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Abstract

This paper is concerned with amoeba median filtering, a structure-adaptive morphological image filter. It has been introduced by Lerallut et al. in a discrete formulation. Experimental evidence shows that iterated amoeba median filtering leads to segmentation-like results that are similar to those obtained by self-snakes, an image filter based on a partial differential equation. We establish this correspondence by analysing a space-continuous formulation of iterated median filtering. We prove that in the limit of vanishing radius of the structuring elements, iterated amoeba median filtering indeed approximates the partial differential equation of self-snakes. This result holds true under very general assumptions on the metric used to construct the amoebas. We present experiments with discrete iterated amoeba median filtering that confirm qualitative and quantitative predictions of our analysis.

Keywords: morphological amoebas, self-snakes, median filtering, mathematical morphology, partial differential equations

1 Introduction

Since its beginning in the 60s [23], mathematical morphology has developed into a powerful theory that provides useful operators for e.g. image denoising, structure enhancement, and shape simplification [12, 27, 28]. More recently, several adaptive approaches [5, 22, 32] have been introduced in order to support the preservation of important image structures by denoising filters.

Introduced by Lerallut et al. [21, 22], morphological amoebas stand out as a class of morphological image filters in which structuring elements adapt to image structures with a maximum of flexibility. The key idea of the amoeba construction is that the structuring elements adapt locally to the variation of grey (or colour) values, also taking into account the distance to the origin pixel. Thereby, large deviations in the image values are penalised, so that the amoebas may grow around corners or along anisotropic image structures. The resulting shape then takes the role of a structuring element that can be used in conjunction with a variety of morphological filters. Our investigation in this paper concentrates on amoeba median filtering (AMF).

It is typical for the use of a classical median filter that it can be used iteratively. This is also true for amoeba median filtering. Here, iterated application can be carried out in different ways. In [21], a *pilot image* is used to steer the iterated processes via an alternating procedure. This works as follows. A smoothed version of the original image f is used for constructing amoebas for all pixels. The median filter is applied using the corresponding structuring elements. In subsequent iterations, new amoebas are constructed in every step from the previous *filtered* image. These amoebas are then used as structuring elements to filter the *original* image f. For the purpose of the present paper, we concentrate instead on a more straightforward iterative procedure for AMF that acts analogous to classical iterated median filtering [19, 30]. In each iteration, the following two steps are carried out pixelwise on the previous filtered image: (i) amoeba construction, and (ii) median filtering using the amoeba as structuring element.

For iterated median filtering with a fixed structuring element, work by Guichard and Morel [11] has brought out that, in the continuous-scale limit, it approximates the partial differential equation (PDE) $u_t = |\nabla u| \operatorname{div}\bigl(\nabla u / |\nabla u|\bigr)$, known as *(mean) curvature motion* [1]. In this sense, iterated discrete median filtering with a fixed structuring element can be understood as a specific discretisation of that PDE [17].

Iterated AMF simplifies images towards a cartoon-like appearance with homogeneous regions separated by sharp contours. Even corners are preserved fairly well, in contrast to median filtering with a fixed structuring element. Using PDE approaches, similar segmentations can be achieved e.g. by so-called self-snakes [26, 34]. These are filters that stand in close relationship to curvature motion, with the difference that the evolution is modulated by an edge-stopping function depending on the local image gradient. Thereby the displacement of edges is avoided, and edges are sharpened. In the light of Guichard and Morel's above-mentioned result it is therefore natural to ask whether there exists a similar correspondence between a continuous-scale limit case of amoeba filters and a self-snakes-like PDE.

In the following, we address this question. We prove that iterated amoeba filters can indeed be understood as a discrete approximations of curvature-based PDE image filters. We discuss how different choices for the distance measures involved in the amoeba definition influence the limit case. In one of the settings we discuss, the self-snakes PDE is recovered.

Our results extend the framework of known correspondences between discrete and PDE formulations of morphological filters. The study of these relationships helps to gain a unified view on image filtering methods and to combine advantages of both approaches. In particular, they also allow to use amoeba procedures as discretisations of structure-adaptive PDE filters similar to [17].

Preliminary results of this research have been published in [33], where some coefficients in the derived PDEs were unfortunately flawed due to a mistake in the derivation, precluding the correct interpretation of the results. In the present paper, we describe in more detail the derivation of the PDE, thereby correcting also the mistake from [33]. Moreover, while in [33] already two amoeba metrics of potential interest were introduced, most of the considerations were carried out only for the simpler of the two, namely an L^2 type (Euclidean) metric. In the present paper, a full analysis of general amoeba metrics is carried out, thereby also covering the practically interesting L^1 metric case.

Related work. Median filtering in its non-adaptive form goes back to Tukey [30] and became common as a structure-preserving image filter in the 90s [10, 19].

On the PDE side, (mean) curvature motion for image smoothing has been proposed by Alvarez et al. [1], already together with a "modulated" variant in which the right-hand side of the original PDE is multiplied with a decreasing function of the image gradient. Sapiro [26] proposed a variant of this idea, named *self-snakes*, in which the edge-stopping factor is placed *within* the divergence expression. While curvature motion smoothes in level-line direction only, Caselles et al. [7] defined for image interpolation purposes a process that smoothes exclusively in gradient flow line direction, called *adaptive monotone Lipschitz extension* (AMLE). The general principle to write curvature- and diffusion-based image filter PDEs as mixtures of smoothing along level line and gradient flow line directions, which is also an important ingredient of our analysis, has been established by Carmona and Zhong [6].

The representation of an image as a manifold embedded in the product space of image domain and grey-value range has been introduced in PDE-based image filtering with the so-called Beltrami framework by Kimmel et al. [18] and Yezzi [35].

Since the seminal paper by Guichard and Morel [11] further cross-relationships between discrete and PDE-based image filters have been studied. For example, van den Boomgaard [31] proved a PDE approximation result for the Kuwahara-Nagao operator [20, 25]. Didas and Weickert [9] studied correspondences between adaptive averaging and a class of generalised curvature motion filters. Barash [2] and Chui and Wang [8] considered PDE limits of bilateral filters [29].

With regard to future improvements in the algorithmic realisation of amoeba filters, we mention also digital distance transforms, in particular the work by Borgefors [3, 4] and Ikonen et al. [15, 16].

Using a discrete filter as discretisation of a PDE by virtue of an equivalence result like [11] can also be seen in the context of other unconventional discretisations of continuous filters that are tailored to preserve certain important qualitative properties of PDEs. This includes, for example, mimetic discretisations [13, 14] as well as so-called nonstandard schemes [24].

Structure of the paper. The paper is organised as follows. In Section 2 we describe the discrete algorithm. Our main contribution, namely the derivation of PDEs corresponding to AMF, follows in Section 3. In Section 4, we show some numerical results that illustrate the theoretical findings. The paper is finished with a conclusion in Section 5.

2 The Discrete Amoeba Construction

The basic procedure is described in Lerallut et al.'s papers [21, 22]. Here, we give a brief account of the algorithm in the form we have implemented, which is slightly modified in a few points that will be pointed out in the sequel, compare also our conference paper [33].

In the following, we work with images f whose pixels are numbered by integers, such that f_i denotes the grey-value of the pixel with index i. The coordinates of this pixel are denoted by (x_i, y_i). We distinguish the initial image f from the iterated images $u^{(n)}$, where n denotes the iteration number. For starting the iterative process, we set $u^{(0)} := f$. On the amoebas whose construction is described below the standard median filter is applied.

Description of the algorithm. For each pixel i_0 with (x, y)-coordinates (x_{i_0}, y_{i_0}), an adaptive structuring element is determined as follows. We consider pixels i^* within a prescribed maximal Euclidean distance ϱ of pixel i_0. The number ϱ represents the maximal size of the shape of the amoeba, since it will also be used for limiting the allowed *amoeba distance*. For the so pre-selected pixels we consider paths $(i_0, i_1, \ldots, i_k \equiv i^*)$ that connect i_0 with i^* via a sequence of pixels in which each two subsequent pixels i_j, i_{j+1} are neighbours. Among all these, we determine the shortest path P with respect to the amoeba distance $L(P)$. If the amoeba distance is below ϱ for P, the pixel i^* is accepted as a member of the amoeba structuring element.

It remains to specify the amoeba distance as well as the neighbourhood relation between subsequent pixels. In [21, 22], the amoeba distance is given by

$$L_L^{(n)}(P) = \sum_{m=0}^{k-1} 1 + \sigma \sum_{m=0}^{k-1} \left| u_{i_{m+1}}^{(n)} - u_{i_m}^{(n)} \right| , \tag{1}$$

where $\sigma > 0$ is a parameter that penalises large deviations in grey-value data, and each pixel is required to be in the 4-neighbourhood of its predecessor, i.e. a horizontal or vertical neighbour. Note that this definition involves the measurement of spatial distances by the L^1 distance (city-block metric), since the first sum in (1) counts the pixels in the path P (without the starting pixel i_0). Moreover, spatial and tonal distances (i.e. grey-value differences) are combined via an l_1 sum.

In our implementation, we use a metric that better approximates the Euclidean distance in space. To this end, we use 8-neighbourhoods that include horizontal, vertical, and diagonal neighbours, and use the Euclidean distance on these pixel pairs. This results in shorter paths compared to the procedure of Lerallut et al., as well, conceptually, in an improvement in terms of rotational invariance. For the way how spatial and tonal distances are combined we implement either a Euclidean sum, or an l_1 sum like in (1), which leads finally to two alternative amoeba distance measures L_2 and L_1 given by

$$L_2^{(n)}(P) = \sum_{m=0}^{k-1} \sqrt{\left(x_{i_{m+1}} - x_{i_m}\right)^2 + \left(y_{i_{m+1}} - y_{i_m}\right)^2 + \sigma^2 \left(u_{i_{m+1}}^{(n)} - u_{i_m}^{(n)}\right)^2}, \tag{2}$$

$$L_1^{(n)}(P) = \sum_{m=0}^{k-1} \left(\sqrt{\left(x_{i_{m+1}} - x_{i_m}\right)^2 + \left(y_{i_{m+1}} - y_{i_m}\right)^2} + \sigma \left| u_{i_{m+1}}^{(n)} - u_{i_m}^{(n)} \right| \right) . \tag{3}$$

While our implementation and experiments concentrate on the metrics (2), (3), in our theoretical framework these two cases will be embedded into a more general context.

We remark that the distance measurements could be improved: To better approximate the spatial Euclidean distance one could use the digital distance transforms discussed in [3], see also [4]. Alternative digital distance transforms approximating spatial-tonal Euclidean distance or L^1 distance and efficient algorithms are discussed in [15, 16].

3 Space-Continuous Amoeba Model

For our further investigation, we need a space-continuous formulation of AMF. We base this on the representation of a (smooth) image u by its graph $\Gamma = \Gamma_{u,\sigma} :=$

$\{p(x,y)=(x,y,\sigma u(x,y)) \mid (x,y)\in\Omega\}$ where $\Omega\subset\mathbb{R}^2$ is the image domain, and σ a scaling parameter for grey-values as in (1)–(3). Note that this embedding is analogous to the Beltrami framework, compare [35]. The surface Γ is equipped with a metric d which can be obtained by restricting the Euclidean metric of the embedding space $\mathbb{R}^3$, i.e.

$$d(p_1,p_2)\equiv d_2(p_1,p_2)=\min\int\limits_0^1\sqrt{x'(s)^2+y'(s)^2+\sigma^2u'(s)^2}\,\mathrm{d}s \qquad (4)$$

where the minimum is taken over all curves $[0,1]\to\Gamma$ that start in $p_1:=p(x_1,y_1)$ and end in $p_2:=p(x_2,y_2)$, and u is treated as a function of the curve parameter, $u(s)\equiv u(x(s),y(s))$ such that $u'(s)=u_x(x(s),y(s))x'(s)+u_y(x(s),y(s))y'(s)$.

Alternatively, and closer to the setting of [21], one can use an l_1 sum of the Euclidean distance in space and the grey-value distance,

$$d(p_1,p_2)\equiv d_1(p_1,p_2)=\min\int\limits_0^1\left(\sqrt{x'(s)^2+y'(s)^2}+\sigma\left|u'(s)\right|\right)\mathrm{d}s\;. \qquad (5)$$

Of course, one could even use an l_p sum for arbitrary $p\geq 1$; this would include in the limit $p\to\infty$ an amoeba metric that measures the maximum of the spatial and weighted tonal distance. For our theoretical considerations, we will use as the most general setting a metric

$$d(p_1,p_2)\equiv d_\varphi(p_1,p_2)=\min\int\limits_0^1\varphi\left(\sqrt{x'(s)^2+y'(s)^2},\sigma\left|u'(s)\right|\right)\mathrm{d}s \qquad (6)$$

where φ is a homogeneous C^2 function of degree 1, strictly increasing in both variables, and fulfils the triangle inequality $\varphi(s+v,t+w)\leq\varphi(s,t)+\varphi(v,w)$.

One step of amoeba filtering then reads as follows. For a given location (x_0,y_0) in the image domain, an amoeba structuring element $\mathcal{A}(x_0,y_0)$ is constituted by all locations (x,y) for which $d(p(x_0,y_0),p(x,y))$ does not exceed a given radius ϱ. Typical shapes of amoeba structuring elements with the metrics d_2 and d_1 are shown in Figure 1. It is worth noticing that with the metric $d\equiv d_2$ from (4) the boundary of $\mathcal{A}(x_0,y_0)$ crosses the level line through (x_0,y_0) orthogonally and smoothly, while e.g. with $d\equiv d_1$ as given by (5) it has kinks at the intersection points, giving the structuring element a digonal overall shape in contrast to the elliptical contour with (4).

Once the structuring element has been constructed, the median of all grey-values within the structuring element is taken, i.e. the value μ whose level line (the curve along which $u(x,y)=\mu$ holds) cuts $\mathcal{A}(x_0,y_0)$ into two parts of equal area. In the filtered image, μ becomes the new grey-value at location (x_0,y_0).

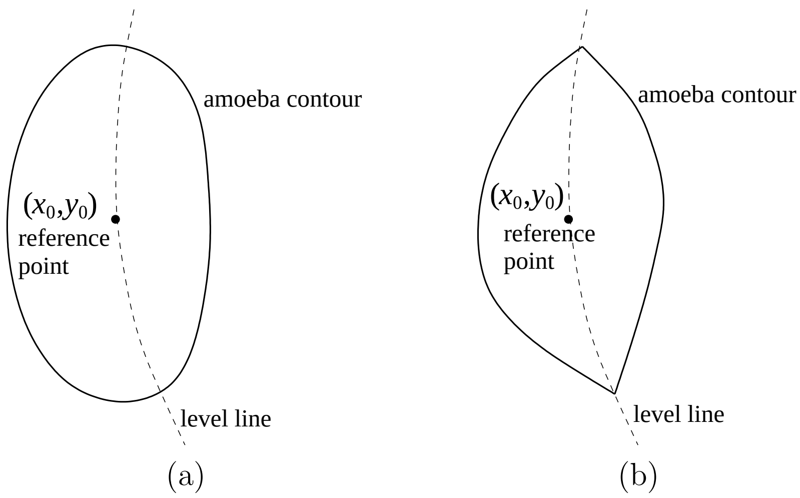


Figure 1: Amoeba structuring elements. **(a)** Typical amoeba with metric $d \equiv d_2$ from (4). **(b)** Typical amoeba with metric $d \equiv d_1$ from (5).

3.1 Analysis with Euclidean Amoeba Metric

We analyse the amoeba median filter now in a manner similar to Guichard and Morel's approach [11], and focus first on the case of the metric $d \equiv d_2$, see (4).

Without loss of generality, we assume that we are dealing with the location $(x_0, y_0) = (0, 0)$. We assume further that $u(x_0, y_0) = 0$, and that the image gradient at (x_0, y_0) is given by $\nabla u(x_0, y_0) = (\alpha/\sigma, 0)^{\mathrm{T}}$ with some positive α. Then σu possesses the Taylor expansion

$$\sigma u(x, y) = \alpha x + \beta x^2 + \gamma xy + \delta y^2 + \mathcal{O}(\varrho^3) \tag{7}$$

within $\mathcal{A} = \mathcal{A}(x_0, y_0)$, where we have used that $x, y = \mathcal{O}(\varrho)$.

Consider now a value $z = \mathcal{O}(\varrho)$. We are interested in the level line of u corresponding to the grey-value z/σ, restricted to $\mathcal{A}$. On this line, $\sigma u(x, y) = z$ holds. Due to the prescribed gradient direction of u, level lines of u within $\mathcal{A}$ are roughly oriented in y direction. We can therefore express the level line by writing x as a function of y. To this end, we read $\sigma u(x, y) = z$ as a quadratic equation for x with the solutions

$$x_{1,2} = \frac{1}{2\beta}\left(-\alpha - \gamma y \pm \sqrt{(\alpha + \gamma y)^2 - 4\beta(\delta y^2 - z)}\right) + \mathcal{O}(\varrho^3) \tag{8}$$

We drop the "−" solution which is outside $\mathcal{A}$ if ϱ is small enough, and use the

Taylor expansion $\sqrt{1+v} = 1 + \frac{1}{2}v - \frac{1}{8}v^2 + \mathcal{O}(v^3)$ to obtain

$$\begin{aligned} x &= \frac{1}{2\beta}\left(-\alpha - \gamma y + \alpha\sqrt{1 + \frac{2\gamma}{\alpha}y + \frac{4\beta}{\alpha^2}z + \frac{\gamma^2}{\alpha^2}y^2 - \frac{4\beta\delta}{\alpha^2}y^2}\right) + \mathcal{O}(\varrho^3) \\ &= \frac{1}{2\beta}\left(-\alpha - \gamma y + \alpha + \gamma y + \frac{2\beta z}{\alpha} + \frac{\gamma^2}{2\alpha}y^2 - \frac{2\beta\delta}{\alpha}y^2\right. \\ &\qquad\qquad \left. - \frac{\gamma^2}{2\alpha}y^2 - \frac{2\beta\gamma}{\alpha^2}yz - \frac{2\beta^2}{\alpha^3}z^2\right) + \mathcal{O}(\varrho^3) \end{aligned} \tag{9}$$

and thus

$$x = x(y) = x(y,z) = \left(\frac{z}{\alpha} - \frac{z^2\beta}{\alpha^3}\right) - \frac{z\gamma}{\alpha^2}y - \frac{\delta}{\alpha}y^2 + \mathcal{O}(\varrho^3)\;. \tag{10}$$

We will now compute the length of the level line segment within $\mathcal{A}$ because it contributes to the weight with which the value $u = z/\sigma$ enters the computation of the median μ. The end points of this segment are obtained by equating $d_2(p(x_0,y_0), p(x(y),y))$ to ϱ. Approximating d_2 by the Euclidean distance within $\mathbb{R}^3$, this equation becomes $x(y)^2 + y^2 + z^2 = \varrho^2$, and because of $y, z = \mathcal{O}(\varrho)$ we have

$$\left(1 - \frac{2\delta}{\alpha^2}z\right)y^2 - \frac{2\gamma}{\alpha^3}z^2 y + \left(\frac{1+\alpha^2}{\alpha^2}z^2 - \varrho^2 - \frac{2\beta}{\alpha^4}z^3 + \mathcal{O}(\varrho^4)\right) = 0\;, \tag{11}$$

a quadratic equation for y. For its two solutions y_1, y_2 one easily checks that $y_1, y_2 = \mathcal{O}(\varrho)$ and $\frac{y_1+y_2}{2} = \mathcal{O}(\varrho^2)$.

The difference $|y_1 - y_2|$ yields up to $\mathcal{O}(\varrho^3)$ the desired length $L(z)$ of the level line segment within $\mathcal{A}$, i.e.

$$L(z) = 2\varrho\sqrt{1 - \frac{z^2(1+\alpha^2)}{\varrho^2\alpha^2}}\left(1 + \frac{z\delta}{\alpha^2} + \frac{z^3\beta}{\alpha^2\bigl(\alpha^2\varrho^2 - z^2(1+\alpha^2)\bigr)}\right) + \mathcal{O}(\varrho^3)\;. \tag{12}$$

To determine the median μ, we consider the equation

$$\int_{Z_-}^{\sigma\mu} L(\zeta)\,\frac{\partial x}{\partial z}(\zeta)\,\mathrm{d}\zeta = \int_{\sigma\mu}^{Z_+} L(\zeta)\,\frac{\partial x}{\partial z}(\zeta)\,\mathrm{d}\zeta\;, \tag{13}$$

where Z_+ and Z_- are the smallest positive and largest negative values for which $L(Z_+) = L(Z_-) = 0$, and the derivative $\frac{\partial x}{\partial z}(\zeta) = \frac{\partial x}{\partial z}(\frac{y_1+y_2}{2}, \zeta)$ is taken at the midpoint of the corresponding level line segment, $\frac{y_1+y_2}{2} = \mathcal{O}(\varrho^2)$. To interpret (13), we notice that each side integrates the lengths $L(\zeta)$ of level line segments over some range of grey-values, weighted with the inverse density $\frac{\partial x}{\partial z}(\zeta)$ of these level lines. As a consequence, each integral measures the area of the portion of the structuring element with grey-values in its integration domain, and (13) as a whole expresses the condition that the portions with grey-values below and above μ have equal area, which is exactly the characterisation of the median.

From (10) it follows that

$$\frac{\partial x}{\partial z}(\zeta) = \frac{1}{\alpha} - \frac{2\beta}{\alpha^3}\zeta + \mathcal{O}(\varrho^3) \; . \tag{14}$$

For the integration bounds in (13) one has $Z_+, -Z_- = Z_* + \mathcal{O}(\varrho^3)$ with $Z_* = \varrho\alpha/\sqrt{1+\alpha^2}$.

Provided that $\mu = \mathcal{O}(\varrho^2)$, the equality (13) can then be transformed into

$$\int\limits_0^{Z_*} \left(L(\zeta)\cdot\frac{\partial x}{\partial z}(\zeta) - L(-\zeta)\cdot\frac{\partial x}{\partial z}(-\zeta)\right)\,\mathrm{d}\zeta = 2\sigma\mu L(0)\cdot\frac{\partial x}{\partial z}(0) + \mathcal{O}(\varrho^4) \; , \tag{15}$$

thus

$$\int\limits_0^{Z_*} \left(L(\zeta)\left(1-\frac{2\beta}{\alpha^2}\zeta\right) - L(-\zeta)\left(1+\frac{2\beta}{\alpha^2}\zeta\right)\right)\mathrm{d}\zeta = 2\sigma\mu L(0) + \mathcal{O}(\varrho^4) \tag{16}$$

(note that we have multiplied here with α to eliminate the factor $\frac{\partial x}{\partial z}(0) = \frac{1}{\alpha}$). Up to higher order terms the left-hand side equals

$$\begin{aligned}
&\frac{4\varrho(\delta-2\beta)}{\alpha^2}\int\limits_0^{Z_*}\zeta\sqrt{1-\frac{\zeta^2(1+\alpha^2)}{\varrho^2\alpha^2}}\,\mathrm{d}\zeta + \frac{4\beta}{\varrho\alpha^4}\int\limits_0^{Z_*}\frac{\zeta^3}{\sqrt{1-\frac{\zeta^2(1+\alpha^2)}{\varrho^2\alpha^2}}}\,\mathrm{d}\zeta \\
&= \frac{4\varrho^3(\delta-2\beta)}{1+\alpha^2}\int\limits_0^1 \xi\sqrt{1-\xi^2}\,\mathrm{d}\xi + \frac{4\varrho^3\beta}{(1+\alpha^2)^2}\int\limits_0^1\frac{\xi^3}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi \\
&= \frac{4\varrho^3}{3(1+\alpha^2)}(\delta-2\beta) + \frac{8\varrho^3}{3(1+\alpha^2)^2}\beta \\
&= \frac{4\varrho^3}{3(1+\alpha^2)}\delta - \frac{8\varrho^3\alpha^2}{3(1+\alpha^2)^2}\beta \; .
\end{aligned} \tag{17}$$

Together with $L(0) = 2\varrho + \mathcal{O}(\varrho^3)$, this implies

$$\mu = \frac{\varrho^2}{3\sigma}\left(\frac{\delta}{1+\alpha^2} - \frac{2\beta\alpha^2}{(1+\alpha^2)^2}\right) + \mathcal{O}(\varrho^3) \tag{18}$$

which by virtue of (7) can be restated in terms of spatial derivatives of u as

$$\mu = \frac{\varrho^2}{6}\left(\frac{u_{yy}}{1+\sigma^2 u_x^2} - \frac{2\sigma^2 u_x^2 u_{xx}}{(1+\sigma^2 u_x^2)^2} + \mathcal{O}(\varrho)\right) . \tag{19}$$

We have therefore obtained a result of the same type as the equivalence from [11]: One amoeba median filter step acts approximately like one time step of an explicit scheme for the PDE

$$u_t = \frac{u_{\xi\xi}}{1+\sigma^2 |\nabla u|^2} - \frac{2\sigma^2 |\nabla u|^2\, u_{\eta\eta}}{\left(1+\sigma^2 |\nabla u|^2\right)^2} \tag{20}$$

with time step size $\tau = \varrho^2/6$. On the right-hand side, second order derivatives are taken in the directions of the normalised gradient vector $\eta := \nabla u / |\nabla u|$ and the perpendicular vector $\xi := \eta^\perp$, the tangential vector of the local level line of u.

When ϱ tends to zero, the iterated amoeba median filter therefore converges to the PDE (20). The first summand of the right-hand side of (20) can obviously be interpreted as right-hand side of curvature motion $u_t = u_{\eta\eta}$ multiplied by an edge-stopping factor $g(|\nabla u|) := \left(1+\sigma^2 |\nabla u|^2\right)^{-1}$. The second summand corresponds to an AMLE evolution weighted with $h(|\nabla u|) := -2\sigma^2 |\nabla u|^2 (1 + \sigma^2 |\nabla u|^2)^{-2}$, compare Figure 2(a). However, as h takes negative values, this term represents rather an edge-enhancing backward AMLE, i.e. a one-dimensional backward diffusion process.

A closer look reveals that $sg'(s) = h(s)$. Thus the second summand can be expressed as $|\nabla u|\, g'(|\nabla u|)\, u_{\eta\eta}$ which transforms further into $\langle \nabla g, \nabla u\rangle$.

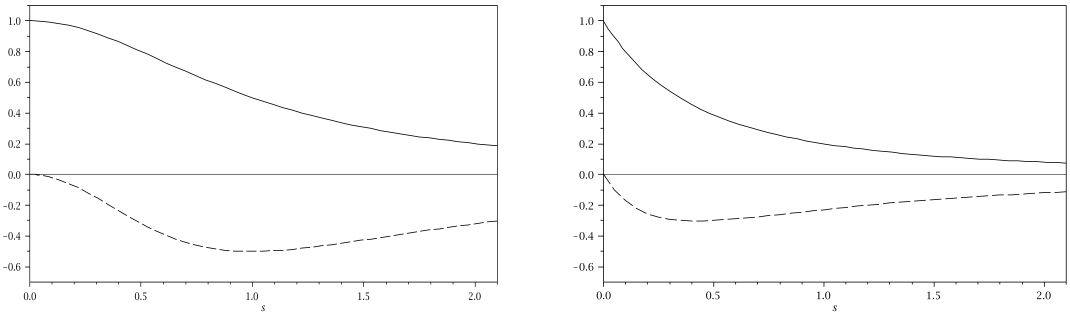
As a result, we see that (20) is nothing else than the self-snakes PDE [26, 34]

$$u_t = |\nabla u| \operatorname{div}\left(g(|\nabla u|)\,\frac{\nabla u}{|\nabla u|}\right) = g(|\nabla u|) u_{\xi\xi} + \langle \nabla g(|\nabla u|), \nabla u\rangle . \tag{21}$$

3.2 Analysis with General Amoeba Metric

Replacing the d_2 amoeba metric (4) with the d_φ metric from (6), we can carry out a very similar analysis like in the previous subsection.

We start again from the Taylor ansatz (7). The end points of the level line segment within the amoeba for $u = z/\sigma$ now need to fulfil $d_\varphi(p(x_0, y_0), p(x(y), y)) = \varrho$, where d_φ is approximated by the corresponding distance in 3D space, thus



(a)

(b)

Figure 2: Edge-stopping functions in PDEs approximated by iterated amoeba median filtering. For visualisation, σ is fixed to 1. **(a)** Weight functions $g(s) = \big(1+s^2\big)^{-1}$ for the curvature motion term (solid line), $h(s) = -2s^2\big(1+s^2\big)^{-2}$ for the shock term (dashed line) from the PDE (20) based on the Euclidean amoeba metric (4). **(b)** Corresponding weight functions for the amoeba metric (5).

$\varphi\left(\sqrt{x(y)^2+y^2}, |z|\right) = \varrho$. Using the homogeneity of φ we can write $\varphi(v,w) = w\varphi(\frac{v}{w},1) =: w\psi(\frac{v}{w})$. Thus we have $|z|\,\psi\left(\frac{\sqrt{x(y)^2+y^2}}{|z|}\right) = \varrho$, and finally

$$x(y)^2 + y^2 = z^2\psi^{-2}\left(\frac{\varrho}{|z|}\right) \tag{22}$$

where ψ^{-2} denotes the square of the inverse function of ψ. Substituting (10) into the condition, we obtain instead of (11) the quadratic equation

$$\left(1-\frac{2\delta}{\alpha^2}z\right)y^2 - \frac{2\gamma}{\alpha^3}z^2y + \left(\frac{1}{\alpha^2} - \psi^{-2}\left(\frac{\varrho}{|z|}\right)\right)z^2 - \frac{2\beta}{\alpha^4}z^3 + \mathcal{O}(\varrho^4) = 0\,, \tag{23}$$

from which we conclude by analogous considerations as before that

$$L(z) = 2\,|z|\sqrt{\psi^{-2}\left(\frac{\varrho}{|z|}\right) - \frac{1}{\alpha^2}}\left(1 + \frac{\delta}{\alpha^2}z + \frac{\beta}{\alpha^4\psi^{-2}\left(\frac{\varrho}{|z|}\right) - \alpha^2}z\right) + \mathcal{O}(\varrho^3) \tag{24}$$

By inserting this and observing the new integration bound $Z_* = \varrho/\psi(1/\alpha)$ (remember that $\alpha > 0$) the left-hand side of (16) becomes

$$\begin{aligned} &\frac{4(\delta-2\beta)}{\alpha^2}\int_0^{Z_*}\zeta^2\sqrt{\psi^{-2}\left(\frac{\varrho}{|\zeta|}\right) - \frac{1}{\alpha^2}}\,\mathrm{d}\zeta + \frac{4\beta}{\alpha^4}\int_0^{Z_*}\frac{\zeta^2}{\sqrt{\psi^{-2}\left(\frac{\varrho}{|\zeta|}\right) - \frac{1}{\alpha^2}}}\,\mathrm{d}\zeta \\ &= \frac{4\varrho^3(\delta-2\beta)}{\alpha^2\psi^3(1/\alpha)}I_1(\alpha) + \frac{4\varrho^3\beta}{\alpha^4\psi^3(1/\alpha)}I_2(\alpha) \end{aligned} \tag{25}$$

where the two integrals

$$I_1(\alpha) := \int_0^1 \xi^2 R_\psi(\xi,\alpha)\,\mathrm{d}\xi \tag{26}$$

$$I_2(\alpha) := \int_0^1 \frac{\xi^2}{R_\psi(\xi,\alpha)}\,\mathrm{d}\xi \tag{27}$$

depend on the expression

$$R_\psi(\xi,\alpha) := \sqrt{\psi^{-2}\left(\frac{1}{\xi}\psi\left(\frac{1}{\alpha}\right)\right) - \frac{1}{\alpha^2}}\ . \tag{28}$$

By analogous reasoning as for the d_2 metric, it is finally proven that one step of amoeba median filtering approximates a time step of size $\tau = \varrho^2/6$ of an explicit scheme for the PDE

$$u_t = g(|\nabla u|)u_{\xi\xi} + h(|\nabla u|)u_{\eta\eta} \tag{29}$$

with the two weight functions

$$g(s) = \frac{3I_1(\sigma s)}{(\sigma s)^2\psi^3(1/(\sigma s))}\ , \tag{30}$$

$$h(s) = \frac{3I_2(\sigma s)}{(\sigma s)^4\psi^3(1/(\sigma s))} - 2g(s)\ . \tag{31}$$

Again, the right-hand side consists of a curvature motion $u_{\xi\xi}$ modulated by the weight g and an edge-sharpening shock term given by the AMLE expression $u_{\eta\eta}$ with the weight h.

As an example, we show in Figure 2(b) the functions g and h for the case of the d_1 amoeba metric, (5).

While for general functions ψ the integrals I_1 and I_2 may not be solvable in closed form, it is possible to compute these integrals, and thereby the weight functions g and h, numerically and thereby to implement a numerical scheme for the evolution PDE (29). In calculating the integrals, however, care must be taken of the (weak) singularity of the integrands at $\xi = 1$.

It is easy to show that the derivative of the integrand of I_1 w.r.t. s is bounded over the product $[a,b]\times[0,1]$ of closed intervals $[a,b]$ $(0 < a < b)$ for s and $[0,1]$ for ξ such that it is uniformly continuous. It follows further that its integral is uniformly convergent w.r.t. s. As a consequence, one can compute $g'(s)$ by exchanging differentiation and integration, and derive the expression

$$sg'(s) - h(s) = \frac{3\psi'\left(\frac{1}{\sigma s}\right)}{(\sigma s)^3 \psi^4\left(\frac{1}{\sigma s}\right)} I_3(\sigma s) \tag{32}$$

with the integral

$$I_3(\alpha) := \int_0^1 \left(3\xi^2 R_\psi(\xi,\alpha) - \frac{\xi \psi^{-1}\left(\frac{1}{\xi}\psi\left(\frac{1}{\alpha}\right)\right)\psi\left(\frac{1}{\alpha}\right)}{\psi'\left(\psi^{-1}\left(\frac{1}{\xi}\psi\left(\frac{1}{\alpha}\right)\right)\right) R_\psi(\xi,\alpha)} \right) \mathrm{d}\xi\ . \tag{33}$$

Using the antiderivative

$$\int \left(3\xi^2 R_\psi(\xi,\alpha) - \frac{\xi \psi^{-1}\left(\frac{1}{\xi}\psi\left(\frac{1}{\alpha}\right)\right)\psi\left(\frac{1}{\alpha}\right)}{\psi'\left(\psi^{-1}\left(\frac{1}{\xi}\psi\left(\frac{1}{\alpha}\right)\right)\right) R_\psi(\xi,\alpha)} \right) \mathrm{d}\xi = \xi^3 R_\psi(\xi,\alpha) \tag{34}$$

one concludes that $I_3(\alpha) = 0$ and $sg'(s) = h(s)$ holds exactly as in the d_2 case. The PDE (29) can therefore be identified as a self-snakes evolution (21) with the edge-stopping function g given by (30).

4 Experiments

We present two experiments that confirm the behaviour suggested by the analytical results from the previous section.

The *House* experiment. We use first a comparatively simple image in order to investigate the effect of the different parameters of the amoeba median filtering algorithm, see Figure 3, and to compare results against self-snakes, see Figure 4.

Figure 3(a) shows the original image. Subfigure (b) depicts the steady state of standard median filtering with a fixed (3×3) structuring element. As usual with median filtering, the shape of edges is rounded, and the facade of the depicted house is quite non-uniform in its grey-value distribution. The use of a larger non-adaptive structuring element will distort the shape of important image features.

In Figure 3(c–f) we compare the results of iterated AMF using the L_2 amoeba distance together with varying parameters.

We start with a relatively strong penalisation of grey-value differences given by $\sigma = 0.25$, see (c, d). This leads to a visible image simplification in which almost

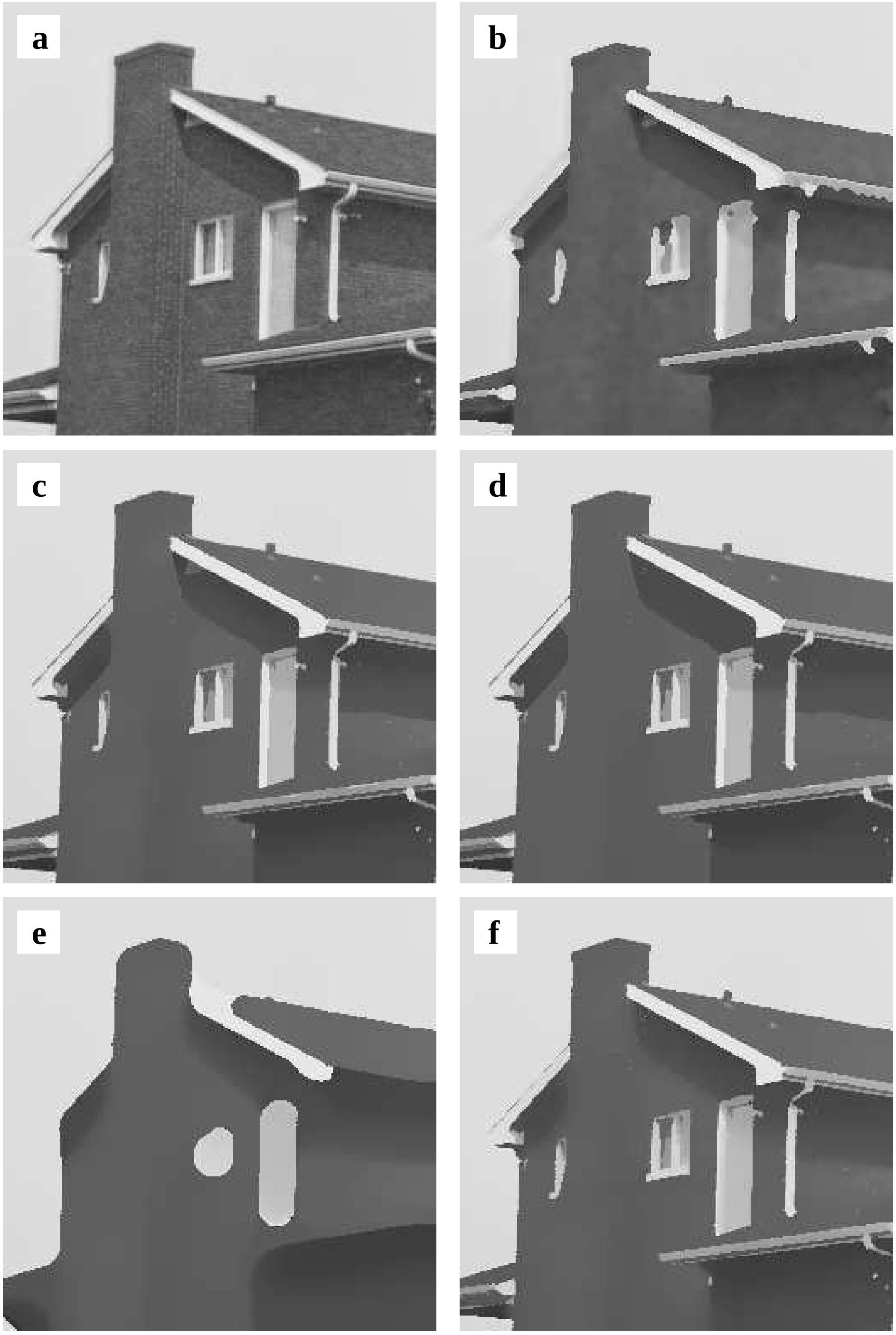


Figure 3: The ***House*** experiment. **Top row: (a)** Original image. **(b)** Filtered with iterated median filter, 3×3 stencil, 40 iterations. **Middle row: (c)** Iterated AMF, $\varrho = 10$, $\sigma = 0.25$, 4 iterations. **(d)** Same as in (c) but 20 iterations. **Bottom row: (e)** Iterated AMF, $\varrho = 10$, $\sigma = 0.02$, 10 iterations. **(f)** Iterated AMF, $\varrho = 20$, $\sigma = 0.25$, 1 iteration.

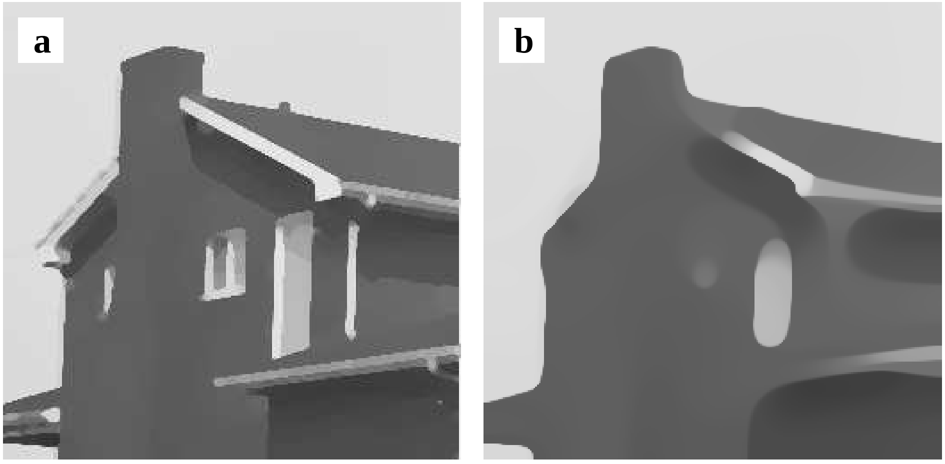


Figure 4: The ***House*** experiment with self-snakes. **(a)** Image from Figure 3(a) filtered by self-snakes (21) with Euclidean edge-stopping function, $\sigma = 0.25$, time step $\tau = 0.25$, 267 iterations. **(b)** Same but $\sigma = 0.02$, $\tau = 0.25$, 667 iterations.

flat image regions are flattened even more. Edges are sharpened but stay fairly well in place.

When a very small σ is used, as in Figure 3(e), image contrast has little influence on the amoeba shapes such that the filter gets close to a non-adaptive median filter with large structuring elements. The typical rounding of corners and disappearance of small-scale structures can be observed well. Translating to the approximated self-snakes evolution, the edge-stopping function now takes values close to 1, implying a smoothing behaviour very similar to curvature motion.

In Figure 3(f) we return to the sensitive tonal weight setting of (c) but increase the amoeba parameter ϱ relative to (c, d). From the analytic point of view, this corresponds to a larger time step size: Due to the quadratic relationship $\tau = \varrho^2/6$ we can expect that a single iteration with structure element radius $\varrho = 20$ should roughly correspond to four iterations with a structure element of half the radius, $\varrho = 10$. Indeed, the results shown in Figure 3(c) and (f) look very similar. We observe especially that the transition zones at the shadows are located very similarly. The self-snake-like sharpening, however, appears somewhat more prominent in the image processed with four iterations. The similarity between the two results also gives rise to the expectation that for the simple test image used here the qualitative behaviour of amoeba median filtering with structure element radius $\varrho = 10$ is still comparable to the continuous limit $\varrho \to 0$.

For comparison, Figure 4 displays results of numerical evaluation of the self-snakes PDE (21) with Euclidean edge-stopping function $g(s) = (1 + \sigma^2 s^2)^{-1}$. An explicit time-stepping scheme was used wherein the curvature-motion component

$g(|\nabla u|)u_{\xi\xi}$ was discretised using central differences, while the backward AMLE contribution was discretised by an upwind-like discretisation.

The parameters for Figure 4(a) have been adjusted to match those of the AMF experiments in Figure 3(c) or (f), and indeed the filtering result is very similar. Figure 4(b) should approximate Figure 3(e). Here the approximation is somewhat worse: On one hand, the discretisations of derivatives in the PDE scheme inevitably introduce some additional blurring and accelerate the structure simplification process. On the other hand, one has to be aware that the structuring elements of the amoeba evolution are fairly large such that visible deviations from the limit case behaviour can be expected.

The *Head* experiment. In our second experiment (Figure 5) we use an MR image of a human head which abounds in details of different contrast and scale. The original image is shown in Subfigure (a), a non-adaptive median filter result in (b). In (c, d) iterated AMF results both with L_2 and L_1 amoeba distance are displayed. It can be seen that both distance measures lead to similar results. Moreover, we observe even clearer than in the *House* experiment the good quality of segmentation that is achieved in spite of the relative simplicity of the filtering approach.

We contrast the results once more with numerical computations based on the self-snakes PDE, where we use this time the edge-stopping function (30) corresponding to the L^1 amoeba metric (5). A precomputed lookup table with linear interpolation was used for the edge-stopping function.

Since the present test image contains a multitude of small-scale structures, the differences between a conventional PDE discretisation and amoeba median filtering become evident: Figure 5(e) clearly demonstrates an over-simplification due to numerical blurring effects. To give an impression of the influence of the latter, we have re-computed the self-snakes evolution on a finer grid: Using a mesh size of 0.25 instead of 1, the resulting image, Figure 5(f), retains much more structure and shows in the coarser image structures a comparable degree of simplification as the AMF result in Subfigure (d). However, small-scale structures still retain more contrast in the AMF result.

5 Conclusion

In this paper we have presented an analysis of iterated amoeba median filtering which demonstrates that even highly adaptive discrete morphological image filters can be interpreted in terms of PDE-based evolutions. An equivalence re-

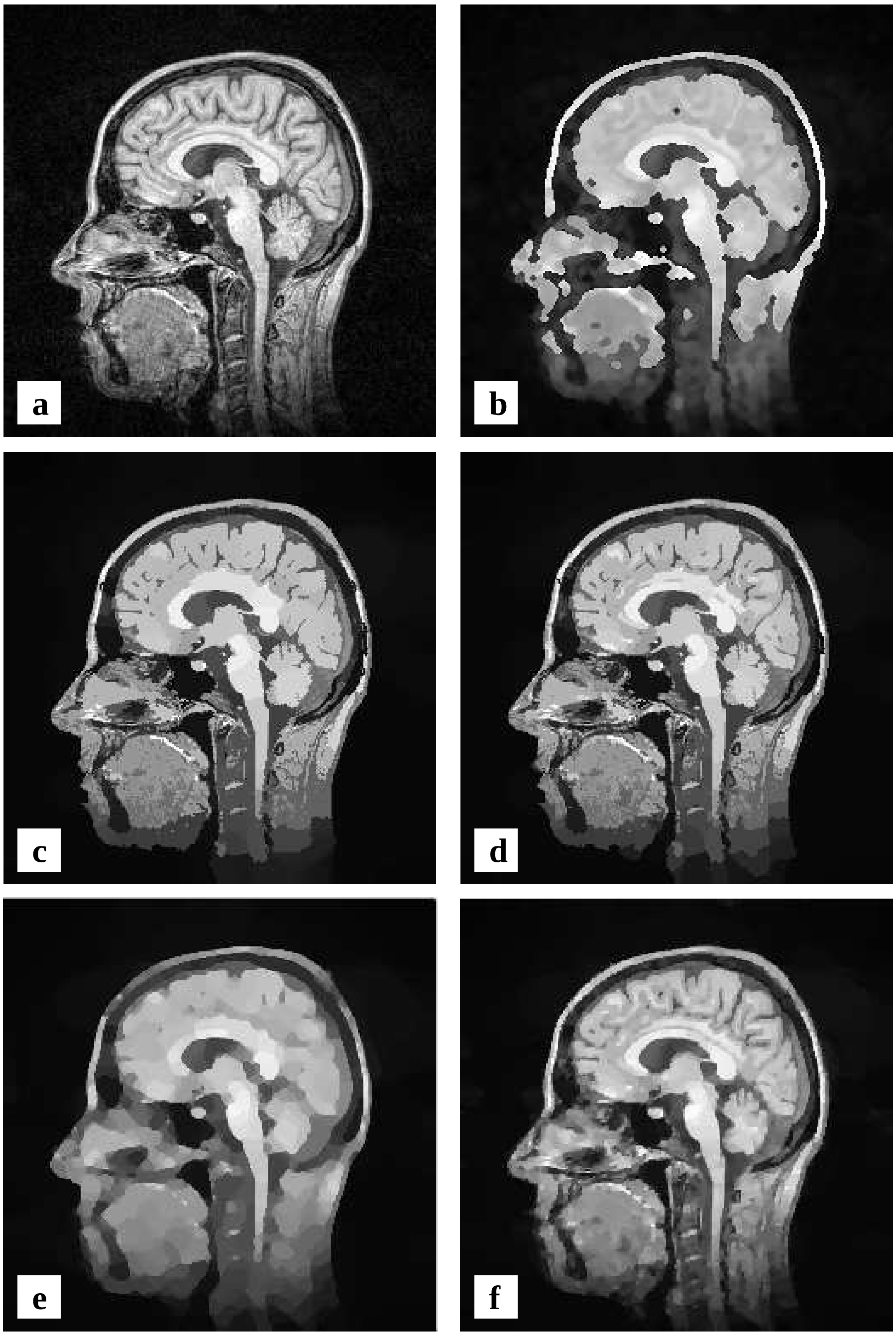


Figure 5: The ***Head*** experiment. **Top row:** (**a**) Original image. (**b**) Steady state of iterated median filter. **Middle row:** (**c**) Iterated AMF, $\varrho = 10$, $\sigma = 0.25$, 10 iterations, L_2 amoeba distance. (**d**) Same but with L_1 amoeba distance. **Bottom row:** (**e**) Self-snakes with edge-stopping function based on L^1 metric (5), $\sigma = 0.25$, time step size $\tau = 0.25$, 667 iterations. (**f**) Same self-snakes evolution computed on a grid with spatial mesh size 0.25, $\sigma = 0.25$, $\tau = 0.015625$, 10667 iterations.

sult has been proven that links iterated amoeba median filtering to self-snakes, a well-known PDE image filter with segmentation behaviour. Thereby a clear explanation of qualitative properties of iterated AMF was given, together with predictions that could be experimentally checked. Our experiments confirm the predictions of the theoretical investigation, while demonstrating at the same time the difficulties of conventional discretisations of the self-snakes PDE in reproducing its favourable theoretical properties.

It appears therefore attractive to use iterated AMF itself as a discretisation of the self-snakes PDE. Together with equivalence results as by Guichard and Morel this may lead to establishing a dictionary of unconventional discretisations of image filtering PDEs which retain advantageous qualitative characteristics of the PDEs better than standard approaches.

This emphasises a common perspective on discrete and PDE-based image filters, which may help to fuse both formerly disparate branches of image processing.

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