Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 278

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Saarbrücken 2010

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 278
submitted: August 2, 2010

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Abstract
We prove $C^{1,\alpha}$-regularity for local minimizers of functionals

$$\int_\Omega \varphi(|\nabla u|)\,dx,$$

where φ is a Young function. In order to generalize the results of [Fu1] and [DSV], we assume for the function φ only the (Δ_2)-condition and

$$\varphi''(t) \geq \widehat{\epsilon}\,\frac{\varphi'(t)}{t}$$

for a positive constant $\widehat{\epsilon}$ (and of course a Hölder-condition for the second derivatives).

AMS Subject Classification: 49 N 60

Keywords: vector-valued problems, local minimizers, nonstandard growth, regularity

1 Introduction

We consider local minimizers $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^M$, defined on an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, of the variational integral

$$\mathcal{F}(w,\Omega) = \int_\Omega H(\nabla w)\,dx. \tag{1.1}$$

In the vectorial case, in the sixties De Giorgi, Giusti and Miranda provide counterexamples to the full regularity until K. Uhlenbeck ,[Uh], proved in 1974 everywhere regularity for a radial functional, of p-growth, $p \geq 2$. Since then a lot of generalizations have been made in the power case, in the almost linear case and for general growth. The model case is:

$$H(Z) = \varphi(|Z|), \quad Z \in \mathbb{R}^{nM}, \tag{1.2}$$

for a function $\varphi : [0,\infty) \to [0,\infty)$ being C^2 away from 0. Integrands of this particular form with essential contributions to the question of interior regularity have been studied by many authors: the case $\varphi(t) = t^p$ with $p \geq 2$ was investigated first by Uhlenbeck [Uh] and later extended by Giaquinta and Modica [GM], the subquadratic case by [AF]. For a more complete list of references see the survey [Min]. Marcellini first investigate the so-called (p, q)-growth conditions, i.e.,

$$c|Z|^{p-2}|P|^2 \le D^2H(Z)(P,P) \le C(1+|Z|^{q-2})|P|^2$$

for $Z, P \in \mathbb{R}^{nM}$, constants $c, C > 0$ and exponents $1 < p \le q < \infty$. On account of counterexamples of Giaquinta [Gi] and Hong [Ho] we know that there is no hope for regularity if p and q are to far apart. Note that the best known bound in the (p, q)-growth situation is

$$q < p + 2,$$

which is due to the work of Esposito, Leonetti and Mingione [ELM] as well as Bildhauer and Fuchs [BF] and they assume that the minimizer is locally bounded.

More general functions φ are the subject of Marcellini's work (see [Ma1]-[Ma3] and also [MP]).

To get a model, which is more flexible, additional assumptions are necessary. This leads to variational problems with φ-growth, where φ is a Young function (compare [Ad]). This means we have to assume that φ is strictly increasing and convex together with

$$\lim_{t\to 0}\frac{\varphi(t)}{t} = \lim_{t\to\infty}\frac{t}{\varphi(t)} = 0.$$

Minimizers of (1.1) under the assumption (1.2), where φ is a Young-function, were intensively studied by Fuchs [Fu1], [Fu2] and Marcellini, Marcellini-Papi and Diening, Stroffolini and Verde [DSV] (where one can find additional references). In a comparison of this works, one notice that in all cases the authors assume a (Δ_2)-condition for the function φ, i.e.,

$$\varphi(2t) \le c\varphi(t) \quad \text{for all } t \ge 0. \tag{A1}$$

Note that in [DSV] (A1) is not directly supposed, but follows from $\varphi''(t) \le \widehat{h}\frac{\varphi'(t)}{t}$. This excludes some extreme cases like exponential growth (we remark that this situation is studied in [MP]) and guarantees that the Orlicz class $K_\varphi(\Omega;\mathbb{R}^M)$ and the Orlicz space $L_\varphi(\Omega;\mathbb{R}^M)$ coincide (see [Ad], Chapter VIII, for notation). Moreover, we can introduce the Orlicz-Sobolev space $\mathring{W}^{1,\varphi}(\Omega;\mathbb{R}^M)$ of functions with zero traces in the usual way. Furthermore they suppose a (∇_2)-condition, which is defined as the (Δ_2)-condition for the conjugate Young function

$$\varphi^*(t) := \sup_{s\ge 0}[st - \varphi(s)],$$

which is a Young function itself. As shown in [RR], if $\varphi \in C^2(0,\infty)$, the (∇_2)-condition follows from

$$\varphi''(t) \geq \widehat{\epsilon}\frac{\varphi'(t)}{t} \quad \text{for all } t \geq 0 \tag{1.3}$$

for a constant $\widehat{\epsilon} > 0$. (1.3) is supposed explicitly in [Fu1] and [DSV] (note that Fuchs [Fu1] works under the more restrictive assumption $\varphi''(t) \geq \frac{\varphi'(t)}{t}$ but notes in Remark 3.1 that (1.3) is enough to prove his result). Both, (Δ_2)- and (∇_2)-condition, together are equivalent to the reflexivity of the corresponding Orlicz-Sobolev space $W^{1,\varphi}(\Omega;\mathbb{R}^N)$ (compare [Ad]), hence it seems to be quite natural to assume $(\Delta_2) \cap (\nabla_2)$ for the function φ. This corresponds to the assumption $1 < p < \infty$ in the power-growth situation. On the other hand the (∇_2)-condition excludes almost linear growth, for example

$$\varphi(t) = t\log(1+t).$$

But this situation is already intensively studied in [FS], [EM], [FM] and [MS].
Let us have a look at the differences of the papers [Fu1] and [DSV]: As mentioned above both assume a (Δ_2)-condition and (1.3), but in addition Fuchs supposes that φ is C^2 up to 0 and

$$\varphi''(0) > 0. \tag{1.4}$$

(1.4) implies that the minimization problem is non-degenerated at the origin. This is quite restrictive because simple examples like

$$\varphi(t) = t^p$$

for $1 < p < 2$ or

$$\varphi(t) = t^p\log(1+t)$$

are excluded. In the approach of [DSV] this examples can be considered . But on the other hand in this work the condition

$$\varphi''(t) \leq \widehat{h}\frac{\varphi'(t)}{t} \quad \text{for all } t > 0 \tag{1.5}$$

is supposed. This is restrictive, too, which can be easily seen by consideration of the example taken from Marcellini-Papi [MP]:

$$\varphi(t) = \begin{cases} t^p & , \quad t \leq \tau_0, \\ t^{\frac{p+q}{2}+\frac{q-p}{2}\sin\log\log\log t}, & t > \tau_0; \end{cases}$$

where τ_0 is such that $\sin\log\log\log\tau_0 = -1$. Hence the aim of this paper is to prove full $C^{1,\alpha}$-regularity only under the assumption $\varphi \in (\Delta_2)$ and assumption (1.3) and to get rid of (1.4) and (1.5).
Let us formulate our regularity result.

THEOREM 1.1 *Let $u \in W^{1,\varphi}_{loc}(\Omega;\mathbb{R}^M)$ be a local minimizer of (1.1) under the assumption (1.2) and $\varphi \in C^1[0,\infty) \cap C^2(0,\infty)$ a Young function satisfying (A1) and for an exponent $\omega \geq 0$ and constants $\widehat{\epsilon}, a > 0$*

$$\widehat{\epsilon}\frac{\varphi'(t)}{t} \leq \varphi''(t) \leq a(1+t^2)^{\frac{\omega}{2}}\frac{\varphi'(t)}{t}. \tag{A2}$$

a) Then ∇u is locally bounded.

b) If we suppose, in addition, the existence of $\beta \in (0,1]$ such that

$$|\varphi''(s+t) - \varphi''(t)| \leq c\varphi''(t)\left(\frac{|s|}{t}\right)^{\beta} \tag{A3}$$

for all $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$, then there exists $\sigma > 0$ such that $u \in C^{1,\sigma}(\Omega;\mathbb{R}^M)$.

REMARK 1.1 *a) Note that from the first inequality in (A2) one can follow the (∇_2)-condition of φ, compare [Ad]. Whereas the second inequality is always fulfilled, since we have no upper bound for the value of ω.*

b) Condition (A3) shows that φ'' is locally Hölder continuous off the diagonal. Such a condition appears in every full regularity theorem for vector valued functions (compare [GM]).

c) Theorem 1.1 shows that the only assumptions which are necessary for full regularity of functionals with φ-growth are the (Δ_2)-condition and $\varphi''(t) \geq \widehat{\epsilon}\frac{\varphi'(t)}{t}$. This improves significantly the results of [Fu1] and [DSV].

Let us give some comments about properties of the function φ which follows from our assumptions.

REMARK 1.2 *a) From (A1) follows the existence of an exponent $q > 1$ such that*

$$\varphi(t) \leq ct^q \quad \textit{for } t \geq 1. \tag{1.6}$$

b) From the first inequality in (A2) we can deduce that for $p = 1 + \widehat{\epsilon}$

$$\varphi(t) \geq ct^p \quad \textit{for } t \geq 1. \tag{1.7}$$

b) By convexity and (Δ_2)-condition we obtain

$$\varphi'(t)t \approx \varphi(t). \tag{1.8}$$

2 Lipschitz regularity

The first step is to regularize the problem. Since our problem is degenerated at the origin we can not use the standard regularization (compare, for example, [BF] and the references therein) as done in [Fu1]. In order to overcome this problem we work with an approximation, which was introduced in [DE] and also used in [DSV]. Firstly we define for $\lambda > 0$ the shifted Young function $\varphi_\lambda(t) := \int_0^t \varphi'_\lambda(s)\,ds$ with

$$\varphi'_\lambda(t) := \frac{\varphi'(\lambda + t)}{\lambda + t} t \tag{2.1}$$

for $t \geq 0$. For the function φ_λ we obtain the following properties.

LEMMA 2.1 *Let φ be a Young function with the properties (A1) and (A2) and φ_λ, $0 < \lambda \leq 1$, defined as in (2.1). Then we have*

a) *φ_λ fulfills a uniform (Δ_2)-condition, i.e.,*

$$\varphi_\lambda(2t) \leq c\varphi_\lambda(t), \quad t \geq 0,$$

where c does not depend on λ.

b) *assumption (A2) extends to φ_λ uniformly, i.e.,*

$$\widehat{\epsilon}\frac{\varphi'_\lambda(t)}{t} \leq \varphi''_\lambda(t) \leq a(1 + t^2)^{\frac{\omega}{2}} \frac{\varphi'_\lambda(t)}{t},$$

where $\widehat{\epsilon}, a$ do not depend on λ.

c) *$\varphi_\lambda \in C^2[0, \infty)$ and*

$$\varphi''_\lambda(0) > 0.$$

d) *(1.6) and (1.7) extend uniformly to φ_λ, i.e.*

$$c_1 t^p \leq \varphi_\lambda(t) \leq c_2 t^q, \quad t \geq 1,$$

where $c_1, c_2 > 0$ and $1 < p \leq q < \infty$ are all independent of λ.

e) *(1.8) extends uniformly to φ_λ, i.e.*

$$c_1 \varphi'_\lambda(t) t \leq \varphi_\lambda(t) \leq \varphi'_\lambda(t) t$$

where $c_1 > 0$ is independent of λ.

f) for $t \geq 1$ we have

$$c_1\varphi_\lambda(t) \leq \varphi(t) \leq c_2\varphi_\lambda(t),$$

where c_1, c_2 are positive constants independent of λ.

Proof: For the first statement we use (A1) and (1.8) to get

$$\begin{aligned}\varphi_\lambda(2t) &= \int_0^{2t} \frac{\varphi'(\lambda+s)}{\lambda+s} s\,ds = 4\int_0^t \frac{\varphi'(\lambda+2s)}{\lambda+2s} s\,ds \leq c\int_0^t \frac{\varphi(\lambda+2s)}{(\lambda+2s)^2} s\,ds \\ &\leq c\int_0^t \frac{\varphi(\lambda+s)}{(\lambda+2s)^2} s\,ds \leq c\int_0^t \frac{\varphi(\lambda+s)}{(\lambda+s)^2} s\,ds \leq c\int_0^t \frac{\varphi'(\lambda+s)}{(\lambda+s)} s\,ds \\ &= c\varphi_\lambda(t).\end{aligned}$$

Hence the first inequality in e) follows, whereas the second one as a consequence of the convexity of φ_λ. We have a look at b):

$$\begin{aligned}\frac{\varphi_\lambda'(t)}{t} &= \frac{\varphi'(\lambda+t)}{\lambda+t}; \\ \varphi_\lambda''(t) &= \frac{\varphi'(\lambda+t)}{\lambda+t} + \frac{\varphi''(\lambda+t)(\lambda+t) - \varphi'(\lambda+t)}{(\lambda+t)^2} t.\end{aligned}$$

Hence we obtain by the first inequality in (A2)

$$\begin{aligned}\varphi_\lambda''(t) &\geq \frac{\varphi'(\lambda+t)(\lambda+t) + (\widehat{\epsilon}-1)\varphi'(\lambda+t)t}{(\lambda+t)^2} \\ &= (\lambda+\widehat{\epsilon}t)\frac{\varphi'(\lambda+t)}{(\lambda+t)^2} \geq \min\{1,\widehat{\epsilon}\}\frac{\varphi'(\lambda+t)}{(\lambda+t)} \\ &= \min\{1,\widehat{\epsilon}\}\frac{\varphi_\lambda'(t)}{t}.\end{aligned}$$

The second inequality in b) is a consequence of

$$\begin{aligned}\frac{\varphi''(\lambda+t)(\lambda+t) - \varphi'(\lambda+t)}{(\lambda+t)^2} t &\leq \varphi''(\lambda+t) \leq c(1+(\lambda+t)^2)^{\frac{\omega}{2}} \frac{\varphi'(\lambda+t)}{\lambda+t} \\ &\leq c(1+t^2)^{\frac{\omega}{2}} \frac{\varphi_\lambda'(t)}{t},\end{aligned}$$

where we used (A2). Finally we receive

$$\varphi_\lambda''(0) = \frac{\varphi'(\lambda)}{\lambda} > 0.$$

The first inequality in the last statement follows easily by direct calculations using the Δ_2-property of φ. For the second one we calculate by convexity of φ

$$\varphi_\lambda(t) \geq \int_{\frac{t}{2}}^t \frac{\varphi'(\lambda+s)}{\lambda+s} s\,ds \geq \varphi'\left(\frac{t}{2}\right)\frac{t}{2}\int_{\frac{t}{2}}^t \frac{1}{\lambda+s}\,ds \geq c\varphi(t),$$

using (Δ_2) and (1.8). By (1.6) and (1.7) we can deduce d) from the last statement. Now the proof is complete. $\square$

With this preparations we are able to define a useful regularization: we define for $H_\lambda(Z) := \varphi_\lambda(|Z|)$ the functional

$$\mathcal{F}_\lambda(u, B) := \int_B H_\lambda(\nabla u_\delta)\, dx$$

for $B \Subset \Omega$ and u_λ as the unique minimizer of $\mathcal{F}_\lambda(\cdot, B)$ in $u + \overset{\circ}{W}{}^{1,\varphi}(B, \mathbb{R}^M)$ (note that on account of Lemma 2.1 f) the spaces $W^{1,\varphi}(B, \mathbb{R}^M)$ and $W^{1,\varphi_\lambda}(B, \mathbb{R}^M)$ coincide). For u_λ we obtain:

LEMMA 2.2

- *u_λ is uniformly bounded in $W^{1,\varphi}(B, \mathbb{R}^M)$ and $\sup_\lambda \mathcal{F}_\lambda(u_\lambda, B) < \infty$.*
- *$\nabla u_\lambda \in L^\infty_{loc}(\Omega, \mathbb{R}^{nM})$ and $\Psi_{\lambda,s} \in W^{1,2}_{loc}(B)$ for every $s < \infty$, where*

$$\Psi_{\lambda,s} := \int_0^{|\nabla u_\lambda|} \left(\frac{\varphi_\lambda'(t)}{t} t^s \right)^{\frac{1}{2}} dt.$$

Proof: The first statement follows from

$$\mathcal{F}_\lambda(u_\lambda, B) \le \mathcal{F}_\lambda(u, B)$$

and Lemma 2.1 f) (note that $\varphi_\lambda(t) \le \varphi(2)$ for $t \le 1$). The second one is a consequence of [Fu1], Theorem 1.1, Remark 3.1 and (2.3), since all conditions mentioned there are fulfilled by Lemma 2.1, especially the problem is non-degenerated at the origin on account of Lemma 2.1 c). $\square$

LEMMA 2.3 *Under the assumptions of Theorem 1.1 a) ∇u_λ is uniformly bounded in $L^t_{loc}(B, \mathbb{R}^{nM})$ for every $t < \infty$.*

Proof: We follow the main ideas of [Fu1]. Let $\eta \in C_0^\infty(B)$ be a cut-off function and $s_k \ge 0$, we define $\Gamma_\lambda := 1 + |\nabla u_\lambda|^2$ and

$$\Psi_{\lambda,k} := \int_0^{|\nabla u_\lambda|} \left(\frac{\varphi_\lambda'(t)}{t} t^{s_k} \right)^{\frac{1}{2}} dt,$$

where we will specify s_k later. From [Fu1], equation (2.3), we quote

$$\int_B \eta^2 |\nabla \Psi_{\lambda,k}|^2\, dx \le c(\eta) \int_{supp(\eta)} \varphi_\lambda(|\nabla u_\lambda|) \Gamma_\lambda^{\frac{s_k}{2}}\, dx. \qquad (2.2)$$

By Lemma 2.1 e), straightforward calculations show

$$\sqrt{\varphi_\lambda(t)t^s} \approx \int_0^t \sqrt{\frac{\varphi'_\lambda(\tau)}{\tau}\tau^s}\, d\tau, \tag{2.3}$$

hence

$$\int_B \eta^2 \left(|\Psi_{\lambda,k}|^2 + |\nabla \Psi_{\lambda,k}|^2\right)\, dx \leq c(\eta) \int_{supp(\eta)} \varphi_\lambda(|\nabla u_\lambda|)\Gamma_\lambda^{\frac{s_k}{2}}\, dx. \tag{2.4}$$

Since $\varphi_\lambda(t) \geq ct$ for $t \geq 1$ (compare Lemma 2.1 d)) we have with the help of (2.3) and Lemma 2.1 f)

$$\begin{aligned}\varphi_\lambda(|\nabla u_\lambda|)|\nabla u_\lambda|^{\frac{2}{n-2}}|\nabla u_\lambda|^{s_k\frac{n}{n-2}} &\leq c\left(\varphi_\lambda(|\nabla u_\lambda|)^{\frac{n}{n-2}}|\nabla u_\lambda|^{s_k\frac{n}{n-2}} + 1\right)\\ &\leq c\left(\Psi_{\lambda,k}^{\frac{2n}{n-2}} + 1\right).\end{aligned}$$

Plugging all together we have (note that $c(\eta)$ does not depend on λ)

$$\int_B \eta^2 \left(|\Psi_{\lambda,k+1}|^2 + |\nabla \Psi_{\lambda,k+1}|^2\right)\, dx \leq c(\eta)\left\{\int_{supp(\eta)} \Psi_{\lambda,k}^{\frac{2n}{n-2}}\, dx + 1\right\} \tag{2.5}$$

$$s_{k+1} := \frac{2}{n-2} + s_k\frac{n}{n-2}. \tag{2.6}$$

We start the iteration with $s_0 := 0$, $s_1 := \frac{2}{n-2}$. Using (2.3) and (2.4) together with Lemma 2.2 we can deduce uniform $W^{1,2}_{loc}$-bounds on $\Psi_{\lambda,0}$. By (2.5) it follows iteratively

$$\Psi_{\lambda,k} \in W^{1,2}_{loc}(B) \text{ uniform for all } k \in \mathbb{N}. \tag{2.7}$$

One easily checks that $s_k \to \infty$, $k \to \infty$, hence the claim of Lemma 2.3 is a consequence of (2.3) and Lemma 2.1 d). □

LEMMA 2.4 *Under the assumptions of Theorem 1.1 a) ∇u_λ is uniformly bounded in $L^\infty_{loc}(B, \mathbb{R}^{nM})$.*

Proof: We consider for $\widetilde{q} := q + \omega$ (q and ω from Lemma 2.1) the function

$$\tau(k,r) := \int_{A(k,r)} \Gamma_\lambda^{\frac{\widetilde{q}-2}{2}}(\omega_\lambda - k)^2\, dx$$

where we abbreviated $\omega_\lambda := \log \Gamma_\lambda$ and $A(k,r) := B_r \cap [|\nabla u_\lambda| > k]$. We want to show

$$\tau(h,r) \leq \frac{c}{(\widehat{r} - r)^\gamma (h-k)^\theta}\tau(k,\widehat{r})^\mu \tag{2.8}$$

for $0 < h < k$, $0 < r < \widehat{r} < R_0$ with exponents $\gamma, \theta > 0$ and $\mu > 1$ (and, of course, c independent of λ). From (2.8) we arrive at uniform L^∞_{loc}-bounds on ∇u_λ using Stampacchia's Lemma ([St], Lemma 5.1, p. 219), details are given in [Bi]. Note that uniform bounds for τ (which are necessary) follow from Lemma 2.3.
Combining Lemma 2.1 b), d) and e) we conclude

$$c(1+|Z|^2)^{\frac{p-2}{2}}|P|^2 \le D^2 H_\lambda(Z)(P,P) \le C(1+|Z|^2)^{\frac{\tilde{q}-2}{2}}|P|^2 \quad \text{for } |Z| \ge 1,$$

where $c, C > 0$ do not depend on λ. If we consider only the situation $|\nabla u_\lambda| \ge 1$ (which is no restriction) we can apply the arguments from [Bi], Thm. 5.22, to calculate τ and obtain as mentioned there (2.8). Note that Bildhauer consider in [Bi], Thm. 5.22, the case $M = 1$, but due to the structure condition (1.2) we can use [Bi] ((32), p. 62) and procedure in the same way. Hence the claim of Lemma 2.4 follows. $\square$

The next step is to transfer the result to the minimizer u of (1.1). To do this, on account of Lemma 2.3 and 2.4, it is enough to show

LEMMA 2.5 *Under the assumptions of Theorem 1.1 a) we have the convergence*

$$u_\lambda \rightharpoonup u \text{ in } W^{1,\varphi}(B, \mathbb{R}^M), \quad \lambda \to 0.$$

Proof: From Lemma 2.2 a) we deduce (at least for a subsequence, note that due to our assumptions $W^{1,\varphi}(B, \mathbb{R}^M)$ is reflexive)

$$u_\lambda \rightharpoonup: v \text{ in } W^{1,\varphi}(B, \mathbb{R}^M), \quad \lambda \to 0. \tag{2.9}$$

We have to show that v minimizes the original functional. Considerering the isomorphism

$$W_\lambda : \mathbb{R}^{nM} \to \mathbb{R}^{nM}, \xi \mapsto \int_0^{|\xi|} \sqrt{\frac{\varphi'_\lambda(t)}{t}}\, dt$$

we obtain from the proof of Lemma 2.3 uniform $W^{1,2}_{loc}$-bounds on $W_\lambda(\nabla u_\lambda)$. This means

$$W_\lambda(\nabla u_\lambda) \rightharpoonup: H \text{ in } W^{1,2}_{loc}(B, \mathbb{R}^{nM}),$$
$$W_\lambda(\nabla u_\lambda) \to: H \text{ in } L^2_{loc}(B, \mathbb{R}^{nM})$$

for subsequences. Since $W_\lambda \to W_0$ uniformly we have $\nabla u_\lambda \to W_0^{-1}(H)$ almost everywhere, thus $W_0^{-1}(H) = \nabla v$ by (2.9). Now we know $\varphi_\lambda(|\nabla u_\lambda|) \to \varphi(|\nabla v|)$ almost everwhere, hence using Fatou's Lemma

$$\int_B \varphi(|\nabla v|)\, dx \le \liminf_\lambda \int_B \varphi_\lambda(|\nabla u_\lambda|)\, dx \le \liminf_\lambda \int_B \varphi_\lambda(|\nabla u|)\, dx$$

by definition of u_λ. On account of dominated convergence (compare Lemma 2.1 f)) we can change limes and integral and obtain

$$\int_B \varphi(|\nabla v|)\,dx \le \int_B \varphi(|\nabla u|)\,dx,$$

which give us the claim due to uniqueness. $\square$

We note that the arguments from Lemma 2.4 are note necessary for the proof of Lemma 2.5, it is enough to have uniform $W^{1,2}_{loc}$-bounds on $W_\lambda(\nabla u_\lambda)$.

3 Full regularity

In this section we prove that $u \in C^{1,\sigma}(\Omega, \mathbb{R}^M)$ for a $\sigma > 0$. The main idea is the construction in the following lemma, which is motivated by [MS].

LEMMA 3.1 *Let $\varphi \in C^1[0,\infty) \cap C^2(0,\infty)$ a Young function satisfying (A1), (A2) and (A3) and $\Theta \ge 1$. Then there is a Young function $\varphi_\Theta \in C^1[0,\infty) \cap C^2(0,\infty)$ such that*

$$\varphi_\Theta(t) = \varphi(t) \quad \textit{for} \quad t \le \Theta$$

and

$$c(\Theta)\frac{\varphi'_\Theta(t)}{t} \le \varphi''_\Theta(t) \le C(\Theta)\frac{\varphi'_\Theta(t)}{t}, \tag{3.1}$$

$$|\varphi''_\Theta(s+t) - \varphi''_\Theta(t)| \le C(\Theta)\varphi''_\Theta(t)\left(\frac{|s|}{t}\right)^\beta \tag{3.2}$$

for all $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$.

The result of the last section is

$$\|\nabla u\|_{\infty,B} \le K$$

on a ball $B \Subset \Omega$ for a positive constant K. If we choose $\Theta > K$, then u is a minimizer of the functional

$$\mathcal{F}_\Theta(w,B) := \int_B \varphi_\Theta(|\nabla w|)\,dx,$$

since it is a solution of the Euler equation and $\mathcal{F}_\Theta$ is a convex functional. Due to the r.h.s. of (3.1) in Lemma 3.1 all assumptions of [DSV] are fullfiled.

This means we can quote their result and obtain the claim of Theorem 1.1 b).

Proof of Lemma 3.1: We define

$$\begin{aligned} \varphi_\Theta(t) &:= \varphi(t) + \psi_\Theta(t), \\ \psi_\Theta(t) &:= \max\{t-\Theta, 0\}^{q+\omega}, \end{aligned}$$

where q and ω are defined in (1.6) resp. (A2). Assuming $q+\omega > 3$ we reach that ψ_Θ is a C^3-function, hence $\varphi_\Theta \in C^1[0,\infty) \cap C^2(0,\infty)$.
Let us start with inequality (3.1).
For $t < \Theta$ we have $\psi_\Theta = 0$ and by (A2)

$$\varphi''(t) \le a(1+t^2)^{\frac{\omega}{2}} \frac{\varphi'(t)}{t} \le a(1+\Theta^2)^{\frac{\omega}{2}} \frac{\varphi'(t)}{t}.$$

For $\Theta \le t \le 2\Theta$ we first observe that

$$\inf_{\Theta \le t \le 2\Theta} \varphi''(t) \ge c \inf_{\Theta \le t \le 2\Theta} \frac{\varphi'(t)}{t} \ge c\frac{\varphi'(\Theta)}{2\Theta} = C(\Theta) > 0$$

and so

$$\varphi''_\Theta(t) = \varphi''(t) + c(t-\Theta)^{q+\omega-2} \le c(1+4\Theta^2)^{\frac{\omega}{2}} \frac{\varphi'(t)}{t} + \Theta^{q+\omega-2} \frac{\varphi'(t)}{t} \frac{t}{\varphi'(t)} \le C(\Theta) \frac{\varphi'(t)}{t}.$$

Moreover we obtain

$$\frac{\psi'_\Theta(t)}{t} \le c \le c\varphi''(t)$$

and, as a consequence,

$$c(\Theta)\frac{\varphi'_\Theta(t)}{t} \le \varphi''_\Theta(t).$$

Last, if $t > 2\Theta$

$$\varphi''_\Theta(t) \le c(1+t)^\omega \frac{\varphi'(t)}{t} + c(t-\Theta)^{q+\omega-2} \le c(t-\Theta)^{q+\omega-2} \le c\frac{(t-\Theta)^{q+\omega-1}}{t}\frac{t}{t-\Theta}$$

Now, $\frac{t}{t-\Theta} \to 1$ as $t \to \infty$ therefore fix $\varepsilon = \frac{1}{2}$, there exists T_2 such that fot $t > T_2$ we have $\frac{1}{2} < \frac{t}{t-\Theta} < \frac{3}{2}$ and obviously for $2\Theta < t < T_2$ we have $C_1(\Theta) < \frac{t}{t-\Theta} < C_2(\Theta)$ we conclude (3.1) also in this case.
We now prove (3.2). It is sufficient to prove the following inequality:

$$|\psi''_\Theta(s+t) - \psi''_\Theta(t)| \le C(\Theta)\varphi''_\theta(t)\left(\frac{|s|}{t}\right)^\beta \text{ for } |s| < \frac{1}{2}t$$

In the first case $t < \Theta, s + t > \Theta$ we compute (see (1.7) and (1.8))

$$\begin{aligned}\psi''_\Theta(s+t) &= c(s+t-\Theta)^{q+\omega-2} \le cs(s+t-\Theta)^{q+\omega-3} \le c\frac{|s|}{t}t^{q+\omega-2} \\ &\le c\left(\frac{|s|}{t}\right)^\beta t^{p-2}C(\Theta) \le C\left(\frac{|s|}{t}\right)^\beta \frac{\varphi'(t)}{t} \le c\left(\frac{|s|}{t}\right)^\beta \varphi''(t).\end{aligned}$$

Now we suppose $\Theta \le t \le 2\Theta$: we have for all $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$

$$\begin{aligned}|\psi''_\Theta(s+t) - \psi''_\Theta(t)| &\le \sup_{\frac{\Theta}{2}\le w\le 3\Theta} \psi'''_\Theta(w)\,|s| \le C(\Theta)\left(\frac{|s|}{t}\right)^\beta \\ &\le C(\Theta)\varphi''(t)\left(\frac{|s|}{t}\right)^\beta,\end{aligned}$$

since $\inf_{\Theta\le t\le 2\Theta}\varphi''(t) > 0$. Hence

$$|\varphi''_\Theta(s+t) - \varphi''_\Theta(t)| \le C(\Theta)\varphi''_\Theta(t)\left(\frac{|s|}{t}\right)^\beta.$$

In the last situation $t \ge 2\Theta$ we clearly have

$$|\psi''_\Theta(s+t) - \psi''_\Theta(t)| \le c\psi''_\Theta(t)\left(\frac{|s|}{t}\right)^\beta,$$

since $\psi_\Theta(t) \approx t^{q+\omega}$, hence by (A3)

$$|\varphi''_\Theta(s+t) - \varphi''_\Theta(t)| \le c\varphi''_\Theta(t)\left(\frac{|s|}{t}\right)^\beta.$$

□

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