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Abstract

Let $\Gamma = \mathrm{GL}(2, \mathbb{F}_q[T])$ be the Drinfeld modular group, which acts on the rigid-analytic upper half-plane Ω. We determine the zeroes of the coefficient modular forms ${}_a\ell_k$ on the standard fundamental domain $\mathcal{F}$ for Γ on Ω, along with the dependence of $|{}_a\ell_k(z)|$ on $z \in \mathcal{F}$.

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Introduction. In this article we study Drinfeld modular forms on the full modular group $\Gamma = \mathrm{GL}(2, \mathbb{F}[T])$, where $\mathbb{F}$ is a finite field with q elements. Such forms arise from different sources, e.g.

- coefficients of the “defining equations” of Drinfeld modules,
- Eisenstein series,
- para-Eisenstein series (coefficients of the exponential functions of rank-2 lattices),
- coefficients of the division polynomials of Drinfeld modules.

As in the case of classical elliptic modular forms for $\mathrm{SL}(2, \mathbb{Z})$, the basic interest is in their relations, expansions at infinity, congruence properties, zeroes, special values.

Serious work about such questions for the first two types of modular forms started in the late seventies (see e.g. [?], [?], [?]) and was continued with the description of their expansions at infinity and congruence properties modulo primes [?]. Later, Cornelissen [?] and the author [?] described the zeroes of Eisenstein series as well as the corresponding j-invariants. Like their classical counterparts, the zeroes z in the natural fundamental domain $\mathcal{F}$ of Γ satisfy $|z| = 1$.

Motivated from arithmetical questions like the determination of the ramification of Galois representations attached to Drinfeld modules, the third family of modular forms (baptized para-Eisenstein series in [?]) was studied in [?]. In the present paper, making use of techniques introduced in [?] and [?] (which relate the growth/decay of modular forms with their zero distribution pattern via non-archimedean contour integration), we start the study of the fourth group of modular forms ${}_a\ell_k$, where $a \in A = \mathbb{F}[T]$, $0 \le k \le 2 \deg a$, see (1.15) for the precise definition. These forms are crucial for the understanding of the interplay between the arithmetic of a Drinfeld module and the geometry of the lattice by which it is uniformized. They are the coefficients of the division polynomial ϕ_a of a "generic" Drinfeld module ϕ, and are thus referred to as *coefficient forms.*

We succeed in determining the absolute values $|z|$ of the zeroes z of ${}_a\ell_k$ in $\mathcal{F}$ (Theorem 5.1) as well as of the associated $|j(z)|$ (Theorem 4.11). In contrast with the case of Eisenstein series (but similar to para-Eisenstein series), these $|z|$ are in general larger than 1, but always of shape q^ℓ with $0 \leq \ell \leq k-1$. The crucial part of the proof is in section 4, where, by a complicated induction argument, we determine the Newton polygon of the companion polynomial (see Definition 1.13) of the form ${}_{T^d}\ell_k$.

As a by-product of our study, but of independent interest, we find the description Theorem 3.4 of the modular j-invariant function restricted to $\mathcal{F}_k = \{z \in \mathcal{F} \mid |z| = q^k\}$.

We apply the results about the zeroes to derive the decay properties of the ${}_a\ell_k$ on the fundamental domain $\mathcal{F}$. In Theorems 6.7 and 6.11 we give precise formulas for the spectral norm (the supremum, which in fact is a maximum, and in most cases even the constant absolute value) of ${}_a\ell_k$ on $\mathcal{F}_n$, in case $k \leq d = \deg a$ (where $|{}_a\ell_k|$ becomes eventually constant), and in case $k > d$ (where ${}_a\ell_k$ is a cusp form and thus decreases fast at infinity), respectively. As a consequence we find (Theorem 6.16) that, after a suitable normalization, the ${}_a\ell_k$ for $\deg a \to \infty$ tend to the para-Eisenstein series of weight $q^k - 1$.

The present paper seems to be the first one that systematically studies the zero patterns and growth/decay properties of families of Drinfeld modular forms. It possibly sheds new light also to similar questions about classical modular forms.

The paper was finished in the fall of 2009, when I was on sabbatical leave at the National Center for Theoretical Sciences in Hsinchu, Taiwan. With great pleasure I acknowledge the hospitality of that institution, and I would like to heartily thank the colleagues in Hsinchu and in particular the director of the NCTS, Professor Wen-Ching Li, both for valuable discussions and for their support during my visit.

Notation. Throughout the paper, we use without further definition the following notation, which is largely compatible with e.g. the articles [?] [?] [?] [?].

$\mathbb{F}$ is the finite field with q elements, with algebraic closure $\overline{\mathbb{F}}$,
$\mathbb{F}^{(k)}$ the extension of degree k of $\mathbb{F}$ in $\overline{\mathbb{F}}$,
$A = \mathbb{F}[T]$ the polynomial ring in an indeterminate T, with field of fractions $K = \mathbb{F}(T)$ and $K_\infty = \mathbb{F}((T^{-1}))$, its completion at infinity,

$|\,.\,|$ the absolute value on K_∞ normalized by $|T| = q$ and uniquely extended to C_∞, the completed algebraic closure of K_∞,
$|\,.\,|_i$ the "imaginary part" function on C_∞, $|z|_i = \inf_{x \in K_\infty} |z - x|$,
$\Omega = \mathbb{P}^1(C_\infty) - \mathbb{P}^1(K_\infty) = C_\infty - K_\infty$ the Drinfeld upper half-plane, equipped with the action of $\Gamma = \mathrm{GL}(2, A)$ through fractional linear transformations,
$C_\infty\{\tau\}$ the non-commutative polynomial ring over C_∞ with commutation rule $\tau x = x^q \tau$ for $x \in C_\infty$, identified with the ring of $\mathbb{F}$-linear polynomials $\sum a_i X^{q^i}$ $(\tau^i = X^{q^i})$.

Similar notation will be used for $R\{\tau\}$ ("polynomials" with coefficients in the subring R of C_∞) and $C_\infty\{\{\tau\}\}$ ("power series" in τ).

The following denote special elements of $A = \mathbb{F}[T]$:
$[k] = T^{q^k} - T$, $k = 0, 1, 2, \ldots$
$D_k = [k][k-1]^q \cdots [1]^{q^{k-1}}$, $k = 1, 2, \ldots$, $D_0 := 1$.

1. Basic concepts [?] [?]. An *A-lattice* Λ in C_∞ is a finitely generated (hence free) discrete A-submodule of C_∞ of some rank $r \in \mathbb{N}$. With Λ we associate its *exponential function*

$$z \longmapsto e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} (1 - \frac{z}{\lambda}),$$

an entire, surjective and $\mathbb{F}$-linear function from C_∞ to C_∞ with kernel Λ. It may be expanded as

$$e_\Lambda(z) = \sum_{k \geq 0} \alpha_k z^{q^k}, \tag{1.1}$$

which we regard as the element $\sum \alpha_k \tau^k$ of $C_\infty\{\{\tau\}\}$. (Note that the non-commutative multiplication $f \circ g$ in $C_\infty\{\{\tau\}\}$ corresponds to inserting the function $g(z)$ into $f(z)$.) It satisfies the functional equation

$$e_\Lambda(Tz) = \phi_T(e_\Lambda(z)) \tag{1.2}$$

with some polynomial $\phi_T \in C_\infty\{\tau\}$ of shape

$$\phi_T(X) = TX + g_1 X^q + \cdots + g_r X^{q^r} = T + g_1\tau + \cdots + g_r \tau^r, \tag{1.3}$$

where $r = \mathrm{rank}(\Lambda)$ and $g_r \neq 0$. The $\mathbb{F}$-algebra homomorphism

$$\begin{array}{rccc} \phi: & A & \longrightarrow & C_\infty\{\tau\} \\ & a & \longmapsto & \phi_a \end{array} \tag{1.4}$$

uniquely determined by ϕ_T yields a Drinfeld A-module of rank r over C_∞, and the above describes bijections between the sets of A-lattices of rank r in

C_∞, of polynomials ϕ_T of shape (1.3), and of Drinfeld A-modules of rank r over C_∞, respectively.

(1.5) The most simple example is the *Carlitz module* ρ given by $\rho_T = TX + X^q = T + \tau$, i.e., when the attached lattice $\Lambda = \overline{\pi}A =: L$ has rank one. Here the constant $\overline{\pi} \in C_\infty$, determined up to a $q-1$-th root of unity (i.e., up to an element of $\mathbb{F}^*$), is such that (1.2) holds, that is,

$$e_L(Tz) = Te_L(z) + e_L(z)^q.$$

Comparing coefficients, we find

$$e_L(z) = \sum_{k \geq 0} D_k^{-1} z^{q^k}. \tag{1.6}$$

The reader should think of $\overline{\pi}$ as an analogue of $2\pi i$, with $\pi = 3.14159\ldots$. We have (see [?] 4.9–4.11 for similar formulas)

$$\overline{\pi}^{q-1} = -[1] \prod_{i \geq 1} (1 - [i]/[i+1])^{q-1}, \tag{1.7}$$

in particular, $|\overline{\pi}^{q-1}| = q^q$. For later purposes, we put

$$t(z) := \frac{1}{e_L(\overline{\pi}z)} = \frac{1}{\overline{\pi}e_A(z)},\ s(z) = t(z)^{q-1}, \tag{1.8}$$

which are invertible holomorphic functions on the Drinfeld upper half-plane Ω.

Rank-two lattices Λ are up to scaling of the form $\Lambda_\omega = \overline{\pi}(A\omega + A)$, where $\omega \in \Omega$. The associated polynomial (1.3) is

$$\phi_T^{(\omega)} = T + g(\omega)\tau + \Delta(\omega)\tau^2, \tag{1.9}$$

where g and Δ are holomorphic functions of Ω with Δ invertible. In fact, they are modular forms of respective weights $q-1$ and q^2-1. Here a *Drinfeld modular form of weight k* for the group $\Gamma = \mathrm{GL}(2, A)$ is a holomorphic function $f : \Omega \longrightarrow C_\infty$ subject to

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \quad \text{for } \begin{pmatrix} a\,b \\ c\,d \end{pmatrix} \in \Gamma \tag{1.10}$$

and that has an expansion (convergent for small values of s, i.e., large values of $|z|_i$)

$$f(z) = \sum_{k \geq 0} a_k s(z)^k \tag{1.11}$$

in the uniformizer $s(z)$ at infinity. We often abuse notation and write simply $f = \sum a_k s^k$. The form f is a *cusp form* if its zeroth coefficient a_0 vanishes. The modular forms of weight k form a C_∞-vector space $M_k(\Gamma)$; we write $M(\Gamma) = \oplus_k M_k(\Gamma)$ for the algebra of all modular forms. Then $M(\Gamma)$ is the polynomial ring $C_\infty[g, \Delta]$ in the algebraically independent forms $g \in M_{q-1}$ and $\Delta \in M_{q^2-1}(\Gamma)$. The *modular invariant* is defined as

$$j(\omega) := g(\omega)^{q+1}/\Delta(\omega); \tag{1.12}$$

it is invariant under Γ and yields a biholomorphic identification $j : \Gamma \setminus \Omega \xrightarrow{\cong} C_\infty$ of the quotient space $\Gamma \setminus \Omega$ with the affine line over C_∞. Each modular form of weight k has a unique presentation

$$f(z) = \varphi_f(j(z))\Delta(z)^a g(z)^b \tag{1.13}$$

with a polynomial $\varphi_f \in C_\infty[X]$, $a \in \mathbb{N}_0$, $0 \leq b \leq q$, and $a(q^2-1)+b(q-1) = k$. We call φ_f the *companion polynomial* of f. The j-values $j(z)$ where $f(z) = 0$ (i.e., the zeroes of $\varphi_f(X)$, and $j = 0$ if $b > 0$) are briefly called *j-zeroes* of f.

The main examples of modular forms are:

(1.14) The forms g, Δ as above. They are normalized such that their s-expansions have coefficients in A, with starting terms

$$g = 1 - [1]s - [1]s^{q^2-q+1} + \cdots, \quad \Delta = -s + s^q - [1]s^{q+1} + \cdots$$

(see [?] 10.11, 10.3. The present normalization agrees with g_{new}, Δ_{new} *loc. cit.* 6.4).

(1.15) More generally, let $\phi^{(\omega)}$ be the Drinfeld module corresponding to $\omega \in \Omega$ (i.e., to the lattice Λ_ω, cf. (1.9)), let $a \in A$, and write

$$\phi_a^{(\omega)} = \sum_{0 \leq k \leq 2 \deg a} {}_a\ell_k(\omega)\tau^k.$$

Then ${}_a\ell_k$ is modular of weight $q^k - 1$ for Γ, a *coefficient form*. The purpose of the present paper is to determine its zeroes, j-zeroes, and decay properties.

(1.16) The *Eisenstein series* of weight k

$$E_k(\omega) := \sum_{(0,0) \neq (a,b) \in A^2} \frac{1}{(a\omega + b)^k}$$

is well-defined (i.e., convergent) and non-zero if $0 < k \equiv 0(q-1)$, and defines an element of $M_k(\Gamma)$ [?].

(1.17) Write the exponential function e_ω of Λ_ω as

$$e_\omega(z) = \sum_{k\geq 0} \alpha_k(\omega) z^{q^k}.$$

Then $\alpha_k(\omega)$ is modular of weight $q^k - 1$, a so-called *para-Eisenstein series*.

The principal arithmetical properties of the E_k (mainly when $k = q^i - 1$ for some i) and the α_k and their zeroes are studied in [?], [?], [?] and [?]. Except for section 6 (Theorem 6.16), they play no major role in the present article.

2. The fundamental domain $\mathcal{F}$. We put

$$(2.1) \qquad \begin{array}{lcl} \mathcal{F} & := & \{z \in \Omega \mid |z| = |z|_i \geq 1\}, \text{ and for } n \in \mathbb{N}_0, \\ \mathcal{F}_n & := & \{z \in \Omega \mid |z| = |z|_i = q^n\}, \end{array}$$

which are open analytic subspaces of Ω. Further, for a holomorphic function f on $\mathcal{F}_n$, we let

$$\|f\|_n := \sup\{|f(x)| \mid z \in \mathcal{F}_n\} = \max\{|f(z)| \mid z \in \mathcal{F}_n\}$$

be the *spectral norm* on $\mathcal{F}_n$ (see [?], our general reference for rigid analytic geometry). Due to the presence of many elements of finite order in $\Gamma = \mathrm{GL}(2, A)$, we cannot expect a fundamental domain for Γ on Ω in the proper sense (i.e., a subset of Ω with a "reasonable" topological structure and which maps bijectively onto $\Gamma \setminus \Omega$). However:

2.2 Proposition.

(i) Each element z of Ω is Γ-equivalent to at least one and at most finitely many elements of $\mathcal{F}$.

(ii) If $\gamma \in \Gamma$ satisfies $\gamma(\mathcal{F}_n) \cap \mathcal{F} \neq \emptyset$ then $\gamma(\mathcal{F}_n) = \mathcal{F}_n$ and $\gamma \in \Gamma_n$, where $\Gamma_0 = \mathrm{GL}(2, \mathbb{F})$ and $\Gamma_n = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma \mid a, d \in \mathbb{F}^*, \deg b \leq n\}$, $n \geq 1$. Conversely, each $\gamma \in \Gamma_n$ stabilizes $\mathcal{F}_n$.

Proof. [?] 6.5. $\square$

It is important to understand the behavior of some basic functions on $\mathcal{F}$. The following table is a compilation of [?] 2.13–2.18, as far as g, Δ, j are concerned, and follows for s by direct calculation from its definition. It gives the values $\log_q \|f\|_n$ for the functions $f = s, g, \Delta$ and j.

2.3 Table.

f	s	g	Δ	j
$\log_q \|f\|_n$	$-q^{n+1}$	0	$-q^{n+1}$	q^{n+1}

Each of the above $|f|$ is constant on $\mathcal{F}_n$ ($n = 0, 1, 2, \ldots$) with the exception of $f = g$ or j on $\mathcal{F}_0$, in which case we have only the inequalities $\log_q |g(z)| \leq 0$ and $\log_q |j(z)| \leq q$ for $z \in \mathcal{F}_0$.

2.4 Remark. For our given functions f, the values of $\log_q |f|$ "interpolate linearly" for $z \in \mathcal{F}$, $z \notin \bigcup \mathcal{F}_n$. Viz, let $|z| = |z|_i = q^{n-\epsilon}$ with $n \in \mathbb{N}_0$, $0 < \epsilon < 1$; then a tedious but straightforward calculation gives e.g.

$$\log_q |s(z)| = -q^{n+1}(1 - \frac{q-1}{q}\epsilon),$$

and similar formulas hold for the other functions under consideration. This can best be understood by relating Ω with the Bruhat-Tits tree $\mathcal{T}$ of K_∞, where $\mathcal{F}$ corresponds to the standard fundamental domain of Γ on $\mathcal{T}$ and the distinguished subsets $\mathcal{F}_n$ to its vertices, see [**?**] sect. 1. It will turn out that, moreover, the modular forms we consider have their zeroes in $\mathcal{F}$ in fact in $\bigcup \mathcal{F}_n$ (see (2.5), (2.6) below). This justifies to focus attention to the $\mathcal{F}_n$.

2.5 Example. Let g_k be the Eisenstein series of weight $g^k - 1$, normalized with absolute term 1 in its s-expansion. Then ([**?**], [**?**] Theorem 8.5):

(i) All the zeroes of g_k are simple;

(ii) if $z \in \mathcal{F}$ is a zero of g_k then $z \in \mathcal{F}_0$;

(iii) for each $z_0 \in \mathbb{F}^{(k+1)} - \mathbb{F} \subset \mathcal{F}_0$ there exists a unique zero $z \in \mathcal{F}_0$ of g_k with $|z - z_0| < 1$, and these are all the zeroes of g_k in $\mathcal{F}$.

2.6 Example. Let α_k be the para-Eisenstein series of weight $q^k - 1$ (see (1.17)). Then ([**?**] Corollary 8.12):

(i) All the zeroes of α_k are simple;

(ii) if $z \in \mathcal{F}$ is a zero of α_k then $z \in \mathcal{F}_n$, where $0 \leq n \leq k-1$, $n \equiv k+1$ (2);

(iii) for each n as in (ii) there exist precisely $(q-1)q^k$ zeroes of α_k in $\mathcal{F}_n$.

3. Description of j restricted to $\mathcal{F}_n$.

3.1 Definition. *Let $z \in C_\infty$ satisfy $|z| = q^n$ with $n \in \mathbb{Z}$. Then $z = T^n \cdot u \cdot (1 + x)$ with well-defined $u \in \overline{\mathbb{F}} - \{0\}$ and $x \in C_\infty$, $|x| < 1$. We call $L(z) := u$ the* leading coefficient *of z.*

The map $L : \mathcal{F}_0 \longrightarrow \overline{\mathbb{F}} - \mathbb{F}$ is well-defined and surjective, as is

$$L : \{z \in C_\infty \mid \log_q |z| = q^n\} =: U_{q^n} \longrightarrow \overline{\mathbb{F}} - \{0\}.$$

Since moreover $T^n : \mathcal{F}_0 \xrightarrow{\cong} \mathcal{F}_n$ and j maps $\mathcal{F}_n$ onto $U_{q^{n+1}}$ if $n \geq 1$, we may define a map $j_n : \overline{\mathbb{F}} - \mathbb{F} \longrightarrow \overline{\mathbb{F}}$ through the commutative diagram

$$\text{(3.2)} \qquad \begin{array}{ccc} \mathcal{F}_n & \xrightarrow{j} & U_{q^{n+1}} \\ L \downarrow & & \downarrow L \\ \overline{\mathbb{F}} - \mathbb{F} & \xrightarrow{j_n} & \overline{\mathbb{F}}\ , \end{array}$$

provided that $L(j(z))$ depends only on $L(z)$. In case $n = 0$, we must replace (3.2) by

$$\text{(3.3)} \qquad \begin{array}{cccl} \mathcal{F}_0 & \longrightarrow & U'_q & := \{z \in C_\infty \mid \log_q |z| \leq q\} \\ L \downarrow & & \downarrow L' & \\ \overline{\mathbb{F}} - \mathbb{F} & \xrightarrow{j_0} & \overline{\mathbb{F}}\ , & \end{array}$$

where L' restricted to U_q agrees with L and $L'(z) = 0$ for $\log_q |z| < q$.

The goal of the present section is to derive the following explicit description of j_n.

3.4 Theorem. *The maps j_n are well-defined and given by*

$$\begin{aligned} j_n(z) &= (z^q - z)^{q^n(q-1)} \ \textit{for } n \geq 1 \\ &= \frac{(z^{q^2} - z)^{q+1}}{(z^q - z)^{q^2+1}} \quad \textit{for } n = 0. \end{aligned}$$

Remark. The formula for $n = 0$ differs in sign from [?] 3.6, due to a different normalization taken there.

For the proof we need to collect a number of facts about the functions in question.

(3.5) The power series Δ/s and $\frac{g-1}{[1]}$ belong to $A[[s]]$ and, writing Δ/s or $\frac{g-1}{[1]}$ as $\sum_{i\geq 0} a_i s^i$, the a_i satisfy $\deg a_i \leq i$ ([?] Proposition 6.7). In particular, the series converge on $\mathcal{F}$ (see (2.3)) and yield the right values $g(z)$, $\Delta(z)$, respectively.

Suppose that $z \in \mathcal{F}_n$ with $n > 0$. We see by (2.3) and (3.5) that in both $g = 1 - [1]s - [1]s^{q^2-q+1} + \cdots$ and $\Delta = -s + s^q - [1]s^{q+1} + \cdots$ the leading terms 1 resp. $-s$ dominate and thus determine $L(g(z))$ and $L(\Delta(z))$. Therefore, for $z \in \mathcal{F}_n$ with $n \geq 1$,

$$\text{(3.6)} \qquad L(j(z)) = L\left(\frac{g^{q+1}(z)}{\Delta(z)}\right) = L(-s^{-1}(z)) = -L(s^{-1}(z))$$

holds. Let's next consider $z \in \mathcal{F}_0$, where some more terms must be taken into account. We need the following sharpening of (3.5).

3.7 Lemma. *Write $g = \sum_{i \geq 0} a_i s^i$ with $a_i \in A$. Then for $i \geq 2$, the strict inequality $\deg a_i < i + q$ holds.*

Proof. This follows from combining the following facts from [?] (with the notation given there, which we will not repeat here):

- $g = 1 - [1] \sum_{a \in A \text{ monic}} t_a^{q-1}$ (*loc. cit.* 6.4)
- $\sum_{\substack{a \in A \text{ monic} \\ \text{of degree } d}} t_a^{q-1} = s^{q^d} \sum (1/f_a)^{q-1}$ (by definition (4.6) of f_a)
- f_a and $1/f_a$ as power series in s satisfy property $(*)$ of (6.7).

□

3.8 Corollary. *For $z \in \mathcal{F}_0$ we have $L'(g(z)) = L'(1 - [1]s(z))$. (Here again as in* (3.3) *we write $L'(w) = L(w)$ if the argument w has maximal possible absolute value* 1 *and $L'(w) = 0$ otherwise.)*

Proof. By (3.7) and (2.3) the higher terms $a_i s^i(z)$ of the s-expansion of $g(z)$ don't contribute to $L'(g(z))$. □

Since $|\Delta(z)|$ is constant on $\mathcal{F}_0$ with dominating term $-s$, we have $L(\Delta(z)) = L(-s(z)) = -L(s(z))$ for $z \in \mathcal{F}_0$, and so

$$L'(j(z)) = -L'(1 - [1]s(z))^{q+1} / L(s(z)). \tag{3.9}$$

We are thus reduced to determining the right hand sides of (3.6) for $n \geq 1$ and of (3.9) for $n = 0$, respectively, on $\mathcal{F}_n$.

Let's start with the case $\boxed{n \geq 1}$.

We have $s(z)^{-1} = e_L(\overline{\pi} z)^{q-1}$, where $e_L(z)$ is given by (1.6), thus

$$s(z)^{-1} = \overline{\pi}^{q-1} \Big(\sum_{i \geq 0} \frac{\overline{\pi}^{q^i - 1}}{D_i} z^{q^i} \Big)^{q-1}.$$

3.10 Lemma. *Write d_i for $\frac{\overline{\pi}^{q^i-1}}{D_i} z^{q^i}$. Then for $z \in \mathcal{F}_n$,*

(i) $|d_n| = |d_{n+1}| = |d_n + d_{n+1}|$;

(ii) if i is different from n, $n+1$, then $|d_i| < |d_n|$.

Proof. $|d_n| = |d_{n+1}| > |d_i|$ for $i \neq n$, $n+1$ follows from $\log_q |z| = n \geq 1$, $\log_q |D_i| = iq^i$ and $\log_q |\overline{\pi}^{q-1}| = q$. By (1.7) we have $L(\overline{\pi}^{q-1}) = -1$, thus $L(d_i) = (-1)^i L(z)^{q^i}$. Since $L(z) \notin \mathbb{F}$, no cancellation can take place in $d_n + d_{n+1}$, so $|d_n + d_{n+1}| = |d_n| = |d_{n+1}|$. $\square$

By the lemma and (3.6) we have

$$\begin{aligned} L(j(z)) &= -L(s(z)^{-1}) = +(L(d_n) + L(d_{n+1}))^{q-1} \\ &= ((-1)^n L(z)^{q^n} + (-1)^{n+1}(L(z)^{q^{n+1}})^{q-1} = (L(z)^q - L(z))^{q^n(q-1)}, \end{aligned}$$

which at the same time shows the well-definedness of j_n and the formula asserted by the thoerem for $n \geq 1$.

There remains the case $\boxed{n=0}$.

For $z \in \mathcal{F}_0$ we write $s(z) = \overline{\pi}^{1-q} e_A(z)^{1-q} = \overline{\pi}^{1-q} \sum_{a\in A} \frac{1}{(z-a)^{q-1}}$ ([?] 2.2(v)+3.4(v)).

Now $L(\sum_{a\in A} \frac{1}{(z-a)^{q-1}}) = L(\sum_{a\in\mathbb{F}} \frac{1}{(z-a)^{q-1}})$, $\sum_{a\in\mathbb{F}} \frac{1}{(z-a)^{q-1}} = (\frac{1}{z^q - z})^{q-1}$, $\overline{\pi}^{1-q}[1] = -1+$ smaller terms, hence from (3.9),

$$L'(j(z)) = L'(1 + (\frac{1}{z^q - z})^{q-1})^{q+1} / L((\frac{1}{z^q - z})^{q-1}).$$

This shows first that $L'(j(z))$ depends only on $L(z)$, so j_0 is well-defined. If further $z = L(z)$, i.e., $z \in \overline{F} - \mathbb{F}$, we get

$$L'(j(z)) = (1 + (\frac{1}{z^q - z})^{q-1})^{q+1} (z^q - z)^{q-1},$$

which, upon simplifying and recollecting terms, equals $\frac{(z^{q^2} - z)^{q+1}}{(z^q - z)^{q^2+1}}$. Theorem 3.4 is proved. $\square$

3.11 Corollary. *For $z \in \mathcal{F}$ we have the following equivalences:*

(i) $\log_q |j(z)| < q \Leftrightarrow z \in \mathcal{F}_0$ *and* $L(z) \in \mathbb{F}^{(2)}$

(ii) $\log_q |j(z)| = q$ *and* $L(j(z)) = 1 \Leftrightarrow z \in \mathcal{F}_0$ *and* $L(z) \in \mathbb{F}^{(3)}$

(iii) $\log_q |j(z)| = q$ *and* $L(j(z)) \neq 1 \Leftrightarrow z \in \mathcal{F}_0$ *and* $[\mathbb{F}(L(z)) : \mathbb{F}] \geq 4$

(iv) $\log_q |j(z)| = q^{n+1}$ $(n \geq 1)$ *and* $L(j(z)) = -1 \Leftrightarrow z \in \mathcal{F}_n$ *and* $L(z) \in \mathbb{F}^{(2)}$

(v) $\log_q |(j(z)| = q^{n+1}$ $(n \geq 1)$ *and* $L(j(z)) \neq -1 \Leftrightarrow z \in \mathcal{F}_n$ *and* $L(z) \notin \mathbb{F}^{(2)}$.

Proof. (iii) is a consequence of (i) and (ii), (v) a consequence of (iv).
(i) By (2.3), $\log_q |j(z)| \le q \Leftrightarrow z \in \mathcal{F}_0$. We have strict inequality if and only if $j_0(L(z)) = 0$, which is equivalent with $L(z) \in \mathbb{F}^{(2)}$.
(ii) For $z \in \mathcal{F}_0$, $L'(j(z)) = 1 \Leftrightarrow j_0(L(z)) = 1 \Leftrightarrow L(z) \in \mathbb{F}^{(3)}$, where the last equivalence results from an easy calculation.
(iv) As $\log_q |j(z)| = q^{n+1} \Leftrightarrow z \in \mathcal{F}_n$, we are reduced to showing that for $z \in \overline{F} - \mathbb{F}$, the equivalence $j_n(z) = -1 \Leftrightarrow z \in \mathbb{F}^{(2)}$ holds. Now $j_n(z) = (\frac{z^{q^2}-z^q}{z^q-z})^{q^n}$ equals $-1 \Leftrightarrow \frac{z^{q^2}-z^q}{z^q-z} = -1 \Leftrightarrow z^{q^2} = z$. $\square$

4. The polynomials $\beta_{d,k}(X)$ and their Newton polygons. We wish to determine the zeroes of the forms ${}_a\ell_k$ defined in (1.15). To this end, we study their companion polynomials ${}_a\varphi_k$, restricting first to the case where $a = T^d$.

4.1 Definition. *Let X be an indeterminate, let ϕ be the Drinfeld module over the polynomial ring $A[X]$ defined by $\phi_T = T + X\tau + \tau^2$, and write*

$$\phi_{T^d} = \sum_{0 \le k \le 2d} \beta_{d,k}(X)\tau^k.$$

The first few of the polynomials $\beta_{d,k}$ are given by
(4.2)

$$\begin{aligned}
\beta_{1,k} &= T, X, 1 \textit{ for } k = 0,1,2,\\
\beta_{2,k} &= T^2, (T^q+T)X, X^{q+1} + (T^{q^2}+T), X^{q^2}+X, 1 \textit{ for } k = 0,1,\ldots,4\\
\beta_{3,k} &= T^3, (T^{2q}+T^{q+1}+T^2)X, (T^{q^2}+T^q+T)X^{q+1} + (T^{2q^2}+T^{q^2+1}+T^2),\\
&\quad X^{q^2+q+1} + (T^{q^3}+T^{q^2}+T)X^{q^2} + (T^{q^3}+T^q+T)X,\\
&\quad X^{q^3+q^2} + X^{q^3+1} + X^{q+1} + (T^{q^4}+T^{q^2}+T), X^{q^4}+X^{q^2}+X, 1\\
&\quad \textit{for } k = 0,1,\ldots 6.
\end{aligned}$$

As we will see in a moment, $\beta_{d,k}$ almost agrees with the companion polynomial φ_f of $f = {}_{T^d}\ell_k$. Its elementary properties are given in the next proposition. We always write "deg" for the degree of $a \in A$ in T, and "$\deg_X$" for the degree of a polynomial in X.

4.3 Proposition. *Putting $\beta_{d,k} = 0$ for $k < 0$, we have for $d, k \ge 0$:*

(i) (recursion I) $\beta_{d+1,k} = T^{q^k}\beta_{d,k} + X^{q^{k-1}}\beta_{d,k-1} + \beta_{d,k-2}$;

(ii) (recursion II) $\beta_{d+1,k} = T\beta_{d,k} + X\beta_{d,k-1}^q + \beta_{d,k-2}^{q^2}$;

(iii) $\beta_{d,k}(X) = X^{\chi(k)}\varphi_{d,k}(X^{q+1})$ with a polynomial $\varphi_{d,k} \in A[X]$, where $\chi(k) = 0$ (resp. 1) if k is even (resp. odd);

(iv) $\deg_X \beta_{d,k} \leq d_k := \frac{q^k-1}{q-1}$;

(v) *suppose $k \leq d$. Then equality holds in (iv), and the leading coefficient of $\beta_{d,k}$ is monic of degree $(d-k)q^k$ as an element of A;*

(vi) *suppose $k \geq d$. Then $\beta_{d,k}$ is monic of degree*

$$\deg_X \beta_{d,k} = d_k - \frac{q^{2(k-d)}-1}{q-1};$$

(vii) *if k is even, the absolute term of $\beta_{d,k}$ is monic of degree $(d-\frac{k}{2})q^k$, if k is odd, the linear term of $\beta_{d,k}$ is monic of degree $(d-\frac{k+1}{2})q^k$.*

Proof. (i) and (ii) follow from comparing coefficients in $\phi_{T^d} \circ \phi_T = \phi_{T^{d+1}} = \phi_T \circ \phi_{T^d}$, respectively. The proofs of (iii) to (vii) are excercises in induction, using (i) and (ii). □

Since the X1modular invariant j equals g^{q+1}/Δ, property (iii) above shows that $\varphi_{d,k}$ is the companion polynomial φ_f of the modular form $f = {}_{T^d}\ell_k$. Hence knowing the zeroes of $\beta_{d,k}$ is as good as knowing the zeroes of ${}_{T^d}\ell_k$ itself. We will determine the Newton polygon of $\beta_{d,k}$ over K_∞.

In the following, we write $k = 2m-1$ for odd and $k = 2m$ for even k. If $\beta_{d,k} \neq 0$ then $d \geq m$. We aim to describe $\beta_{d,k}$ for such d, k.

4.4 Definition. *Let* $k = \left\{ \begin{array}{c} 2m-1 \\ 2m \end{array} \right\}$ *be given and* $m \leq d \leq k$. *A number* $\ell \in \mathbb{N}_0$ *is a* critical exponent *for* $\beta_{d,k}$ *with associated* critical degree δ *if:*

$\boxed{\textit{case } k = 2m-1 \textit{ odd}}$

$$\begin{array}{r|c} l & \delta \\ \hline 1 & (d-m)q^k \\ q^{k-1} & (d-m)q^k \\ q^{k-1}+q^{k-2}+q^{k-3} & (d-m-1)q^k \\ q^{k-1}+q^{k-2}+q^{k-3}+q^{k-4}+q^{k-5} & (d-m-2)q^k \\ \vdots & \vdots \\ q^{k-1}+q^{k-2}+\ldots+q^{2(k-d)} & 0 \cdot q^k \end{array} \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} d-m \textit{ rows}$$

$\boxed{\text{case } k = 2m \text{ even}}$

l	δ	
0	$(d-m)q^k$	
$q^{k-1}+q^{k-2}$	$(d-m-1)q^k$	
$q^{k-1}+q^{k-2}+q^{k-3}+q^{k-4}$	$(d-m-2)q^k$	$d-m$ rows
$\vdots$	$\vdots$	
$q^{k-1}+q^{k-2}+\ldots+q^{2(k-d)}$	$0\cdot q^k$	

The critical exponents for $\beta_{d,k}$ with $d > k$ are the same as those of $\beta_{k,k}$, with critical degrees augmented by $(d-k)q^k$.

Note that the largest critical exponent of $\beta_{d,k}$ equals the degree $\deg_X \beta_{d,k}$.

For given (d,k) with $d \geq m$, we produce a polygonal chain as follows: For each critical exponent ℓ with critical degree δ, draw the *critical point* $(\ell, -\delta)$ in the euclidean plane, and connect neighboring critical points through a straight line. If is immediate from the definition that the slopes of these lines strictly increase, i.e., for consecutive c.exponents $\ell_1 < \ell_2 < \ell_3$, the slope $\frac{\delta_2-\delta_3}{\ell_3-\ell_2}$ is strictly larger than $\frac{\delta_1-\delta_2}{\ell_2-\ell_1}$. In other words, our polygonal chain is convex, and will in fact turn out to be the Newton polygon of $\beta_{d,k}$ over the valued field K_∞ (we use the notation and conventions of [?] II sect. 6). More precisely, $\beta_{d,k}$ enjoys the properties described in the next proposition, which is our principal technical result.

4.5 Proposition. *Let (d,k) with $d \geq m$ be as before, and write*

$$\beta_{d,k}(X) = \sum_\ell b_\ell X^\ell$$

with $b_\ell \in A$.

(i) If ℓ is a critical exponent for $\beta_{d,k}$ with c.degree δ, then $\deg b_\ell = \delta$, and b_ℓ is monic as an element of A.

(ii) If ℓ is non-critical and $\ell_1 < \ell < \ell_2$ with neighboring c.exponents ℓ_1, ℓ_2, then $\deg b_\ell$ interpolates sub-linearly:

$$\deg b_\ell \leq \deg b_{\ell_1} + \frac{\ell-\ell_1}{\ell_2-\ell_1}(\deg b_{\ell_2} - \deg b_{\ell_1}).$$

Proof. We induce on d, where the cases $d=1$ and $d=2$ may be directly read off from (4.2). Let now be $d \geq 2$ and assume the assertion is shown for $d' \leq d$. We want to show that $\beta_{d+1,k}$ has the properties stated, where without restriction $k \geq 1$. Write $k = \left\{ \begin{array}{c} 2m-1 \\ 2m \end{array} \right\}$ as before.

If $\boxed{d+1 < m}$ then $\beta_{d+1,k} = 0$ and nothing has to be shown.

If $\boxed{d+1 = m}$ then $k = \left\{ \begin{array}{c} 2(d+1)-1 \\ 2(d+1) \end{array} \right\}$ and the assertion is $\left\{ \begin{array}{c} \text{immediate} \\ \text{trivial} \end{array} \right\}$ since $\beta_{d+1,k} = \left\{ \begin{array}{c} X + X^{q^2} + X^{q^4} + \cdots + X^{q^{k-1}} \\ 1 \end{array} \right\}$.

Suppose $\boxed{m \leq d \leq k-1}$. We first treat the case $\boxed{k \text{ odd}}$, using recursion I of (4.3), labelled RI. Below we list the critical exponents/degrees for $\beta_{d,k}$, $\beta_{d,k-1}$, $\beta_{d,k-2}$, and $\beta_{d+1,k}$. (Note that the quantity "m" for $\beta_{d,k-1}$, and $\beta_{d,k-2}$ must be replaced by $m' = m-1$. Further, if $k=1$, the term $\beta_{d,k-2}$ must be omitted.)

(4.6) Critical exponents and degrees (k odd).

$\beta_{d,k}$		$\beta_{d,k-1}$ $(m' = m-1)$		$\beta_{d,k-2}$ $(m' = m-1)$	
c.exponents	c.degrees	c.exponents	c.degrees	c.exponents	c.degrees
1	$(d-m)q^k$			1	$(d-m+1)q^{k-2}$
q^{k-1}	$(d-m)q^k$	0	$(d-m+1)q^{k-1}$	q^{k-3}	$(d-m+1)q^{k-2}$
$q^{k-1}+q^{k-2}+q^{k-3}$	$(d-m-1)q^k$	$q^{k-2}+q^{k-3}$	$(d-m)q^{k-1}$	$q^{k-3}+q^{k-4}+q^{k-5}$	$(d-m)q^{k-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q^{k-1}+\ldots+q^{2(k-d)}$	$0\cdot q^k$	$q^{k-2}+\ldots+q^{2(k-d)}$	$1\cdot q^{k-1}$		
		$q^{k-2}+\ldots+q^{2(k-1-d)}$	$0\cdot q^{k-1}$	$q^{k-3}+\ldots+q^{2(k-2-d)}$	$0\cdot q^{k-2}$
	$\uparrow$ add q^k	$\uparrow$ add q^{k-1}	in recursion I		

$$\beta_{d+1,k}$$

$$\begin{array}{c|c}
\text{c.exponents} & \text{c.degrees} \\
\hline
1 & (d-m+1)q^k \\
q^{k-1} & (d-m+1)q^k \\
\end{array}$$

$$\left.\begin{array}{c|c}
q^{k-1}+q^{k-2}+q^{k-3} & (d-m)q^k \\
\vdots & \vdots \\
q^{k-1}+\ldots+q^{2(k-1-d)} & 0\cdot q^k
\end{array}\right\} d-m+1 \text{ rows}$$

By induction hypothesis, the c.degrees of $\beta_{d,k}$, $\beta_{d,k-1}$, $\beta_{d,k-2}$ are the degrees of the coefficients with the corresponding c.exponents as subscripts. RI prescribes that the c.exponents of $\beta_{d,k-1}$ must be enlarged by q^{k-1} when entering into $\beta_{d+1,k}$ (the corresponding c.degrees remaining unchanged), while the c.exponents of $\beta_{d,k}$ remain, but their c.degrees are enlarged by q^k.

Comparing the four tables and taking RI into account shows that all the critial coefficients of $\beta_{d+1,k}$ (those that corresponds to c. exponents) have the degrees predicted. (Note that $\beta_{d,k-2}$ can at most contribute to the linear term of $\beta_{d+1,k}$, but not to other c. coefficients.) Moreover, as by induction hypothesis the critical coefficients of $\beta_{d,k}$, $\beta_{d,k-1}$, $\beta_{d,k-2}$ are monic, the same is true for $\beta_{d+1,k}$.

Let now $\ell \in \mathbb{N}$ satisfy $\ell_1 < \ell < \ell_2$ with neighboring c.exponents ℓ_1 and ℓ_2 of $\beta_{d+1,k}$. Let for the moment r, s, t, u be the relevant coefficients of $\beta_{d+1,k}$, $\beta_{d,k}$, $\beta_{d,k-1}$, $\beta_{d,k-2}$, corresponding to terms of order $\ell, \ell, \ell - q^{k-1}, \ell$, respectively. By RI, we have

$$r = T^{q^k}s + t + u.$$

Distinguish the cases:

$\boxed{\ell_1 = 1,\ \ell_2 = q^{k-1}}$ Here $t = 0$, $\deg u \le (d-m+1)q^{k-2} < \deg(T^{q^k}s)$ (provided $s \neq 0$), and the *sublinear interpolation property* SIP of $\deg s$ extends to $\deg r$.

$\boxed{\ell_1 = q^{k-1} + \cdots + q^{2(k-d)},\ \ell_2 = \ell_1 + q^{2(k-d)-1} + q^{2(k-1-d)}}$ Here $s = 0 = u$, and the SIP extends from $\deg t$ to $\deg r$.

$\boxed{\text{all other cases}}$ Here still $u = 0$. Moreover, ℓ_1 and ℓ_2 are neighboring c.exponents of $\beta_{d,k}$, $\ell_1 - q^{k-1}$ and $\ell_2 - q^{k-1}$ are neighboring c.exponents of $\beta_{d,k-1}$ and the SIP of $\deg s$ and $\deg t$ is inherited by $\deg r$.

Still in the situation $m \le d \le k-1$, the case $\boxed{k \text{ even}}$ is handled in the same fashion: RI implies that the asserted properties of $\beta_{d,k}$, $\beta_{d,k-1}$, $\beta_{d,k-2}$ turn over to $\beta_{d+1,k}$. We omit the details (which would require presenting a table similar to (4.6) for even k).

There remains the case where $\boxed{d \ge k}$. Here $\beta_{d+1,k}$ and $\beta_{d,k}$ have the same critical exponents and in particular the same degrees as polynomials in X. Again distinguishing the subcases $\boxed{k \text{ odd}/k \text{ even}}$, RI shows that the degrees of the coefficients at critical exponents are augmented by q^k under $\beta_{d.k} \rightsquigarrow \beta_{d+1,k}$, preserving monicity, and the degrees of the other coefficients still interpolate sublinearly. Hence $\beta_{d+1,k}$ enjoys the stated properties for all k, and the induction step is finished. $\square$

We let $NP(\beta_{d,k})$ be the Newton polygon of $\beta_{d,k}$ as described in (4.4), with breaks $(\ell, -\delta)$, where ℓ is a critical exponent with corresponding critical degree δ. We write $(a,b)-(c,d)$ for the line segment that joins two points (a,b) and (c,d) in the plane and $NP+(a,b)$ for the Newton polygon shifted by the vector (a,b).

We have actually shown (recall that $k = \left\{ \begin{array}{c} 2m-1 \\ 2m \end{array} \right\}$):

(4.7)
$$\begin{array}{l} d < m: \ \beta_{d,k} = 0, \ NP(\beta_{d,k}) = \emptyset \\ d = m, \ k = 2m \text{ even}: \ \beta_{m,2m} = 1, \ NP(\beta_{m,2m}) = \{(0,0)\} \\ d = m, \ k = 2m-1 \text{ odd}: NP(\beta_{m,2m-1}) = (1,0) - (q^{k-1}, 0) \\ m \le d < k: \ NP(\beta_{d+1,k}) \text{ results from appending the line} \\ \text{segment } (\deg_X \beta_{d,k}, -q^k) - (\deg_X \beta_{d,k} + q^{2k-2d-1} + q^{2k-2d-2}, 0) \\ \text{to } NP(\beta_{d,k}) + (0, -q^k). \\ d \ge k: \ NP(\beta_{d+1,k}) = NP(\beta_{d,k}) + (0, -q^k) \end{array}$$

4.8 Corollary.

(i) For $d, k \ge 0$, the Newton polygon $NP(\beta_{d,k})$ lies strictly above $NP(\beta_{d+1,k})$.

(ii) Let $a = \sum_i a_i T^i$ ($a_i \in \mathbb{F}$) be an arbitrary element of A, of degree d, and write $\phi_a = \sum_{0 \le k \le 2d} {}_a\beta_k(X)\tau^k$ for the Drinfeld module ϕ of (4.1). The Newton polygon $NP({}_a\beta_k)$ of ${}_a\beta_k(X) \in A[X]$ satisfies

$$NP({}_a\beta_k) = NP(\beta_{d,k}).$$

Proof. (i) results from the description given in (4.7). Therefore, the Newton polygon of ${}_a\beta_k = \sum_i a_i \beta_{i,k}$ depends only on the largest term $a_d T^d$, which gives (ii). $\square$

Together with the relation $\beta_{d,k}(X) = X^{\chi(k)}\varphi_{d,k}(X^{q+1})$ of (4.3)(iii) we find that the companion polynomial ${}_a\varphi_k$ of ${}_a\ell_k$ is given by

$$ {}_a\varphi_k = \sum_i a_i \varphi_{i,k}(X). \tag{4.9} $$

For the reader's convenience, we list the $\varphi_{d,k}$ for $d \leq 3$, derived from (4.2).

(4.10)

$$\begin{aligned}
\varphi_{1,k} &= T, 1, 1 \text{ for } k = 0, 1, 2 \\
\varphi_{2,k} &= T^2,\ T^q + T,\ X + (T^{q^2} + T),\ X^{q-1} + 1,\ 1 \text{ for } k = 0, 1, \ldots 4 \\
\varphi_{3,k} &= T^3,\ T^{2q} + T^{q+1} + T^2,\ (T^{q^2} + T^q + T)X + (T^{2q^2} + T^{q^2+1} + T^2), \\
&\quad X^q + (T^{q^3} + T^{q^2} + T)X^{q-1} + (T^{q^3} + T^q + T),\ X^{q^2} + X^{q^2-q+1} + X + \\
&\quad (T^{q^4} + T^{q^2} + T),\ X^{q^3-q^2+q-1} + X^{q-1} + 1,\ 1 \text{ for } k = 0, 1, \ldots 6.
\end{aligned}$$

Combining the preceding with the properties of the Newton polygon ([?] II sect. 6), we arrive at the following description of the j-zeroes of ${}_a\ell_k$, counted with multiplicity.

4.11 Theorem. *Let ${}_a\ell_k(z)$ be the coefficient modular form defined in* (1.15), *where $a \in A$ has degree d, and write $k = \left\{ \begin{array}{c} 2m-1 \\ 2m \end{array} \right\}$. The pattern of j-zeros of ${}_a\ell_k$ as described below depends only on d.*

If $\boxed{d < m}$ then ${}_a\ell_k = 0$.

If $\boxed{m \leq d \leq k = 2m-1}$, there are

$$\begin{array}{lccl}
\frac{q^{k-1}-1}{q+1} & j\text{-zeroes } x \text{ of } {}_a\ell_k \text{ with } \log_q |x| & = & 0 \\
q^{k-3} & '' & = & q^3 \\
q^{k-5} & '' & = & q^5 \\
\vdots & \vdots & & \\
q^{2k-2d} & '' & = & q^{2d-k}
\end{array}$$

and, not to forget, there is the trivial simple j-zero $x = 0$.

If $\boxed{m \leq d \leq k = 2m}$, there are

$$\begin{array}{lccl}
q^{k-2} & j\text{-zeroes } x \text{ of } {}_a\ell_k \text{ with } \log_q |x| & = & q^2 \\
q^{k-4} & '' & = & q^4 \\
\vdots & \vdots & & \\
q^{2k-2d} & '' & = & q^{2d-k}.
\end{array}$$

These are all the j-zeroes of ${}_a\ell_k$.

If $\boxed{d > k}$ then we have the same pattern as with $d' = k$. $\square$

5. Location of the zeroes of ${}_a\ell_k$ in $\mathcal{F}$. We keep the notation of the last section. The j-zeroes of ${}_a\ell_k$ described in Theorem 4.11 correspond to z-zeros in $\mathcal{F}$ which lie in $\mathcal{F}_n$, where $0 \leq n \leq \min(k-1, 2d-k-1)$, $n \equiv k+1 (\bmod 2)$. The mapping j restricted to $\mathcal{F}_n$ is $q^{n+1}(q-1)$ to 1 if $n \geq 1$, $(q^2-1)q$ to 1 on non-elliptic points (i.e., $j \neq 0$) of $\mathcal{F}_0$, and $(q-1)q$ to 1 on elliptic points of $\mathcal{F}_0$. Thus we get the next result.

5.1 Theorem. *Let ${}_a\ell_k$ be as in Theorem 4.11. Counted with multiplicities, it has $q^k(q-1)$ zeroes in $\mathcal{F}_n$ for each n that satisfies $0 \leq n \leq \min(k-1, 2d-k-1)$, $n \equiv k+1 (\bmod 2)$, and these are all the zeroes of ${}_a\ell_k$ in $\mathcal{F}$ ($d = \deg a$). For k odd, these $q^k(q-1)$ zeroes z in $\mathcal{F}_0$ either satisfy $|j(z)| = 1$ ($(q^k-q)(q-1)$ many) or $z \in \mathbb{F}^{(2)} - \mathbb{F}$ (q^2-q many).* $\square$

¿From (3.11) we find:

5.2 Corollary. *Let k be odd. The $q^k(q-1)$ zeroes z of ${}_a\ell_k$ in $\mathcal{F}_0$ satisfy $L(z) \in \mathbb{F}^{(2)}$.* $\square$

Let now again $a \in A$ have degree $d \in \mathbb{N}$, and suppose $2 \leq k \leq d$. As follows from Theorem 4.11, the form ${}_a\ell_k$ has a unique j-zero ${}_ax_k$ with maximal possible $\log_q |{}_ax_k|$. It lies in K_∞ and corresponds to the segment of width 1 and slope q^k of $NP({}_a\varphi_k)$, i.e., the segment of width $q+1$ and slope $\frac{q^k}{q+1}$ rightmost to the Newton polygon of ${}_a\beta_k$, see (4.7). The two relevant breaks of $NP({}_a\beta_k) = NP(\beta_{d,k})$ have abscissas $\ell_1 = \frac{q^k-1}{q-1} - (q+1)$ and $\ell_2 = \frac{q^k-1}{q-1}$ and, as the coefficients b_ℓ of $\beta_{d,k}(X)$ with these indices are monic as elements of A (see 4.5), we have $L({}_ax_k) = -1$. As a consequence of (3.11) we find:

5.3 Corollary. *Let $a \in A$ have degree d, where $2 \leq k \leq d$, and let ${}_ax_k \in K_\infty$ be the unique maximal j-zero of ${}_a\ell_k$ as described above. Then $L({}_ax_k) = -1$, and the corresponding z-zeroes $z \in \mathcal{F}$ lie in $\mathcal{F}_{k-1}$ and satisfy $L(z) \in \mathbb{F}^{(2)} - \mathbb{F}$.*
$\square$

In what follows we assume that k is odd and derive similar properties of the j-zeroes of ${}_a\ell_k$ with $|x| = 1$.

5.4 Lemma. *Let k be odd and let $1 < \ell < q^{k-1}$ be such that the degree $\deg b_\ell$ (where ${}_a\beta_k(X) = \sum b_\ell X^\ell$) interpolates linearly (i.e., we have equality in 4.5 (ii)). Then $\ell \equiv 0 (\bmod q)$.*

Proof. Without restriction, $a = T^d$. Then we proceed as in the proof of (4.5), using induction on d and starting with $d = m$. Then $\beta_{m,k} = \beta_{m,2m-1} = X + X^{q^2} + \cdots X^{q^{k-1}}$, for which the assertion is true. Further, the assertion is stable under $d \rightsquigarrow d+1$, as follows from the argument used in the case

$\boxed{\ell_1 = 1,\, \ell_2 = q^{k-1}}$ in the second half of the proof of (4.5), where we showed the stability of SIP under $d \rightsquigarrow d+1$. $\square$

We now consider the polynomial ${}_a\beta_k$ up to terms of order q^{k-1}, which corresponds to the leftmost segment of $NP({}_a\beta_k)$ between the abscissas $\ell_1 = 1$ and $\ell_2 = q^{k-1}$. Since the slope of that segment is zero,

$$ {}_a\beta_k(X) = \sum_{1 \leq \ell \leq q^{k-1}} b_\ell X^\ell + \sum_{\ell > q^{k-1}} b_\ell X^\ell $$

with $b_\ell \in K_\infty$, $|b_1| = |b_{q^{k-1}}|$, $|b_\ell| < |b_1|$ for $\ell > q^{k-1}$, and for $1 < \ell < q^{k-1}$, either $|b_\ell| < |b_1|$ or $|b_\ell| = |b_1|$ and $\ell \equiv 0 (\mathrm{mod}\, q)$ holds. But this implies that the $q^{k-1} - 1$ zeroes of ${}_a\beta_k$ corresponding to this segment are simple and lie in an unramified extension of K_∞. (Divide the above equation by b_1. The resulting polynomial reduced modulo the maximal ideal of O_∞ has constant derivative $\neq 0$, and we may apply the trivial case of Hensel's lemma.) Taking ${}_a\beta_k = X\, {}_a\varphi_k(X^{q+1})$ into account, where ${}_a\varphi_k(X)$ is the companion polynomial of ${}_a\ell_k$, we have shown our next result.

5.5 Theorem. *Let k be odd.*

(i) The $\frac{q^{k-1}-1}{q+1}$ j-zeroes x of ${}_a\ell_k$ with $|x| = 1$ are simple as zeroes of ${}_a\varphi_k(x)$. Correspondingly, the $(q^k - q)(q-1)$ zeroes $z \in \mathcal{F}_0$ of ${}_a\ell_k$ with $|j(z)| = 1$ are all different and simple.

(ii) All the x in (i) lie in an unramified extension of K_∞. $\square$

Remark. This is at least partially analogous with properties of the zeroes of Eisenstein series of weight $q^k - 1$, see Example 2.5. On the other hand, each of the zeroes z from 5.5 (i) with $|j(z)| = 1$ generates an extension of K_∞ ramified with index divisible by $q+1$, as is seen from $1 = |j(z)| = |g(z)|^{q+1}/\Delta(z)$ and $|\Delta(z)| = q^{-q}$ for $z \in \mathcal{F}_0$ (see 2.3). Quite generally, the fields $K(x)$ (resp. $K_\infty(z)$) generated over K (resp. K_∞) by j-zeroes x (zeroes z) of such modular forms seem to deserve more investigation. Moreover, (5.5) raises the

5.6 Question. Are all the roots of ${}_a\varphi_k$ simple, i.e., are the polynomials ${}_a\varphi_k$ separable?

6. The decay of ${}_a\ell_k$ along $\mathcal{F}$. We conclude with describing the decay of the absolute value $|{}_a\ell_k(z)|$ along the sets $\mathcal{F}_n \subset \mathcal{F}$, where we exploit the ideas developed in [**?**] and [**?**] (especially (6.11) and (8.13) pp.).

Recall that $\|f\|_n := \sup\{|f(z)| \mid z \in \mathcal{F}_n\}$ is the spectral norm of the holomorphic function f on $\mathcal{F}_n$. It is, by standard properties of rigid geometry, in fact a maximum, which even agrees with the constant absolute value $|f(z)|$ for $z \in \mathcal{F}_n$, provided that f has no zeroes on $\mathcal{F}_n$ [?] [?].

¿From now on we let f be a modular form of weight k subject to the condition:

(6.1) If $f(z) = 0$ for $z \in \mathcal{F}$ then $z \in \bigcup_{n \in \mathbb{N}_0} \mathcal{F}_n$.

Due to our results so far, all the forms g_k (2.5), α_k (2.6) and ${}_a\ell_k$ satisfy (6.1). As a consequence of (6.1), as follows from the considerations in [?] pp. 93–94, and [?] 1.7.4, the quantity $\log_q |f(z)|$ interpolates linearly (see (2.4)) for $z \in \mathcal{F}$, $n < \log_q |z| = \log_q |z|_i < n+1$ between its extreme points $\log_q \|f\|_n$ and $\log_q \|f\|_{n+1}$. We may therefore restrict to considering $\|f\|_n$ for $n \in \mathbb{N}_0$.

For $0 \neq f \in M_k$ and $n \geq 0$, define

$$r_n(f) := \log_q \frac{\|f\|_{n+1}}{\|f\|_n}, \tag{6.2}$$

which lies in $\mathbb{Z}$ and satisfies $r_n(f_1 \cdot f_2) = r_n(f_1) + r_n(f_2)$. (To be in line with [?], the present $r_n(f)$ is $r(f)(e_n)$ as defined *loc. cit.* 6.9.)

We further put

$$z_n(f) \;:=\; \text{number of zeroes of } f \text{ on } \mathcal{F}_n, \text{ counted with multiplicities.} \tag{6.3}$$

¿From the formalism of rigid-analytic contour integration ([?] pp. 93–95) we find the following relations between $r_n(f)$ and $z_n(f)$:

$$z_0(f) = (q+1)r_0(f) + qk \tag{6.4}$$

and for $n \geq 1$

$$z_n(f) = r_n(f) - q r_{n-1}(f). \tag{6.5}$$

Details would require the introduction of a bulk of new objects and terminology. Suffice it to say that the common source of both identities is [?] 6.11, along with the description *loc. cit.* 7.5 of $r_n(cz+d)$ and the action of our groups Γ_n (see (2.2)) on $\mathcal{F}_n$ and on the neighborhood of the vertex v_n in the Bruhat-Tits tree $\mathcal{T}$ (*loc. cit.* section 6).

Note in particular that the weight k enters in (6.4) only, but not in (6.5). These formulas enable us to relate $\|f\|_n$ and $\|f\|_{n'}$ for $n < n'$, provided we know the zeroes "in between".

To make this effective, we first observe:

6.6 Proposition. *Let $f \in M_k$ satisfy* (6.1) *and suppose that its zero-th s-coefficient $a_0 = a_0(f)$ doesn't vanish. Let n_0 be maximal such that $\mathcal{F}_{n_0}$ contains a zero of f. Then $|f(z)| = |a_0|$ for $z \in \mathcal{F}$, $|z| > q^{n_0}$ and $\|f\|_{n_0} = a_0$.*

Proof. In the neighborhood of ∞, that is for $z \in \mathcal{F}$ with $s(z)$ small (or $|z| = |z|_i$ large), we have $|f(z)| = |a_0|$ by standard non-archimedean properties. But this means $r_n(f) = 0$ for $n \gg 0$.
¿From (6.5) we find that actually $r_n(f) = 0$ for $n \geq n_0$, which, together with the properties stated after (6.1), gives the result. $\square$

(6.6) applies to the forms $f = {}_a\ell_k$ with $a \in A$ of degree d and $\boxed{k \leq d}$. In this case, the coefficient $a_0({}_a\ell_k)$ is the corresponding coefficient of the Carlitz module (1.5). That is, write

$$\rho_a = a + \sum_{1 \leq k \leq d} {}_ac_k \tau^k \quad \text{with } {}_ac_k \in A,$$

then ${}_ac_k = a_0({}_a\ell_k)$. Its degree is $\log_q |a_0({}_a\ell_k)| = \deg {}_ac_k = (d-k)q^k$ ([**?**] 4.5). Combining the preceding, we find:

6.7 Theorem. *Let $a \in A$ have degree d. For $1 \leq k \leq d$ the modular form ${}_a\ell_k$ satisfies*

$$\begin{aligned} \log_q \|{}_a\ell_k\|_n &= (d-k)q^k, \quad n \geq k-1 \\ &= (d-k)q^k - \sum_{i=n}^{k-2} r_i({}_a\ell_k), \quad 0 \leq n < k-1. \end{aligned}$$

Here the $r_i({}_a\ell_k)$ may be recursively solved (stepping down from $i = k-1$) from (6.5) and (6.4), using $r_{k-1}({}_a\ell_k) = 0$ and $z_i({}_a\ell_k) = 0$ resp. $q^k(q-1)$ if $i \equiv k (\mathrm{mod}\, 2)$ resp. $i \not\equiv k (\mathrm{mod}\, 2)$. $\square$

With a bit of labor we can work out e.g. $\|{}_a\ell_d\|_0$ for the "middle" coefficient form ${}_a\ell_d$, $d = \deg a$, an exercise in summing up multiple geometric series. Here is the result.

6.8 Example. Let $a \in A$ have degree $d \geq 1$. Then

$$\begin{aligned} \|{}_a\ell_d\|_0 &= \frac{d}{2}q^d - q\left(\frac{q^d - 1}{q^2 - 1}\right), \quad d \text{ even} \\ &= \frac{d+1}{2}q^d - q\left(\frac{q^{d+1} - 1}{q^2 - 1}\right), \quad d \text{ odd.} \end{aligned}$$

The procedure just employed fails for ${}_a\ell_k$ with $\boxed{k > d = \deg a}$ since then ${}_a\ell_k$ is a cusp form. However we know that its largest zeroes $z \in \mathcal{F}$ satisfy $\log_q |z| = 2d - k - 1$. Hence $\|{}_a\ell_k\|_n$ for $n \geq 2d - k - 1$ is determined by the equation

$${}_a\ell_k(z) = {}_a\varphi_k(j(z))\Delta(z)^{\mu(k)} g(z)^{\chi(k)},$$

$\chi(k) = 0/1$ if k is even/odd, $\mu(k) = \frac{q^k - q^{\chi(k)}}{q^2 - 1}$. Now

- the companion polynomial ${}_a\varphi_k$ is monic of degree $(q^k - q^{2(k-d)} - \chi(k)(q-1))/(q^2-1)$, as results from (4.3) (iv) and (4.9);
- $\log_q \|j(z)\|_n = q^{n+1}$;
- $|j(z)| \geq |x|$ for all the zeroes x of ${}_a\varphi_k$ if $z \in \mathcal{F}_n$, $n \geq 2d-k-1$.

Putting these together, we find for $n \geq 2d-k-1$:

$$\log_q \|{}_a\ell_k\|_n = -q^{n+1}\left(\frac{q^{2(k-d)}-1}{q^2-1}\right) \tag{6.9}$$

and consequently

$$r_n({}_a\ell_k) = -q^{n+1}\left(\frac{q^{2(k-d)}-1}{q+1}\right). \tag{6.10}$$

This yields as in (6.7) the remaining spectral norms.

6.11 Theorem. *Let $a \in A$ have degree d. For $d < k \leq 2d$ the modular form ${}_a\ell_k$ satisfies*

$$\begin{aligned} \log_q \|{}_a\ell_k\|_n &= -q^{n+1}\left(\frac{q^{2(k-d)}-1}{q^2-1}\right), \quad n \geq 2d-k-1 \\ &= \log_q \|{}_a\ell_k\|_{2d-k-1} - \sum_{i=n}^{2d-k-2} r_i({}_a\ell_k), \quad n < 2d-k-1, \end{aligned}$$

where the $r_i({}_a\ell_k)$ may be recursively determined from (6.5) *and* (6.4). □

We restrict to presenting two examples, where the first is immediate from (6.11).

6.12 Example. Let $d \geq 2$ and $k = 2d-1$. Then

$$\log_q \|{}_a\ell_{2d-1}\|_n = -q\left(\frac{q^{2d-2}-1}{q^2-1}\right) \text{ for } n \geq 0.$$

6.13 Example. Let $d \geq 3$ and $k = 2d-2$. For $f = {}_a\ell_{2d-2}$ we have $\log_q \|f\|_1 = -q^2(\frac{q^{2d-4}-1}{q^2-1})$ and $r_1(f) = -q^2(\frac{q^{2d-4}-1}{q+1})$. As $z_1(f) = q^{2d-2}(q-1)$, (6.5) yields $r_0(f) = -q(\frac{q^{2d-2}-1}{q+1})$. (Alternatively, we could use (6.4) and the facts $z_0(f) = 0$, $q^{2d-2}-1 =$ weight of f.) Therefore,

$$\log_q \|f\|_0 = \log_q \|f\|_1 - r_0(f) = (q^{2d} - q^{2d-1} - q^{2d-2} + q)/(q^2-1).$$

Our last result is about the relationship between the forms ${}_a\ell_k$ and the α_k from (1.17). For $d = \deg a \geq k$, let ${}_a\tilde{\ell}_k$ be the normalized multiple of ${}_a\ell_k$;

similarly, let $\tilde{\alpha}_k$ be α_k normalized, so that ${}_a\tilde{\ell}_k$ and $\tilde{\alpha}_k$ have value 1 at infinity. From the considerations preceding Theorem 6.7, we find

$$ {}_a\tilde{\ell}_k = {}_ac_k^{-1}\,{}_a\ell_k \tag{6.14}$$

with the coefficients ${}_ac_k$ of the Carlitz module. Similarly,

$$\tilde{\alpha}_k = D_k\alpha_k, \tag{6.15}$$

as follows from (1.6).

6.16 Theorem. *Write ${}_a\tilde{\varphi}_k$ (resp. μ_k) for the companion polynomial of ${}_a\tilde{\ell}_k$ (resp. $\tilde{\alpha}_k$).*

(i) As $d = \deg a$ tends to infinity, ${}_a\tilde{\ell}_k$ tends, locally uniformly on Ω, to the para-Eisenstein series $\tilde{\alpha}_k$.

(ii) In the space of polynomials of degree less or equal to $\deg {}_a\tilde{\varphi}_k = \deg \mu_k$, we have $\lim_{d\to\infty} {}_a\tilde{\varphi}_k = \mu_k$.

6.17 Remarks. (i) Both ${}_a\tilde{\varphi}_k$ and μ_k have their coefficients in K, hence the convergence in (ii) takes place in a finite-dimensional vector space over the locally compact field K_∞.
(ii) The polynomial μ_k is known to be separable (see Example 2.6), which thus holds too for ${}_a\tilde{\varphi}_k$ resp. ${}_a\varphi_k$ if $d = \deg a$ is large enough. This gives a partial (though insatisfactory) answer to Question 5.6. *Proof* of (6.16). Let $\mathcal{F}^{(k)} = \{z \in \mathcal{F} \mid |z| \geq q^k\}$, an open analytic subspace of $\mathcal{F}$. The commutation rule

$$e_\Lambda(aw) = \phi_a(e_\Lambda(w))$$

for the generic Drinfeld module ϕ associated with the lattice $\Lambda = \Lambda_z$ (see (1.9)) implies the identity

$$(a^{q^k} - a)\alpha_k = \sum_{1\leq i\leq k-1} {}_a\ell_i\alpha_{k-i}^{q^i} + {}_a\ell_k.$$

For sufficiently large d, all the functions appearing have constant absolute values on $\mathcal{F}^{(k)}$ (for the α_i, see [**?**] 8.13, note the different normalizations), given by

$$\begin{array}{rclcl} \log_q \|\alpha_i\|_{\mathcal{F}^{(k)}} & = & \log_q |\alpha_i(\infty)| & = & -iq^i, \\ \log_q \|{}_a\ell_i\|_{\mathcal{F}^{(k)}} & = & \log_q |{}_ac_i| & = & (d-i)q^i,\ 1 \leq i \leq k. \end{array}$$

Plugging in, we see that the $\log_q$ of $(a^{q^k} - a)\alpha_k$ and of ${}_a\ell_k$ grow of order $(d-k)q^k$ with $d \longrightarrow \infty$, while the $\log_q$ of the other terms grow of order less or equal to $(d-k+1)q^{k-1}$. That is, upon normalization ($\tilde{\ }$), ${}_a\tilde{\ell}_k$ tends to $\tilde{\alpha}_k$ uniformly on $\mathcal{F}^{(k)}$. Since on the finite-dimensional C_∞-vector space $M_{q^k-1}(\Gamma)$ to which our forms belong all norms are equivalent, part (i) of the theorem follows. (Consider spectral norms on affinoid subdomains of Ω!) Part (ii) is then a trivial consequence. $\square$

Concluding remark. The companion polynomials ${}_a\varphi_k$ of the coefficient forms ${}_a\ell_k$ have integral coefficients, i.e., in A. If $a = \mathfrak{p}$ is irreducible of degree d, then ${}_\mathfrak{p}\varphi_d$ reduced $(\mathrm{mod}\,\mathfrak{p})$ yields the supersingular polynomial $(\mathrm{mod}\,\mathfrak{p})$, which encodes the supersingular invariants. This is but an example of its interesting arithmetical properties. Thus, besides the ∞-adic study of the ${}_a\varphi_k$ carried out in the present paper, it is desirable to investigate their $\mathfrak{p}$-adic properties and relate them e.g. to the results of [**?**], sect. 12.

Both the ∞-adic and the $\mathfrak{p}$-adic properties of Drinfeld modular forms parallel established or conjectured properties of modular forms in the classical framework. Therefore, a better understanding of the present case could also reveal properties of elliptic modular forms undetected so far.

References

[1] Cornelissen, Gunther: Sur les zéros des séries d'Eisenstein de poids q^k-1 pour $\mathrm{GL}_2(\mathbb{F}_q[T])$. C.R. Acad. Sci. Paris Sér. I Math. 321 (1995), no. 7, 817–820.

[2] Fresnel, Jean; van der Put, Marius: Rigid analytic geometry and its applications. Progress in Mathematics 218. Birkhäuser Boston 2004.

[3] Gekeler, Ernst-Ulrich: A product expansion for the discriminant function of Drinfeld modules of rank two. J. Number Theory 21 (1985), 135–140.

[4] Gekeler, Ernst-Ulrich: On the coefficients of Drinfeld modular forms. Invent. Math. 93 (1988), 667–700.

[5] Gekeler, Ernst-Ulrich: On the Drinfeld discriminant function. Compositio Math. 106 (1997), 182–202.

[6] Gekeler, Ernst-Ulrich: Some new results on modular forms for $\mathrm{GL}(2, \mathbb{F}_q[T])$. Recent progress in algebra (Taejon/Seoul 1997), 111–141, Contemp. Math. 224, AMS, Providence, RI, 1999.

[7] Gekeler, Ernst-Ulrich: A survey on Drinfeld modular forms. Turkish J. Math. 23 (1999), 485–518.

[8] Gekeler, Ernst-Ulrich: Finite modular forms. Finite Fields Appl. 7 (2001) 553–572.

[9] Gekeler, Ernst-Ulrich: Para-Eisenstein series for the modular group $\mathrm{GL}(2, \mathbb{F}_q[T])$. Submitted.

[10] Gekeler, Ernst-Ulrich; Reversat, Marc: Jacobians of Drinfeld modular curves. J. Reine Angew. Math. 476 (1996), 27–93.

[11] Gerritzen, Lothar; van der Put, Marius: Schottky groups and Mumford curves. Lecture Notes in Mathematics 817, Springer, Berlin 1980.

[12] Goss, David: Modular forms for $\mathbb{F}_r[T]$. J. Reine Angew. Math. 317 (1980), 16–39.

[13] Goss, David: π-adic Eisenstein series for function fields. Compositio Math. 41 (1980), 3–38.

[14] Neukirch, Jürgen: Algebraic number theory. Grundlehren der Math. Wiss. 322, Springer, Berlin 1999.