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Spherical contractions and interpolation problems on the unit ball

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Abstract

In this note fractional representations of multipliers on vector-valued functional Hilbert spaces are used to give a proof of Arveson's version of von Neumann's inequality for n-contractions on the unit ball. We prove a commutant lifting theorem for operators on the classical Hardy space over the unit ball in $\mathbb{C}^n$. As applications we obtain interpolation results for functions in the Schur class, we deduce a Toeplitz corona theorem on the unit ball, and we give a simplified definition of Arveson's curvature invariant for n-contractions with finite-dimensional defect space. In the final part we describe a solution of the operator-valued Nevanlinna-Pick problem with uniform bounds on uniqueness sets in the unit ball.*

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A commuting tuple $T \in L(H)^n$ of bounded linear operators on a complex Hilbert space is called a spherical contraction if $\sum_{i=1}^n T_i^* T_i \leq 1_H$. It is well known that the direct analogue of von Neumann's inequality for spherical contractions fails in dimension $n > 1$. It was shown by Drury [18] and Arveson [7] that, for any integer $n > 1$, there is a spherical contraction $T \in L(H)^n$ and a sequence of polynomials $p_k (k \geq 1)$ in the unit ball of $H^\infty(\mathbb{B})$ such that the sequence $(\|p_k(T)\|)_{k \in \mathbb{N}}$ is unbounded. Indeed one can choose T as the adjoint of the standard n-shift studied by Arveson in [7].
The supremum norm of a polynomial p on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ can be regarded as the norm of p as an element in the multiplier space $H^\infty(\mathbb{B})$ of the classical Hardy space $H^2(\mathbb{B})$ on the unit ball. In [18] Drury shows that von Neumann's inequality remains true if the supremum norm is replaced by the multiplier norm of p on the reproducing kernel Hilbert space $H(\mathbb{B})$ given by the positive definite kernel $K : \mathbb{B} \times \mathbb{B} \to \mathbb{C}$, $K(z, w) = (1 - \langle z, w \rangle)^{-1}$. More precisely, the inequality $\|p(T)\| \leq \|p(S)\| = \|p(S^*)\|$ holds for all polynomials p in n variables if S denotes the standard n-shift, i.e. the tuple consisting of the multiplication operators with the coordinate functions on $H(\mathbb{B})$.
According to Arveson a commuting n-tuple $T = (T_1, \ldots, T_n) \in L(H)^n$ is called an n-contraction if its adjoint $T^* = (T_1^*, \ldots, T_n^*)$ is a spherical contraction. Let $\mathcal{A} \subset L(H(\mathbb{B}))$ be the algebra of all polynomials in $S_1, \ldots, S_n$. In [7] Arveson extends the above von Neumann inequality by proving that, for each n-contraction $T \in L(H)^n$, the map

$$\mathcal{A} \to L(H), \quad p(S) \mapsto p(T)$$

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defines a completely contractive representation of $\mathcal{A}$. In the papers cited above the proof of von Neumann's inequality is based on dilation results for contractions on the Euclidean unit ball which can be seen as generalizations of the Sz.-Nagy-Foias dilation theory [34] to the multivariable case.

Let $\mathcal{E}$ be a given Hilbert space. In the present paper we use fractional representations of multipliers on the vector-valued functional Hilbert space $H(\mathcal{E}) = H(\mathbb{B}) \otimes \mathcal{E}$ to give a direct proof of von Neumann's inequality. We prove a commutant lifting theorem for operators on the classical Hardy space $H^2(\mathbb{B}, \mathcal{E})$ over the unit ball. We apply our results to prove interpolation results for functions in the Schur class, to deduce a Toeplitz corona theorem on the ball, and to give a simplified definition of Arveson's curvature invariant for n-contractions with finite-dimensional defect space. In the last part we indicate a solution of the operator-valued Nevanlinna-Pick problem with uniform norms on uniqueness subsets of $\mathbb{B}$.

In Section 1 we define, for given Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, the Schur norm on the space of all analytic $L(\mathcal{E}, \mathcal{E}_*)$)-valued functions on $\mathbb{B}$ as the smallest norm such that the vector-valued analytic functional calculus map $\mathcal{O}(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*)) \to L(H \otimes \mathcal{E}, H \otimes \mathcal{E}_*)$ is contractive for all strict spherical contrations $T \in L(H)^n$. The space $\mathcal{S}(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*))$ of all functions $f \in \mathcal{O}(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*))$ with finite Schur norm is in a natural way a Banach space. Replacing the class of spherical contractions by the class of their adjoints, one obtains the dual Schur norm. Using realizations of multipliers as fractional transforms of suitable unitary operator matrices we show that the resulting Banach space $\mathcal{S}^*(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*))$ coincides isometrically with the multiplier space

$$M(\mathcal{E}, \mathcal{E}_*) = \{f \in \mathcal{O}(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*));\ fH(\mathcal{E}) \subset H(\mathcal{E}_*)\}.$$

The choice $\mathcal{E} = \mathcal{E}_* = \mathbb{C}^n$ leads to a new proof of Arveson's version of the von Neumann inequality for n-contractions. The above results on the Schur class extend corresponding observations of Agler [1], Ball, Li, Timotin and Trent [12], and others, from the case of the unit polydisc to the unit ball.

Let $M \subset H^2(\mathbb{B}, \mathcal{E})$ and $M_* \subset H^2(\mathbb{B}, \mathcal{E}_*)$ be closed linear subspaces of the vector-valued Hardy spaces on $\mathbb{B}$ that are invariant under the adjoints of the n-tuple given by multiplication with the coordinate functions. In Section 2 we give a characterization of those bounded operators $X : M \to M_*$ intertwining the compressions of M_z onto M and M_*, respectively, that possess a lifting to an analytic Toeplitz operator $T_\varphi : H^2(\mathcal{E}) \to H^2(\mathcal{E}_*)$ with a dual Schur-class symbol φ. In the single variable case our result reduces to a standard version of the commutant lifting theorem. A corresponding commutant lifting theorem on the unit polydisc can be found in [12]. A com-

mutant lifting theorem for operators over the space $H(\mathbb{B})$ can be deduced from non-commutative results of Popescu (cf. [6]) by symmetrization.
In Section 3 we apply our results to prove interpolation results of Nevanlinna-Pick type for functions in the Schur class. We use the abstract interpolation results proved in Section 1 to deduce a criterion for the solvability of the corona problem within the Schur class. We apply the results obtained in Section 2 for the classical Hardy space on $\mathbb{B}$ to give a simplified definition of the curvature invariant for n-contractions with a finite-dimensional defect space which was introduced by Arveson in [8].
It is well known that the classical Nevanlinna-Pick theorem for bounded analytic functions on the unit disc admits no direct generalization to the multivariable case. In Section 4 we extend an idea of Koranyi and Pukanszki [23] for interpolation by analytic functions with prescribed values on uniqueness sets in the polydisc to the case of vector-valued bounded analytic functions on the unit ball. This extends at the same time corresponding results of Szafraniec [33] and Beatrous and Burbea [13] to the operator-valued case.

After submitting this paper we learnt that a fractional representation of multipliers in $M(\mathcal{E}, \mathcal{E}_*)$ was independently obtained by J.A.Ball, T.T.Trent and V.Vinnikov in [11]. We are gratefull to the referee for valuable comments on the first version of this paper.

0 Preliminaries

Let H be a complex Hilbert space and let $L(H)$ be the algebra of all bounded linear operators on H. A commuting tuple $T = (T_1, \ldots, T_n) \in L(H)^n$ is a spherical contraction if $\sum_{i=1}^n T_i^* T_i \leq 1_H$. Following Arveson [7] we call T an n-contraction if the adjoint tuple $T^* = (T_1^*, \ldots, T_n^*) \in L(H)^n$ is a spherical contraction. Throughout this paper we use the notation $\mathbb{B} = \{z \in \mathbb{C}^n; |z| < 1\}$ for the open Euclidean unit ball in $\mathbb{C}^n$.
Let Λ be an arbitrary set. An operator-valued function $K : \Lambda \times \Lambda \to L(H)$ is called positive definite if $\sum_{i,j=1}^s \langle K(\lambda_i, \lambda_j) c_i, c_j \rangle \geq 0$ whenever s is a positive integer, $\lambda_1, \ldots, \lambda_s \in \Lambda$ and $c_1, \ldots, c_s \in H$. By a well-known theorem of Kolmogorov and Aronszajn (Theorem I.5.1 in [24]) a function $K : \Lambda \times \Lambda \to L(H)$ is positive definite if and only if there is a Hilbert space G and a function $k : \Lambda \to L(H, G)$ such that $K(z, w) = k(w)^* k(z)$ for $z, w \in \Lambda$. We shall denote by $H(\mathbb{B})$ the functional Hilbert space on $\mathbb{B}$ given by the

reproducing kernel

$$K : \mathbb{B} \times \mathbb{B} \to \mathbb{C}, \quad K(z, w) = (1 - \langle z, w \rangle)^{-1},$$

where $\langle z, w \rangle$ is the usual scalar product in $\mathbb{C}^n$. For any Hilbert space $\mathcal{E}$, we denote by $H(\mathcal{E}) = H(\mathbb{B}) \otimes \mathcal{E}$ the Hilbertian tensor product of $H(\mathbb{B})$ and $\mathcal{E}$. We identify $H(\mathcal{E})$ with the $\mathcal{E}$-valued functional Hilbert space given by the reproducing kernel $K_{\mathcal{E}}(z, w) = K(z, w) 1_{\mathcal{E}}$. It is well known (cf. [7] or [26]) that the functions in $H(\mathcal{E})$ can be represented by convergent power series on $\mathbb{B}$. More precisely,

$$H(\mathcal{E}) = \{f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha; \quad a_\alpha \in \mathcal{E} \text{ and } \|f\|^2 = \sum_{\alpha \in \mathbb{N}^n} \|a_\alpha\|^2 / \gamma_\alpha < \infty\}$$

where $\gamma_\alpha = |\alpha|!/\alpha!$. We denote by $H^2(\mathcal{E})$ the $\mathcal{E}$–valued Hardy space on the unit sphere in $\mathbb{C}^n$, i.e. the norm closure of the set of all polynomials in n variables with coefficients in $\mathcal{E}$ formed in $L^2(\sigma, \mathcal{E})$, where σ is the surface measure on the unit sphere $\partial\mathbb{B}$ in $\mathbb{C}^n$ (cf. [32]). As in the scalar–valued case one can identify $H^2(\mathcal{E})$ with the space

$$H^2(\mathbb{B}, \mathcal{E}) = \{f \in \mathcal{O}(\mathbb{B}, \mathcal{E});\ \|f\|_2^2 = \sup_{0<r<1} \int_{\partial\mathbb{B}} \|f(rz)\|^2 d\sigma(z) < \infty\}.$$

More precisely, each function $f \in H^2(\mathbb{B}, \mathcal{E})$ has radial limits almost everywhere and the map associating with each function $f \in H^2(\mathbb{B}, \mathcal{E})$ its boundary function f^* yields an isometric isomorphism $H^2(\mathbb{B}, \mathcal{E}) \cong H^2(\mathcal{E})$.
For given Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, we consider the multiplier space

$$M(\mathcal{E}, \mathcal{E}_*) = \{\varphi \in \mathcal{O}(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*));\ \varphi H(\mathcal{E}) \subset H(\mathcal{E}_*)\}.$$

By the closed graph theorem each function $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ induces a continuous linear multiplication operator $M_\varphi : H(\mathcal{E}) \to H(\mathcal{E}_*)$, $f \mapsto \varphi f$. The linear subspace $M(\mathcal{E}, \mathcal{E}_*) \subset \mathcal{O}(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*))$ becomes a Banach space relative to the norm

$$\|\varphi\| = \|M_\varphi\| = \sup\{\|\varphi f\|_{H(\mathcal{E}_*)}; \quad \|f\|_{H(\mathcal{E})} \leq 1\}.$$

For a given commuting tuple $T = (T_1, \ldots, T_n) \in L(H)^n$ on a Hilbert (or Banach) space H, we write $\sigma(T)$ for the Taylor spectrum of T, and we denote by $\Phi : \mathcal{O}(\sigma(T)) \to L(H)$, $f \mapsto f(T)$, Taylor's analytic functional calculus (cf. [19] or [35]).
For given Hilbert spaces H and K and any continuous linear operator $A \in L(H, K)$, we define the multiplication operator

$M_A : L(H) \to L(K)$, $X \to AXA^*$. If $A = (A_1, \dots, A_n) \in L(H)^n$ is commuting, then we use the induced operators $\Delta_A^m = (I - \sum_{i=1}^n M_{A_i})^m \in L(L(H))$ for $m = 0, 1, 2, \dots$. It is easy to see that

$$\Delta_A^m(X) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \frac{m!}{\alpha!(m - |\alpha|)!} A^\alpha X A^{*\alpha}.$$

1 The Schur class

Let $\mathcal{E}, \mathcal{E}_*$ be Hilbert spaces. In this section we give different characterizations of multipliers in $M(\mathcal{E}, \mathcal{E}_*)$. It is well known that multipliers and the multiplier norm on functional Hilbert spaces can be characterized via the positive definiteness of suitable derived kernels.

Theorem 1.1 *Let $\varphi : \mathbb{B} \to L(\mathcal{E}, \mathcal{E}_*)$ be given. Then $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ if and only if there is a constant $c \ge 0$ such that the map*

$$\mathbb{B} \times \mathbb{B} \to L(\mathcal{E}_*), \quad (z, w) \mapsto K(w, z)(c^2 1_{\mathcal{E}_*} - \varphi(w)\varphi(z)^*)$$

is positive definite. In this case, the norm $\|M_\varphi\|$ is the minimum of all possible constants $c \ge 0$. □

For the scalar-valued case $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$, the above characterization of multipliers can be found in [14]. The generalization to the operator-valued case is straightforward and will be left to the reader.
Our next aim is to show that multipliers $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ with $\|M_\varphi\| \le 1$ possess a natural fractional representation determined by unitary operator matrices. For any function $\varphi : S \to L(\mathcal{E}, \mathcal{E}_*)$ defined on a subset $S \subset \mathbb{B}$, we consider the map

$$K_\varphi : S \times S \to L(\mathcal{E}_*), \quad (z, w) \mapsto K(w, z)(1_{\mathcal{E}_*} - \varphi(w)\varphi(z)^*).$$

If L is any complex Hilbert space, then for $z \in \mathbb{B}$, we shall use the operator $Z : L^n \to L$, $(x_i) \mapsto \sum_{i=1}^n z_i x_i$.

Proposition 1.2 *Let $S \subset \mathbb{B}$ and $\varphi : S \to L(\mathcal{E}, \mathcal{E}_*)$ be given. Then the following are equivalent:*

(i) the map $K_\varphi : S \times S \to L(\mathcal{E}_*)$ is positive definite;

(ii) there exists a Hilbert space L and a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L \oplus \mathcal{E}, L^n \oplus \mathcal{E}_*)$$

such that $\varphi(z) = D + C(1_L - ZA)^{-1}ZB$ *for* $z \in S$;

(iii) φ *extends to a multiplier* $\Phi \in M(\mathcal{E}, \mathcal{E}_*)$ *with* $\|\Phi\| \leq 1$.

Proof. Suppose that K_φ is positive definite. By Kolmogorov's theorem (Theorem I.5.1 in [24]) there is a Hilbert space G and an operator-valued map $k : S \to L(\mathcal{E}_*, G)$ such that $K_\varphi(z, w) = k(w)^* k(z)$ for $z, w \in S$. By rewriting this identity in the form

$$\sum_{\nu=1}^{n} (\overline{w}_\nu k(w))^* (\overline{z}_\nu k(z)) + 1 = k(w)^* k(z) + \varphi(w)\varphi(z)^*$$

we see that there is a Hilbert space $L \supset G$ and a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L \oplus \mathcal{E}, L^n \oplus \mathcal{E}_*)$$

such that, for all $z \in S$ and $x \in \mathcal{E}_*$,

$$U^* \begin{pmatrix} \begin{pmatrix} \overline{z}_1 k(z)x \\ \vdots \\ \overline{z}_n k(z)x \end{pmatrix} \\ x \end{pmatrix} = \begin{pmatrix} k(z)x \\ \varphi(z)^* x \end{pmatrix}.$$

Solving the first of the two equations

$$A^* Z^* k(z) + C^* = k(z), \quad B^* Z^* k(z) + D^* = \varphi(z)^*$$

for $k(z)$, and substituting $k(z)$ by the result in the second equation, yields

$$\varphi(z)^* = D^* + B^* Z^* (1 - A^* Z^*)^{-1} C^* \quad (z \in S).$$

Suppose that there is a fractional representation of φ on S as in condition (ii). Then obviously φ extends to a holomorphic map $\Phi \in \mathcal{O}(\mathcal{B}, L(\mathcal{E}, \mathcal{E}_*))$ which is given by

$$\Phi(z) = D + C(1_L - ZA)^{-1}ZB \quad (z \in \mathbb{B}).$$

Using the identity $UU^* = I$ one easily obtains that

$$1 - \Phi(w)\Phi(z)^* = C(1 - WA)^{-1}[(1 - WA)(1 - A^* Z^*)$$

$$+(1-WA)A^*Z^* + WA(1-A^*Z^*) - WBB^*Z*](1-A^*Z^*)^{-1}C^*$$

$$= C(1-WA)^{-1}(1-WZ^*)(1-A^*Z^*)^{-1}C^*$$

$$= K(w,z)^{-1}C(1-WA)^{-1}(1-A^*Z^*)^{-1}C^*.$$

Hence the map K_Φ is positive definite, and the proof is complete by Theorem 1.1. □

In [7] Arveson proved that the multiplication tuple $S = M_z \in L(H(\mathbb{B}))^n$ consisting of the multiplication operators by the coordinate functions $M_{z_i}: H(\mathbb{B}) \to H(\mathbb{B})$, $f \mapsto z_i f$, is a universal n–contraction in the sense that, for each n–contraction $T \in L(H)^n$ on a Hilbert space H, the map $p(S) \mapsto p(T)$ defines a completely contractive representation of the algebra of all polynomials in S. Below we give an independent proof of this result.
Let H be a separable infinite–dimensional complex Hilbert space. If $T \in L(H)^n$ is a commuting tuple with Taylor spectrum contained in the closed unit ball, then for each real number r with $0 < r < 1$, we denote by

$$\Psi_r : \mathcal{O}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*)) \to L(H \otimes \mathcal{E}, H \otimes \mathcal{E}_*)$$

the unique continuous linear map with $\Psi_r(f \otimes S) = f(rT) \otimes S$ for $f \in \mathcal{O}(\mathbb{B})$ and $S \in L(\mathcal{E},\mathcal{E}_*)$. To simplify the notation we write again $f(rT)$ instead of $\Psi_r(f)$ for $f \in \mathcal{O}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*))$, where we allow the case $r = 1$ when $\sigma(T) \subset \mathbb{B}$. Define

$$\mathcal{C} = \{T \in L(H)^n;\ T \text{ is a spherical contraction}\}$$

and $\mathcal{C}^* = \{T^*;\ T \in \mathcal{C}\}$. For f in $\mathcal{O}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*))$, we call

$$\|f\|_{\mathcal{S}} = \sup\{\|f(rT)\|;\ 0 < r < 1 \text{ and } T \in \mathcal{C}\}$$

the Schur norm of f. Since, for each point $z \in \mathbb{B}$, the n–tuple $z1 = (z_1 1_H, \ldots, z_n 1_H)$ is a spherical contraction on H such that $f(z1) = 1_H \otimes f(z)$ for $f \in \mathcal{O}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*))$, it follows that $\|f\|_{\mathcal{S}} \geq \|f\|_{\infty,\mathbb{B}}$ for each such function f. A straightforward argument, using the fact that the analytic functional calculus behaves naturally with respect to unitary equivalence, shows that the Schur norm is independent of the choice of the separable infinite-dimensional Hilbert space H.

The linear space

$$\mathcal{S}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*)) = \{f \in \mathcal{O}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*));\ \|f\|_{\mathcal{S}} < \infty\}$$

becomes a Banach space if equipped with the Schur norm $\|\cdot\|_{\mathcal{S}}$. Its unit ball

$$\mathcal{S}_{\mathbb{B}}(\mathcal{E},\mathcal{E}_*) = \{f \in \mathcal{S}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*));\ \|f\|_{\mathcal{S}} \leq 1\}$$

is called the $L(\mathcal{E},\mathcal{E}_*)$–valued Schur class over $\mathbb{B}$. Analogously the dual Schur norm $\|f\|_{\mathcal{S}^*} = \sup\{\|f(rT)\|;\ 0 < r < 1 \text{ and } T \in \mathcal{C}^*\}$ and the dual Schur class $\mathcal{S}^*_{\mathbb{B}}(\mathcal{E},\mathcal{E}_*) = \{f \in \mathcal{O}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*));\ \|f\|_{\mathcal{S}^*} \leq 1\}$ can be defined.
An analytic function $f \in \mathcal{O}(\mathbb{B}, L(\mathcal{E},\mathcal{E}_*))$ belongs to the Schur class $\mathcal{S}_{\mathbb{B}}(\mathcal{E},\mathcal{E}_*)$ if and only if the function $\tilde{f} \in \mathcal{O}(\mathbb{B}, L(\mathcal{E}_*,\mathcal{E}))$ defined by $\tilde{f}(z) = f(\overline{z})^*$ belongs to the dual Schur class $\mathcal{S}^*_{\mathbb{B}}(\mathcal{E}_*,\mathcal{E})$. To check this it suffices to observe that, for each spherical contraction $T \in L(H)^n$ and each real number $0 < r < 1$, the identity $f(rT)^* = \tilde{f}(rT^*)$ holds.

In view of the cited result of Arveson from [7] it is clear that there is a close relationship between the multiplier space $M(\mathcal{E},\mathcal{E}_*)$ and the dual Schur class. To demonstrate the usefulness of the fractional representation of multipliers obtained above, we give a direct argument.
Let L, L', K, K' be Hilbert spaces and let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L \oplus K,\ L' \oplus K')$$

be a contraction. We shall use the well-known, and quite elementary, fact that, for any bounded operator $X \in L(L', L)$ with $\|X\| < 1$, the operator

$$D + C(1_L - XA)^{-1}XB \in L(K, K')$$

is a contraction again (cf. [12]).

Theorem 1.3 *Let $S \subset \mathbb{B}$ and $\varphi : S \to L(\mathcal{E},\mathcal{E}_*)$ be given. Then the following are equivalent:*

(i) φ extends to a map in $\mathcal{S}^*_{\mathbb{B}}(\mathcal{E},\mathcal{E}_*)$;

(ii) the kernel $K_\varphi : S \times S \to L(\mathcal{E}_*)$ is positive definite;

(iii) there is a Hilbert space L and a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L \oplus \mathcal{E}, L^n \oplus \mathcal{E}_*)$$

such that $\varphi(z) = D + C(1_L - ZA)^{-1}ZB$ for $z \in S$;

(iv) the same as (iii), but with U only supposed to be a contraction;

(v) φ extends to a map $\Phi \in M(\mathcal{E}, \mathcal{E}_*)$ with $\|\Phi\| \leq 1$.

Proof. Since we know that (ii), (iii) and (v) are equivalent, it suffices to show that (i) implies (v) and that (iv) implies (i).
To prove that (i) implies (v) we may suppose that $S = \mathbb{B}$. Fix $\varphi \in \mathcal{S}^*_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$. Let $S = M_z \in L(H(\mathbb{B}))^n$ be the standard n–contraction as above. It is well known that $S \in \mathcal{C}^*$ ([7]). Let $U \supset \overline{\mathbb{B}}$ be open. For each function $g \in \mathcal{O}(U, L(\mathcal{E}, \mathcal{E}_*))$ and each $f \in H(\mathbb{B}) \otimes \mathcal{E}$,

$$(g(S)f)(\lambda) = g(\lambda)f(\lambda) \quad (\lambda \in \mathbb{B}).$$

For fixed λ, it suffices to check this identity for the case of elementary tensors $g = g_0 \otimes A$ and $f = f_0 \otimes x$, where it obviously holds. An application of this remark to the functions $\varphi_r(z) = \varphi(rz)(0 < r < 1)$ yields

$$\sup_{0<r<1} \|M_{\varphi_r}\| = \sup_{0<r<1} \|\varphi_r(S)\| = \sup_{0<r<1} \|\varphi(rS)\| \leq \|\varphi\|_{\mathcal{S}^*} \leq 1.$$

A compactness argument allows us to choose a net $(r_i)_{i \in I}$ converging to one such that $A = \mathrm{WOT} - \lim\limits_{i \to \infty} M_{\varphi_{r_i}}$ exists. It easily follows that

$$(Af)(\lambda) = \varphi(\lambda)f(\lambda) \quad (f \in H(\mathcal{E}), \lambda \in \mathbb{B}).$$

Hence $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ and $\|M_\varphi\| = \|A\| \leq 1$.
To prove the remaining implication, let φ be given by a contractive operator matrix U as in condition (iv). Then φ obviously extends to a holomorphic map $\Phi \in \mathcal{O}(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*))$ represented in the same way by U on all of $\mathbb{B}$. Let T be an n-contraction on a complex Hilbert space H. Using standard properties of the analytic functional calculus, we obtain that

$$\Phi(rT) = 1_H \otimes D + 1_H \otimes C(1_{H \otimes L} - Z(rT)1_H \otimes A)^{-1} Z(rT)1_H \otimes B$$

for $0 < r < 1$. Since the operator

$$\begin{pmatrix} 1_H \otimes A & 1_H \otimes B \\ 1_H \otimes C & 1_H \otimes D \end{pmatrix} \cong 1_H \otimes U \in L(H \otimes (L \oplus \mathcal{E}), H \otimes (L^n \oplus \mathcal{E}_*))$$

remains a contraction and since, for $0 < r < 1$, the map

$$Z(rT) \cong (rT_1, \ldots, rT_n) \otimes 1_L \in L(H^n \otimes L, H \otimes L)$$

is a strict contraction, the remark preceding Theorem 1.3 implies that $\varphi \in \mathcal{S}^*_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$. $\square$

Since a function $\varphi \in \mathcal{O}(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*))$ belongs to the Schur class $\mathcal{S}_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ if and only if $\tilde{\varphi}$ belongs to the dual Schur class, we obtain an analogous characterization of the Schur class.

Corollary 1.4 *Let $S \subset \mathbb{B}$ and $\varphi : S \to L(\mathcal{E}, \mathcal{E}_*)$ be given. Then the following are equivalent:*

(i) *φ extends to a map in $\mathcal{S}_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$;*

(ii) *the kernel $S \times S \to L(\mathcal{E})$, $(z,w) \mapsto K(z,w)(1 - \varphi(w)^*\varphi(z))$, is positive definite;*

(iii) *there is a Hilbert space L and a unitary operator (or, equivalently, a contraction)*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L^n \oplus \mathcal{E}, L \oplus \mathcal{E}_*)$$

such that $\varphi(z) = D + CZ^t(1_L - AZ^t)^{-1}B \quad (z \in S)$;

(iv) *$\tilde{\varphi} : \tilde{S} = \{\overline{z}; z \in S\} \to L(\mathcal{E}_*, \mathcal{E})$, $\tilde{\varphi}(z) = \varphi(\overline{z})^*$, extends to a map $\tilde{\Phi} \in M(\mathcal{E}_*, \mathcal{E})$ with $\|\tilde{\Phi}\| \leq 1$.*

□

Since a scalar–valued function is positive definite if and only if its complex conjugate is positive definite, a comparison of the second condition in Theorem 1.3 with the second condition in Corollary 1.4 shows that, for $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$, the Schur class $\mathcal{S}_{\mathbb{B}}$ and the dual Schur class $\mathcal{S}^*_{\mathbb{B}}$ coincide. Using the observation that the class of spherical contractions is not self-dual one can show that in general the Schur class $\mathcal{S}_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ and the dual Schur class $\mathcal{S}^*_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ are different.

The proof of Theorem 1.3 shows that $\mathcal{S}^*(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*)) = M(\mathcal{E}, \mathcal{E}_*)$ isometrically. As mentioned above this identity can also be proved by using results of Arveson on the universal role of the n–contraction $S = M_z \in L(H(\mathbb{B}))^n$. To demonstrate the usefulness of the fractional representation of multipliers described above, we indicate how the corresponding parts of Arveson's results follow from Theorem 1.3. Let $T \in L(H)^n$ be an n–contraction on a

Hilbert space H (that is, $T \in \mathcal{C}^*$), and let $\mathcal{A}$ be the algebra of all polynomials in $S_1, \dots, S_n$. The case $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$ of Theorem 1.3 implies the von Neumann–type inequality (cf.[18])

$$\|p(T)\| = \lim_{r \uparrow 1} \|p(rT)\| \leq \|p\|_{\mathcal{S}^*} = \|p(S)\| \quad (p \in \mathbb{C}[z]).$$

If, for each $N \in \mathbb{N}$, one equips the algebra $M_N(\mathcal{A}) \subset L(H(\mathbb{B})^N)$ with the norm structure that it inherits from the C^*-algebra $L(H(\mathbb{B})^N)$, then the case $\mathcal{E} = \mathcal{E}_* = \mathbb{C}^N$ of Theorem 1.3 implies that the map

$$\rho_N : M_N(\mathcal{A}) \to M_N(L(H)), \quad (p_{ij}(S)) \mapsto (p_{ij}(T))$$

is a contraction. Indeed, for each matrix (p_{ij}) of polynomials, we have

$$\|(p_{ij}(T))\| = \|(p_{ij})(T)\| = \lim_{r \uparrow 1} \|(p_{ij})(rT)\| \leq \|(p_{ij})\|_{\mathcal{S}^*} = \|(p_{ij}(S))\|.$$

Thus we have proved Arveson's version of the von Neumann inequality for n–contractions (Theorem 8.1 in [7]).

Corollary 1.5 (Arveson) *Let $T \in L(H)^n$ be an n–contraction on a complex Hilbert space H, and let $\mathcal{A} \subset L(H(\mathbb{B}))$ be the algebra of polynomials in $S_1, \dots, S_n$. Then the map $\varphi : \mathcal{A} \to L(H)$, $p(S) \mapsto p(T)$, defines a completely contractive representation of $\mathcal{A}$.* □

We conclude this section with an interpolation result which generalizes parts of Corollary 1.4.

Theorem 1.6 *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be complex Hilbert spaces and let $S \subset \mathbb{B}$ be arbitrary. Suppose that $\alpha : S \to L(\mathcal{E}, \mathcal{F})$ and $\beta : S \to L(\mathcal{E}, \mathcal{G})$ are given operator–valued functions. Then there is a function $\varphi \in \mathcal{S}_{\mathbb{B}}(\mathcal{F}, \mathcal{G})$ with*

$$\varphi(z)\alpha(z) = \beta(z) \quad (z \in S)$$

if and only if the mapping

$$K_{\alpha,\beta} : S \times S \to L(\mathcal{E}), \quad K_{\alpha,\beta}(z, w) = \frac{\alpha(w)^*\alpha(z) - \beta(w)^*\beta(z)}{1 - \langle z, w \rangle}$$

is positive definite.

Proof. Suppose that $\varphi \in \mathcal{S}_{\mathbb{B}}(\mathcal{F}, \mathcal{G})$ satisfies $\varphi(z)\alpha(z) = \beta(z)$ for $z \in S$. Then Corollary 1.4 and the identity

$$\alpha(w)^*\alpha(z) - \beta(w)^*\beta(z) = \alpha(w)^*(1 - \varphi(w)^*\varphi(z))\alpha(z) \quad (z, w \in S)$$

show that the operator–valued kernel $K_{\alpha,\beta}$ is positive definite.

Conversely, suppose that $K_{\alpha,\beta}$ is positive definite. Then there is a Hilbert space M and a map $k : S \to L(\mathcal{E}, M)$ such that $K_{\alpha,\beta}(z, w) = k(w)^*k(z)$ for $z, w \in S$. Rewriting this equation as

$$\sum_{\nu=1}^{n}(w_\nu k(w)^*(z_\nu k(z)) + \alpha(w)^*\alpha(z) = k(w)^*k(z) + \beta(w)^*\beta(z)$$

one can deduce that there is a Hilbert space $L \supset M$ and a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L^n \oplus \mathcal{F}, L \oplus \mathcal{G})$$

such that, for all $x \in \mathcal{E}$ and $z \in S$,

$$U \begin{pmatrix} \begin{pmatrix} z_1 k(z)x \\ \vdots \\ z_n k(z)x \end{pmatrix} \\ \alpha(z)x \end{pmatrix} = \begin{pmatrix} k(z)x \\ \beta(z)x \end{pmatrix}.$$

If we regard k as a map with values in $L(\mathcal{E}, L)$, then solving the first of the two equations

$$AZ^t k(z) + B\alpha(z) = k(z), \quad CZ^t k(z) + D\alpha(z) = \beta(z)$$

for $k(z)$ and substituting $k(z)$ in the second equation by the resulting solution gives

$$\beta(z) = [D + CZ^t(1 - AZ^t)^{-1}B]\alpha(z) \quad (z \in S).$$

The function in square brackets extends to a holomorphic map φ on $\mathbb{B}$ given by the same formula. By Corollary 1.4 the map φ belongs to the Schur class $\mathcal{S}_{\mathbb{B}}(\mathcal{F}, \mathcal{G})$.

$\square$

We shall apply Theorem 1.6 to solve concrete interpolation problems on the unit ball in Section 3.

2 A commutant lifting theorem on the ball

In this section we prove a commutant lifting theorem for the classical Hardy space on the unit ball in $\mathbb{C}^n$. Corresponding results for the unit polydisc have been obtained in [12]. A version of the commutant lifting theorem over the space $H(\mathbb{B})$ can also be obtained by specializing corresponding non-commutative results of Popescu to the commutative case (cf. [6]).
Let $\mathcal{E}$ and $\mathcal{E}_*$ be complex Hilbert spaces. We write $S \in L(H^2(\mathcal{E}))^n$ for the tuple with components given by the multiplication with the coordinate functions

$$S_i : H^2(\mathcal{E}) \to H^2(\mathcal{E}), \quad (S_i f)(z) = z_i f(z) \quad (i = 1, \ldots, n).$$

Let $S_* \in L(H^2(\mathcal{E}_*))^n$ be the corresponding multiplication tuple on $H^2(\mathcal{E}_*)$. Suppose that $M \subset H^2(\mathcal{E})$ and $M_* \subset H^2(\mathcal{E}_*)$ are closed linear subspaces invariant under the adjoint tuples S^* and S_*^*, respectively. We denote by $T \in L(M)^n$ and $T_* \in L(M_*)^n$ the compressions of S and S_* onto M and M_*, i.e. the commuting n–tuples with components

$$T_i = PS_i|M, \quad T_{*i} = P_*S_{*i}|M_*,$$

where P and P_* are the orthogonal projections of $H^2(\mathcal{E})$ onto M and $H^2(\mathcal{E}_*)$ onto M_*. We are interested in characterizing those operators $X : M \to M_*$ intertwining T and T_* that possess a lifting to an analytic Toeplitz operator $T_\varphi : H^2(\mathcal{E}) \to H^2(\mathcal{E}_*)$ with a dual Schur–class symbol φ.

Recall from the preliminaries that, for a commuting tuple $A = (A_1, \ldots, A_n) \in L(H)^n$ on a Hilbert space H, the operators Δ_A^m (m= 0,1,2, ...) are defined as $\Delta_A^m = (I - \sum_{i=1}^n M_{A_i})^m \in L(L(H))$. Positivity conditions formulated in terms of these operators have been used in [26] to construct functional models and normal boundary dilations for spherical contractions. The reader should be aware that in our definition of the operator M_A we have exchanged the roles of A and A^*.

Lemma 2.1 *Let $X \in L(M, M_*)$ be an operator with $T_{*i}X = XT_i$ for $i = 1, \ldots, n$. For $h, k \in M_*$, we have*

$$\langle (X^*h)(0), (X^*k)(0) \rangle = \langle \Delta_{T_*}^n (XX^*)h, k \rangle.$$

Proof. We first show that the identity $\Delta_S^n(1)h = h(0)$ holds for all $h \in H^2(\mathcal{E})$. Since $H^2(\mathcal{E})$ is the closed linear hull of the elements $k_{w,x}$ $(w \in \mathbb{B}, x \in \mathcal{E})$ with

$$k_{w,x}(z) = \frac{x}{(1 - \langle z, w \rangle)^n} \quad (z \in \mathbb{B})$$

and since $S_i^* k_{w,x} = \overline{w}_i k_{w,x}$, the assertion follows from the observation that, for $w \in \mathbb{B}$ and $x \in \mathcal{E}$,

$$\Delta_S^n(1)(k_{w,x}) = \sum_{|\alpha| \leq n} c_\alpha S^\alpha S^{*\alpha}(k_{w,x})$$
$$= \Big(\sum_{|\alpha| \leq n} c_\alpha z^\alpha \overline{w}^\alpha \Big) k_{w,x} = (1 - \langle z, w \rangle)^n k_{w,x} = x$$

with suitably chosen constants c_α.

Since X intertwines T and T_*, we obtain the relation

$$\Delta_{T_*}^n(XX^*) = \Delta_{T_*}^n M_X(1_M) = M_X \Delta_T^n(1_M).$$

Using the notations $M_T = (M_{T_1}, \ldots, M_{T_n})$ and $M_S = (M_{S_1}, \ldots, M_{S_n})$, we obtain the identity

$$\langle M_T^j(1_M)h, k \rangle = \langle T^j T^{*j} h, k \rangle = \langle M_S^j(1_{H^2(\mathcal{E})})h, k \rangle$$

for all $j \in \mathbb{N}^n$ and $h, k \in M$. To complete the proof it suffices to observe that

$$\langle \Delta_{T_*}^n(XX^*)h, k \rangle = \langle \Delta_T^n(1_M) X^* h, X^* k \rangle = \langle \Delta_S^n(1)(X^* h), X^* k \rangle = \langle (X^* h)(0), (X^* k)(0) \rangle.$$

$\square$

Our characterization of Toeplitz operators with dual Schur–class symbol is based on the following elementary observation (cf. Lemma 3.2 in [12]).

Lemma 2.2 *Let H be a Hilbert space and let*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(H \oplus \mathcal{E}, H^n \oplus \mathcal{E}_*)$$

be a contraction. Then the map $\Omega : H \to H^2(\mathbb{B}, \mathcal{E}_*)$, $\Omega(\xi)(z) = C(1_H - ZA)^{-1}\xi$, is a well–defined linear contraction. The functions

$$\Omega_i : H^n \to H^2(\mathbb{B}, \mathcal{E}_*), \quad (\xi_\nu) \mapsto \Omega(\xi_i) \quad (i = 1, \ldots, n)$$

satisfy the identity

$$\left[\sum_{i=1}^n S_{*i} \Omega_i(\xi) \right](z) = C(1_H - ZA)^{-1} Z\xi$$

for $\xi \in H^n$ and $z \in \mathbb{B}$.

Proof. The complex–valued function F defined on the open unit ball by

$$F(z) = \langle (1_H + ZA)(1_H - ZA)^{-1}\xi, \xi\rangle$$

has the property that, for any real number $0 < r < 1$,

$$\int_{\partial\mathbb{B}} \text{Re } F(rz) d\sigma = \text{Re } F(0) = \|\xi\|^2.$$

But, for $z \in \mathbb{B}$,

$$\text{Re } F(z) = \langle (1_H - A^*Z^*)^{-1}(1_H - A^*Z^*ZA)(1_H - ZA)^{-1}\xi, \xi\rangle$$

$$\geq \|C(1_H - ZA)^{-1}\xi\|^2 + \langle (1_{H^n} - Z^*Z)A(1_H - ZA)^{-1}\xi, A(1_H - ZA)^{-1}\xi\rangle$$

$$\geq \|C(1_H - ZA)^{-1}\xi\|^2.$$

Therefore Ω is a well–defined contraction. The second part of the lemma is obvious.

□

For a commuting n–tuple $A = (A_1, \dots, A_n) \in L(H)^n$ on a Hilbert space H, let us consider the operator

$$\Sigma_A = \sum_{i=1}^{n} M_{A_i} \in L(L(H)).$$

In [26] (Lemma 7) it is proved that

$$\text{SOT} - \lim_{k\to\infty} (\Sigma_S^k)(1_{H^2(\mathcal{E})}) = 0,$$

where S is as before the tuple consisting of the multiplication operators with the coordinate functions on $H^2(\mathcal{E})$. An inspection of the proof given in [26] shows that even

$$\text{SOT} - \lim_{k\to\infty} (\Sigma_S^k)(X) = 0$$

for each operator $X \in L(H^2(\mathcal{E}))$.

Lemma 2.3 *Let $M \subset H^2(\mathcal{E})$ be a closed invariant subspace for S^*.*

a) For each operator $X \in L(M)$, we have $\text{SOT} - \lim_{k\to\infty}(\Sigma_T^k)(X) = 0$.

b) The operator $\Delta_T = I - \Sigma_T$ is injective.

c) If $X \in L(M)$ is a positive operator with $\Delta_T(X) \leq 0$, then $X = 0$.

Proof. Let $X \in L(M)$ be arbitrary. Since, for $h \in M$ and $k \in \mathbb{N}$,

$$(\Sigma_T^k)(X)h = \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^\alpha X T^{*\alpha} h = P(\Sigma_S^k)(XP)h,$$

part a) follows from the remarks preceding the lemma. If $\Delta_T(X) \leq 0$, then

$$(I - \Sigma_T^{N+1})(X) = \sum_{j=0}^{N} \Sigma_T^j(\Delta_T(X)) \leq 0$$

for each $N \in \mathbb{N}$ and hence $X \leq 0$. The same identity, combined with part a), also shows that Δ_T is injective.

□

The following result is a multivariable version of the commutant lifting theorem for the Hardy space on the unit ball.

Theorem 2.4 *Let $X \in L(M, M_*)$ be a continuous linear operator with $T_{*i}X = XT_i$ for $i = 1, \ldots, n$. Then the following conditions are equivalent:*

(i) there exists a symbol $\varphi \in \mathcal{S}_{\mathbb{B}}^*(\mathcal{E}, \mathcal{E}_*)$ such that $XP = P_* T_\varphi$;

(ii) there exists a Hilbert space L and a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L \oplus \mathcal{E}, L^n \oplus \mathcal{E}_*)$$

such that if $\varphi(z) = D + C(1_L - ZA)^{-1}ZB$, then $XP = P_* T_\varphi$;

(iii) $\Delta_{T_*}^{n-1}(1 - XX^*) \geq 0$.

Proof. The equivalence of the first two conditions follows from Theorem 1.3.

To prove the implication (iii) $\Rightarrow$ (ii) suppose that the operator $\Gamma = \Delta_{T_*}^{n-1}(1 - XX^*) \in L(M_*)$ is positive. According to Lemma 2.1 the identity

$$\|h(0)\|^2 - \|X^* h(0)\|^2 = \langle \Delta_{T_*}^n (1 - XX^*)h, h\rangle = \|\Gamma^{1/2}h\|^2 - \sum_{i=1}^n \|\Gamma^{1/2} T_{*i}^* h\|^2$$

holds for all $h \in M_*$. Hence the map

$$\left(\begin{pmatrix} \Gamma^{1/2} T_{*1}^* h \\ \vdots \\ \Gamma^{1/2} T_{*n}^* h \end{pmatrix} \atop h(0) \right) \longmapsto \begin{pmatrix} \Gamma^{1/2} h \\ (X^* h)(0) \end{pmatrix}$$

defines an isometry V from $\{(\Gamma^{1/2}T_{*1}^* h, \dots, \Gamma^{1/2}T_{*n}^* h, h(0));\ h \in M_*\}$ into $M_* \oplus \mathcal{E}$. Let L be a separable Hilbert space containing M_* such that V can be extended to a unitary operator

$$U^* : L^n \oplus \mathcal{E}_* \longrightarrow L \oplus \mathcal{E}.$$

We write U as an operator matrix

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L \oplus \mathcal{E}, L^n \oplus \mathcal{E}_*).$$

According to Theorem 1.3 the matrix U determines a function

$$\varphi(z) = D + C(1_L - ZA)^{-1} ZB$$

in the dual Schur class $\mathcal{S}_{\mathbb{B}}^*(\mathcal{E}, \mathcal{E}_*)$. Our aim is to show that $XP = P_* T_\varphi$. By Lemma 2.2 we know that

$$\langle (T_\varphi^* h)(0), x \rangle_{\mathcal{E}} = \langle T_\varphi^* h, x \rangle_{H^2(\mathcal{E})}$$

$$= \langle h, Dx + \sum_{i=1}^n S_{*i} \Omega_i (Bx) \rangle_{H^2(\mathcal{E}_*)}$$

$$= \langle D^* h(0) + \sum_{i=1}^n B^* \Omega_i^* S_{*i}^* h, x \rangle_{\mathcal{E}}$$

for all $x \in \mathcal{E}$ and $h \in H^2(\mathcal{E}_*)$. It follows that

$$(T_\varphi^* h)(0) = D^* h(0) + \sum_{i=1}^n B^* \Omega_i^* S_{*i}^* h \tag{1}$$

for all $h \in H^2(\mathcal{E}_*)$.

Secondly, the identities

$$\langle \Omega^* h, \xi \rangle_L = \langle h, C(1_L - ZA)^{-1}((1_L - ZA)\xi + ZA\xi) \rangle_{H^2(\mathcal{E}_*)}$$

$$= \langle h, C\xi + \sum_{i=1}^{n} S_{*i}\Omega_i(A\xi)\rangle_{H^2(\mathcal{E}_*)} = \langle C^*h(0) + \sum_{i=1}^{n} A^*\Omega_i^* S_{*i}^* h, \xi\rangle_L$$

valid for all $h \in H^2(\mathcal{E}_*)$ and $\xi \in L$, show that

$$\Omega^* h = C^* h(0) + \sum_{i=1}^{n} A^* \Omega_i^* S_{*i}^* h \tag{2}$$

for all $h \in H^2(\mathcal{E}_*)$.

Let $j : M_* \hookrightarrow L$ be the inclusion. Define $\Phi = j \circ \Gamma^{1/2} \in L(M_*, L)$. Then our choice of U^* yields

$$U^* \left(\begin{array}{c} \begin{pmatrix} \Phi T_{*1}^* h \\ \vdots \\ \Phi T_{*n}^* h \end{pmatrix} \\ h(0) \end{array} \right) = \begin{pmatrix} \Phi h \\ (X^* h)(0) \end{pmatrix}$$

for all $h \in M_*$. Since

$$U^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \in L(L^n \oplus \mathcal{E}_*, L \oplus \mathcal{E}),$$

the identities (1) and (2) above show that, on the other hand,

$$U^* \left(\begin{array}{c} \begin{pmatrix} \Omega^* S_{*1}^* h \\ \vdots \\ \Omega^* S_{*n}^* h \end{pmatrix} \\ h(0) \end{array} \right) = \begin{pmatrix} \Omega^* h \\ (T_\varphi^* h)(0) \end{pmatrix}$$

for all $h \in H^2(\mathcal{E}_*)$.

We want to show that the difference $\Psi = \Phi - (\Omega^* | M_*) \in L(M_*, L)$ is the zero operator. Since $T_*^* = S_*^* | M_*$, we obtain the identity

$$A^* \begin{pmatrix} \Psi T_{*1}^* h \\ \vdots \\ \Psi T_{*n}^* h \end{pmatrix} = \Psi h$$

for $h \in M_*$. Since A^* is a contraction, it follows that

$$\Psi^* \Psi \leq T_{*1} \Psi^* \Psi T_{*1}^* + \ldots + T_{*n} \Psi^* \Psi T_{*n}^*,$$

or equivalently, that $\Delta_{T_*}(\Psi^*\Psi) \leq 0$. By Lemma 2.3 it is clear that $\Psi = 0$.

Since $\Phi = \Omega^*$ on M_*, we know also that

$$X^*h(0) = (T_\varphi^* h)(0) \quad (h \in M_*).$$

Consequently, for $x \in \mathcal{E}$ and $h \in M_*$, we obtain

$$\langle T_\varphi^* h,\, S^\alpha x\rangle = \langle T_\varphi^* S_*^{*\alpha} h, x\rangle = \langle X^* S_*^{*\alpha} h, x\rangle = \langle X^* T_*^{*\alpha} h, x\rangle$$

$$= \langle T^{*\alpha} X^* h, x\rangle = \langle S^{*\alpha} X^* h, x\rangle = \langle X^* h, S^\alpha x\rangle$$

for all $\alpha \in \mathbb{N}^n$. Since the vectors $S^\alpha x$ $(x \in \mathcal{E}, \alpha \in \mathbb{N}^n)$ span the space $H^2(\mathcal{E})$, we deduce that $X^*h = T_\varphi^* h$ for $h \in M_*$, or equivalently, that $P_* T_\varphi = XP$.

To complete the proof we show that (ii) implies (iii). For this purpose, let φ and U be given as in condition (ii). We first consider the case that $M = H^2(\mathcal{E})$, $M_* = H^2(\mathcal{E}_*)$, and $X = T_\varphi$. Let $\Omega : L \to H^2(\mathcal{E}_*)$ be the operator associated with U as in Lemma 2.2. Recall that the operators

$$\Omega_i = \Omega \circ \pi_i : L^n \to H^2(\mathcal{E}_*) \quad (i = 1, \ldots, n),$$

where π_i is the projection of L^n onto its i–th component, satisfy the equation

$$[\sum_{i=1}^n S_{*i}\Omega_i(\xi)](z) = C(1_L - ZA)^{-1} Z\xi \quad (\xi \in L^n).$$

Denote by $j_i = \pi_i^* : L \to L^n$ the inclusion mappings and define

$$\Gamma = \Delta_{S_*}^{n-1}(1 - T_\varphi T_\varphi^*) \in L(H^2(\mathcal{E}_*)).$$

According to Lemma 2.1 the identity

$$\|(T_\varphi^* h)(0)\|^2 = \|h(0)\|^2 - \langle \Gamma h, h\rangle + \sum_{i=1}^n \langle \Gamma S_{*i}^* h, S_{*i}^* h\rangle$$

holds for all $h \in H^2(\mathcal{E}_*)$. Exactly as in the proof of the implication (iii) $\Rightarrow$ (ii) we obtain that

$$(T_\varphi^* h)(0) = D^* h(0) + \sum_{i=1}^n B^* \Omega_i^* S_{*i}^* h$$

for each function $h \in H^2(\mathcal{E}_*)$. Using this identity and the relation $UU^* = I$ we deduce that

$$\|(T_\varphi^* h)(0)\|^2 = \|h(0)\|^2 - \|C^* h(0)\|^2$$

$$-2\mathrm{Re}\langle AC^* h(0), \sum_{i=1}^n \Omega_i^* S_{*i}^* h\rangle + \sum_{i=1}^n \langle \Omega\Omega^* S_{*i}^* h, S_{*i}^* h\rangle$$

$$-\langle A^* \sum_{i=1}^n \Omega_i^* S_{*i}^* h, A^* \sum_{i=1}^n \Omega_i^* S_{*i}^* h\rangle.$$

Again by the proof of the implication (iii) $\Rightarrow$ (ii) we know that

$$A^* \sum_{i=1}^n \Omega_i^* S_{*i}^* h = \Omega^* h - C^* h(0)$$

for all $h \in H^2(\mathcal{E}_*)$. Thus, for all such functions h,

$$\|(T_\varphi^* h)(0)\|^2 = \|h(0)\|^2 - \langle \Omega\Omega^* h, h\rangle + \sum_{i=1}^n \langle \Omega\Omega^* S_{*i}^* h, S_{*i}^* h\rangle.$$

Hence $\Delta_{S_*}(\Omega\Omega^* - \Gamma) = 0$ and by Lemma 2.3 it follows that $\Gamma = \Omega\Omega^*$ is positive.

In the general case, it suffices to observe that, for $g \in M_*$,

$$\langle \Delta_{T_*}^{n-1}(1 - XX^*)g, g\rangle = \sum_{|\alpha| \le n-1} c_\alpha (\|T_*^{*\alpha} g\|^2 - \|X^* T_*^{*\alpha} g\|^2)$$
$$= \sum_{|\alpha| \le n-1} c_\alpha (\|S_*^{*\alpha} g\|^2 - \|T_\varphi^* S_*^{*\alpha} g\|^2) = \langle \Delta_{S_*}^{n-1}(1 - T_\varphi T_\varphi^*)g, g\rangle$$

with suitable constants c_α.

□

Remark 2.5 (a) Let $X \in L(M, M_*)$ be a continuous linear map with $T_{*i} X = X T_i$ for $i = 1, \dots, n$ such that $\Delta_{T_*}^{n-1}(1 - XX^*) \ge 0$. Let V be the isometry constructed in the proof of the implication (iii) $\Rightarrow$ (ii) of Theorem 2.4. Fix a Hilbert space L containing M_* and a contraction $U^* : L^n \oplus \mathcal{E}_* \to L \oplus \mathcal{E}$ extending the isometry V. If U is represented by the operator matrix

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(L \oplus \mathcal{E}, L^n \oplus \mathcal{E}_*),$$

then as in the previous proof it follows that

$$\varphi(z) = D + C(1_L - ZA)^{-1}ZB$$

defines a function in the dual Schur class $\mathcal{S}^*_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ such that $XP = P_* T_\varphi$.
(b) By applying Theorem 2.4 to the case $M = H^2(\mathcal{E})$ and $M_* = H^2(\mathcal{E}_*)$ one can see that a function $\varphi \in H^\infty(\mathbb{B}, L(\mathcal{E}, \mathcal{E}_*))$ belongs to the dual Schur class $\mathcal{S}^*_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ if and only if $\Delta^{n-1}_{S_*}(1 - T_\varphi T^*_\varphi) \geq 0$.

3 Applications

As a first application of Theorem 2.4 we prove an interpolation result of Nevanlinna–Pick type for Schur–class functions on $\mathbb{B}$.

In Section 1 we saw that the dual Schur class $\mathcal{S}^*_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ can be interpreted as the closed unit ball in the multiplier space $M(\mathcal{E}, \mathcal{E}_*)$ (see Theorem 1.3). Via this identification our interpolation results are generalizations of corresponding results of Agler and McCarthy [3] for the matrix-valued case. Non-commutative operator-valued interpolation problems of Nevanlinna-Pick type on the unit ball have been solved by Popescu [29], Arias and Popescu [6], and Davidson and Pitts [17].
Let $\mathcal{E}$ and $\mathcal{E}_*$ be separable complex Hilbert spaces, and let $w^{(1)}, \ldots, w^{(r)} \in \mathbb{B}$ be pairwise distinct points in the open unit ball. For any given vectors $x^{(1)}_*, \ldots, x^{(r)}_* \in \mathcal{E}_* \setminus \{0\}$ and $x^{(1)}, \ldots, x^{(r)} \in \mathcal{E}$, we are looking for conditions that guarantee the existence of a function $\varphi \in \mathcal{S}^*_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ with

$$\varphi(w^{(j)})^* x^{(j)}_* = x^{(j)} \quad (j = 1, \ldots, r).$$

We use the kernel functions $k_{w,x} : \mathbb{B} \to \mathcal{E}$ defined by

$$k_{w,x}(z) = \frac{x}{(1 - \langle z, w \rangle)^n} \quad (w \in \mathbb{B}, x \in \mathcal{E})$$

as well as the corresponding $\mathcal{E}_*$–valued functions. As an abbreviation we write $k_j = k_{w^{(j)}, x^{(j)}}, \quad k^*_j = k_{w^{(j)}, x^{(j)}_*}$ for $j = 1, \ldots, r$. The spaces

$$M = LH\{k_j;\ j = 1, \ldots, r\} \subset H^2(\mathcal{E}), \quad M_* = LH\{k^*_j;\ j = 1, \ldots, r\} \subset H^2(\mathcal{E}_*)$$

are finite–dimensional invariant subspaces for S^* and S^*_*, respectively. As in Section 2 we write T and T_* for the compressions of S and S_* on M and M_*.

In this setting the positivity condition occurring in part (iii) of Theorem 2.4 has a very natural meaning.

Lemma 3.1 *Let $X : M \to M_*$ be the linear map with adjoint defined by*

$$X^* k_j^* = k_j \quad (j = 1, \dots, r).$$

For $k \in \mathbb{N}$, the condition $\Delta_{T_*}^k (1 - XX^*) \geq 0$ holds if and only if the $(r \times r)$–matrix

$$\left(\frac{\langle x_*^{(i)}, x_*^{(j)} \rangle - \langle x^{(i)}, x^{(j)} \rangle}{(1 - \langle w^{(j)}, w^{(i)} \rangle)^{n-k}} \right)_{1 \leq i,j \leq r}$$

is positive semidefinite.

Proof. Fix $k \in \mathbb{N}$. For $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, define $c_\alpha = (-1)^{|\alpha|} k!/(\alpha!(k - |\alpha|)!)$. Then

$$(1 - \langle z, w \rangle)^k = \sum_{|\alpha| \leq k} c_\alpha z^\alpha \overline{w}^\alpha \quad (z, w \in \mathbb{C}^n).$$

Since the elements k_j^* $(j = 1, \dots, r)$ form a basis of M_*, the condition that $\Delta_{T_*}^k (1 - XX^*) \geq 0$ is equivalent to the fact that the $(r \times r)$–matrix with (i, j)–th entry equal to

$$\langle \Delta_{T_*}^k (1 - XX^*) k_i^*, \, k_j^* \rangle = \sum_{|\alpha| \leq k} c_\alpha \Big(\langle T_*^{*\alpha} k_i^*, \, T_*^{*\alpha} k_j^* \rangle - \langle X^* T_*^{*\alpha} k_i^*, \, X^* T_*^{*\alpha} k_j^* \rangle \Big)$$

$$= \sum_{|\alpha| \leq k} c_\alpha (w^{(j)})^\alpha (\overline{w}^{(i)})^\alpha \Big(\frac{\langle x_*^{(i)}, x_*^{(j)} \rangle - \langle x^{(i)}, x^{(j)} \rangle}{(1 - \langle w^{(j)}, w^{(i)} \rangle)^n} \Big) = \frac{\langle x_*^{(i)}, x_*^{(j)} \rangle - \langle x^{(i)}, x^{(j)} \rangle}{(1 - \langle w^{(j)}, w^{(i)} \rangle)^{n-k}}$$

is positive semidefinite.

□

For X as in the preceding lemma and any function $\varphi \in \mathcal{S}_{\mathbb{B}}^*(\mathcal{E}, \mathcal{E}_*)$, the condition $XP = P_* T_\varphi$ holds if and only if $T_\varphi^* M_* \subset M$ and $X^* = T_\varphi^* | M_*$ or, equivalently, if $T_\varphi^* k_i^* = k_i \quad (i = 1, \dots, r)$. But $T_\varphi^* k_{w,z} = k_{w, \varphi(w)^* z}$ for $w \in \mathbb{B}$ and $z \in \mathcal{E}_*$. Hence a direct application of Theorem 2.4 yields the following interpolation result.

Corollary 3.2 *Let $w^{(1)}, \dots, w^{(r)} \in \mathbb{B}$ be pairwise distinct. For given vectors $x_*^{(1)}, \dots, x_*^{(r)}$ in $\mathcal{E}_* \setminus \{0\}$ and $x^{(1)}, \dots, x^{(r)}$ in $\mathcal{E}$, there is a function $\varphi \in \mathcal{S}_{\mathbb{B}}^*(\mathcal{E}, \mathcal{E}_*)$ in the dual Schur class with*

$$\varphi(w^{(i)})^* x_*^{(i)} = x^{(i)} \quad (i = 1, \dots, r)$$

if and only if the $(r \times r)$–matrix

$$\left(\frac{\langle x_*^{(i)}, x_*^{(j)} \rangle - \langle x^{(i)}, x^{(j)} \rangle}{1 - \langle w^{(j)}, w^{(i)} \rangle} \right)_{1 \le i,j \le r}$$

is positive semidefinite. □

Using the fact that $\varphi \in \mathcal{S}_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ if and only if $\tilde{\varphi} \in \mathcal{S}_{\mathbb{B}}^*(\mathcal{E}_*, \mathcal{E})$, one obtains that, for $w^{(1)}, \dots, w^{(r)}$ as above and given vectors $x^{(1)}, \dots, x^{(r)} \in \mathcal{E} \setminus \{0\}$, $x_*^{(1)}, \dots, x_*^{(r)} \in \mathcal{E}_*$, there is a function $\varphi \in \mathcal{S}_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$ with

$$\varphi(w^{(i)}) x^{(i)} = x_*^{(i)} \quad (i = 1, \dots, r)$$

if and only if the $(r \times r)$–matrix

$$\left(\frac{\langle x^{(i)}, x^{(j)} \rangle - \langle x_*^{(i)}, x_*^{(j)} \rangle}{1 - \langle w^{(i)}, w^{(j)} \rangle} \right)_{1 \le i,j \le r}$$

is positive semidefinite.

A closely related interpolation result follows as an application of Theorem 1.6.

Theorem 3.3 *Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be Hilbert spaces, let $S = \{w_1, \dots, w_r\} \subset \mathbb{B}$ be a finite subset and let $A_i \in L(\mathcal{E}, \mathcal{F})$, $B_i \in L(\mathcal{E}, \mathcal{G})$ $(1 \le i \le r)$ be given operators. Then there is a function $\varphi \in \mathcal{S}_{\mathbb{B}}(\mathcal{F}, \mathcal{G})$ with $\varphi(w_i) A_i = B_i$ for $i = 1, \dots, r$ if and only if the matrix-operator*

$$\left(\frac{A_i^* A_j - B_i^* B_j}{1 - \langle w_j, w_i \rangle} \right)_{1 \le i,j \le r} \in L(\mathcal{E}^r)$$

is positive.

Proof. By Theorem 1.6 there is a function φ with the stated properties if and only if the associated kernel $K_{(A_i),(B_i)} : S \times S \to L(\mathcal{E})$ is positive definite. This condition is easily seen to be equivalent to the positivity of the matrix-operator considered above. □

Since $\varphi \in \mathcal{S}_{\mathbb{B}}(\mathcal{F}, \mathcal{G})$ if and only if $\tilde{\varphi} \in \mathcal{S}^*_{\mathbb{B}}(\mathcal{G}, \mathcal{F})$ (which is the unit ball of $M(\mathcal{G}, \mathcal{F})$) one obtains that, for $S = \{w_1, \ldots, w_r\} \subset \mathbb{B}$ and any given operators $A_i \in L(\mathcal{F}, \mathcal{E})$, $B_i \in L(\mathcal{G}, \mathcal{E})$, there is a multiplier φ in $M(\mathcal{G}, \mathcal{F})$ with $\|\varphi\| \leq 1$ such that $A_i \varphi(w_i) = B_i$ for $i = 1, \ldots, r$ if and only if the matrix-operator

$$\left(\frac{A_i A_j^* - B_i B_j^*}{1 - \langle w_i, w_j \rangle} \right)_{1 \leq i,j \leq r} \in L(\mathcal{E}^r)$$

is positive. A non-commutative interpolation result which specializes in the commutative case to the last observation has been obtained by Popescu [28] (Theorem 4.1). Similar results can be found in Arias-Popescu [6], Davidson and Pitts [17] and Meyer [25].
As a second application of Theorem 1.6 we give a criterion for the solvability of the corona problem within the Schur class. In the one–variable case our result reduces to a version of the Toeplitz corona theorem (see [31] and [22]). In [10] a corresponding result over the unit polydisc is proved.

Theorem 3.4 *Let $f_1, \ldots, f_r : S \to \mathbb{C}$ be complex-valued functions on an arbitrary subset $S \subset \mathbb{B}$ and let $\delta > 0$ be given. Then there are functions $g_1, \ldots, g_r$ in $\mathcal{O}(\mathbb{B})$ such that $(g_1, \ldots, g_r) \in (1/\delta)\mathcal{S}_{\mathbb{B}}(\mathbb{C}^r, \mathbb{C})$ and such that $\sum_{i=1}^r f_i(z) g_i(z) = 1$ for $z \in S$ if and only if the kernel*

$$K_f : S \times S \to \mathbb{C}, \quad K(z, w) = \frac{\sum\limits_{i=1}^{r} \overline{f_i(w)} f_i(z) - \delta^2}{1 - \langle z, w \rangle}$$

is positive definite.

Proof. It suffices to apply Theorem 1.6 with $\mathcal{E} = \mathbb{C}$, $\mathcal{F} = \mathbb{C}^r$, $\mathcal{G} = \mathbb{C}$ to the analytic functions $\beta \equiv \delta$ and $\alpha : S \to L(\mathbb{C}, \mathbb{C}^r)$, $\alpha(z) = (f_1(z), \ldots, f_r(z))^t$. □

As a third application we indicate how the results from Section 1 and Section 2 can be used to give an elementary definition of the curvature invariant introduced by Arveson for n-contractions with finite-dimensional defect space

[8]. Let $T \in L(H)^n$ be an n–contraction such that the defect operator

$$\Delta = (1 - \sum_{i=1}^{n} T_i T_i^*)^{1/2}$$

has finite–dimensional range ΔH. In [8] Arveson used the operator–valued function $F : \mathbb{B} \to L(\Delta H)$,

$$F(z)\xi = \Delta(1 - \sum_{i=1}^{n} z_i T_i^*)^{-1}(1 - \sum_{i=1}^{n} \overline{z}_i T_i)^{-1}\Delta\xi \quad (\xi \in \Delta H),$$

to define the curvature invariant of T. More precisely, the limits

$$K_0(z) = \lim_{r\to 1}(1 - r^2) \text{ trace } F(rz)$$

are shown to exist for almost every $z \in \partial\mathbb{B}$, and the curvature invariant $K(T)$ of T is defined by averaging K_0 over the unit sphere

$$K(T) = \int_{\partial\mathbb{B}} K_0(z) d\sigma(z).$$

Arveson proves that the curvature invariant is an integer for large classes of n–contractions and that this integer is closely related to the Fredholm index of T (see [8] and [9]). The first result was extended to the class of all pure n–contractions by Greene, Richter and Sundberg in [21].

Below we indicate how the results of the previous sections can be used to give a simplified, and very natural, definition of the curvature invariant. In the setting described above define $\mathcal{E}_* = \Delta H$. Regard $C = \Delta \in L(H, \mathcal{E}_*)$ as an operator with values in the finite-dimensional space $\mathcal{E}_*$. Let $A = (T_i^*)_{1\le i\le n} \in L(H, H^n)$ be the column operator with components T_i^*. The operators A and C form the entries of an isometry $V \in L(H, H^n \oplus \mathcal{E}_*)$. Define $\mathcal{E} = (H^n \oplus \mathcal{E}_*) \ominus \operatorname{Im} V$ and consider the unitary matrix–operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(H \oplus \mathcal{E}, H^n \oplus \mathcal{E}_*),$$

where the second column is by definition the inclusion map from $\mathcal{E}$ into $H^n \oplus \mathcal{E}_*$. By Theorem 1.3 the function $\varphi : \mathbb{B} \to L(\mathcal{E}, \mathcal{E}_*)$, $\varphi(z) = D + C(1_H - ZA)^{-1}ZB$, belongs to the dual Schur class $\mathcal{S}^*_{\mathbb{B}}(\mathcal{E}, \mathcal{E}_*)$. In the proof of Proposition 1.2 it was shown that the associated positive definite kernel $K_\varphi : \mathbb{B} \times \mathbb{B} \to L(\mathcal{E}_*)$ has the representation

$$K_\varphi(z, w) = C(1 - \sum_{i=1}^{n} w_i T_i^*)^{-1}(1 - \sum_{i=1}^{n} \overline{z}_i T_i)^{-1} C^*.$$

By restricting both sides to the diagonal we obtain the relation (cf. Theorem 1.2 in [8])

$$1 - \varphi(z)\varphi(z)^* = (1 - |z|^2)F(z) \quad (z \in \mathbb{B}).$$

Since the space $L(\mathcal{E}, \mathcal{E}_*)$ is topologically isomorphic to a Hilbert space, the bounded analytic function φ has radial limits $\tilde{\varphi}(z) = \lim_{r\to 1} \varphi(rz)$ for almost every $z \in \partial\mathbb{B}$. The resulting bounded measurable function $\tilde{\varphi} : \partial\mathbb{B} \to L(\mathcal{E}, \mathcal{E}_*)$ satisfies the condition

$$1 - \tilde{\varphi}(z)\tilde{\varphi}(z)^* = \lim_{r\to 1}(1 - r^2)F(rz)$$

for almost every $z \in \partial\mathbb{B}$. Hence $K_0 = \text{trace}\ (1 - \tilde{\varphi}\tilde{\varphi}^*)$ and the curvature invariant has the representation

$$K(T) = \int_{\partial\mathbb{B}} \text{trace}(1 - \tilde{\varphi}(z)\tilde{\varphi}(z)^*)d\sigma(z).$$

Let (e_k) be an orthonormal basis of $\mathcal{E}$. Since, for $z \in \partial\mathbb{B}$,

$$\text{trace}_{\mathcal{E}_*}(\tilde{\varphi}(z)\tilde{\varphi}(z)^*) = \text{trace}_{\mathcal{E}}(\tilde{\varphi}(z)^*\tilde{\varphi}(z)) = \sum_k \|\tilde{\varphi}(z)e_k\|^2,$$

it follows that $K(T) = \dim(\mathcal{E}_*) - \sum_k \|\varphi e_k\|^2_{H^2(\mathcal{E}_*)}$. By choosing an orthonormal basis (b_j) of $H^2(\mathcal{E}_*)$, the series occurring on the right can be rewritten as

$$\sum_k \sum_j |\langle b_j, T_\varphi e_k\rangle|^2 = \sum_j \|T_\varphi^* b_j(0)\|^2 = \text{trace}\Delta^n_{S_*}(T_\varphi T_\varphi^*),$$

where we have used Lemma 2.1 with $X = T_\varphi$. The proof of Lemma 2.1 shows that the operator $\Delta^n_{S_*}(1_{H^2(\mathcal{E}_*)})$ is the orthogonal projection from $H^2(\mathcal{E}_*)$ onto the closed subspace consisting of all constant functions. Thus we have shown that $K(T) = \text{trace}\ \Delta^n_{S_*}(1 - T_\varphi T_\varphi^*)$. According to the proof of the implication (ii) $\Rightarrow$ (iii) of Theorem 2.4 we have

$$\Delta^n_{S_*}(1 - T_\varphi T_\varphi^*) = \Delta_{S_*}(\Gamma) = \Delta_{S_*}(\Omega\Omega^*),$$

where $\Omega : H \to H^2(\mathcal{E}_*)$ is the contraction $\Omega(\xi)(z) = C(1_H - ZA)^{-1}\xi$ defined in Lemma 2.2.

Theorem 3.5 (Arveson) *For every n–contraction $T \in L(H)^n$ with finite–dimensional defect space $\mathcal{E}_* = \operatorname{Im}\sqrt{1 - \sum_{i=1}^n T_i T_i^*}$, the operator $\Delta_{S_*}(\Omega\Omega^*)$ belongs to the trace class $\mathcal{C}^1(H^2(\mathcal{E}_*))$, and we have*

$$K(T) = \text{trace } \Delta_{S_*}(\Omega\Omega^*).$$

□

The above result should be compared with Theorem C in Arveson [8]. The interested reader will have no difficulties to show that the operator $\Delta_{S_*}(\Omega\Omega^*)$ is precisely the curvature operator introduced by Arveson in [8] (see Definition 3.5 and the subsequent remarks).

4 Interpolation in $H^\infty(\mathbb{B})$ on uniqueness subsets of $\mathbb{B}$

It is well known that the interpolation results presented in Section 3 do not hold for the full class of H^∞–functions on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, see for instance [2], [10]. In this respect it is surprising to find a simple Nevanlinna-Pick type theorem on small subsets of $\mathbb{B}$ valid within the class $H^\infty(\mathbb{B})$. A similar phenomenon was observed in the note [20]. A different proof of these results in the scalar–valued case can be found in Beatrous and Burbea [13] and in Szafraniec [33].

Since a conformal map on the range of analytic maps $f : \mathbb{B} \longrightarrow \mathbb{D}$ does not change the interpolation problem, we shall work with analytic maps $f : \mathbb{B} \longrightarrow \mathbb{C}_r$ which take values in the right half plane

$$\mathbb{C}_r = \{z \in \mathbb{C};\ \operatorname{Re} z \geq 0\}.$$

For such maps, an analogue of the classical Riesz-Herglotz formula is well known (see [23], [4]). More specifically, there exists a bijection between analytic maps $f : \mathbb{B} \longrightarrow \mathbb{C}_r$ and positive Borel measures μ on the unit sphere $\partial\mathbb{B}$ which vanish on the monomials

$$z^\alpha \overline{z}^\beta, \quad \exists i, j \in \{1, 2, \ldots, n\} : (\alpha_i - \beta_i)(\alpha_j - \beta_j) < 0,$$

$$z^\alpha \overline{z}^{\alpha+\beta}(\alpha_k + \beta_k + 1 - (|\alpha| + |\beta| + n)|z_k|^2),$$

$$z^{\alpha+\beta} \overline{z}^\alpha(\alpha_k + \beta_k + 1 - (|\alpha| + |\beta| + n)|z_k|^2),$$

where $\alpha, \beta \in \mathbb{N}^n$ and $1 \leq k \leq n$. The bijective correspondence is given by the formula

$$f(z) = \int_{\partial\mathbb{B}} (\frac{2}{(1 - \langle z, u \rangle)^n} - 1) d\mu(u) + i \operatorname{Im} f(0) \qquad (z \in \mathbb{B}). \tag{4.1}$$

Note that $\frac{1}{(1-\langle z,w\rangle)^n}$ is the Szegö kernel of the unit ball and that the measure μ is the distributional boundary limit of $\operatorname{Re}(f)$.

In this case an elementary computation shows that the kernel

$$K_f(z, \overline{w}) = \frac{f(z) + \overline{f(w)}}{(1 - \langle z, w \rangle)^n} = \int_{\partial\mathbb{B}} \frac{d\mu(u)}{(1 - \langle z, u \rangle)^n (1 - \langle u, w \rangle)^n}$$

defines a positive definite function $K_f : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{C}$.

Viceversa, if K_f is a positive definite kernel, then taking its restriction to the diagonal $z = w$ we obtain that $\operatorname{Re}(f) \geq 0$. As Koranyi and Pukanszky [23] observed in the case of the unit polydisc, it is sufficient to ask the positivity of the kernel K_f on a much smaller set. The next result gives an extension of their original theorem to the case of the unit ball.

A subset $S \subset \mathbb{B}$ is called a uniqueness set for analytic functions if there exists a point $p \in S$ with the property that for all $0 < r < 1$, the only function $f \in \mathcal{O}(B(p, r))$ that vanishes on the set $S \cap B(p, r)$ is the trivial function $f = 0$.

Theorem 4.1 *Let $S \subset \mathbb{B}$ be a uniqueness set for analytic functions. Let $\phi : S \longrightarrow \mathbb{C}_r$ be a map with the property that the kernel*

$$\frac{\phi(z) + \overline{\phi(w)}}{(1 - \langle z, w \rangle)^n}$$

is positive definite on $S \times S$. Then there exists an analytic map $f : \mathbb{B} \longrightarrow \mathbb{C}_r$ such that $f(z) = \phi(z)$ for all $z \in S$.

Proof. According to Kolmogorov's factorization theorem, there exists a Hilbert space H and a vector–valued map $k : S \longrightarrow H$ with

$$\frac{\phi(z) + \overline{\phi(w)}}{(1 - \langle z, w \rangle)^n} = \langle k(z), k(w) \rangle \qquad (z, w \in S). \tag{4.2}$$

By applying a suitable Moebius transform we can reduce the assertion to the case that $p = 0$ is the point of analytic uniqueness for the set S.

By evaluating equation (4.2) successively at $z = 0$ and $w = 0$ we obtain the identity

$$(1 - \langle z, w \rangle)^n \langle k(z), k(w) \rangle = \langle k(z), k(0) \rangle + \langle k(0), k(w) \rangle - \langle k(0), k(0) \rangle. \quad (4.3)$$

Let us write

$$(1 - \langle z, w \rangle)^n = \sum_{|\alpha| \leq n} (-1)^{|\alpha|} c_\alpha z^\alpha \overline{w}^\alpha$$

with suitable constants $c_\alpha \geq 0$. Then equation (4.3) is easily seen to be equivalent to

$$\langle k(z) - k(0), k(w) - k(0) \rangle + \sum_{\alpha \in E(n)} c_\alpha \langle z^\alpha k(z), w^\alpha k(w) \rangle = \quad (4.4)$$

$$\sum_{\alpha \in O(n)} c_\alpha \langle z^\alpha k(z), w^\alpha k(w) \rangle,$$

where $E(n)$ consists of all multiindices $\alpha \in \mathbb{N}^n$ with $0 \neq |\alpha| \leq n$ even and $O(n)$ contains all multiindices $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq n$ odd.

Let $H_e = H \oplus \bigoplus_{E(n)} H$ and $H_o = \bigoplus_{O(n)} H$ be corresponding direct sums of copies of the Hilbert space H. It follows that there exists a contraction $V : H_o \longrightarrow H_e$ with the property that

$$V((\sqrt{c_\alpha} z^\alpha k(z))_{\alpha \in O(n)} = (k(z) - k(0)) \oplus (\sqrt{c_\alpha} z^\alpha k(z))_{\alpha \in E(n)} \qquad (z \in S).$$

In particular, by taking projections onto the first factor, we obtain bounded linear operators $T_\alpha : H \longrightarrow H$ such that

$$k(z) - k(0) = \sum_{|\alpha| \ odd} \sqrt{c_\alpha} z^\alpha T_\alpha k(z) \qquad (z \in S).$$

Thus there exists $0 < \rho < 1$ such that

$$\| \sum_{|\alpha| \ odd} \sqrt{c_\alpha} z^\alpha T_\alpha \| < 1 \qquad (|z| \leq \rho).$$

Consequently, for $|z|$ small enough, we obtain the formula

$$k(z) = [I - \sum_{|\alpha| \ odd} \sqrt{c_\alpha} z^\alpha T_\alpha]^{-1} k(0).$$

But this equation defines an analytic extension of the map $k : S \cap B(0,\rho) \longrightarrow H$ to a map $\ell : B(0,\rho) \longrightarrow H$.

In virtue of the analytic uniqueness property assumed at $p = 0$, equation (4.3) yields

$$(1 - \langle z, w\rangle)^n \langle \ell(z), \ell(w)\rangle = \langle \ell(z), \ell(0)\rangle + \langle \ell(0), \ell(w)\rangle - \langle \ell(0), \ell(0)\rangle \quad (|z|, |w| \leq \rho). \tag{4.5}$$

Recall that so far we have $\ell(z) = k(z)$ only if $z \in S \cap B(0,\rho)$. Our next aim is to show that actually the function ℓ extends analytically to the whole unit ball $\mathbb{B}$.

To this aim, let $\ell(z) = \sum_\alpha \ell_\alpha z^\alpha$, be the Taylor expansion of ℓ at $z = 0$. By evaluating both sides of (4.5) in z and $\overline{w}$ and by comparing the coefficients of the resulting power series in (z, w), one obtains the identity

$$\sum_\alpha \|\ell_\alpha\|^2 z^\alpha w^\alpha = \|\ell_0\|^2/(1 - \langle z, \overline{w}\rangle)^n$$

for $(z, w) \in B(0,\rho)^2$. Since the right–hand side is analytic on $\mathbb{B}^2$, the power series on the left converges absolutely for $(z, w) \in \mathbb{B}^2$. Using the observation that

$$\sum_\alpha \|\ell_\alpha\|^2 \; |z^\alpha|^2$$

converges for all $z \in \mathbb{B}$, one easily obtains that the series

$$\ell(z) = \sum_{\alpha \in \mathbb{N}^n} \ell_\alpha z^\alpha$$

converges on $\mathbb{B}$ and defines a holomorphic extension of the function $\ell : B(0,\rho) \to H$ which satisfies equation (4.5) for all $z, w \in \mathbb{B}$.

By using equation (4.3) and the analytic uniqueness of $B(0,\rho) \cap S$ for $\mathcal{O}(\mathbb{B})$ we obtain, for all $z \in \mathbb{B}$ and $w \in S$,

$$(1 - \langle z, w\rangle)^n \langle \ell(z), k(w)\rangle = \langle \ell(z), k(0)\rangle + \langle k(0), k(w)\rangle - \langle k(0), k(0)\rangle. \tag{4.6}$$

Now it suffices to choose $z = w \in S$ and to substract twice the real part of (4.6) from the sum of equations (4.3) and (4.5) to see that

$$(1 - |w|^2)^n \|\ell(w) - k(w)\|^2 = 0 \qquad (w \in S).$$

To complete the proof observe that $f(z) = \langle \ell(z), \ell(0) \rangle - \overline{\varphi(0)}$ defines an analytic function with non–negative real part on $\mathbb{B}$ which extends ϕ.

$\square$

If one replaces the field of complex numbers by the algebra $L(H)$ of all bounded operators on a given Hilbert space H, one obtains easily an operator–valued version of the last result.

Theorem 4.2 *Let H be a Hilbert space and let $S \subset \mathbb{B}$ be a uniqueness set for analytic functions. Then a map $\phi : S \to L(H)$ can be extended to a holomorphic function $f : \mathbb{B} \to L(H)$ with positive real part*

$$\mathrm{Re} f(z) \geq 0 \qquad (z \in \mathbb{B})$$

if and only if the function $K : S \times S \to L(H)$ defined by

$$K(z, w) = \frac{\phi(z) + \phi(w)^*}{(1 - \langle z, w \rangle)^n}$$

is positive definite. $\square$

To prove Theorem 4.2 it suffices to apply the operator–valued version of Kolmogorov's factorization theorem and then to argue exactly as in the previous proof. The reader will have no problems to fill in the details. A version of Theorem 4.1 for the polydisc can be found in [20]. In [13] and [33] corresponding results are proved for Hilbert spaces of analytic functions given by suitable reproducing kernels.

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