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MEASURE THEORY: TRANSPLANTATION THEOREMS FOR INNER PREMEASURES

HEINZ KÖNIG

Abstract. The main result is a new transplantation theorem for the inner $\star$ premeasures of the author, with a few related theorems. These results have basic implications for example for the construction of Radon measures. They received a certain inspiration from the treatment of Radon measures in the treatise of Fremlin on measure theory.

1. Introduction

The present article is part of the author's new systematization in measure and integration initiated in [3], of which the latest accounts are [8][10]. As before we concentrate on the *inner* version. We recall that its basic concepts are the *inner* $\bullet$ *premeasures* and their *maximal inner* $\bullet$ *extensions* ($\bullet = \star\sigma\tau$ with $\star =$ finite, $\sigma =$ sequential, $\tau =$ nonsequential), and that its basic devices are the *inner* $\bullet$ *envelopes*. We shall often make free use of the concepts and results set up so far.

One of the final chapters in [3] was devoted to *transplantation theorems* for inner premeasures. The central results were for $\star$ premeasures and hence of a certain *simple* character: but these results often appear in combination with topological compactness or with set-theoretic $\bullet = \sigma\tau$ compactness, and thus lead to consequences for Radon measures and in the new $\bullet = \sigma\tau$ theories. We recall the former main theorem [3] 18.10 = [6] 2.3, which also explains the word *transplantation*. Let X denote a nonvoid set.

1.1 Theorem. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$ in X such that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$, and let $\psi : \mathfrak{T} \to [0,\infty[$ be an inner $\star$ premeasure. If $\vartheta : \mathfrak{S} \to [0,\infty[$ is isotone with $\vartheta(\varnothing) = 0$ and supermodular such that $\vartheta_\star|\mathfrak{T} = \psi$, then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi \geqq \vartheta$ such that $\varphi_\star|\mathfrak{T} = \psi$.*

Besides the consequences in [3] there are related results in [5]. Then an essential step forward was the transplantation theorem [6] 2.4. It was the main tool for the new versions of the Prokhorov and Kolmogorov projective limit theorems obtained in [6], of which the latter one in the complemented form of [7] then constituted the core of a new concept of stochastic processes. We recall [6] 2.4 in the modified version initiated in the first few lines of its proof. Let $\top$ denote the usual *transporter*.

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1.2 Theorem. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$ in X such that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$, and let $\psi : \mathfrak{T} \to [0,\infty[$ be an inner $\star$ premeasure. If $\inf\limits_{V \in \mathfrak{T}\top\mathfrak{S}} \psi_\star(V') = 0$, then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ such that $\varphi_\star|\mathfrak{T} = \psi$.*

At this point we turn to the monumental treatise of Fremlin [2], written in the usual terms of measure theory. This work presents in section 416 a sequence of important results 416J-416P for the construction of Radon measures. The initial one 416J is contained in the famous theorem of Kisyński 1968, and 416N is a version of the theorem of Henry 1969 which also appears in [3] as the consequence 18.22 of 18.10. But then 416O, called a *deep* theorem in Bogachev [1] Vol.II p.83, is a consequence of the above [6] 2.4, and was even described as a certain inspiration for this result in [6].

However, the deepest and most comprehensive of the assertions 416J-416P appears to be the final 416P. After quite some time the present author was able to extend [6] 2.4 to an abstract transplantation theorem which in fact furnishes 416P. This will be the main theorem of the present paper, to be formulated and proved in section 2. Then section 3 will be devoted to the concrete implications of the main theorem.

After this there remain the assertions 416K and 416L with its consequence 416M. It will be seen in the final section 4 that 416K and 416L are likewise consequences of transplantation type theorems, this time of results in [5].

2. The Main Theorem

As above we fix a nonvoid set X which contains all set systems under consideration. We start with two useful equivalences. The first is the case $\bullet = \star$ of [6] 1.8. It allows a reformulation of the assertions in 1.1 and 1.2 which will be relevant in the sequel.

2.1 Proposition. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$ such that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$, and let $\psi : \mathfrak{T} \to [0,\infty[$ be an inner $\star$ premeasure. For each $\varphi : \mathfrak{S} \to [0,\infty[$ isotone with $\varphi(\varnothing) = 0$ and supermodular then*

$$\varphi_\star|\mathfrak{T} = \psi \iff \Phi := \varphi_\star|\mathfrak{C}(\varphi_\star) \text{ is an extension of } \Psi := \psi_\star|\mathfrak{C}(\psi_\star).$$

2.2 Proposition. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$, and let $\psi : \mathfrak{T} \to [0,\infty[$ be an inner $\star$ premeasure with $\Psi := \psi_\star|\mathfrak{C}(\psi_\star)$. For each $T \in \mathfrak{T}$ then*

$$\inf_{V \in \mathfrak{T}\top\mathfrak{S}} \psi_\star(T \cap V') = 0 \iff \psi(T) = \sup_{V \in \mathfrak{T}\top\mathfrak{S}} \Psi^\star(T \cap V).$$

Note that the second relation for $T \in \mathfrak{T}$ can be written

$$\psi(T) = \sup\{\Psi^\star(S) : S \in \mathfrak{S} \text{ with } S \subset T \text{ and } S \in \mathfrak{T}\top\mathfrak{S}\};$$

in case $\mathfrak{T} \subset \mathfrak{S}\top\mathfrak{S}$ (which is equivalent to $\mathfrak{S} \subset \mathfrak{T}\top\mathfrak{S}$) the relation reads $\psi(T) = \sup\{\Psi^\star(S) : S \in \mathfrak{S} \text{ with } S \subset T\}$.

Proof. First note that $\psi_\star = \Psi_\star$, because this holds true on $\mathfrak{C}(\psi_\star)$ and hence partout, because both sides are inner regular $\mathfrak{C}(\psi_\star)$. Now [9] 1.1 asserts for $T \in \mathfrak{T}$ that

$$\psi(T) = \Psi(T) = \psi_\star(T \cap V') + \Psi^\star(T \cap V) \quad \text{for all } V \subset X.$$

In view of $\psi(T) < \infty$ the assertion follows. $\square$

We also need the assertions which follow.

2.3 Proposition. *Let $\varphi, \psi : \mathfrak{S} \to [0,\infty[$ be inner $\star$ premeasures. Then* 1) *$\vartheta := \varphi + \psi$ is an inner $\star$ premeasure with $\vartheta_\star = \varphi_\star + \psi_\star$.* 2) *If $\varphi \leqq \psi$ then $\psi - \varphi$ is an inner $\star$ premeasure with $\psi_\star = \varphi_\star + (\psi - \varphi)_\star$.*

Proof. 1) is contained in [4] 6.1. 2) We consider the restrictions $\alpha := \Phi|\mathfrak{A}$ and $\beta := \Psi|\mathfrak{A}$ of $\Phi := \varphi_\star|\mathfrak{C}(\varphi_\star)$ and $\Psi := \psi_\star|\mathfrak{C}(\psi_\star)$ to the algebra $\mathfrak{A} := \mathfrak{C}(\varphi_\star) \cap \mathfrak{C}(\psi_\star) \supset \mathfrak{S}$. Then $\varphi_\star \leqq \psi_\star$ and hence $\alpha \leqq \beta$. After [4] 1.4 we form the content $\beta \setminus \alpha : \mathfrak{A} \to [0,\infty]$. We have $\alpha + (\beta \setminus \alpha) = \beta$, and $\beta \setminus \alpha$ is inner regular $\mathfrak{S}$. Thus $\beta \setminus \alpha = \psi - \varphi$ on $\mathfrak{S}$ implies that $\psi - \varphi$ is an inner $\star$ premeasure, and the last assertion follows from 1). $\square$

We turn to the main theorem. The proof proceeds via the two subsequent lemmata.

2.4 Theorem. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$, and let $\psi : \mathfrak{T} \to [0,\infty[$ be an inner $\star$ premeasure with $\Psi := \psi_\star|\mathfrak{C}(\psi_\star)$. If $\psi(T) = \sup\limits_{V \in \mathfrak{T}\top\mathfrak{S}} \Psi^\star(T \cap V)$ for all $T \in \mathfrak{T}$ and $\Psi^\star|\mathfrak{S} < \infty$, then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ such that $\Phi := \varphi_\star|\mathfrak{C}(\varphi_\star)$ is an extension of Ψ.*

The assumption $\Psi^\star|\mathfrak{S} < \infty$ is clear in case that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$. Thus in this case in view of 2.1-2.2 the theorem is an extension of 1.2.

2.5 Lemma. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$ such that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$, and let $\psi : \mathfrak{T} \to [0,\infty[$ be an inner $\star$ premeasure. Define $\Delta = \Delta(\mathfrak{S}, \mathfrak{T}, \psi)$ to consist of the set functions $\varphi : \mathfrak{S} \to [0,\infty[$ which are isotone with $\varphi(\varnothing) = 0$ and supermodular, and such that $\varphi_\star|\mathfrak{T} \leqq \psi$ and $\varphi_\star|\mathfrak{T}$ is an inner $\star$ premeasure* (note that Δ is nonvoid because it contains $\varphi = 0$). *Then* 1) *Δ is upward inductive in the argumentwise order.* 2) *Each maximal member of Δ is an inner $\star$ premeasure.*

Proof. 1) Let $H \subset \Delta$ be nonvoid and totally ordered, and put $\varepsilon := \sup\limits_{\varphi \in H} \varphi$. Then i) $\varepsilon : \mathfrak{S} \to [0,\infty]$ is isotone with $\varepsilon(\varnothing) = 0$. And $\varepsilon_\star = \sup\limits_{\varphi \in H} \varphi_\star$ is an obvious consequence of the definition. Thus $\varepsilon_\star|\mathfrak{T} \leqq \psi < \infty$ and hence $\varepsilon : \mathfrak{S} \to [0,\infty[$. ii) ε is supermodular. In fact, let $A, B \in \mathfrak{S}$. For $\varphi, \psi \in H$ we can assume that $\varphi \leqq \psi$ and hence

$$\varphi(A) + \psi(B) \leqq \psi(A) + \psi(B) \leqq \psi(A \cup B) + \psi(A \cap B) \leqq \varepsilon(A \cup B) + \varepsilon(A \cap B),$$

and the assertion follows. iii) $\varepsilon_\star|\mathfrak{T}$ is an inner $\star$ premeasure. In fact, let $P \subset Q$ in $\mathfrak{T}$. For $\varphi \in H$ then

$$\varphi_\star(Q) = \varphi_\star(P) + (\varphi_\star|\mathfrak{T})_\star(Q \setminus P) \leqq \varepsilon_\star(P) + (\varepsilon_\star|\mathfrak{T})_\star(Q \setminus P),$$

and hence $\varepsilon_\star(Q) \leqq \varepsilon_\star(P) + (\varepsilon_\star|\mathfrak{T})_\star(Q \setminus P)$ from i). Moreover $\varepsilon_\star$ is supermodular by ii) and [10] 1.5.1.Inn). Thus $\varepsilon_\star|\mathfrak{T}$ is an inner $\star$ premeasure by [10] 4.2.

2) Let $\vartheta \in \Delta$ be a maximal member of Δ. From the above 1.1 applied to $\vartheta_\star|\mathfrak{T}$ instead of ψ we obtain an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi \geqq \vartheta$ and $\varphi_\star|\mathfrak{T} = \vartheta_\star|\mathfrak{T}$. Thus $\varphi \in \Delta$ with $\varphi \geqq \vartheta$, and hence $\varphi = \vartheta$ since ϑ is maximal. $\square$

2.6 LEMMA. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$ such that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$, and let $\psi : \mathfrak{T} \to [0,\infty[$ be an inner $\star$ premeasure. If $\inf\limits_{V \in \mathfrak{T}\top\mathfrak{S}} \psi_\star(T \cap V') = 0$ for all $T \in \mathfrak{T}$, then each maximal member $\varphi : \mathfrak{S} \to [0,\infty[$ of $\Delta = \Delta(\mathfrak{S},\mathfrak{T},\psi)$ fulfils $\varphi_\star|\mathfrak{T} = \psi$.*

Proof. By definition $\varphi_\star|\mathfrak{T}$ is an inner $\star$ premeasure $\leqq \psi$, and by 2.5.2) φ is an inner $\star$ premeasure. By 2.3.2) then $\eta := \psi - (\varphi_\star|\mathfrak{T})$ is an inner $\star$ premeasure with $\psi_\star = (\varphi_\star|\mathfrak{T})_\star + \eta_\star$. Now assume that there exists $M \in \mathfrak{T}$ with $\varphi_\star(M) < \psi(M)$, that is $\eta(M) > 0$. After [6] 1.9 we form the inner $\star$ premeasure $\eta^M : \mathfrak{T} \to [0,\infty[$ defined to be $\eta^M(T) = \eta(M \cap T)$ for $T \in \mathfrak{T}$. Then $(\eta^M)_\star(A) = \eta_\star(M \cap A)$ for all $A \subset X$. Thus we have

$$\inf_{V \in \mathfrak{T}\top\mathfrak{S}} (\eta^M)_\star(V') = \inf_{V \in \mathfrak{T}\top\mathfrak{S}} \eta_\star(M \cap V') \leqq \inf_{V \in \mathfrak{T}\top\mathfrak{S}} \psi_\star(M \cap V') = 0.$$

Therefore the above 1.2 furnishes an inner $\star$ premeasure $\xi : \mathfrak{S} \to [0,\infty[$ such that $\xi_\star|\mathfrak{T} = \eta^M$. From 2.3.1) we obtain the inner $\star$ premeasure $\vartheta := \varphi + \xi$ which fulfils $\vartheta_\star = \varphi_\star + \xi_\star$. For $T \in \mathfrak{T}$ therefore

$$\begin{aligned}\vartheta_\star(T) = \varphi_\star(T) + \xi_\star(T) = \varphi_\star(T) + \eta^M(T) &= \varphi_\star(T) + \eta(M \cap T) \\ &\leqq \varphi_\star(T) + \eta(T) = \psi(T),\end{aligned}$$

that is $\vartheta_\star|\mathfrak{T} \leqq \psi$. Also $\vartheta_\star|\mathfrak{T}$ is an inner $\star$ premeasure by 2.3.1). Thus $\vartheta \in \Delta$ and $\vartheta \geqq \varphi$, and hence $\vartheta = \varphi$ since φ is maximal. It follows that $\xi = 0$ and hence $\eta^M = 0$. In particular $0 = \eta^M(M) = \eta(M)$, which is a contradiction. $\square$

Proof of theorem 2.4. i) Define $\mathfrak{s} := \mathfrak{S} \cap (\sqsubset \mathfrak{T})$ to consist of those $S \in \mathfrak{S}$ which are contained in some member of $\mathfrak{T}$. Thus $\mathfrak{s}$ is a lattice with $\varnothing$ which is upward enclosable $\mathfrak{T}$. It is obvious that $\mathfrak{T}\top\mathfrak{S} = \mathfrak{T}\top\mathfrak{s}$. Thus $\inf\limits_{V \in \mathfrak{T}\top\mathfrak{s}} \psi_\star(T \cap V') = 0$ from 2.2, and 2.6 asserts that each maximal member $\gamma : \mathfrak{s} \to [0,\infty[$ of $\Delta(\mathfrak{s},\mathfrak{T},\psi)$ is an inner $\star$ premeasure such that $\gamma_\star|\mathfrak{T} = \psi$, which by 2.1 means that $\Gamma := \gamma_\star|\mathfrak{C}(\gamma_\star)$ is an extension of Ψ.

ii) Next we assert that $\gamma_\star|\mathfrak{S} < \infty$. In fact, for $S \in \mathfrak{S}$ we have by assumption $\Psi^\star(S) < \infty$, so that there exists $A \in \mathfrak{C}(\psi_\star)$ with $S \subset A$ and $\Psi(A) < \infty$. It follows that $A \in \mathfrak{C}(\gamma_\star)$ with $\gamma_\star(S) \leqq \gamma_\star(A) = \Gamma(A) = \Psi(A) < \infty$, as claimed.

iii) Now $\mathfrak{s} \subset \mathfrak{S} \subset \mathfrak{s}\top\mathfrak{s}$. Thus from [6] 1.6 for $\bullet = \star$ applied to γ we see that $\varphi := \gamma_\star|\mathfrak{S}$ is an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ which fulfils $\varphi_\star = \gamma_\star$. Therefore $\Phi = \Gamma$, and hence Φ is an extension of Ψ. $\square$

3. CONSEQUENCES OF THE MAIN THEOREM

Our main consequence reads as follows.

3.1 THEOREM. *Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a measure on the σ algebra $\mathfrak{A}$, and let $\mathfrak{T} \subset \mathfrak{A}$ be a lattice with $\varnothing$ such that α is inner regular $\mathfrak{T}$. Assume that $\mathfrak{S}$ is a lattice with $\varnothing$ and with $\alpha^\star|\mathfrak{S} < \infty$ and $\mathfrak{T} \subset \mathfrak{S}\top\mathfrak{S}$, and such that*

for each $A \in \mathfrak{A}$ with $\alpha(A) > 0$ there exists $S \in \mathfrak{S}$ with $\alpha^\star(A \cap S) > 0$.

Then 1) α is inner regular $\mathfrak{t} := \mathfrak{T} \cap [\alpha < \infty] \subset [\alpha < \infty] \subset \mathfrak{A}$. Thus $\psi := \alpha|\mathfrak{t}$ is an inner $\star$ premeasure $\psi : \mathfrak{t} \to [0,\infty[$ such that $\Psi := \psi_\star|\mathfrak{C}(\psi_\star)$ is an extension of α. 2) There exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ such that $\Phi := \varphi_\star|\mathfrak{C}(\varphi_\star)$ is an extension of Ψ and hence of α.

Proof. i) For each $A \in \mathfrak{A}$ with $\alpha(A) > 0$ there exists $S \in \mathfrak{S}$ with $S \subset A$ and $\alpha^\star(S) > 0$. In fact, there is $T \in \mathfrak{T} \subset \mathfrak{A}$ with $T \subset A$ and $\alpha(T) > 0$, and then $S \in \mathfrak{S}$ with $\alpha^\star(T \cap S) > 0$. But $T \cap S \in \mathfrak{S}$ in view of $\mathfrak{T} \subset \mathfrak{S}\top\mathfrak{S}$, and $T \cap S \subset T \subset A$, so that the assertion follows.

ii) We claim that $\alpha(A) = \sup\{\alpha^\star(S) : S \in \mathfrak{S} \text{ with } S \subset A\}$ for all $A \in \mathfrak{A}$. In fact, assume that $A \in \mathfrak{A}$ fulfils $\alpha(A) > c := \sup\{\alpha^\star(S) : S \in \mathfrak{S} \text{ with } S \subset A\}$. We fix an increasing sequence $(S_n)_n$ in $\mathfrak{S}$ with $S_n \subset A$ and $\alpha^\star(S_n) \uparrow c$, and then a sequence $(A_n)_n$ in $\mathfrak{A}$ with $S_n \subset A_n \subset A$ and $\alpha(A_n) < \alpha^\star(S_n) + \frac{1}{n}$. Then the $D_n := \bigcap\limits_{l \geqq n} A_l$ form an increasing sequence $(D_n)_n$ in $\mathfrak{A}$ with $S_n \subset D_n \subset A_n \subset A$ and hence $D_n \uparrow D \in \mathfrak{A}$ with $D \subset A$ and $\alpha(D) = c$. Thus $A \setminus D \in \mathfrak{A}$ has $\alpha(A \setminus D) = \alpha(A) - c > 0$. We fix an $S \in \mathfrak{S}$ with $S \subset A \setminus D$ and $\alpha^\star(S) > 0$. Then the $S_n \cup S \in \mathfrak{S}$ with $S_n \cup S \subset A$ fulfil $\alpha^\star(S_n \cup S) \geqq \alpha^\star(S_n) + \alpha^\star(S)$, because each $E \in \mathfrak{A}$ with $E \supset S_n \cup S$ fulfils $E \supset (E \cap D) \cup (E \cap (A \setminus D))$ with $E \cap D \supset S_n$ and $E \cap (A \setminus D) \supset S$ and hence $\alpha(E) \geqq \alpha^\star(S_n) + \alpha^\star(S)$. It follows that $\alpha^\star(S_n \cup S) > c$ for almost all $n \in \mathbb{N}$ and hence a contradiction.

iii) From ii) we obtain $\alpha(A) = \sup\{\alpha(E) : E \in [\alpha < \infty] \text{ with } E \subset A\}$ for all $A \in \mathfrak{A}$, so that α is semifinite. It follows that α is inner regular $\mathfrak{t} := \mathfrak{T} \cap [\alpha < \infty]$ and hence 1). Moreover we obtain $\Psi^\star \leqq \alpha^\star$ and hence $\Psi^\star|\mathfrak{S} < \infty$.

iv) We have $\psi_\star = \Psi_\star$ as in the proof of 2.2. Likewise $\psi_\star = \alpha_\star$ on $\mathfrak{A} \supset \mathfrak{t}$ and hence partout, because both sides are inner regular $\mathfrak{A}$. Thus $\Psi_\star = \alpha_\star$. Now [9] 1.1 asserts for all $V \subset X$ that

$$\Psi(E) = \psi_\star(E \cap V') + \Psi^\star(E \cap V) \quad \text{for } E \in \mathfrak{C}(\psi_\star),$$
$$\alpha(E) = \psi_\star(E \cap V') + \alpha^\star(E \cap V) \quad \text{for } E \in \mathfrak{A},$$

and hence $\Psi^\star(E \cap V) = \alpha^\star(E \cap V)$ for $E \in [\alpha < \infty]$. From ii) and $\mathfrak{t} \subset \mathfrak{S}\top\mathfrak{S}$ we conclude for $T \in \mathfrak{t}$ that

$$\psi(T) = \alpha(T) = \sup\{\alpha^\star(S) : S \in \mathfrak{S} \text{ with } S \subset T\}$$
$$= \sup_{V \in \mathfrak{t}\top\mathfrak{S}} \alpha^\star(T \cap V) = \sup_{V \in \mathfrak{t}\top\mathfrak{S}} \Psi^\star(T \cap V).$$

Thus our main theorem 2.4 implies the assertion 2). $\square$

The specialization of 3.1 which follows is the principal implication in the basic theorem [2] 416P discussed above (and even a somewhat fortified version).

3.2 Consequence. *Let X be a Hausdorff topological space, and let $\alpha : \mathfrak{A} \to [0, \infty]$ be a measure on the σ algebra $\mathfrak{A}$ in X which is inner regular* $\mathrm{Cl}(X) \cap \mathfrak{A}$. *Assume that* $\alpha^\star|\mathrm{Comp}(X) < \infty$ (which is obvious when α is *locally finite* in the sense of [2] 411F(a))*, and that*

for each $A \in \mathfrak{A}$ with $\alpha(A) > 0$ there exists $S \in \mathrm{Comp}(X)$ with $\alpha^\star(A \cap S) > 0$.

Then there exists a Radon premeasure $\varphi : \mathrm{Comp}(X) \to [0, \infty[$ such that its maximal Radon measure $\Phi := \varphi_\star|\mathfrak{C}(\varphi_\star)$ is an extension of α.

4. Two Further Theorems

The present section wants to derive from the earlier paper [5] two related theorems of transplantation type, which are abstract versions of the results [2] 416K and 416L discussed above.

The basic aim of [5] was to represent certain set functions as upper envelopes of inner $\star$ premeasures. We recall a few concepts and results, as before on a fixed nonvoid set X. Let $\mathfrak{S}$ be a lattice with $\varnothing$, and let $\beta : \mathfrak{S} \to [0,\infty[$ be isotone with $\beta(\varnothing) = 0$. Define $M(\beta)$ to consist of the isotone and supermodular set functions $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi \leqq \beta$. Then [5] 3.5 asserts that 1) $M(\beta)$ is upward inductive in the argumentwise order, and 2) if β is submodular then each maximal member of $M(\beta)$ is modular.

Next let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$. For an isotone set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ one defines $\hat{\varphi} := (\varphi_\star|\mathfrak{T})^\star|\mathfrak{S}$. Thus $\hat{\varphi} : \mathfrak{S} \to [0,\infty]$ is isotone with $\hat{\varphi}(\varnothing) = 0$ as well and $\varphi \leqq \hat{\varphi}$. Now the main result [5] 3.6 reads as follows: *Assume that* $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$ *, and that* $\mathfrak{T}$ *separates* $\mathfrak{S}$ (in the usual sense). *If β as above is submodular with $\beta = \hat{\beta}$, then each maximal member $\varphi \in M(\beta)$ is an inner $\star$ premeasure with $\varphi = \hat{\varphi}$.*

4.1 Theorem. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$ such that $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$ and that $\mathfrak{T}$ separates $\mathfrak{S}$. Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a content on a ring $\mathfrak{A}$, and assume that $\mathfrak{S}$ is upward enclosable $\mathfrak{T} \cap \mathfrak{A}$. Then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ with $\Phi := \varphi_\star|\mathfrak{C}(\varphi_\star)$ such that $\Phi = \varphi \geqq \alpha$ on $\mathfrak{S} \cap \mathfrak{A}$ and $\Phi \leqq \alpha$ on $\mathfrak{T} \cap \mathfrak{A}$.*

Proof. We put $\beta := (\alpha|\mathfrak{T} \cap \mathfrak{A})^\star|\mathfrak{S}$ and $\xi := (\alpha|\mathfrak{S} \cap \mathfrak{A})_\star|\mathfrak{S}$. Thus $\beta : \mathfrak{S} \to [0,\infty[$ is isotone with $\beta(\varnothing) = 0$ and submodular, and $\xi : \mathfrak{S} \to [0,\infty[$ is isotone with $\xi(\varnothing) = 0$ and supermodular with $\xi \leqq \beta$. Hence $\xi \in M(\beta)$. From [5] 2.2 we have $\beta = \hat{\beta}$. Now let $\varphi \in M(\beta)$ be a maximal member with $\varphi \geqq \xi$. We know from the above that φ is an inner $\star$ premeasure with $\varphi = \hat{\varphi}$. And 1) on $\mathfrak{S} \cap \mathfrak{A}$ one has $\Phi = \varphi \geqq \xi = \alpha$. 2) For $S \in \mathfrak{S}$ and $A \in \mathfrak{T} \cap \mathfrak{A}$ with $S \subset A$ we have by definition $\beta(S) \leqq \alpha(A)$, so that $\beta_\star \leqq \alpha$ on $\mathfrak{T} \cap \mathfrak{A}$. Also note that $\varphi_\star \leqq \beta_\star$ from $\varphi \leqq \beta$, and $\mathfrak{T} \cap \mathfrak{A} \subset \mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp \subset \mathfrak{C}(\varphi_\star)$. On $\mathfrak{T} \cap \mathfrak{A}$ therefore $\Phi = \varphi_\star \leqq \beta_\star \leqq \alpha$. $\square$

In order to obtain the former result [2] 416K let X be a Hausdorff topological space with $\mathfrak{S} := \mathrm{Comp}(X)$ and $\mathfrak{T} := \mathrm{Op}(X)$, and assume that $\mathfrak{T} \cap \mathfrak{A}$ is an open cover of X. Then $\mathfrak{S}$ is in fact upward enclosable $\mathfrak{T} \cap \mathfrak{A}$. Thus one obtains a Radon premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ such that its maximal Radon measure $\Phi := \varphi_\star|\mathfrak{C}(\varphi_\star)$ is as required.

We turn to the second theorem of the section.

4.2 Theorem. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$ such that $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$ and $\mathfrak{S} \subset (\mathfrak{T}\top\mathfrak{T})\perp$ and that $\mathfrak{T}$ separates $\mathfrak{S}$. Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$ and*

$$\textit{subadditive}: \ \varphi(A \cup B) \leqq \varphi(A) + \varphi(B) \ \text{ for all } A, B \in \mathfrak{S}, \ \text{ and}$$
$$\textit{additive}: \ \varphi(A \cup B) = \varphi(A) + \varphi(B) \ \text{ for all } A, B \in \mathfrak{S} \text{ with } A \cap B = \varnothing.$$

Assume that $\hat{\varphi} := (\varphi_\star|\mathfrak{T})^\star|\mathfrak{S}$ is $< \infty$. Then $\hat{\varphi}$ is an inner $\star$ premeasure.

Proof. i) By [5] 2.5.1) each pair $A, B \in \mathfrak{T}$ is *coseparated* $\mathfrak{S}$, defined to mean that for each $K \in \mathfrak{S}$ with $K \subset A \cup B$ there exists a pair $P, Q \in \mathfrak{S}$ with $P \subset A$ and $Q \subset B$ such that $K \subset P \cup Q$.

ii) We form $\psi := \varphi_\star|\mathfrak{T}$, so that $\psi : \mathfrak{T} \to [0, \infty]$ isotone with $\psi(\varnothing) = 0$ and $\hat{\varphi} = \psi^\star|\mathfrak{S}$. We first claim that ψ is *submodular*. In fact, fix $U, V \in \mathfrak{T}$, and then $K, D \in \mathfrak{S}$ with $K \subset U \cup V$ and $D \subset U \cap V$. To be shown is $\varphi(K) + \varphi(D) \leqq \varphi_\star(U) + \varphi_\star(V)$. To see this note that $K \subset U \cup (V \setminus D)$ and $D' \in \mathfrak{S}\perp \subset \mathfrak{T}\top\mathfrak{T}$ and hence $V \setminus D = V \cap D' \in \mathfrak{T}$. Thus from i) we obtain $P, Q \in \mathfrak{S}$ with $P \subset U$ and $Q \subset V \setminus D$ and $K \subset P \cup Q$. It follows from the assumptions that in fact

$$\begin{aligned}\varphi(K) + \varphi(D) &\leqq \varphi(P \cup Q) + \varphi(D) \leqq \varphi(P) + \varphi(Q) + \varphi(D) \\ &= \varphi(P) + \varphi(Q \cup D) \leqq \varphi_\star(U) + \varphi_\star(V).\end{aligned}$$

iii) We next claim that ψ is *additive*. After ii) it remains to prove that $\varphi_\star(U) + \varphi_\star(V) \leqq \varphi_\star(U \cup V)$ for all $U, V \in \mathfrak{T}$ with $U \cap V = \varnothing$. But this is an obvious consequence of the assumption that φ be additive.

iv) From iii) we conclude that $\psi^\star(A \cup B) \geqq \psi^\star(A) + \psi^\star(B)$ for all $A, B \in \mathfrak{S}$ with $A \cap B = \varnothing$. In fact, there are $U, V \in \mathfrak{T}$ with $U \supset A$ and $V \supset B$ and $U \cap V = \varnothing$. For $T \in \mathfrak{T}$ with $T \supset A \cup B$ therefore

$$\psi(T) \geqq \psi(T \cap (U \cap V)) = \psi(T \cap U) + \psi(T \cap V) \geqq \psi^\star(A) + \psi^\star(B),$$

and hence the assertion.

v) Now [5] 2.3 asserts that $\hat{\varphi} = \psi^\star|\mathfrak{S} < \infty$ fulfils $\hat{\varphi}(B) \leqq \hat{\varphi}(A) + (\hat{\varphi})_\star(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$. Moreover iv) implies that

$$\hat{\varphi}(B) \geqq \hat{\varphi}(A \cup S) \geqq \hat{\varphi}(A) + \hat{\varphi}(S) \ \text{ for the } S \in \mathfrak{S} \text{ with } S \subset B \setminus A,$$

and hence $\hat{\varphi}(B) \geqq \hat{\varphi}(A) + (\hat{\varphi})_\star(B \setminus A)$. Thus $\hat{\varphi}(B) = \hat{\varphi}(A) + (\hat{\varphi})_\star(B \setminus A)$, so that $\hat{\varphi}$ is an inner $\star$ premeasure after [10] 4.2. □

In order to obtain the former result [2] 416L let X be a Hausdorff topological space with $\mathfrak{S} := \mathrm{Comp}(X)$ and $\mathfrak{T} := \mathrm{Op}(X)$. It follows that the former assumption that X be *regular* can be dispensed with.

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