\phi \in C_0^\infty\left(\mathbb{R}^{d+1};\, \mathbb{R}\right)$ shall hold

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \left[ u(\mathbf{x},t)\,\phi_t(\mathbf{x},t) + \sum_{l=1}^d f_l(u(\mathbf{x},t)) \frac{\partial}{\partial x_l}\phi(\mathbf{x},t) \right] \mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \; -\int_{\mathbb{R}^d} u_0(\mathbf{x})\,\phi_0(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_0^\infty \int_{\mathbb{R}^d} q(\mathbf{x},t)\,\phi(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\mathrm{d}t\,. \end{aligned} \tag{11}$$

It is well known that weak solutions are in general not unique, see e.g. [13]. In order to ensure uniqueness, a so-called *entropy condition* has to be introduced. The entropy condition due to Kružkov [1] which guarantees the uniqueness of a solution of (3), (4) takes the form

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \Bigg[ |u(\mathbf{x},t) - \xi|\phi_t(\mathbf{x},t) \\ &\qquad + \sum_{l=1}^d \mathsf{sgn}\,(u(\mathbf{x},t) - \xi)\,[f_l(u(\mathbf{x},t)) - f_l(k)]\,\frac{\partial}{\partial x_l}\phi(\mathbf{x},t) \Bigg]\,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &\geq \; -\int_{\mathbb{R}^d} |u_0(\mathbf{x}) - \xi|\phi_0(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &\qquad - \int_0^\infty \int_{\mathbb{R}^d} \mathsf{sgn}\,[u(\mathbf{x},t) - \xi]\,q(\mathbf{x},t)\,\phi(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\mathrm{d}t\,, \end{aligned} \tag{12}$$

for all $\phi \in C_0^\infty\left(\mathbb{R}^{d+1};\, \mathbb{R}\right)$ with $\phi \geq 0$ and for all $\xi \in \mathbb{R}$. Thereby, $\mathsf{sgn}(\cdot)$ denotes the signum function.

## 2.2 HCLs with differentiable fluxes and non-linear source terms

We also deal with the IVP

$$\frac{\partial}{\partial t}u(\mathbf{x},t) + \sum_{l=1}^d \frac{d}{dx_l} f_l(\mathbf{x},t,u(\mathbf{x},t)) \;=\; q \quad \text{on} \quad \mathbb{R}^d \times (0,T)\,, \tag{13}$$

$$u(\mathbf{x},0) \;=\; u_0(\mathbf{x}) \quad \text{on} \quad \mathbb{R}^d, \tag{14}$$

where $T$ is again a fixed positive number. As the fluxes can now be dependent on space and time, one needs to employ the total differential

$$\frac{d}{dx_l} f_l \equiv (f_l)_{x_l} + (f_l)_u \, u_{x_l} \, . \tag{15}$$

In comparison to the first scenario from Section 2.1, especially different assumptions on the fluxes and the source terms are imposed. As in (8), there is no particular condition on the initial data. The *flux functions* are now assumed to satisfy

$$f_l(\mathbf{x}, t, u) \in C^1 \left( \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}; \mathbb{R} \right) , \quad l = 1, \, \ldots, \, d. \tag{16}$$

As source terms, functions

$$q \equiv q\left(\mathbf{x}, t, u(\mathbf{x}, t)\right) \in C^1 \left( \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}; \mathbb{R} \right) \tag{17}$$

are considered. Under the conditions (16) and (17), Kružkov [9] proved the uniqueness of the entropy solution of (13), (14).
Comparing the weak formulation of this problem with the weak formulation (11), one has to substitute

$$\int_0^\infty \int_{\mathbb{R}^d} q\left(\mathbf{x}, t, u(\mathbf{x}, t)\right) \phi\left(\mathbf{x}, t\right) \, \mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{for} \quad \int_0^\infty \int_{\mathbb{R}^d} q\left(\mathbf{x}, t\right) \phi\left(\mathbf{x}, t\right) \, \mathrm{d}\mathbf{x}\mathrm{d}t. \tag{18}$$

The Kružkov entropy condition corresponding to (13), (14) then reads as

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \Bigg[ |u\left(\mathbf{x}, t\right) - \xi| \phi_t\left(\mathbf{x}, t\right) \\
&\quad + \sum_{l=1}^{d} \mathsf{sgn}\left(u\left(\mathbf{x}, t\right) - \xi\right) \left[ f_l(\mathbf{x}, t, u\left(\mathbf{x}, t\right)) - f_l(\mathbf{x}, t, k) \right] \frac{\partial}{\partial x_l} \phi\left(\mathbf{x}, t\right) \Bigg] \mathrm{d}\mathbf{x}\mathrm{d}t \\
&\qquad \geq \; - \int_{\mathbb{R}^d} |u_0\left(\mathbf{x}\right) - \xi| \phi_0\left(\mathbf{x}\right) \, \mathrm{d}\mathbf{x} \\
&\qquad\qquad - \int_0^\infty \int_{\mathbb{R}^d} \sum_{l=1}^{d} \mathsf{sgn}\left[u\left(\mathbf{x}, t\right) - \xi\right] \left[ q\left(\mathbf{x}, t, u(\mathbf{x}, t)\right) - f_{l_{x_l}}\left(\mathbf{x}, t, k\right) \right] \phi\left(\mathbf{x}, t\right) \, \mathrm{d}\mathbf{x}\mathrm{d}t
\end{aligned}$$

for all $\phi \in C_0^\infty \left( \mathbb{R}^{d+1}; \mathbb{R} \right)$ with $\phi \geq 0$ and for all $\xi \in \mathbb{R}$.

## 2.3 A HCL with solution of non-compact support

In order to point out the difficulties encountered when approaching the introduced general settings numerically, we briefly discuss a consequence of the basic assumptions from Section 2.1 by a special 1-D example. If the flux function of a nonlinear conservation law is not Lipschitz continuous as allowed in the setting of Section 2.2, it may happen that degenerate waves of infinite speed appear. This is the case in an example given by Kružkov and Panov in [11], which is concerned with the equation

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left(\frac{|u(x,t)|^{\alpha}}{\alpha}\right) = 0\,, \quad \alpha \in (0,1),\, t > 0,\, x \in \mathbb{R}. \tag{19}$$

Given the initial condition

$$u_0(x) \;=\; \begin{cases} 0 \;:\; & x < -1, \\ 1 \;:\; & -1 \le x \le 0, \\ 0 \;:\; & x > 0, \end{cases} \tag{20}$$

the exact solution defined over a time interval depending on the exact choice of $\alpha$ reads

$$u(x,t) \;=\; \begin{cases} 0 & : \quad t > \alpha(x+1), \\ 1 & : \quad x < t \le \alpha(x+1), \\ \left(\frac{t}{x}\right)^{1/(1-\alpha)} & : \quad t \le x. \end{cases} \tag{21}$$

This solution incorporates a rarefaction wave connecting the states $u = 1$ and $u = 0$; the latter is located at infinity after arbitrarily small time. The reason for the appearance of such a wave is, that the flux features a pole at $u = 0$. Let us stress, that the domain of the solution is infinite for the initial condition $u_0(x)$ from (20) which has compact support.

# 3 Monotonicity of numerical schemes for HCLs in multiple dimensions

This section is structured by two parts. In the first paragraph, we clarify the notation and show some basic results important for the proceeding. In the second paragraph, we discuss the main assertions of the paper.

## 3.1 The set-up

Since we want to deal with numerical methods in $d$ spatial dimensions, we will spend some efforts on a general notation. At first we introduce the grid points. For this, we employ uniform grid spacings $\Delta x_l$ corresponding to the space dimensions $l = 1, \ldots, d$, and $\Delta t$ corresponding to time. This results in a countable number of grid points. We introduce a *linear numbering* $J$ of the spatial grid points

$$J = \{0, 1, 2, \ldots\}\,. \tag{22}$$

We also define a bijective mapping

$$\begin{aligned} \tilde{J} \quad : \quad J &\longrightarrow \mathbb{R}^d \\ i &\mapsto (i_1 \Delta x_1,\, i_2 \Delta x_2,\, \ldots,\, i_d \Delta x_d)^\top \quad \text{with} \quad (i_1,\, i_2,\, \ldots,\, i_d)^\top \in \mathbb{Z}^d \end{aligned} \tag{23}$$

In order to describe the indices within the stencil of a numerical method, we define the indices $i \pm \delta l$ that denote the left and right neighbours of the point $i$ w.r.t. the $l$-th space coordinate via

$$i \pm \delta l \ \overset{\tilde{J}}{\longrightarrow} \ (i_1 \Delta x_1,\, i_2 \Delta x_2,\, \ldots,\, (i_l \pm 1)\, \Delta x_l,\, \ldots,\, i_d \Delta x_d)^\top \,. \tag{24}$$

Let $U_j^k$ and $q_j^k$ denote the value of the numerical solution and the value of the source term at the point with the index $j \in J$ at the time level $k\Delta t$, respectively. With these notations, we consider *conservative* $(2d+1)$-point implicit methods in the form

$$U_j^{n+1} \ = \ U_j^n - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ g_l \left( U_j^{n+1},\, U_{j+\delta l}^{n+1} \right) - g_l \left( U_{j-\delta l}^{n+1},\, U_j^{n+1} \right) \right\} + \Delta t q_j^{n+1}. \tag{25}$$

We assume that the numerical flux functions $g_l$ introduced in (25) are *consistent*, i.e.

$$g_l(v,\, v) \ = \ f_l(v) \ \text{ holds for all } v \in \mathbb{R} \text{ and for all } l = 1,\, \ldots,\, d. \tag{26}$$

In the case of the theoretical set-up described in Section 2.2, we simply need to add arguments $(\mathbf{x}_j, t^{n+1})$ within the fluxes. We will not do this explicitly in the following.

We now recall the definition of a *monotone method*, cf. (2). It will be useful to specify the scheme mappings $\mathcal{H}$ and $H_l$ using $\underline{d} = \{1,\, \ldots,\, d\}$ via

$$\begin{aligned} U_j^{n+1} \ &= \ \mathcal{H} \left( l \in \underline{d};\, U_{j-\delta l}^{n+1},\, U_j^{n+1},\, U_{j+\delta l}^{n+1},\, U_j^n \right) && (27) \\ &= \ U_j^n - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ g_l \left( U_j^{n+1},\, U_{j+\delta l}^{n+1} \right) - g_l \left( U_{j-\delta l}^{n+1},\, U_j^{n+1} \right) \right\} + \Delta t q_j^{n+1} \\ &= \ U_j^n + \sum_{l=1}^{d} H_l \left( U_{j-\delta l}^{n+1},\, U_j^{n+1},\, U_{j+\delta l}^{n+1} \right) + \Delta t q_j^{n+1}. && (28) \end{aligned}$$

In what follows, we always consider conservative and consistent methods of the general structure described above.

The first important *property of numerical schemes* is concerned with the numerical flux functions. In what follows, we always consider numerical fluxes $g_l(v, w)$ that satisfy the following *two conditions*:

*(g1).* $g_l(v,w)$ is a monotonously growing function of the variable $v$,

*(g2).* $g_l(v,w)$ is a monotonously decreasing function of the variable $w$.

This set-up is fundamental for our approach. Note that the conditions (g1) and (g2) are analogous to corresponding properties of numerical flux functions for monotone schemes in the explicit case.

An essential technical assertion for our proceeding is the following.

**Lemma 3.1** *Let $a_l$, $\Delta a_l$, $c_l$, $l = 1, \ldots, d$, $b$, $b^*$, $e$ and $\Delta e$ be real numbers, where $\Delta a_l \geq 0$, $\Delta e \geq 0$. Consider corresponding equalities defined as*

$$b = \mathcal{H}\left(l \in \underline{d};\, a_l, b, c_l, e\right) , \tag{29}$$

$$b^* = \mathcal{H}\left(l \in \underline{d};\, a_l + \Delta a_l, b^*, c_l, e + \Delta e\right) . \tag{30}$$

*Then by (g1) and (g2) holds*

$$\mathcal{H}\left(l \in \underline{d};\, a_l + \Delta a_l, b, c_l, e + \Delta e\right) \geq b^* \geq \mathcal{H}\left(l \in \underline{d};\, a_l, b, c_l, e\right) = b . \tag{31}$$

**Proof.** We first show

$$\mathcal{H}\left(l \in \underline{d};\, a_l + \Delta a_l, b, c_l, e + \Delta e\right) \geq \mathcal{H}\left(l \in \underline{d};\, a_l, b, c_l, e\right) . \tag{32}$$

Straight forward computation of the latter expression gives

$$\begin{aligned} &\mathcal{H}\left(l \in \underline{d};\, a_l + \Delta a_l, b, c_l, e + \Delta e\right) - \mathcal{H}\left(l \in \underline{d};\, a_l, b, c_l, e\right) \\ &\overset{(28)}{=} \Delta e - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ [g_l(b, c_l) - g_l(a_l + \Delta a_l, b)] - [g_l(b, c_l) - g_l(a_l, b)] \right\} \\ &= \Delta e + \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left[ g_l(a_l + \Delta a_l, b) - g_l(a_l, b) \right] . \end{aligned} \tag{33}$$

As $\Delta a_l \geq 0$ and $\Delta e \geq 0$ follows using (g1), (g2), that

$$\mathcal{H}\left(l \in \underline{d};\, a_l + \Delta a_l, b, c_l, e + \Delta e\right) - \mathcal{H}\left(l \in \underline{d};\, a_l, b, c_l, e\right) \geq 0 , \tag{34}$$

i.e. the inequality (32) is valid.

Now we consider $\mathcal{H}\left(l \in \underline{d};\, a_l + \Delta a_l, b, c_l, e + \Delta e\right) - b^*$. By the identity (30) we compute:

$$\begin{aligned} &\mathcal{H}\left(l \in \underline{d};\, a_l + \Delta a_l, b, c_l, e + \Delta e\right) - \mathcal{H}\left(l \in \underline{d};\, a_l + \Delta a_l, b^*, c_l, e + \Delta e\right) \\ &= -\sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ [g_l(b, c_l) - g_l(a_l + \Delta a_l, b)] - [g_l(b^*, c_l) - g_l(a_l + \Delta a_l, b^*)] \right\} \\ &= \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \underbrace{[g_l(b^*, c_l) - g_l(b, c_l)]}_{\text{terms (t1)}} + \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \underbrace{[g_l(a_l + \Delta a_l, b) - g_l(a_l + \Delta a_l, b^*)]}_{\text{terms (t2)}} . \end{aligned} \tag{35}$$

Inspecting terms (t1) and (t2), it follows by the properties (g1) and (g2) of the numerical flux function that one of the following two situations must hold:

(s1) $\mathcal{H}\,(l \in \underline{d};\ a_l + \Delta a_l, b, c_l, e + \Delta e) - b^* \geq 0$ for $b^* \geq b$, or

(s2) $\mathcal{H}\,(l \in \underline{d};\ a_l + \Delta a_l, b, c_l, e + \Delta e) - b^* \leq 0$ for $b^* \leq b$.

With respect to the situation (s2), we have:

$$\begin{aligned} &\mathcal{H}\,(l \in \underline{d};\ a_l + \Delta a_l, b, c_l, e + \Delta e) - b^* \leq 0 \\ \Rightarrow \qquad & b^* \geq \mathcal{H}\,(l \in \underline{d};\ a_l + \Delta a_l, b, c_l, e + \Delta e) \overset{(32)}{\geq} \mathcal{H}\,(l \in \underline{d};\ a_l, b, c_l, e) \overset{(29)}{=} b\,, \end{aligned} \tag{36}$$

i.e. $\mathcal{H}\,(l \in \underline{d};\ a_l + \Delta a_l, b, c_l, e + \Delta e) - b^* \leq 0$ as in (s2) implies $b^* \geq b$. However, at the same time (s2) relies on $b^* \leq b$, so that (s2) only addresses the case $b^* = b$. The latter is included in the assertion of the lemma.
By (s1) and making use of (29) we directly obtain the whole inequality chain (31). *qed*

We summarise the analogous assertion concerned with
$\mathcal{H}\,(l \in \underline{d};\ a_l, b, c_l + \Delta c_l, e + \Delta e) - b^*$ via

**Lemma 3.2** *Let $a_l$, $c_l$, $\Delta c_l$, $l = 1, \ldots, d$, $b$, $b^*$, $e$ and $\Delta e$ be real numbers, where $\Delta c_l \geq 0$, $\Delta e \geq 0$. Consider corresponding equalities defined as*

$$b = \mathcal{H}\,(l \in \underline{d};\ a_l, b, c_l, e)\ , \tag{37}$$

$$b^* = \mathcal{H}\,(l \in \underline{d};\ a_l, b^*, c_l + \Delta c_l, e + \Delta e)\ . \tag{38}$$

*Then by (g1) and (g2) holds*

$$\mathcal{H}\,(l \in \underline{d};\ a_l, b, c_l + \Delta c_l, e + \Delta e) \;\geq\; b^* \;\geq\; \mathcal{H}\,(l \in \underline{d};\ a_l, b, c_l, e) \;=\; b\,. \tag{39}$$

The proof is completely analogous to the one of Lemma 3.1, so we do not give it here.
Lemma 3.1 and Lemma 3.2 together allow to circumvent the computation of solutions of the arising implicit scheme formulae for our estimates.

## 3.2 Monotonicity

We now come to the main issues of this section.

**Theorem 3.3** (Monotonicity conditions for implicit methods) *Let $a$, $b$ and $c$ be arbitrarily chosen but fixed real numbers. If for all spatial dimensions $l \in \{1, \ldots, d\}$ the relations*

$$H_l(a+\Delta a, b, c) \quad \geq \quad H_l(a,b,c) \quad \forall\, \Delta a \geq 0\,, \tag{40}$$
$$H_l(a, b, c+\Delta c) \quad \geq \quad H_l(a,b,c) \quad \forall\, \Delta c \geq 0 \tag{41}$$

*hold, then the method is monotone.*

One may have expected an additional condition of the form

$$\mathcal{H}\left(l \in \underline{d};\, U_{j-\delta l}^{n+1},\, U_j^{n+1},\, U_{j+\delta l}^{n+1},\, s+\Delta s\right) \;\geq\; \mathcal{H}\left(l \in \underline{d};\, U_{j-\delta l}^{n+1},\, U_j^{n+1},\, U_{j+\delta l}^{n+1},\, s\right) \tag{42}$$

for all $j \in J$ and all $\Delta s \geq 0$. However, this condition turns out to be redundant. Note also that the monotonicity conditions do not depend on the exact nature of the source terms, i.e. both scenarios addressed in Section 2.1 and 2.2 are included in the set-up above.
It will turn out to be useful to have the following alternative formulation of the monotonicity conditions from Theorem 3.3. By these we clarify the role of the properties (g1), (g2), cf. Section 3.1.

**Theorem 3.4** *The monotonicity conditions stated in Theorem 3.3 are equivalent to the conditions (g1) and (g2) on the numerical flux functions, respectively.*

**Proof.** We begin by showing the equivalence of condition (40) to (g1):

$$\begin{aligned} & H_l(a+\Delta a, b, c) - H_l(a,b,c) \quad \geq \quad 0 \\ \overset{(28)}{\Leftrightarrow} \quad & -\left[g_l(b,c) - g_l(a+\Delta a, b)\right] + \left[g_l(b,c) - g_l(a,b)\right] \quad \geq \quad 0 \\ \Leftrightarrow \quad & g_l(a+\Delta a, b) - g_l(a,b) \quad \geq \quad 0\,. \end{aligned} \tag{43}$$

The other part of the equivalence proof follows analogously. *qed*

We now turn to the proof of Theorem 3.3. In order to give it a convenient structure, we first give the following Lemma.

**Lemma 3.5** *Let a method $\mathcal{H}$ be given which satisfies the conditions (40) and (41). Furthermore, let two data sets $V^n = \left\{V_j^n\right\}_{j\in J}$ and $W^n = \left\{W_j^n\right\}_{j\in J}$ be given. Then from*

$$\exists\, i \in J:\; V_i^n > W_i^n \quad \textit{and} \quad \forall\, j \in J\,(j \neq i):\; V_j^n = W_j^n \tag{44}$$

*follows by application of $\mathcal{H}$ the validity of $V^{n+1} \geq W^{n+1}$ in the sense of the comparison of components.*

**Proof.** By assumption there exists an index $i \in J$ so that $V_i^n > W_i^n$ holds. Without restriction of generality we choose $i = 0$. The proof of the lemma follows by induction over indices of suitable subsets $J_m$ of $J$. This is necessary in order to make the proceeding well-defined, i.e. independent of special choices of index sets. The subset $J_m$ of $J$ shall contain $m$ elements that denote adjacent grid points:

$$\forall\, m_0 \in J_m \,\exists\, m_1 \in J_m \,:\, \Big[\{m_0\} \cap \{p \in J_m \,;\, p = m_1 \pm \delta l,\, l = 1,\, \ldots,\, d\}\Big] \neq \emptyset \tag{45}$$

for $m \geq 2$. The induction takes place over the number of indices in $J_m$.

*Beginning of the induction:* $m = 1$. We choose without restriction of generality $J_1 = \{0\}$. The statement of the lemma is true because of the form of the method (25):

$$\mathcal{H}\left(l \in \underline{d};\, W_{-\delta l}^{n+1},\, W_0^{n+1},\, W_{\delta l}^{n+1},\, s + \Delta s\right) \geq \mathcal{H}\left(l \in \underline{d};\, W_{-\delta l}^{n+1},\, W_0^{n+1},\, W_{\delta l}^{n+1},\, s\right) \tag{46}$$

$\forall\, \Delta s \geq 0$.

*Assumption of the induction:* The statement of the lemma is true for arbitrary but fixed $m > 1$.

*Induction step:* $m \mapsto m + 1$. Let the statement be true for the subsets $\left\{V_i^{n+1}\right\}_{i \in J_m}$ and $\left\{W_i^{n+1}\right\}_{i \in J_m}$ of the sets $V^{n+1}$ and $W^{n+1}$. Then it holds in particular:

$$\begin{aligned} & V_{\tilde m}^{n+1} \geq W_{\tilde m}^{n+1} \quad \text{for an index} \quad \tilde m \in J_m \\ \text{with} \quad & \Big[\{i \in J \,:\, i = \tilde m \pm \delta l,\, l = 1,\, \ldots,\, d\} \cap (J \setminus J_m)\Big] \neq \emptyset\,. \end{aligned} \tag{47}$$

This means, we consider an index $\tilde m$ corresponding to a grid point with at least one neighbour with index not in $J_m$. We choose $l_m$ corresponding to

$$\tilde m \in J_m \quad \text{and} \quad \tilde m + \delta l_m \notin J_m\,. \tag{48}$$

Furthermore, one may distinguish between the situations

$$\text{(i)} \qquad \tilde m \in J_m \quad \text{and} \quad \{\tilde m + \delta l_m,\, \tilde m + 2\delta l_m\} \subset J \setminus J_m\,, \tag{49}$$

$$\text{(ii)} \qquad \{\tilde m,\, \tilde m + 2\delta l_m\} \subset J_m \quad \text{and} \quad \tilde m + \delta l_m \notin J_m\,. \tag{50}$$

Thereby, we implicitly defined the index $\tilde m + 2\delta l_m$ as the index of the node that is fixed via $\tilde m$, $\tilde m + \delta l_m$ and the geometry of the computational stencil of the method, cf. (27) and (28). We now proceed in accordance to (i) and (ii). By $V_{\tilde m}^{n+1} \geq W_{\tilde m}^{n+1}$ and (40) follows

$$H_{l_m}\left(V_{\tilde m}^{n+1},\, W_{\tilde m+\delta l_m}^{n+1},\, W_{\tilde m+2\delta l_m}^{n+1}\right) \;\geq\; H_{l_m}\left(W_{\tilde m}^{n+1},\, W_{\tilde m+\delta l_m}^{n+1},\, W_{\tilde m+2\delta l_m}^{n+1}\right)\,. \tag{51}$$

If the index $\tilde{m} + 2\delta l_m$ is already in $J_m$, we estimate

$$H_{l_m}\left(V^{n+1}_{\tilde{m}},\, W^{n+1}_{\tilde{m}+\delta l_m},\, V^{n+1}_{\tilde{m}+2\delta l_m}\right) \,\geq\, H_{l_m}\left(W^{n+1}_{\tilde{m}},\, W^{n+1}_{\tilde{m}+\delta l_m},\, W^{n+1}_{\tilde{m}+2\delta l_m}\right) \tag{52}$$

by using (41) in addition to (40).
We now make use of these estimates. For this, we also set

$$V^{n+1}_k \;:=\; W^{n+1}_k + \Delta_k \quad \text{for any index } k \in J_m\,, \tag{53}$$

with $\Delta_k \geq 0$. Of course, for $k \notin J_m$ we may retrieve $W^{n+1}_k$ from $W^{n+1}_k + \Delta_k$ via $\Delta_k = 0$. Generalising then in a straight forward fashion the setting from (49) and (50) to $d$ space dimensions, we obtain (at the index $\tilde{m} + \delta l_m$) by Lemma 3.1 and Lemma 3.2:

$$\begin{aligned}
&\mathcal{H}\left(l \in \underline{d};\, W^{n+1}_{(\tilde{m}+\delta l_m)-\delta l} + \Delta_{(\tilde{m}+\delta l_m)-\delta l},\, W^{n+1}_{\tilde{m}+\delta l_m},\, W^{n+1}_{(\tilde{m}+\delta l_m)+\delta l} + \Delta_{(\tilde{m}+\delta l_m)+\delta l}\right)\\
&\quad\geq\;\; V^{n+1}_{\tilde{m}+\delta l_m} \;\geq\; \mathcal{H}\left(l \in \underline{d};\, W^{n+1}_{(\tilde{m}+\delta l_m)-\delta l},\, W^{n+1}_{\tilde{m}+\delta l_m},\, W^{n+1}_{(\tilde{m}+\delta l_m)+\delta l}\right)\\
&\quad=\;\; W^{n+1}_{\tilde{m}+\delta l_m}\,.
\end{aligned} \tag{54}$$

The case $\tilde{m} \in J_m$ and $\tilde{m} - \delta l_m \notin J_m$ can be dealt with analogously. By defining in accordance

$$J_{m+1} := J_m \cup \{\tilde{m} + \delta l_m\} \quad \text{or} \quad J_{m+1} := J_m \cup \{\tilde{m} - \delta l_m\}\,, \tag{55}$$

it follows $V^{n+1}_i \geq W^{n+1}_i$ for all $i \in J_{m+1}$. Since $\tilde{m}$ and $l_m$ were chosen arbitrarily, the proceeding is well-defined and the proof is finished. *qed*

Having shown the validity of Lemma 3.5, we proceed with proving Theorem 3.3.

**The idea of the proof.** The idea of the proof of Theorem 3.3 can be sketched as follows. Let two data sets $W^n$ and $W^{n+1}$ be given with $W^{n+1} = \mathcal{H}\left(W^n, W^{n+1}\right)$. Then the proof proceeds along the following *steps*.

1. A positive perturbation in $W^n_j$ results in a non-negative perturbation in $W^{n+1}_j$.

2. The non-negative perturbation in $W^{n+1}_j$ results in non-negative perturbations in $W^{n+1}_{j\pm\delta l}$ for all $l$.

3. Consider an arbitrary index $i$ and an arbitrary dimensional index $l$. Then, any non-negative perturbation in the neighbouring values $W^{n+1}_{i\pm\delta l}$ give a non-negative perturbation in $W^{n+1}_i$.

For deciding if an induced perturbation is non-negative, we rely on Lemma 3.1 and Lemma 3.2 without stating this explicitly anymore.
As the spatial domain is covered by a grid of a countable number of nodes, the complete set of grid nodes can be addressed by using the *induction principle*. The induction over the spatial indices $k$ shows, that by applying the method $\mathcal{H}$ at two data sets $V^n$ and $W^n$, where $V^n \geq W^n$ holds (componentwise), a comparison principle $V_k^{n+1} \geq W_k^{n+1}$ (where the induction is performed) as in (2) holds. Consequently, the method $\mathcal{H}$ is monotone by definition. While it seems natural to proceed by an induction proof – given the countable number of grid points and the possibility to order them by a suitable linear numbering – this kind of proof structure is very unconventional in the field of numerical methods for HCLs.

**Proof.** (Of Theorem 3.3.) To prove is the validity of (2) by using the assumptions (40) and (41). Therefore, we define the set

$$\hat{J}^n := \{i \in J \mid V_i^n > W_i^n,\ V_i^n \in V^n,\ W_i^n \in W^n\}\ . \tag{56}$$

There are only a few possibilities for the composition of $\hat{J}^n$: It may consist of the empty set, or a finite or infinite subset of the index set $J$. We recall that $J$ contains the indices of all spatial grid points.
Since we have to take into account all these cases, we define

$$\hat{J}^n_m := \hat{J}^n \text{ with } \sharp \hat{J}^n = m\,. \tag{57}$$

The proof of the assertion follows by induction over $m \geq 1$ (the case $m = 0$ is trivial).

*Beginning of the induction:* $\hat{J}^n = \hat{J}^n_1$.
Let $i$ be the index in the arbitrarily chosen but fixed index set $\hat{J}^n_1$. Then the monotonicity follows by Lemma 3.5.

*Assumption:* The assertion holds for all subsets of $\hat{J}^n = \hat{J}^n_m$ for an arbitrarily chosen but fixed number $m > 1$.

*Induction step:* $m \mapsto m + 1$
Now we consider $\hat{J}^n_{m+1}$ with $\hat{J}^n_m \subset \hat{J}^n_{m+1}$. We define two particular indices $m_1$, $m_2$ with

$$m_1 \in \hat{J}^n_m \quad \text{and} \quad m_2 \in \left(\hat{J}^n_{m+1} \setminus \hat{J}^n_m\right)\ . \tag{58}$$

Thereby, the index $m_1$ is chosen arbitrarily but fixed. By the assumption of the induction, it holds

$$V^n \geq W^n \text{ (componentwise)} \quad \overset{\mathcal{H}}{\Rightarrow} \quad V^{n+1} \geq W^{n+1} \text{ (componentwise)} \tag{59}$$

when $V_i^n > W_i^n$ for the indices of the index set $\hat{J}_m^n$.
While it seems clear that the consideration of one more index as here $m_2$ is covered by the proof of Lemma 3.5, there is the open question at this point how the processes corresponding to

$$V_{m_1}^n > W_{m_1}^n \quad \text{and} \quad V_{m_2}^n > W_{m_2}^n \tag{60}$$

interact. By the assumption of the induction the perturbations due to $\hat{J}_{m+1}^n \setminus \{m_1,\, m_2\}$ are non-negative. We denote the latter values by $\bar{V}^{n+1}$, i.e. it holds $\bar{V}^{n+1} \geq W^{n+1}$.
Now, let $\Delta_i^1$ denote a perturbation in $\bar{V}_i^{n+1}$ induced by $V_{m_1}^n - W_{m_1}^n$. Then $\Delta_i^1$ is also always non-negative by the assumption of the induction.
Analogously, let $\Delta_i^2$ denote a perturbation in $\bar{V}_i^{n+1}$ induced by $V_{m_2}^n - W_{m_2}^n$. Then $\Delta_i^2$ is non-negative by applying Lemma 3.5.
There are only two situations of interest left corresponding the mutual effects of such perturbations. For this, we consider an arbitrary but fixed index $\tilde{i}$ and an accordingly arranged index $l_i \in \{1,\, \ldots,\, d\}$. By (40), (41) follows:

$$H_{l_i}\left(\bar{V}_{\tilde{i}-\delta l_i}^{n+1} + \Delta_{\tilde{i}-\delta l_i}^1,\, \bar{V}_{\tilde{i}}^{n+1},\, \bar{V}_{\tilde{i}+\delta l_i}^{n+1} + \Delta_{\tilde{i}+\delta l_i}^2\right) \geq H_{l_i}\left(\bar{V}_{\tilde{i}-\delta l_i}^{n+1},\, \bar{V}_{\tilde{i}}^{n+1},\, \bar{V}_{\tilde{i}+\delta l_i}^{n+1}\right), \tag{61}$$

$$H_{l_i}\left(\bar{V}_{\tilde{i}-\delta l_i}^{n+1} + \Delta_{\tilde{i}-\delta l_i}^2,\, \bar{V}_{\tilde{i}}^{n+1},\, \bar{V}_{\tilde{i}+\delta l_i}^{n+1} + \Delta_{\tilde{i}+\delta l_i}^1\right) \geq H_{l_i}\left(\bar{V}_{\tilde{i}-\delta l_i}^{n+1},\, \bar{V}_{\tilde{i}}^{n+1},\, \bar{V}_{\tilde{i}+\delta l_i}^{n+1}\right). \tag{62}$$

By the same argumentation as in the proof of Lemma 3.5, we conclude for any perturbation $\Delta_k \geq 0$ that

$$V_{\tilde{i}}^{n+1} = \mathcal{H}\left(l \in \underline{d};\, \bar{V}_{\tilde{i}-\delta l_i}^{n+1} + \Delta_{\tilde{i}-\delta l_i},\, V_{\tilde{i}}^{n+1},\, \bar{V}_{\tilde{i}+\delta l_i}^{n+1} + \Delta_{\tilde{i}+\delta l_i}\right) \geq \bar{V}_{\tilde{i}}^{n+1}. \tag{63}$$

Note the arbitrary choice of $m_1$ and $m_2$ by a simultaneous change in the data corresponding to $\hat{J}_m^n \setminus \{m_1,\, m_2\}$. Since there are also no limitations concerning the choices of $\hat{J}_m^n$ and $l_i$, the proceeding is well defined. *qed*

We now ask for an assertion that can be derived by assuming validity of the monotonicity of $\mathcal{H}$.

**Theorem 3.6** *Let a monotone method $\mathcal{H}$ be given. Then the numerical flux functions $g_l(v, w)$ must satisfy (g1) and (g2).*

**Proof.** Let two sets $V^n$, $W^n$ be given with $V^n \geq W^n$ (componentwise). These are mapped to sets $V^{n+1}$ and $W^{n+1}$ by the consistent and conservative method $\mathcal{H}$. By the assumed monotonicity of $\mathcal{H}$ follows $V^{n+1} \geq W^{n+1}$

(componentwise), so that:

$$b+\Delta b \;:=\; \mathcal{H}\left(l \in \underline{d};\; a_l+\Delta a_l, b+\Delta b, c_l+\Delta c_l, e+\Delta e\right) \;\geq\; \mathcal{H}\left(l \in \underline{d};\; a_l, b, c_l, e\right) \;=:\; b \tag{64}$$

with the obvious definitions of participating values, compare the previous proofs.
As the assertion of the theorem is an implication, we may specify values of $\Delta a_l$, $\Delta c_l$, $\Delta e$ as required.
*To (g1).* Specifying $\Delta c_l := 0$ and $\Delta e := 0$ gives

$$\begin{aligned} & \mathcal{H}\left(l \in \underline{d};\; a_l+\Delta a_l, b, c_l, e\right) - \mathcal{H}\left(l \in \underline{d};\; a_l, b, c_l, e\right) \;\geq 0 \\ \overset{(28)}{\Leftrightarrow} \quad & \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l}\left[g_l(a_l+\Delta a_l, b) - g_l(a_l, b)\right] \;\geq 0\,, \end{aligned} \tag{65}$$

cf. (33). Setting moreover all $\Delta a_l = 0$ for all $l$ but one $\tilde{l}$ shows that

$$g_{\tilde{l}}(a_{\tilde{l}}+\Delta a_{\tilde{l}}, b) \;\geq\; g_{\tilde{l}}(a_{\tilde{l}}, b) \tag{66}$$

for any arbitrarily chosen but fixed $\tilde{l}$, and this implies (g1).
The assertion concerned with (g2) follows analogously by considering in a first step $\Delta a_l := 0$ and $\Delta e := 0$. *qed*

# 4 The discrete entropy inequality

We now want to construct on the discrete level the link between a monotone discretisation and the entropy condition, cf. Section 2. As indicated, this is done in a similar fashion as in [6]. Accordingly, we make use of the following definition.

**Definition 4.1 (**Consistency with the Entropy Condition) *An implicit numerical scheme $\mathcal{H}$ is consistent with the entropy condition of Kružkov if there exist for all $l = 1, \ldots, d$ numerical entropy fluxes $G_l$ which satisfy for all $\xi \in \mathbb{R}$ the following assertions:*

1. *Consistency with the entropy flux of Kružkov*

$$G_l(v, v; \xi) = F_l(v; \xi) \quad \forall v \quad \textit{with} \quad F_l(v; \xi) = \mathsf{sgn}(v-\xi)\left[f_l(v) - f_l(\xi)\right]. \tag{67}$$

*2. Validity of a discrete entropy inequality*

$$\begin{aligned}
&\frac{U\left(U_j^{n+1};\xi\right)-U\left(U_j^{n};\xi\right)}{\Delta t} \\
&\quad\le\ -\sum_{l=1}^{d}\frac{G_l\left(U_j^{n+1},U_{j+\delta l}^{n+1};\xi\right)-G_l\left(U_{j-\delta l}^{n+1},U_j^{n+1};\xi\right)}{\Delta x_l} \\
&\qquad +\mathsf{sgn}\left[U_j^{n+1}-\xi\right]q_j^{n+1} \qquad (68)
\end{aligned}$$

*where $U(v;\xi)=|v-\xi|$ is chosen due to Kružkov.*

In the following, let

$$a\vee b:=\max(a,b)\quad\text{and}\quad a\wedge b:=\min(a,b) \qquad (69)$$

hold. The important connection between the numerical entropy fluxes $G_l$ and the numerical flux functions $g_l$ is now established.

**Lemma 4.2** (Format of numerical entropy flux) *Let a monotone implicit scheme $\mathcal{H}$ be given. Then the numerical entropy fluxes defined by*

$$G_l(v,w;\xi)\ :=\ g_l(v\vee\xi,w\vee\xi)-g_l(v\wedge\xi,w\wedge\xi) \qquad (70)$$

*are consistent with the entropy fluxes of Kružkov.*

**Proof.** Because the numerical scheme is consistent and conservative, the statement

$$G_l(v,v;\xi)=g_l(v\vee\xi,v\vee\xi)-g_l(v\wedge\xi,v\wedge\xi)=\mathsf{sgn}(v-\xi)[f_l(v)-f_l(\xi)]$$

holds by use of (69) for all $l=1,\ldots,d$ and all $\xi\in\mathbb{R}$. *qed*

One can now prove the following result, partly by a variation of the procedure given in [6]. We introduce the source term within the proof.

**Theorem 4.3** (Consistency with the entropy condition) *Let a monotone implicit scheme $\mathcal{H}$ be given. Then the scheme is also consistent with the entropy condition of Kružkov.*

Under the same assumptions, we prove later on convergence of the corresponding numerical approximation to the entropy solution. **Proof.** *(Of Theorem 4.3.)* Since the method $\mathcal{H}$ features consistent and conservative numerical flux functions $g_l$, $l = 1, \ldots, d$, one can construct numerical entropy fluxes $G_l$ by applying Lemma 4.2. Thus, the consistency with the entropy fluxes of Kružkov is given.

It is left to show the validity of a discrete entropy inequality. Therefore, let $\xi \in \mathbb{R}$ be chosen arbitrarily but fixed. By using the definition of $G_l$, we derive

$$\begin{aligned}
&-\sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l}\Big\{G_l\left(U_j^{n+1}, U_{j+\delta l}^{n+1}; \xi\right) - G_l\left(U_{j-\delta l}^{n+1}, U_j^{n+1}; \xi\right)\Big\} \\
&\quad = \mathcal{H}\left(l \in \underline{d}, U_{j-\delta l}^{n+1} \vee \xi, U_j^{n+1} \vee \xi, U_{j+\delta l}^{n+1} \vee \xi, U_j^n \vee \xi\right) \\
&\qquad -\mathcal{H}\left(l \in \underline{d}, U_{j-\delta l}^{n+1} \wedge \xi, U_j^{n+1} \wedge \xi, U_{j+\delta l}^{n+1} \wedge \xi, U_j^n \wedge \xi\right) - \left|U_j^n - \xi\right|. \qquad (71)
\end{aligned}$$

Now we estimate the terms involving $\mathcal{H}$ by using the properties (g1) and (g2) of the numerical flux functions of a monotone method, cf. Theorem 3.4. It is necessary to employ a diversion of the cases $U_j^{n+1} \geq \xi$ and $U_j^{n+1} < \xi$.

*(a) Case* $U_j^{n+1} \geq \xi$*:*

$$\begin{aligned}
&H\left(l \in \underline{d}, U_{j-\delta l}^{n+1} \vee \xi, U_j^{n+1} \vee \xi, U_{j+\delta l}^{n+1} \vee \xi, U_j^n \vee \xi\right) \\
&\overset{(a)}{=} \quad U_j^n \vee \xi - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l}\left\{g_l\left(U_j^{n+1}, U_{j+\delta l}^{n+1} \vee \xi\right) - g_l\left(U_{j-\delta l}^{n+1} \vee \xi, U_j^{n+1}\right)\right\} + \Delta t q_j^{n+1} \\
&\geq \quad U_j^n - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l}\left\{g_l\left(U_j^{n+1}, U_{j+\delta l}^{n+1}\right) - g_l\left(U_{j-\delta l}^{n+1}, U_j^{n+1}\right)\right\} + \Delta t q_j^{n+1} \\
&= \quad U_j^{n+1} \quad \overset{(a)}{=} \quad U_j^{n+1} \vee \xi.
\end{aligned}$$

*(b) Case* $U_j^{n+1} < \xi$*:*

$$\begin{aligned}
&H\left(l \in \underline{d}, U_{j-\delta l}^{n+1} \vee \xi, U_j^{n+1} \vee \xi, U_{j+\delta l}^{n+1} \vee \xi, U_j^n \vee \xi\right) \\
&\overset{(b)}{=} \quad U_j^n \vee \xi - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l}\left\{g_l\left(\xi, U_{j+\delta l}^{n+1} \vee \xi\right) - g_l\left(U_{j-\delta l}^{n+1} \vee \xi, \xi\right)\right\} + \Delta t q_j^{n+1} \\
&\geq \quad \xi - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l}\left\{g_l\left(\xi, \xi\right) - g_l\left(\xi, \xi\right)\right\} + \Delta t q_j^{n+1} \\
&= \quad \xi + \Delta t q_j^{n+1} \quad \overset{(b)}{=} \quad U_j^{n+1} \vee \xi + \Delta t q_j^{n+1}.
\end{aligned}$$

*(c) Case $U_j^{n+1} \geq \xi$:*

$$H\left(l \in \underline{d},\, U_{j-\delta l}^{n+1} \wedge \xi,\, U_j^{n+1} \wedge \xi,\, U_{j+\delta l}^{n+1} \wedge \xi,\, U_j^n \wedge \xi\right)$$

$$\begin{aligned}
&\overset{\text{(c)}}{=} \; U_j^n \wedge \xi - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ g_l\left(\xi,\, U_{j+\delta l}^{n+1} \wedge \xi\right) - g_l\left(U_{j-\delta l}^{n+1} \wedge \xi,\, \xi\right)\right\} + \Delta t q_j^{n+1} \\
&\leq \; \xi - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ g_l(\xi,\, \xi) - g_l(\xi,\, \xi)\right\} + \Delta t q_j^{n+1} \\
&= \; \xi + \Delta t q_j^{n+1} \quad \overset{\text{(c)}}{=} \quad U_j^{n+1} \wedge \xi + \Delta t q_j^{n+1}.
\end{aligned}$$

*(d) Case $U_j^{n+1} < \xi$:*

$$H\left(l \in \underline{d},\, U_{j-\delta l}^{n+1} \wedge \xi,\, U_j^{n+1} \wedge \xi,\, U_{j+\delta l}^{n+1} \wedge \xi,\, U_j^n \wedge \xi\right)$$

$$\begin{aligned}
&\overset{\text{(d)}}{=} \; U_j^n \wedge \xi - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ g_l\left(U_j^{n+1},\, U_{j+\delta l}^{n+1} \wedge \xi\right) - g_l\left(U_{j-\delta l}^{n+1} \wedge \xi,\, U_j^{n+1}\right)\right\} + \Delta t q_j^{n+1} \\
&\leq \; U_j^n - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ g_l\left(U_j^{n+1},\, U_{j+\delta l}^{n+1}\right) - g_l\left(U_{j-\delta l}^{n+1},\, U_j^{n+1}\right)\right\} + \Delta t q_j^{n+1} \\
&= \; U_j^{n+1} \quad \overset{\text{(d)}}{=} \quad U_j^{n+1} \wedge \xi.
\end{aligned}$$

By combining all these cases, we obtain from (71) the inequality

$$\begin{aligned}
&-\sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ G_l\left(U_j^{n+1},\, U_{j+\delta l}^{n+1}; \xi\right) - G_l\left(U_{j-\delta l}^{n+1},\, U_j^{n+1}; \xi\right)\right\} + \mathsf{sgn}\left[U_j^{n+1} - \xi\right] \Delta t q_j^{n+1} \\
\geq \;& U_j^{n+1} \vee \xi - U_j^{n+1} \wedge \xi - \mathsf{sgn}\left[U_j^{n+1} - \xi\right] \Delta t q_j^{n+1} \\
&+ \mathsf{sgn}\left[U_j^{n+1} - \xi\right] \Delta t q_j^{n+1} - \left|U_j^n - \xi\right| \\
= \;& \left|U_j^{n+1} - \xi\right| - \left|U_j^n - \xi\right|.
\end{aligned}$$

By construction, the proceeding is well defined. Division by $\Delta t$ gives the desired discrete entropy inequality. *qed*

Concerning the setting described in Section 2.2, the validity of the corresponding discrete entropy inequality follows analogously.

# 5 Convergence

Within this section, we prove convergence for implicit monotone schemes $\mathcal{H}$, of course under the assumption that the conditions for monotonicity are fulfilled. We do this in some detail for the implicit Upwind method since this is demonstrated in the easiest fashion, and we refer to the differences concerning the proofs of convergence with respect to other methods afterwards. We proceed analogously with respect to the two settings described in the Sections 2.1 and 2.2. Let us note that some part of the convergence proof is technically identical to the one-dimensional case without sources described in [3], so that we refer to that work for more details in some instances.

The basic idea of the convergence proofs is the following. Corresponding to sequences $\Delta x_l^k \downarrow 0$ for $k \to \infty$, $l \in \underline{d}$, we construct a monotonously growing sequence of discrete initial data. Then by the monotonicity of the method we get a monotonously growing sequence of numerical solutions. Because of the assumption $U_0 \in L_\infty$ (in a discrete sense) and pointwise integrable sources, we have $L_\infty$-stability. Thus, the function sequence obtained via the monotone scheme is integrable and bounded from above. Then we use the well known *Theorem of Monotone Convergence* of Beppo Levi to show convergence (almost everywhere) to a limit function. Since the approximative sequence satisfies a discrete entropy inequality, convergence to the entropy inequality of Kružkov follows by the established strong convergence almost everywhere of the sequence.

Consequently, we begin this section with showing $L_\infty$-stability.

**Lemma 5.1** ($L_\infty$-Stability) *Let an implicit monotone method $\mathcal{H}$ be given. Then the numerical solution is $L_\infty$-stable over any finite time interval $[0, T]$.*

**Proof.** Let a data set $U^0 \in L_\infty$ be given. We then identify the finite values

$$A := \inf_{j \in J} U_j^0 \quad \text{and} \quad B := \sup_{j \in J} U_j^0 \,. \tag{72}$$

Since the source terms are pointwise bounded over the time intervall $(0, T)$, cf. (10) and (17), they are bounded by a finite number $M$ with

$$\int_0^T ||q||_\infty \, dt \, < \, M.$$

Consequently, by the monotonicity of $\mathcal{H}$ follows that the numerical solution obtained via given data $U_0$ is bounded for all $n$ with $n\Delta t < T$ by $A^n \leq U^n \leq B^n$ with

$$A_j^n := A - M \, (> -\infty) \;\; \forall j \in J \quad \text{and} \quad B_j^n := B + M \, (< \infty) \;\; \forall j \in J.$$

*qed*

The main assertion of this section is then:

**Theorem 5.2 (Convergence of implicit monotone schemes)** *Let* $u_0(\mathbf{x})$ *be in* $L_\infty^{loc}\left(\mathbb{R}^d;\mathbb{R}\right)$*. Consider a sequence of nested grids indexed by* $k = 1, 2, \ldots$*, with mesh parameters* $\Delta t_k \downarrow 0$ *and* $\Delta x_l^k \downarrow 0$*,* $l = 1, \ldots, d$*, as* $k \to \infty$*, and let* $u_k(\mathbf{x}, t)$ *denote the step function obtained via the numerical approximation by a consistent, conservative and monotone scheme in the form of the discussed methods. Then* $u_k(\mathbf{x}, t)$ *converges to the unique entropy solution of the given conservation law as* $k \to \infty$*.*

**Proof.** At first, the convergence to a weak solution of the conservation law is established, followed by the verification that this weak solution is the entropy solution. For brevity of the notation, we omit the arguments $(\mathbf{x}, t)$ when appropriate.
We employ sequences $\Delta t_k \downarrow 0$ and $\Delta x_l^k \downarrow 0$, assuming that the resulting grids are nested in order to compare data sets of values, i.e. refined grids always inherit cell borders.
The most important technical detail is the special discretization of the initial condition $u_0 \in L_\infty^{loc}\left(\mathbb{R}^d;\mathbb{R}\right)$. After a suitable modification on a set of Lebesgue measure zero, the initial condition is discretized on cell $j \in J$, i.e. for

$$\mathbf{x} \in \left((j_1 - 1)\Delta x_1^0, j_1 \Delta x_1^0\right] \times \ldots \times \left((j_d - 1)\Delta x_d^0, j_d \Delta x_d^0\right],$$

by

$$U_j^0 \;:=\; \inf_{\mathbf{x} \text{ in cell } j} u_0(\mathbf{x}). \tag{73}$$

Corresponding to the initial data we also define a piecewise continuous function

$$u_k(\mathbf{x}, 0) := U_j^0,\ \mathbf{x} \text{ in cell } j. \tag{74}$$

It is a matter of classic analysis to verify that the discretisation (73) together with (74) gives on any compact spatial domain a monotonously growing function sequence with

$$\lim_{k\to\infty} u_k(\mathbf{x}, 0) \;=\; u_0(\mathbf{x}) \text{ almost everywhere} \tag{75}$$

by application of the Theorem of Monotone Convergence.
In the classic fashion using point values, we extract discrete test elements $\phi_j^0$ out of a given test function $\phi \in C_0^\infty\left(\mathbb{R}^{d+1};\mathbb{R}\right)$. Also, we define for $n \geq 1$ the step function

$$u_k(x, t) := U_j^n,\ \mathbf{x} \text{ in cell } j,\ t^{n-1} < t \leq t^n.$$

In the following, let the test function $\phi$ be chosen arbitrarily but fixed. We proceed by investigating in detail the implicit upwind scheme, and then by making use of this we refer to other schemes. It is given as

$$U_j^{n+1} = U_j^n - \sum_{l=1}^{d} \frac{\Delta t}{\Delta x_l} \left\{ f_l\left(U_j^{n+1}\right) - f_l\left(U_{j-\delta l}^{n+1}\right) \right\} + \Delta t q_j^{n+1}. \tag{76}$$

Multiplication of (76) with $\Delta t^k \prod_{l=1}^d \Delta x_l^k$ as well as with the discrete test element $\phi_j^{n+1}$, summation over the spatial indices $j \in J$ and the temporal indices $n \geq 0$, and finally summation by parts yields

$$\begin{aligned} &\Delta t^k \prod_{l=1}^{d} \Delta x_l^k \left\{ \sum_{j\in J} \sum_{n\geq 0} \left[ U_j^n \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t^k} + \sum_{l=1}^{d} f_l(U_j^{n+1}) \frac{\phi_{j+\delta l}^{n+1} - \phi_j^{n+1}}{\Delta x_l^k} \right] \right\} \\ = \; & -\prod_{l=1}^{d} \Delta x_l^k \sum_{j\in J} U_j^0 \phi_j^0 + \Delta t^k \prod_{l=1}^{d} \Delta x_l^k \sum_{j\in J} q_j^{n+1} \phi_j^{n+1}. \end{aligned} \tag{77}$$

By the definition of the introduced step functions, (77) is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left[ u_k(\mathbf{x}, t) \frac{\phi_k(\mathbf{x}, t + \Delta t^k) - \phi_k(\mathbf{x}, t)}{\Delta t^k} \right. \\ &\left. + \sum_{l=1}^{d} f_l(u_k(\mathbf{x}, t + \Delta t^k)) \frac{\phi_k(\mathbf{x} + \Delta x_l^k, t + \Delta t^k) - \phi_k(\mathbf{x}, t + \Delta t^k)}{\Delta x_l^k} \right] \mathrm{d}\mathbf{x}\mathrm{d}t \\ = \; & -\int_{\mathbb{R}^d} u_k(\mathbf{x}, 0)\phi_k(\mathbf{x}, 0)\, d\mathbf{x} + \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} q_k(\mathbf{x}, t + \Delta t^k)\phi_k(\mathbf{x}, t + \Delta t^k)\, \mathrm{d}\mathbf{x}\mathrm{d}t \\ & . \end{aligned} \tag{78}$$

We now prove convergence of (78) to the form which implies that $u$ is a weak solution of the original problem. We first investigate the right hand side of (78). Set $\tilde{\Delta} := \max_{l\in \underline{d}} \Delta x_l^0$ and let

$$K := \left\{ (\mathbf{x}, t) \mid \exists (\mathbf{y}, t) \in support(\phi) : \, t = 0 \text{ and } y_l - \tilde{\Delta} \leq x_l \leq y_l + \tilde{\Delta} \text{ for all } l \in \underline{d} \right\}.$$

By construction, K is compact and gives the largest possible spatial domain where non-zero discrete initial data may occur. Adding zeroes, we now cast the problem into a more suitable form, namely

$$\begin{aligned} &\int_{\mathbb{R}^d} u_k(\mathbf{x}, 0)\phi_k(\mathbf{x}, 0)\, \mathrm{d}\mathbf{x} = \int_K u_0(\mathbf{x})\phi(\mathbf{x}, 0)\, \mathrm{d}\mathbf{x} \\ &+ \int_K u_k(\mathbf{x}, 0) \left[\phi_k(\mathbf{x}, 0) - \phi(\mathbf{x}, 0)\right] \mathrm{d}\mathbf{x} + \int_K \left[u_k(\mathbf{x}, 0) - u_0(\mathbf{x})\right] \phi(\mathbf{x}, 0)\, \mathrm{d}\mathbf{x}\,. \end{aligned} \tag{79}$$

Because of $u_0 \in L^\infty(\mathbb{R}^d; \mathbb{R})$ and by our construction, we can estimate the absolute of the second right hand side term in (79) by the help of a constant $M_u < \infty$:

$$\left| \int_K u_k(\mathbf{x}, 0) \left[\phi_k(\mathbf{x}, 0) - \phi(\mathbf{x}, 0)\right] \mathrm{d}\mathbf{x} \right| \leq M_u |K| \sup_{x \in K} |\phi_k(\mathbf{x}, 0) - \phi(\mathbf{x}, 0)| \ . \quad (80)$$

Since $\phi$ is a smooth testfunction, it is a simple but technical exercise to show

$$||\phi_k(\mathbf{x}, 0) - \phi(\mathbf{x}, 0)||_\infty \to 0 \quad \text{for} \quad k \to \infty \,. \quad (81)$$

By (80) and (81), the investigated term tends to zero with $k \to \infty$. Since $\phi$ is continuous and since $u_k(\mathbf{x}, 0)$ approaches $u_0(\mathbf{x})$ from below by construction, we can estimate the absolute of the third right hand side term in (79) with the help of a constant $M_\phi < \infty$ by

$$\left| \int_K \left[u_k(\mathbf{x}, 0) - u_0(\mathbf{x})\right] \phi(\mathbf{x}, 0) \, \mathrm{d}\mathbf{x} \right| \leq M_\phi \int_K u_0(\mathbf{x}) - u_k(\mathbf{x}, 0) \, \mathrm{d}\mathbf{x} \,.$$

The Theorem of Monotone Convergence implies that

$$\int_K u_0(\mathbf{x}) - u_k(\mathbf{x}, 0) \, \mathrm{d}\mathbf{x}$$

vanishes in the limit for $k \to \infty$, i.e. the corresponding term in (79) goes to zero for $k \to \infty$. To condense these results, we obtain

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} u_k(\mathbf{x}, 0) \phi_k(\mathbf{x}, 0) \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} u_0(\mathbf{x}) \phi(\mathbf{x}, 0) \, \mathrm{d}\mathbf{x} \,.$$

It remains to show

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} q_k(\mathbf{x}, t + \Delta t^k) \phi_k(\mathbf{x}, t + \Delta t^k) \, \mathrm{d}\mathbf{x} \mathrm{d}t \overset{k \to \infty}{\longrightarrow} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} q(\mathbf{x}, t) \phi(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} \mathrm{d}t \,.$$

This result can easily be achieved by analogously introducing a compact domain $S \subset \mathbb{R}^d$ including the support of $\phi$ in space and time, setting for $n \geq 1$ ($n = 0$ is not relevant since $q(\cdot, 0) \equiv 0$)

$$q_j^n \ := \ \inf_\Theta q(\mathbf{x}, t) \,,$$

where $\Theta$ is the part of the $(\mathbf{x}, t)$-domain with $\mathbf{x}$ is in cell $j$ and $t$ is in $(t^n - \Delta t^0, t^n]$, using then a similar manipulation as for the terms involving $u_0$.

Concerning the left hand side of (78), adding zeroes and using the attributes of test functions together with the $L_\infty$-stability of $u_k$ yields that we finally have to show

$$\int_S \big| u(\mathbf{x},t) - u_k(\mathbf{x},t) \big| \, |\phi_t(\mathbf{x},t)| \; \mathrm{d}\mathbf{x}\mathrm{d}t \;\overset{k\to\infty}{\longrightarrow}\; 0\,, \tag{82}$$

and also for all $l \in \underline{d}$

$$\int_S \big| f_l(u(\mathbf{x},t)) - f_l(u_k(\mathbf{x},t+\Delta t^k)) \big| \left| \frac{\partial}{\partial x_l}\phi(\mathbf{x},t) \right| \; \mathrm{d}\mathbf{x}\mathrm{d}t \;\overset{k\to\infty}{\longrightarrow}\; 0 \tag{83}$$

in order to prove convergence to a weak solution.
Since $\phi_t$ is continuous on $S$, we can estimate $|\phi_t|$ in (82) by a constant $M_t < \infty$. Since $u_k(\mathbf{x},t)$ grows monotonously with $k \to \infty$ in the sense of pointwise comparison, and since it is positive and bounded from above because of $u_0 \in L_\infty(S)$ and the monotonicity of the method, the function sequence $(u_k(\mathbf{x},t))_{k\in\mathbb{N}}$ converges almost everywhere to an integrable limit function on $S$ by the Theorem of Monotone Convergence.
We set

$$u(\mathbf{x},t) \;:=\; \lim_{k\to\infty} u_k(\mathbf{x},t)\,.$$

Introducing exactly this limit function as the function $u(\mathbf{x},t)$ used up to now, the corresponding term in (82) becomes zero in the limit:

$$\lim_{k\to\infty} \int_S \big| u(\mathbf{x},t) - u_k(\mathbf{x},t) \big| \, |\phi_t(\mathbf{x},t)| \; \mathrm{d}\mathbf{x}\mathrm{d}t \;\le\; M_t \int_S u(\mathbf{x},t) - \lim_{k\to\infty} u_k(\mathbf{x},t) \, \mathrm{d}\mathbf{x}\mathrm{d}t \;=\; 0\,.$$

Note that the pointwise convergence $u_k \to u$ almost everywhere is now established and can be used in the following.
For proving (83), we need some further simple manipulations. We use again the continuity of the derivatives of $\phi$ to introduce constants $M_x^l < \infty$ to obtain by the triangle inequality

$$\begin{aligned} \int_S \big| f_l(u(\mathbf{x},t)) - f_l(u_k(\mathbf{x},t+\Delta t^k)) \big| \left| \frac{\partial}{\partial x_l}\phi(\mathbf{x},t) \right| \; \mathrm{d}\mathbf{x}\mathrm{d}t \;\le \\ M_x^l \int_S |f_l(u_k(\mathbf{x},t)) - f_l(u(\mathbf{x},t))| \; \mathrm{d}\mathbf{x}\mathrm{d}t \\ + M_x^l \int_S \big| f_l(u_k(\mathbf{x},t+\Delta t^k)) - f_l(u_k(\mathbf{x},t)) \big| \; \mathrm{d}\mathbf{x}\mathrm{d}t \end{aligned} \tag{84}$$

for all $l \in \underline{d}$. We now discuss the first right hand side term in (84). Since by construction $u_k$ and $u$ are in $L_\infty(S)$, we can estimate every $|f_l(u_k(\mathbf{x},t)) - f_l(u(\mathbf{x},t))|$

over $S$ from above by a constant $M_f^l < \infty$ because of the continuity of the $f_l$ on the compact set of possible values. Then the functions

$$M_f^l(\mathbf{x},t) := \left\{ \begin{array}{ccc} M_f^l & : & (\mathbf{x},t) \in S \\ 0 & : & \text{else} \end{array} \right. ,$$

are in $L_1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$ and dominate $|f_l(u_k(\mathbf{x},t)) - f_l(u(\mathbf{x},t))|$ for all $l \in \underline{d}$ and all $k$. Because of the established pointwise convergence $u_k \to u$ a.e., we can apply the Theorem of Dominated Convergence by Lebesgue to obtain for all $l$

$$\lim_{k\to\infty} M_x^l \int_S |f_l(u_k(\mathbf{x},t)) - f_l(u(\mathbf{x},t))| \, \mathrm{d}\mathbf{x}\mathrm{d}t = 0\,. \tag{85}$$

Now we discuss the second right hand side term in (84). Since by construction $u_k$ is a step function with finite values on the compact domain $S$, $u_k$ is in $L_1(S)$. Since the $f_l$ are continuous, also $f_l \circ u_k$ are in $L_1(S)$. By the continuity in the mean of $L_1$-functions, there exist $\delta_l(\epsilon)$ for all $\epsilon > 0$ with

$$\int_S \left| f_l(u_k(\mathbf{x}, t+\Delta t^k)) - f_l(u_k(\mathbf{x},t)) \right| \, \mathrm{d}\mathbf{x}\mathrm{d}t \;<\; \epsilon$$

if $\Delta t^k < \delta_l(\epsilon)$. Since $\Delta t^k \downarrow 0$ for $k \to \infty$, $\epsilon$ can be chosen arbitrarily small, i.e.

$$M_x^l \int_S \left| f_l(u_k(\mathbf{x}, t+\Delta t^k)) - f_l(u_k(\mathbf{x},t)) \right| \, \mathrm{d}\mathbf{x}\mathrm{d}t \;\to\; 0 \quad \text{for} \quad k \to \infty \tag{86}$$

holds for all $l \in \underline{d}$. By (85) and (86) the assertion in (83) is proven. Since the test element $\phi$ was chosen arbitrarily, convergence to a weak solution is established.

We have now to show that exactly this weak solution is the unique entropy solution in the sense of Kružkov. Therefore, we derive in a similar fashion as in the derivation of (77) the weak form of the discrete entropy condition (68). For the implicit upwind scheme, it reads as

$$\begin{aligned} & -\Delta t^k \prod_{l=1}^d \Delta x_l^k \sum_{j\in J} |u_j^0 - \xi| \phi_j^0 - \Delta t^k \prod_{l=1}^d \Delta x_l^k \sum_{j\in J} \sum_{n\geq 0} \mathsf{sgn}\left(U_j^{n+1} - \xi\right) q_j^{n+1} \phi_j^{n+1} \\ \leq \;\; & \Delta t^k \prod_{l=1}^d \Delta x_l^k \sum_{j\in J} \sum_{n\geq 0} \Bigg[ |U_j^n - \xi| \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t^k} \\ & \quad + \mathsf{sgn}\left(U_j^{n+1} - \xi\right) \sum_{l=1}^d \left\{ \left[ f_l(U_j^{n+1}) - f_l(\xi) \right] \frac{\phi_{j+1}^{n+1} - \phi_j^{n+1}}{\Delta x_l^k} \right\} \Bigg] . \end{aligned} \tag{87}$$

Using the established convergence $u_k \to u$ a.e. of the function sequence generated by the numerical method for $\Delta t^k \downarrow 0$ and $\Delta x_l^k \downarrow 0$ for all $l \in \underline{d}$, we now prove convergence of (87) towards the form of the entropy condition due to Kružkov, cf. Sections 2.1 and 2.2. Therefore, we have to consider arbitrarily chosen but fixed test elements composed of a test function $\phi$ with $\phi \geq 0$, $\phi \in C_0^\infty(\mathbb{R}^{d+1}; \mathbb{R})$, and a test number $\xi \in \mathbb{R}$.
Using the same notation and applying a similar procedure as in the case of the convergence proof to a weak solution, we first want to prove

$$\lim_{k\to\infty} M_\phi \int_K \big||u_k(\mathbf{x}, 0) - \xi| - |u_0(\mathbf{x}) - \xi|\big| \,\mathrm{d}\mathbf{x} \;=\; 0. \tag{88}$$

Since $\xi$ is fixed and $u_k(\mathbf{x}, 0)$ and $u_0$ bounded, one can find a constant function over the compact intervall $K$ which dominates the integrand for all $\xi$. Then (88) follows by use of the already established convergence $u_k(x, 0) \to u_0(x)$ a.e. and the Theorem of Dominated Convergence. We also have to compute

$$\lim_{k\to\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \mathsf{sgn}\left(u_k(\mathbf{x}, t + \Delta t^k) - \xi\right) q_k(\mathbf{x}, t + \Delta t^k)\phi(\mathbf{x}, t + \Delta t^k) \,\mathrm{d}\mathbf{x}\mathrm{d}t\,.$$

Therefore, we expand the factor $\phi(\mathbf{x}, t + \Delta t^k)$ by adding zeroes in the form

$$\phi(\mathbf{x}, t + \Delta t^k) \;=\; \phi(\mathbf{x}, t + \Delta t^k) - \phi(\mathbf{x}, t) + \phi(\mathbf{x}, t)\,.$$

Convergence of the integrals involving the factor $\phi(\mathbf{x}, t + \Delta t^k) - \phi(\mathbf{x}, t)$ to zero follows by estimating $\mathsf{sgn}$, $u_k$ and $q_k$ from above and using the usual properties of test functions. In a similar fashion, we expand the factor $\mathsf{sgn}\left(u_k(\mathbf{x}, t + \Delta t^k) - \xi\right)$, adding zero in the form $-\mathsf{sgn}\,(u_k(\mathbf{x}, t) - \xi) + \mathsf{sgn}\,(u_k(\mathbf{x}, t) - \xi)$. The proof that the integrals involving $\mathsf{sgn}\left(u_k(\mathbf{x}, t + \Delta t^k) - \xi\right) - \mathsf{sgn}\,(u_k(\mathbf{x}, t) - \xi)$ vanish follows from the continuity in the mean of $L_1$-functions. Again similarly, we expand in the form $q_k(\mathbf{x}, t + \Delta t^k) = q_k(\mathbf{x}, t + \Delta t^k) - q(\mathbf{x}, t + \Delta t^k) + q(\mathbf{x}, t + \Delta t^k)$ and use the concept of monotone convergence due to Beppo Levi to obtain convergence to zero of the integrals involving $q_k(\mathbf{x}, t+\Delta t^k) - q(\mathbf{x}, t+\Delta t^k)$. Lastly, the proof of convergence of $q(\mathbf{x}, t+\Delta t^k)$ to $q(\mathbf{x}, t)$ under the integral follows from the continuity in the mean of $L_1$-functions. The technical details only require to take all expansions obtained via adding suitable zeroes into account and eliminating all integrals which involve discrete notions.
The other terms left to investigate are

$$\int_S |u_k(\mathbf{x}, t) - \xi|\, \phi_t(\mathbf{x}, t) \,\mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{and}$$

$$\int_S \sum_{l=1}^{d} \mathsf{sgn}\left[u_k(\mathbf{x}, t + \Delta t^k) - \xi\right] \left[f_l(u_k(\mathbf{x}, t + \Delta t^k)) - f_l(\xi)\right] \frac{\partial}{\partial x_l}\phi(\mathbf{x}, t) \,\mathrm{d}\mathbf{x}\mathrm{d}t\,.$$

The procedure is the same in both cases. Since the occuring derivatives of $\phi$ are continuous, we can estimate these over the compact domain $S$ by finite constants. Since $\xi$ is a fixed value (and so is $f_l(\xi)$ for all $l \in \underline{d}$), as $u_k(\mathbf{x},t)$ is bounded and because the $f_l$ are continuous over the bounded interval of possible values of $u_k$ (due to the established $L_\infty$-stability), we can also give constants which estimate all the expressions from above that involve $u_k$. Using the product of these finite constants as dominating function over S as well as $u_k \to u$ a.e., we employ the Theorem of Dominated Convergence to receive the desired result for the implicit upwind scheme.
For instance, in the case of the *implicit Lax-Friedrichs (ILF) method*,

$$U_j^{n+1} = U_j^n + \sum_{l=1}^{d} \left\{ \frac{1}{2} \left[ U_{j-\delta l}^{n+1} - 2U_j^{n+1} + U_{j+\delta l}^{n+1} \right] - \frac{\Delta t}{2\Delta x_l} \left[ f_l(U_{j+\delta l}^{n+1}) - f_l(U_{j-\delta l}^{n+1}) \right] \right\}, \tag{89}$$

the differences in the corresponding weak forms are made up from

$$-\frac{\Delta t_k}{2} \int_S u_k(\mathbf{x}, t+\Delta t^k) \hat{\phi}_l \,\mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{and} \quad -\frac{\Delta t_k}{2} \int_S \left| u_k(\mathbf{x}, t+\Delta t^k) - \xi \right| \hat{\phi}_l \,\mathrm{d}\mathbf{x}\mathrm{d}t\,,$$

respectively. Thereby, $\hat{\phi}$ converges in the $L_\infty$-Norm to $\partial^2_{x_l^2}\phi(\mathbf{x},t)$ which is continuous since $\phi \in C_0^\infty(\mathbb{R}^{d+1};\mathbb{R})$. Thus, the corresponding term can be estimated from above by a constant over the compact domain $S$. Since $\xi$ is fixed and $u_k(\mathbf{x},t)$ is bounded as usual, both expressions vanish with $\Delta t^k \downarrow 0$.

Also for any other consistent scheme, the proof can be done exactly in an analogous fashion as above for the ILF scheme, namely to write down the differences in the weak forms to the case of the implicit upwind method. By consistency, these will vanish.
As another example, let us investigate the *implicit Godunov-type (IGT) method* defined via

$$g_l^G(v,w) \;=\; \begin{cases} \min_{v \le u \le w} f_l(u) & \text{for} \quad v \le w, \\ \max_{w \le u \le v} f_l(u) & \text{for} \quad v > w. \end{cases} \tag{90}$$

Here we have as differences to the implicit upwind case

$$\Big\{ \left[ g_l^G(U_j^{n+1} \vee \xi, u_{j+\delta l}^{n+1} \vee \xi) - g_l^G(U_j^{n+1} \wedge \xi, u_{j+\delta l}^{n+1} \wedge \xi) \right] \\ -\mathsf{sgn}\left(U_j^{n+1} - \xi\right) \left[ f_l(U_j^{n+1}) - f_l(\xi) \right] \Big\} \frac{\phi_{j+\delta l}^{n+1} - \phi_j^{n+1}}{\Delta x_l^k} \tag{91}$$

$$\text{and} \quad \left[ g_l^G(U_j^{n+1}, U_{j+1}^{n+1}) - f_l(U_j^{n+1}) \right] \frac{\phi_{j+\delta l}^{n+1} - \phi_j^{n+1}}{\Delta x_l^k}\,. \tag{92}$$

Since $g_G$ is continuous in the components and $u_k(\mathbf{x},t) \in L_1(S)$, the $g_l^G \circ u_k$ are also in $L_1(S)$. After introducing step functions as usual, the expressions incorporating $g_l^G$ from (91) give values $f_l(\eta_l)$ with

$$\begin{aligned} & \eta_l \in \left[u_k(\mathbf{x}, t+\Delta t^k), u_k(\mathbf{x}+\Delta x_l^k, t+\Delta t^k)\right] \\ \text{or} \quad & \eta_l \in \left[u_k(\mathbf{x}+\Delta x_l^k, t+\Delta t^k), u_k(\mathbf{x}, t+\Delta t^k)\right] , \end{aligned}$$

respectively. The integrals over the terms corresponding to (91) then go to zero with $k \to \infty$ because of the continuity in the mean of $g_l^G \circ u_k$. The idea for proving convergence to zero concerning the integral of the expressions corresponding to (92) is the same.

Concerning the setting from Section 2.2, the described strategy is fully transferable. *qed*

# 6 Application of the monotonicity conditions

This section contains the theoretical investigation of a few selected implicit methods. These are: An implicit Upwind (IU) scheme, the ILF scheme from (89), and the IGT method defined via (90).

## 6.1 An implicit upwind method

We now investigate (76) with respect to monotonicity.

*To condition (40):*

$$\begin{aligned} & H_l\,(a+\Delta a,\, b,\, c) - H_l\,(a,\, b,\, c) \\ = \; & \left[-\frac{\Delta t}{\Delta x_l}\left[f_l(b) - f_l(a+\Delta a)\right]\right] - \left[-\frac{\Delta t}{\Delta x_l}\left[f_l(b) - f_l(a)\right]\right] \\ = \; & \frac{\Delta t}{\Delta x_l}\left[f_l(a+\Delta a) - f_l(a)\right] . \end{aligned}$$

The condition (40) is fulfilled if $f_l$ grows monotonously for all $l = 1, \ldots, d$.

*To condition (41):*

$$\begin{aligned} & H_l\,(a, b, c+\Delta c) - H_l\,(a, b, c) \\ = \; & \left[-\frac{\Delta t}{\Delta x_l}\left[f_l(b) - f_l(a)\right]\right] - \left[-\frac{\Delta t}{\Delta x_l}\left[f_l(b) - f_l(a)\right]\right] = 0 \;\; (\geq 0) . \end{aligned}$$

Thus, the condition (41) is always fulfilled and the IU scheme is monotone if all the fluxes $f_l$ grow monotonously. This is a reasonable property of the considered IU scheme as the monotonicity respects the direction of the flow. Note that the $f_l$ do not need to be Lipschitz continuous to ensure the monotonicity of the scheme, cf. [3].

## 6.2 The implicit Lax-Friedrichs method

We investigate the ILF scheme from (89).
*To condition (40):*

$$H_l(a + \Delta a, b, c) - H_l(a, b, c) = \frac{1}{2}\Delta a + \frac{\Delta t}{2\Delta x_l}\left[f_l(a + \Delta a) - f_l(a)\right]. \qquad (93)$$

This expression is not positive or equal to zero without additional requirements.
*To condition (41):*

$$H_l(a, b, c + \Delta c) - H_l(a, b, c) \; = \; \frac{1}{2}\Delta c - \frac{\Delta t}{2\Delta x_l}\left[f_l(c + \Delta c) - f_l(c)\right]. \qquad (94)$$

Again this expression is not automatically positive or equal to zero. The requirements (93) and (94) can be combined to

$$\frac{|f_l(x + \Delta x) - f_l(x)|}{\Delta x_l} \; \leq \; \frac{\Delta x_l}{\Delta t} \quad \forall\, l = 1, \ldots, d \text{ and } \forall\, \Delta x \geq 0.$$

Therefore, the ILF scheme is monotone only for Lipschitz-continuous flux functions with Lipschitz constants $L_l \leq (\Delta x_l/\Delta t)$. This can also be understood as a condition on the time step size which does not depend on the dimension since each single one of the $2l$ conditions (40) and (41) has to be satisfied and no coupling is involved.

This is quite surprising since *(i)* it is usually suggested that the numerical characteristics include the whole domain in the case of implicit methods, and as *(ii)* no dimensional influence on the monotonicity property is obtained.

In order to discuss these points, we investigate the cases of 1-D and 2-D linear advection, respectively.
In the case of a linear flux $f(u) = vu$ in 1-D, with $\lambda := \Delta t/\Delta x$, the linear system defined by the ILF is given as

$$\left[-\frac{1}{2} - v\frac{\lambda}{2}\right] U_{j-1}^{n+1} + 2U_j^{n+1} + \left[-\frac{1}{2} + v\frac{\lambda}{2}\right] U_{j+1}^{n+1} = U_j^n. \qquad (95)$$

We investigate the structure of the tridiagonal matrix $A = (a_{ij})$ defined by (95). Therefore, let $v$ be positive with $v > (1/\lambda)$ so that the formal monotonicity property of the scheme is lost. Then the entries in the lower diagonal $a_{i+1,i}$ always take on negative values while the entries in the upper diagonal $a_{i,i+1}$ are always positive.

We at first eliminate the entries in the lower diagonal $a_{i+1,i}$. The diagonal entries of the matrix have to be modified accordingly, i.e. the diagonal entry in the $i$-th row is modified via

$$a_{ii}^{new} = a_{ii}^{old} - \frac{a_{i,i-1}}{a_{i-1,i-1}} a_{i-1,i}.$$

Thereby, note that we always have the situation

$$a_{i,i-1} < 0,\ a_{i-1,i-1} > 0 \text{ and } a_{i-1,i} > 0,$$

so that $a_{ii}^{new} > a_{ii}^{old}$ is always satisfied. Since the right hand side $(b_i)$ of the investigated system incorporating the given data is modified via

$$b_i = U_i^n - \frac{a_{i,i-1}}{a_{i-1,i-1}} b_{i-1},$$

data sets with $U_k^n \geq 0\ \forall k$ imply only positive possible changes in the values $b_i$. In particular, the values in the upper diagonal $a_{i,i+1}$ remain unchanged and positive.

We now investigate what happens at a jump in given data $u_k^n$ from values 0 to 1 when backward elimination is applied in order to solve the system. Therefore, we fix $U_j^n := 0\ \forall j < i$ and $U_j^n := 1\,\forall j \geq i$. By the described procedure, it is clear that the corresponding entries on the right hand side also show a jump from 0 to 1 after the modification due to elimination of the lower diagonal since $b_{i-1} = U_{i-1}^n = 0$, so that no positive update in $b_i$ takes place. Backward elimination results in

$$U_{i-1}^{n+1} = \underbrace{\frac{1}{a_{i-1,i-1}^{new}}}_{>0} \big( \underbrace{U_{i-1}^n}_{=0} - \underbrace{a_{i-1,i}}_{>0} \underbrace{U_i^n}_{=1} \big) \ < \ 0\,,$$

so that the monotonicity is violated as expected, cf. Figure 1. The violation of the monotonicity property can also be observed at jumps from high to lower values within given data.

Concerning the 2-D situation, we consider the linear advection equation

$$\frac{\partial}{\partial t} u(x,y,t) + \frac{\partial}{\partial x}\left(vu(x,y,t)\right) + \frac{\partial}{\partial y}\left(vu(x,y,t)\right) = 0$$

with grid parameters $\Delta x = \Delta y = 0.1$ and the initial condition

$$u(x,y,0) = \begin{cases} 1 & \text{for} \quad (x,y) \in [0,1] \times [0,1]\,, \\ 0 & \text{else.} \end{cases}$$

The monotonicity condition infers that the chosen time step size $\Delta t = 0.1$ is the largest one allowed for $v = 1.0$ in order to preserve the monotonicity of the scheme, the same as would be in the 1-D case. See Figure 2 for a visualisation of the monotone and monotonicity-violating property of the method.

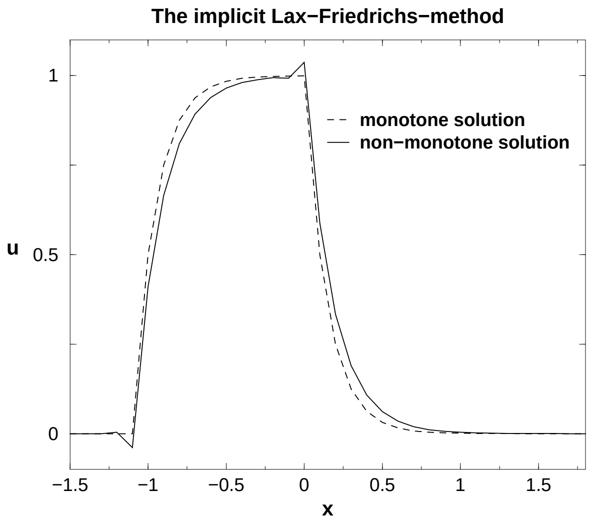


Figure 1: Numerical solutions after one time step of the 1-D linear advection problem. The solution is computed once with $v = 1$ satisfying the monotonicity condition and once with $v = 1.5$, resulting in a monotonicity violation.

## 6.3 An implicit Godunov-type method

In the scalar case, a closed form of the exact solution of a Riemann problem was described by Osher [16]. Using this, a numerical scheme can be defined via the $d$ numerical flux functions from (90). Since the relative values of the test variables have to be compared within the scheme, diversions by cases have to be employed for the monotonicity investigation.

*To condition (40):*
Generally, for $l = 1, \ldots, d$,

$$H_l(a + \Delta a, b, c) - H_l(a, b, c) = \frac{\Delta t}{\Delta x_l}\left[g_l^G(a + \Delta a, b) - g_l^G(a, b)\right]$$

holds. Since only the values $b$, $a$ and $a+\Delta a$ are of importance, it is necessary to investigate three cases for each $l \in \underline{d}$.

1. Case: $b \leq a \leq a + \Delta a$

$$\frac{\Delta t}{\Delta x_l}\left[g_l^G(a+\Delta a, b) - g_l^G(a, b)\right] = \frac{\Delta t}{\Delta x_l}\left[\max_{b \leq u \leq a+\Delta a} f_l(u) - \max_{b \leq u \leq a} f_l(u)\right] \geq 0.$$

2. Case: $a \leq b \leq a + \Delta a$

$$\frac{\Delta t}{\Delta x_l}\left[g_l^G(a+\Delta a, b) - g_l^G(a, b)\right] = \frac{\Delta t}{\Delta x_l}\left[\max_{b \leq u \leq a+\Delta a} f_l(u) - \min_{a \leq u \leq b} f_l(u)\right] \geq 0.$$

3. Case: $a \leq a + \Delta a \leq b$

$$\frac{\Delta t}{\Delta x_l}\left[g_l^G(a+\Delta a, b) - g_l^G(a, b)\right] = \frac{\Delta t}{\Delta x_l}\left[\min_{a+\Delta a \leq u \leq b} f_l(u) - \min_{a \leq u \leq b} f_l(u)\right] \geq 0.$$

Thus, the validity of the condition (40) is guaranteed without any additional condition on the flux function. This can be verified analogously for condition (41), so that the investigated IGT scheme is monotone for general continuous flux functions.

# 7 Summary and conclusion

We have shown how the monotonicity notion needs to be understood in the implicit case. For this, we have employed some new tools in order to deal with the implicit scheme definition.
Our results show, that the usual intuition that implicitness gives unconditional stability does not necessarily hold when dealing with numerical schemes for HCLs. By the details of our proceeding, we conjecture that only upwind-type schemes can really achieve this desirable property.
The question we aim to address in future research is, if there is a good way to investigate other non-linear stability notions like e.g. the TVD notion rigorously for implicit schemes.

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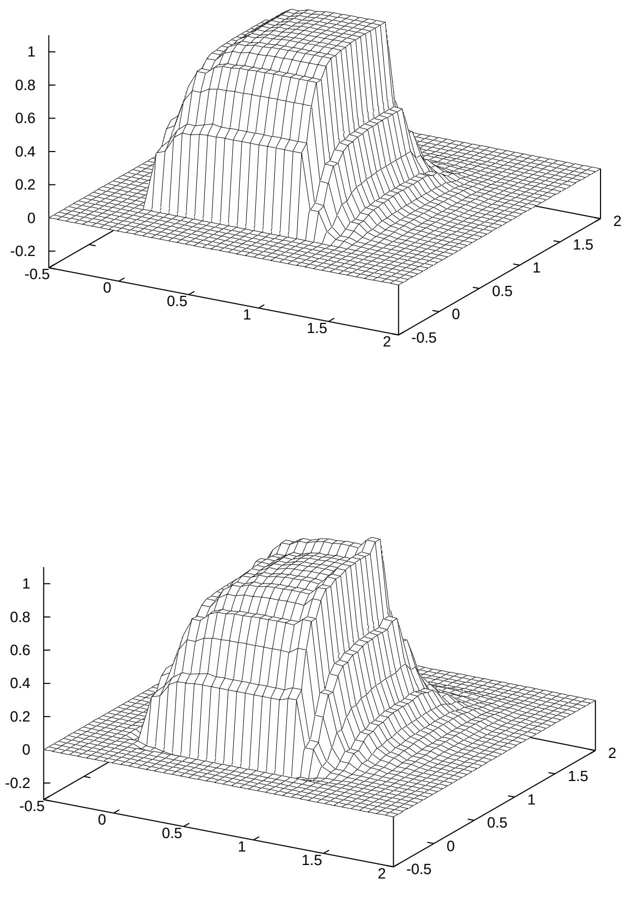


Figure 2: Numerical solutions after one time step of the 2-D linear advection problem. **Top.** The solution is computed with $v = 1$ satisfying the monotonicity condition. **Bottom.** The solution is computed with $v = 1.5$ resulting in monotonicity loss, as in 1-D.