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Abstract

If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, we prove the inequality $\|u\|_1 \leq c(n)\mathrm{diam}(\Omega)\int_\Omega |\varepsilon^D(u)|$ being valid for functions of bounded deformation vanishing on $\partial\Omega$. Here $\varepsilon^D(u)$ denotes the deviatoric part of the symmetric gradient and $\int_\Omega |\varepsilon^D(u)|$ stands for the total variation of the tensor-valued measure $\varepsilon^D(u)$. Further results concern possible extensions of this Poincaré–type inequality.

A technical ingredient of the fundamental research of G. A. Seregin on the regularity theory for problems from plasticity theory (see, e.g., [Se1–12] and [FuS]) is a collection of Poincaré–type inequalities established by Strauss [Str], Temam and Strang [TS] and by Anzellotti and Giaquinta [AG], in which certain integral norms of the deformation u are estimated in terms of the total variation of the strain tensor $\varepsilon(u)$. The purpose of our note is to show that it is sometimes possible to replace $\varepsilon(u)$ in these inequalities through its deviatoric part. To be precise, suppose that we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and a field $u : \Omega \to \mathbb{R}^n$. We introduce the symmetric gradient of u

$$\varepsilon(u) := (\varepsilon_{ij}(u))_{1\leq i,j\leq n}\,,\ \varepsilon_{ij}(u) := \frac{1}{2}(\partial_i u^j + \partial_j u^i)\,,$$

and its deviatoric part

$$\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}(\mathrm{div}\, u)\mathbf{1},\ \mathbf{1} = (\delta_{ij})_{1\leq i,j\leq n}\,,$$

whenever these expressions make sense. We further denote by $W^1_s(\Omega;\mathbb{R}^n)$ the Sobolev space of all fields $u : \Omega \to \mathbb{R}^n$, which together with their first weak partial derivatives are in the Lebesgue class $L^s(\Omega;\mathbb{R}^n)$ for some exponent $s \in [1,\infty)$. Moreover, we consider the subspace $\overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^n)$ of $W^1_s(\Omega;\mathbb{R}^n)$ consisting of all functions vanishing on the boundary of Ω. For a more detailed definition and further properties of these spaces the reader is referred to the monograph of Adams [Ad]. Our first result is in some sense an extension of a Sobolev–Poincaré inequality obtained by Strauss (see [Str], Theorem 1):

THEOREM 1. *There is a constant $c(n)$ such that*

$$\|u\|_{L^1(\Omega)} \leq c(n)\mathrm{diam}(\Omega)\|\varepsilon^D(u)\|_{L^1(\Omega)} \tag{1}$$

holds for any function $u \in \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^n)$. If p is some number in $[1,\frac{n}{n-1})$, then for a suitable constant $c(n,p)$ we have

$$\|u\|_{L^p(\Omega)} \leq c(n,p)\mathrm{diam}(\Omega)^{1-n+\frac{n}{p}}\|\varepsilon^D(u)\|_{L^1(\Omega)} \tag{2}$$

for all fields $u \in \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^n)$.

Let us add some comments:

a) In his work Strauss discusses fields from the space $\overset{\circ}{W}{}^1_{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ and proves ([Str], Theorem 1)

$$\|u\|_{L^{\frac{n}{n-1}}(\Omega)} \le C\|\varepsilon(u)\|_{L^1(\Omega)}\,,$$

whereas in our case only the deviatoric part of the symmetric gradient occurs on the right-hand side. However, on the left-hand side of inequality (2) our techniques do not allow us to include the limit exponent $p = \frac{n}{n-1}$, and so it remains an interesting open question, if (2) is true for this choice of p.

b) Assume that $n \ge 3$ and fix an exponent $s \in (1,\infty)$. Then, according to Theorem 2 of Reshetnyak's deep paper [Re], we have the Korn–type inequality

$$\|v - P(v)\|_{W^1_s(\Omega)} \le C\|\varepsilon^D(v)\|_{L^s(\Omega)} \tag{3}$$

valid for all $v \in W^1_s(\Omega;\mathbb{R}^n)$ with a finite constant C depending on n, s and Ω. Here $P(v)$ denotes the projection of v on the kernel of ε^D (space of Killing vectors), which is of finite dimension. If v is smooth having in addition compact support in Ω, then it can be deduced from the representation formula (2.20) in [Re] that $P(v)$ is constant (cf. proof of Theorem 1 for details), and we infer from (3)

$$\|\nabla v\|_{L^s(\Omega)} \le C\|\varepsilon^D(v)\|_{L^s(\Omega)} \tag{4}$$

for all $v \in \overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^n)$. Combining (4) with Poincaré's inequality we obtain

$$\|v\|_{L^s(\Omega)} \le C\|\varepsilon^D(v)\|_{L^s(\Omega)}$$

again for $v \in \overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^n)$, which is the "$L^s$–variant" of (1) for exponents $s > 1$. We emphasize that it is not possible to derive inequality (1) along these lines, since even $\int_\Omega |\varepsilon(u)|\,dx$ does not dominate each quantity $\int_\Omega |\partial u^i/\partial x_j|\,dx$ for arbitrary fields from $\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^n)$. Counterexamples can be traced in the works of Mitjagin [Mi], de Leeuw and Mirkil [DLM] and of Ornstein [Or].

Next we pass to the space BD(Ω) consisting of all fields $u \in L^1(\Omega;\mathbb{R}^n)$ having bounded deformation introduced by Suquet [Su] and by Matthies, Strang and Christiansen [MSC] and further investigated by e.g. Temam and Strang [TS] and Anzellotti and Giaquinta [AG] in the context of plasticity theory. According to Proposition 1.2 of [AG] it holds

$$\int_{\mathbb{R}^n} |u|\,dx \le c(n)\mathrm{diam}(\mathrm{spt}\,u)\int_{\mathbb{R}^n} |\varepsilon(u)|$$

for $u \in BD(\mathbb{R}^n)$ having compact support, and we can state:

THEOREM 2. *There is a constant $c(n)$ such that*

$$\|u\|_{L^1(\Omega)} \le c(n)\mathrm{diam}(\Omega)\int_\Omega |\varepsilon^D(u)| \tag{5}$$

is satisfied for all fields $u \in BD(\Omega)$ with $u|_{\partial\Omega} = 0$.

Here $u|_{\partial\Omega}$ denotes the trace of the function u in the sense of [TS], Theorem 1.1. The proof of Theorem 2 is easily obtained, if we accept Theorem 1 for the moment and follow the remarks stated in [AG] after the proof of their Theorem 1.3: given u as above, there exists a sequence $u_h \in C^\infty(\Omega;\mathbb{R}^n) \cap \mathrm{BD}(\Omega)$ such that

i) $u_h \to u \quad \text{in } L^1(\Omega;\mathbb{R}^n)$,

ii) $u_h|_{\partial\Omega} = u|_{\partial\Omega} = 0$,

iii) $\int\limits_\Omega |\varepsilon^D(u_h)|\,dx \to \int\limits_\Omega |\varepsilon^D(u)| \quad \text{as } h \to \infty.$

On account of ii) we have inequality (1) for the sequence u_h, and by i), iii) we may pass to the limit $h \to \infty$ in order to obtain our claim (5). Before we present the proof of Theorem 1, we want to mention an additional related result:

THEOREM 3. *For a finite constant* $c(n)$ *we have the inequality* ($\kappa = \kappa(u)$)

$$\|u-\kappa\|_{L^1(\Omega)} \le c(n)\ \mathrm{diam}(\Omega) \int\limits_\Omega |\varepsilon^D(u)|$$

valid for all $u \in BD(\Omega)$. *In case* $n = 2$ κ *denotes a suitable holomorphic function, whereas for* $n \ge 3$ κ *is a Killing vector as explained for example in [Da], p.537. For* $p \in [1, \frac{n}{n-1})$ *we also have*

$$\|u-\kappa\|_{L^p(\Omega)} \le c(n,p)\mathrm{diam}(\Omega)^{1-n+\frac{n}{p}} \int\limits_\Omega |\varepsilon^D(u)|\,.$$

We wish to remark that the estimates from Theorem 3 correspond to the inequalities obtained in [TS] and [AG], in which the BD–distance of fields u from $\mathrm{BD}(\Omega)$ to the space of rigid motions is controlled through the total variation of the tensor–valued measure $\varepsilon(u)$. A proof of Theorem 3 for domains $\Omega \subset \mathbb{R}^2$ and functions u from the space $W^1_1(\Omega;\mathbb{R}^2)$ has been given in [Fu], and from this work the BD–variant follows by approximation. The higher dimensional case will be a consequence of the arguments needed for the proof of Theorem 1.

For proving Theorem 1 we first consider the case $u \in C_0^\infty(B;\mathbb{R}^n)$, B denoting the open unit ball. From Dain's paper [Da] we quote the identity ($i = 1,\dots,n$)

$$\frac{1}{2}\Delta u^i = \sum_{j=1}^n \partial_j \varepsilon^D_{ij}(u) - \left(\frac{1}{2} - \frac{1}{n}\right)\partial_i(\mathrm{div}\,u)\,,$$

which gives in combination with Green's representation formula

$$u^i(x) = \int_B \Gamma(y-x)2\left\{\sum_{j=1}^n \partial_j\varepsilon^D_{ij}(u)(y) - \left(\frac{1}{2}-\frac{1}{n}\right)\partial_i(\mathrm{div}\,u)(y)\right\}\,dy \tag{6}$$

valid for all $x \in B$. Here Γ denotes the normalized fundamental solution of the Laplace equation (see, e.g. [GT], (2.12)). Now, if $n = 2$, the right-hand side of (6) equals

$$-2 \int_B \sum_{j=1}^n \frac{\partial}{\partial y_j} \Gamma(y-x) \varepsilon_{ij}^D(u)(y)\, dy\,,$$

and we can apply the theory of Riesz potentials (compare [Ste] or [GT]) to deduce our claims (1) and (2) for $\Omega = B$ and u as above. Unfortunately this argument does not work in case $n \geq 3$, since then the right-hand side of (6) not only consists of terms involving $\varepsilon^D(u)$. Instead of (6) we now use a different representation, which is due to Reshetnyak [Re]. According to formula (2.43) of this paper it holds

$$u(x) = P_2 u(x) + R_2(Q_2 u)(x),\ x \in B\,, \tag{7}$$

where the quantities on the right-hand side of (7) have the following meaning: $P_2 u$ denotes a suitable Killing vector, i.e. an element of the kernel of ε^D, $Q_2 u$ is just the tensor $\varepsilon^D(u)$ and R_2 is the potential operator being defined in (2.41) of [Re]. According to the structure of R_2 and the representation of its kernel stated in (2.42) of [Re], we can apply the theory of Riesz potentials (see, e.g. [Ste] or [GT]) to deduce

$$\|R_2(Q_2 u)\|_{L^1(B)} \leq c(n) \|Q_2 u\|_{L^1(B)}\,. \tag{8}$$

In order to continue we need more information concerning the projection $P_2 u$. Again we benefit from Reshetnyak's work: we use formula (2.20) and pass to the mean value $\fint_B \dots dy$ with respect to the variable $y \in B$ on the right-hand side. According to the comment given after (2.22) the i^{th} component of $P_2 u(x)$ is the remaining expression on the right-hand side, in which no integration with respect to the variable $t \in [0,1]$ is performed, i.e. we have the identity ($i = 1, \dots, n,\ x \in B$)

$$\begin{aligned} (P_2 u)^i(x) \;=\; & \fint_B u^i dy + \fint_B \sum_{j=1}^n \frac{1}{2} (\partial_j u^i - \partial_i u^j)(y)(x_j - y_j)\, dy \\ & + \fint_B \frac{1}{n} \operatorname{div} u(y)(x_i - y_i)\, dy \\ & + \fint_B \sum_{j=1}^n (x_j - y_j) \frac{1}{n} \partial_j \operatorname{div} u(y)(x_i - y_i)\, dy \\ & - \fint_B \frac{1}{2} |x-y|^2 \frac{1}{n} \partial_i \operatorname{div} u(y)\, dy\,. \end{aligned} \tag{9}$$

Since u has compact support in B, we may integrate by parts on the right-hand side of (9) to get

$$P_2 u \equiv \alpha(n) \overline{u},\ \overline{u} := \fint_B u\, dy\,,\ \alpha(n) := 1 + \frac{n-1}{2} + \frac{n+1}{n}\,. \tag{10}$$

With (10) we return to (7) and take $\fint_B \ldots dx$ on both sides with the result

$$\overline{u} = \alpha(n)\overline{u} + \fint_B R_2(Q_2 u)\, dx\,,$$

hence

$$|\overline{u}| \le \frac{1}{\alpha(n)-1} \fint_B |R_2(Q_2 u)|\, dx\,,$$

and we can apply (8) to get (with another constant $c(n)$)

$$|\overline{u}| \le c(n)\|\varepsilon^D(u)\|_{L^1(B)}\,,$$

which on account of (10) implies

$$\|P_2 u\|_{L^1(B)} \le c(n)\|\varepsilon^D(u)\|_{L^1(B)}\,. \tag{11}$$

By combining (7), (8) and (11) we finally arrive at

$$\|u\|_{L^1(B)} \le c(n)\|\varepsilon^D(u)\|_{L^1(B)}\,. \tag{12}$$

Suppose next that $u \in C_0^\infty(\Omega;\mathbb{R}^n)$ for a bounded Lipschitz domain Ω. Then we have $u \in C_0^\infty(B_R(x_0);\mathbb{R}^n)$ for a suitable ball $B_R(x_0) \supset \Omega$, whose diameter is proportional to diam (Ω), and by using (12) our claim (1) for smooth u just follows by scaling. Finally, if u is from the space $\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^n)$ we apply (1) to a sequence $u_m \in C_0^\infty(\Omega;\mathbb{R}^n)$ such that $\|u_m - u\|_{W_1^1(\Omega)} \to 0$ as $m \to \infty$. In order to verify (2) we only observe that for $1 \le p < \frac{n}{n-1}$ inequality (8) can be replaced by

$$\|R_2(Q_2 u)\|_{L^p(B)} \le c(n,p)\|Q_2 u\|_{L^1(B)}\,,$$

which is a well–known property of Riesz potentials. $\square$

Next we prove Theorem 3 for the case $n \ge 3$: from [Re] we deduce as before (compare (7) and (8))

$$\|u - \kappa\|_{L^1(\Omega)} \le c(n) \text{ diam } (\Omega)\|\varepsilon^D(u)\|_{L^1(\Omega)} \tag{13}$$

at least for smooth fields u with a suitable Killing vector $\kappa = \kappa(u)$. For $u \in$ BD(Ω) we can use the approximation argument of [AG] stated after Theorem 2 (of course ii) now reads $u_h|_{\partial\Omega} = u|_{\partial\Omega}$) with the result that (13) is valid for the sequence u_h with corresponding Killing vectors κ_h. At the same time it holds (see (7))

$$u_h = \kappa_h + R_2(Q_2 u_h)\,,$$

which gives

$$\|\kappa_h\|_{L^1(\Omega)} \le \|u_h\|_{L^1(\Omega)} + c(n,\Omega)\|\varepsilon^D(u_h)\|_{L^1(\Omega)}\,,$$

hence

$$\sup_h \|\kappa_h\|_{L^1(\Omega)} < \infty \ .$$

Since the vectors κ_h belong to a space of finite dimension, this bound is enough to deduce that $\kappa_h \to: \kappa$ in $L^1(\Omega; \mathbb{R}^n)$ at least for a subsequence and a Killing vector κ. This proves our claim. $\square$

We finish our discussion by mentioning an open problem: suppose that Γ is a subset of $\partial\Omega$ having positive $(n-1)$–dimensional measure. Do we have the validity of the inequality

$$\|u\|_{L^1(\Omega)} \leq c(n, \Gamma, \Omega) \int_\Omega |\varepsilon^D(u)| \tag{14}$$

for all $u \in \mathrm{BD}(\Omega)$ such that $u|_\Gamma = 0$? A positive answer would provide a stronger result as stated in Corollary 1.11 of [AG], but if we try to prove (14) by contradiction we do not have enough information to use the continuity of the trace operator (cf. the comments given in [AG] after Theorem 1.4) which would lead to the desired contradiction. So this open problem is in some sense related to the question if there is a reasonable concept of a trace for fields $u \in L^1(\Omega; \mathbb{R}^n)$ whose distributional deviator $\varepsilon^D(u)$ is a tensor–valued measure of finite total variation. However, a meaningful definition of boundary values for fields in this class seems to be impossible: let B denote the open unit disc centered at the origin and let

$$u : B \to \mathbb{C},\ u(z) := \frac{1}{z-1} \, .$$

Then u is in the space $L^1(B; \mathbb{C})$, and $\varepsilon^D(u) = 0$ on B holds, since u is holomorphic on B. If a trace $u|_{\partial B}$ of u in the space $L^1(\partial B; \mathbb{C})$ would exist, then it should hold

$$u(z) = u|_{\partial B}(z)\ \mathcal{H}^1 - \text{a.e. on } \partial B \, ,$$

but this contradicts the fact that

$$\int_{\partial B} \frac{1}{|z-1|} d\mathcal{H}^1(z) = +\infty \, .$$

References

[Ad] Adams, R. A., Sobolev spaces. Academic Press, New York-San Francisco-London (1975).

[AG] Anzellotti, G., Giaquinta, M., Existence of the displacement field for an elasto–plastic body subject to Hencky's law and von Mises yield condition. Manus. Math. 32 (1980), 101–136.

[Da] Dain, S., Generalized Korn's inequality and conformal Killing vectors. Calc. Var. 25(4) (2006), 535–540.

[DLM] de Leeuw, K., Mirkil, H., A priori estimates for differential operators in L_∞ norm. Illinois J. Math. 8 (1964), 112–124.

[Fu] Fuchs, M., An estimate for the distance of a complex valued Sobolev function defined on the unit disc to the class of holomorphic functions. Preprint 247, Universität des Saarlandes (2009).

[FuS] Fuchs, M., Seregin G. A., Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lect. Notes in Mathematics 1749, Springer Verlag Berlin (2000).

[GT] Gilbarg, D., Trudinger, N. S., Elliptic partial differential equations of second order. Springer Verlag, Berlin–Heidelberg–New York (1998).

[MSC] Matthies, H., Strang, G., Christiansen, E., The saddle point of a differential program, in "Energy methods in finite element analysis", volume dedicated to Professor Fraeijs de Veubeke; Glowinski, R., Rodin, E., Zienkiewicz, O. C., eds. John Wiley, New York (1979).

[Mi] Mitjagin, B. S., On the mixed second derivative. Dokl. Akad. Nauk SSR 123 (1958), 606–609.

[Or] Ornstein, D., A non–inequality for differential operators in the L_1 norm. Arch. Rational Mech. Anal. 11 (1962), 40–49.

[Re] Reshetnyak, Y. G., Estimates for certain differential operators with finite dimensional kernel. Sibirsk. Mat. Zh. 11 (1970), 414–428.

[Se1] Seregin, G. A., Variational–difference scheme for problems in the mechanics of ideally elastoplastic media, Zh. Vychisl. Mat. i Mat. Fiz. 25 (1985), 237–352 (in Russian). English translation: U.S.S.R. Comput.Math. and Math.Phys. 25 (1985), 153–165.

[Se2] Seregin, G. A., Differential properties of weak solutions of nonlinear elliptic systems arising in plasticity theory, Mat.Sb. (N.S.) 130 (3), 172 (7) (1986), 291–309 (in Russian). English translation: Math. USSR–Sb. 58 (1987), 289–309.

[Se3] Seregin, G. A., Differentiability of local extremals of variational problems in the mechanics of perfect elastoplastic media, Differentsial'nye Uravneniya 23 (11) (1987), 1981–1991; English transl. Differential Equations 23 (1987), 1349–1358.

[Se4] Seregin, G. A., On differential properties of extremals of variational problems arising in plasticity theory, Differentsial'nye Uravneniya 26 (1990), 1033–1043 (in Russian). English translation: Differential Equations 26 (1990).

[Se5] Seregin, G. A., On regularity of weak solutions of variational problems in plasticity theory, Dokl. Acad. Sci. 314 (1990), 1344-1349 (in Russian). English transplation: Soviet Math. Dokl. 42 (1991).

[Se6] Seregin, G. A., On the regularity of weak solutions of variational problems in plasticity theory, Algebra i Analiz 2 (1990), 121–140 (in Russian). English translation: Leningrad Math. J. 2 (1991).

[Se7] Seregin, G. A., On regularity of minimizers of certain variational problems in plasticity theory, Algebra i Analiz 4 (1992), 181–218 (in Russian). English translation: St. Petersburg Math. J. 4 (1993), 989–1020.

[Se8] Seregin, G. A., On differentiability properties of the stress tensor in Coulomb–Mohr plasticity, Algebra i Analiz 4 (1992), 234–252 (in Russian). English translation: St. Petersburg Math. J. 4 (1993), 1257–1272.

[Se9] Seregin, G. A., Differentiability properties of weak solutions of certain variational problems in the theory of perfect elasticplastic plates, Appl. Math. Optim. 28 (1993), 307–335.

[Se10] Seregin, G. A., Twodimensional variational problems in plasticity theory, Izv. Russian Academy of Sciences 60 (1996), 175–210 (in Russian). English translation in Izvestiya: Mathematics 60, no. 1 (1996), 179–219.

[Se11] Seregin, G. A., On the differentiability of local extremals of variational problems in the mechanics of rigidly viscoplastic media, Izv. Vyssh. Uchebn. Zaved Mat. No. 10 (305) (1987), 23–30 (in Russian). English translation: Sov. Math. (Iz. VUZ) 31 (1987).

[Se12] Seregin, G. A., On differential properties of extremals of variational problems of the mechanics of viscoplastic media, Proc. Stoklov Inst. Math. 3 (1991), 147–157.

[Ste] Stein, E., Singular integrals and differentiability properties of functions. Princeton University Press, Princeton (1970).

[Str] Strauss, M. J., Variations of Korn's and Sobolev inequalities. Berkeley symposium on partial differential equations, AMS Symposia Vol 23 (1971).

[Su] Suquet, P., Existence et régularité des solutions des equations de la plasticité parfaite. Thèse de 3e Cycle, Université de Paris VI (1978). Also: C. R. Acad. Sci. Paris, Ser. D. 286, 1201–1204 (1978).

[TS] Temam, R., Strang, G., Functions of bounded deformation. Arch. Rat. Mech. Anal. 75 (1981), 7–21 .

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