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# Twodimensional Variational Problems With A Wide Range Of Anisotropy

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**Abstract**

We consider local minimizers $u : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^M$ of the variational integral $\int_\Omega H(\nabla u)\, dx$ with density $H$ growing at least quadratically and allowing a very large scale of anisotropy. We discuss higher integrability properties of $\nabla u$ as well as the differentiability of $u$ in the classical sense. Moreover, a Liouville-type theorem is established.

# 1 Introduction

The aim of our note is to give a further analysis of the regularity properties such as local higher integrability of the gradient or differentiability in the classical sense of local minimizers $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^M$ from a suitable energy space of variational integrals like

$$(1.1) \qquad I[u, \Omega] = \int_\Omega H(\nabla u)\, dx\,,$$

provided $\Omega$ is a domain in $\mathbb{R}^2$ and $H$ is a density allowing a wide range of anisotropy, but growing at least quadratically. Let us agree for the moment to the following convention: $H$ is called an anisotropic energy density, if

$$(1.2) \qquad \lambda(|Z|)|Y|^2 \le D^2H(Z)(Y, Y) \le \Lambda(|Z|)|Y|^2$$

holds for all $Y$, $Z \in \mathbb{R}^{nM}$ with functions $\lambda$, $\Lambda : [0, \infty) \to [0, \infty)$, which can not be chosen in such a way that

$$c_1 \le \Lambda/\lambda \le c_2$$

is true with positive constants $c_1$ and $c_2$. Starting with the pioneering work of Marcellini [Ma1–4] various authors exhibited sufficient conditions on $\lambda$, $\Lambda$ and $H$ implying

I. full interior regularity, if $M = 1$ (scalar case)
or if $H = H(|\nabla u|)$ (dependence on the modulus) and

II. interior partial regularity in case $n \ge 3$ together with $M \ge 2$.

For I. we again refer to Marcellini's papers and his recent collaboration with Papi [MP]. We also mention the contributions of Choe [Ch], Fusco and Sbordone [FS] and of Mingione and Siepe [MS] as well as the references quoted by these authors. Further contributions to I. are given in [ABF] and [Fu]. In connection with II. the reader should consult for example the papers of Acerbi and Fusco [AF], Cupini, Guidorzi and Mascolo [CGM], Esposito, Leonetti and Mingione [ELM1,2] and of Passarelli Di Napoli and Siepe [PS] together with the references cited by these authors. We further refer to [BF1,2]. One very popular hypothesis for proving the results stated in I. and II. is the assumption of anisotropic $(p, q)$–growth, which means that the functions $\lambda$ and $\Lambda$ occurring in (1.2) behave like

$$\lambda(t) \approx t^{p-2},\ \Lambda(t) \approx t^{q-2} \tag{1.3}$$

with exponents $1 < p \le q < \infty$. If $p$ and $q$ are too far apart, then Giaquinta's counterexample [Gi] shows that even in the scalar case singular minimizers can occur. On the other hand, if we assume (1.3), then I. and II. are true, provided we additionally impose a bound of the form

$$q < c(n)p\,, \tag{1.4}$$

where $c(n)$ can be chosen rather large for low dimensions $n$, but $c(n) \to 1$ as $n \to \infty$. Let us remark that it is possible to drop (1.4) in the case $M = 1$ and to weaken this condition for $M \ge 2$, if $H$ is of splitting type and if we restrict ourselves to locally bounded local minimizers (cf. [BFZ], [BF3-5]).

Up to now the dimension $n$ was arbitrary, but the experience in regularity theory gives rise to the hope, that in case $n = 2$ better results for vectorial minimizers can be obtained, which means that in principle we expect the same behaviour as in the scalar case. In fact, if (1.2) holds with $\lambda$, $\Lambda$ from (1.3), then in [BF6] we could show interior $C^1$–regularity under the assumption

$$q < 2p\,, \tag{1.5}$$

and for the splitting case this is even true for any exponents $2 \le p \le q < \infty$ without the limitation (1.5), we refer to [BF3]. Further details are presented in [BF7]. In this note we want to discuss the twodimensional case for densities $H : \mathbb{R}^{2M} \to [0, \infty)$ of class $C^2$ assuming that $p$ is equal to 2, but imposing no upper bound on $D^2H$. It turns out that under these weak assumptions minimizers already belong to the Sobolev class $W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^M)$. If in addition we have (1.2) and (1.3) with $p = 2$ and $q$ arbitrary large, then there is an open set $\Omega_0 \subset \Omega$ (depending on the minimizer $u$) such that $u \in C^1(\Omega_0; \mathbb{R}^M)$ and $\mathcal{H} - \dim(\Omega - \Omega_0) = 0$. Let us now give a precise formulation of our assumptions imposed on $H$: without loss of generality let $H(0) = 0$ and $DH(0) = 0$. Moreover it should hold:

$$\exists a > 0 : |\xi||DH(\xi)| \le aH(\xi),\ \xi \in \mathbb{R}^{2M}\,; \tag{A1}$$

$$\lambda_0 := \inf_{\xi \in \mathbb{R}^{2M}} \frac{|DH(\xi)|}{|\xi|} > 0\,; \tag{A2}$$

$$\exists A > 0 : \frac{|DH(\xi)|}{|\xi|} \le A \inf_{|\sigma|=1} D^2H(\xi)(\sigma, \sigma),\ \xi \in \mathbb{R}^{2M}\,. \tag{A3}$$

**REMARK 1.1.** *From (A2) and (A3) it follows that we have the first inequality in (1.2) with $\lambda(t) \equiv \lambda_0/A$, which corresponds to the case "$p = 2$". In particular there exists a positive constant $c$ such that*

$$H(\xi) \geq c|\xi|^2, \ \xi \in \mathbb{R}^{2M}. \tag{1.6}$$

*On the other hand, hypothesis (A1) provides an upper bound for $H(\xi)$, more precisely we have*

$$H(\xi) \leq C(|\xi|^a + 1), \ \xi \in \mathbb{R}^{2M}, \tag{1.7}$$

*for a suitable constant $C$ and with exponent $a$ from (A1). Combining (1.6) and (1.7) we see that $a \geq 2$, and both inequalities show that the density $H$ itself is of anisotropic $(2, a)$-growth. In order to prove (1.7) we choose $\xi \in \mathbb{R}^{2M}$ such that $|\xi| \geq 1$ and let $H_0 := \sup\limits_{|\sigma|=1} H(\sigma)$. Then it holds*

$$\begin{aligned}
\ell n H(\xi) &\leq \ell n H(\xi) - \ell n H\left(\frac{\xi}{|\xi|}\right) + \ell n H_0 \\
&= \int_1^{|\xi|} \frac{d}{dt} \ell n H\left(t\frac{\xi}{|\xi|}\right) dt + \ell n H_0 \\
&= \int_1^{|\xi|} \frac{1}{H\left(t\frac{\xi}{|\xi|}\right)} DH\left(t\frac{\xi}{|\xi|}\right) : \frac{\xi}{|\xi|}\, dt + \ell n H_0 \\
&\overset{(A1)}{\leq} \int_1^{|\xi|} \frac{a}{t}\, dt + \ell n H_0 = a \ell n |\xi| + \ell n H_0, \ \textit{i.e.} \\
H(\xi) &\leq \exp(a \ell n |\xi| + \ell n H_0) = H_0 |\xi|^a,
\end{aligned}$$

*and (1.7) is established.*

**REMARK 1.2.** *With respect to the growth condition (1.6) a natural space for local minimizers $u$ of the functional $I$ defined in (1.1) is the class*

$$\mathcal{C} := \left\{ v \in W^1_{2,\mathrm{loc}}(\Omega; \mathbb{R}^M) : I[v, \Omega'] < \infty \ \forall\, \Omega' \Subset \Omega \right\},$$

*where here and in what follows $W^k_{p,\mathrm{loc}}(\dots)$ is the Sobolev space of functions as introduced for example in Adam's book [Ad]. By definition $u \in \mathcal{C}$ is a local $I$–minimizer iff $I[u, \Omega'] \leq I[v, \Omega']$ holds for all subdomains $\Omega' \Subset \Omega$ and any function $v \in \mathcal{C}$ such that $\mathrm{spt}(u - v) \subset \Omega'$.*

**REMARK 1.3.** *Obviously $H$ is a (strictly) convex function, thus in addition to (A1) we have the upper bound $H(\xi) \leq |DH(\xi)||\xi|$ for all $\xi \in \mathbb{R}^{2M}$.*

Our results are as follows:

**THEOREM 1.1.** *Let $H$ satisfy (A1-3) and consider a local $I$–minimizer $u \in \mathcal{C}$. Then it holds:*

*a)* $\int_{\Omega'} D^2H(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\, dx \le c(\Omega') < \infty$ *for any subdomain* $\Omega' \Subset \Omega$ *and for* $\alpha = 1, 2$.
*In particular* $u$ *is of class* $W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^M)$*, and this is also true in the case* $\Omega \subset \mathbb{R}^n$ *with* $n \ge 3$.

*b)* $\int_{\Omega_1} H(\nabla u)^2\, dx \le c(\Omega_1, \Omega_2)(\int_{\Omega_2} H(\nabla u)\, dx)^2$ *for arbitrary subdomains* $\Omega_1 \Subset \Omega_2 \Subset \Omega$.

**THEOREM 1.2.** *In addition to (A1-3) suppose that*

$$\text{(A4)} \qquad D^2H(\xi)(\sigma, \sigma) \le \Lambda_0(1 + |\xi|^2)^{\frac{q-2}{2}} |\sigma|^2, \xi, \sigma \in \mathbb{R}^{2M};$$

*is true for some* $\Lambda_0 > 0$ *and an exponent* $q \in [2, \infty)$*. Let* $u \in \mathcal{C}$ *denote a local minimizer. Then there is an open subset* $\Omega_0$ *of* $\Omega$*, whose complement is of zero Hausdorff–dimension, and* $u \in C^{1,\mu}(\Omega_0; \mathbb{R}^M)$ *for any* $\mu < 1$.

**REMARK 1.4.** *We conjecture that even under the hypotheses of Theorem 1.1 we have interior* $C^1$*–regularity of local minimizers.*

From the proof of Theorem 1.1 we will deduce the following Liouville-type result:

**THEOREM 1.3.** *Suppose that* $H$ *satisfies (A1-3) and let* $u$ *denote an entire local minimizer of the functional* $I$*, i.e.* $u$ *is a local minimizer on the whole plane. Then, if* $\int_{\mathbb{R}^2} H(\nabla u)\, dx$ *is finite,* $u$ *is a constant function. If the same situation is considered on* $\Omega = \mathbb{R}^n$ *with* $n \ge 3$*, then* $u$ *is affine. Moreover, if the condition of the finiteness of the energy is replaced by the requirement that the entire local minimizer* $u$ *is a bounded function, then in the* $2D$*-case* $u$ *again must be constant.*

Our paper is organized as follows: in Section 2 we present some examples of densities satisfying (A1-4). In Section 3 we will prove Theorem 1.1 and 1.3, and Section 4 is devoted to the proof of Theorem 1.2.

# 2 Some examples

We start with a construction borrowed from [BF7]. Let

$$\text{(2.1)} \qquad h(t) := \int_0^t sg(s)\, ds, \ t \ge 0,$$

with

$$\text{(2.2)} \qquad g(t) := 1 + \int_0^t \Theta(s)\, ds, \ t \ge 0,$$

where the continuous function $\Theta : [0, \infty) \to [0, \infty)$ is given as follows: suppose that we have fixed an arbitrary large exponent $\alpha$ and a sequence $\{a_i\}$ such that $0 < a_i < a_{i+1}$,

$\lim\limits_{i\to\infty} a_i = \infty$. Next we choose small positive numbers $\varepsilon_i$ with the properties $I_i := (a_i - \varepsilon_i, a_i + \varepsilon_i) \subset [0,\infty)$, $I_i \cap I_j = \phi$ for $i \neq j$ and

$$\sum_{i=1}^{\infty} \varepsilon_i a_i^{\alpha-1} < \infty\,. \tag{2.3}$$

Then we let

$$\Theta(t) := \begin{cases} 0 \text{ on } [0,\infty) - \bigcup\limits_{i=1}^{\infty} I_i\,, \\ \text{affine linear on } (a_i - \varepsilon_i, a_i) \quad \text{and on} \\ (a_i, a_i+\varepsilon_i) \text{ with value } a_i^{\alpha-1} \text{ at } t = a_i,\ i \in \mathbb{N}\,. \end{cases}$$

**Lemma 2.1.** *The function $h$ defined in (2.1) and (2.2) is of class $C^2([0,\infty))$, strictly increasing and convex together with $h''(0) = 1$ and $\lim\limits_{t\searrow 0} \frac{h(t)}{t} = 0$. With suitable positive constants $c_\ell$ it holds for all $t \geq 0$*

*a)* $h(2t) \leq c_1 h(t)$;

*b)* $\frac{h'(t)}{t} \leq h''(t)$, *and the function* $\frac{h'(t)}{t}$ *is increasing;*

*c)* $h'(t)t \leq c_2 h(t)$;

*d)* $h''(t) \leq c_3 (1+t^2)^{\frac{\alpha}{2}} \frac{h'(t)}{t}$ ;

*e)* $c_4 t^2 \leq h(t) \leq c_5 t^2$.

*Moreover it is not possible to replace $\alpha$ in d) by a smaller number.*

**REMARK 2.1.** *With $h$ from above let us introduce the energy density $H_0 : \mathbb{R}^{2M} \to [0,\infty)$, $H_0(Z) := h(|Z|)$. According to e) $H_0$ is of quadratic growth, and from b) and d) using $\frac{h'(t)}{t} \geq 1$ we infer for all $Y$, $Z \in \mathbb{R}^{2M}$*

$$|Y|^2 \leq D^2 H_0(Z)(Y,Y) \leq c_3 (1+|Z|^2)^{\frac{\alpha}{2}} g(|Z|)|Y|^2\,.$$

*Since $g$ is a bounded function (see below), this estimate shows that $H_0$ is $(p,q)$–elliptic in the sense of (1.3) with $p := 2$ and $q := 2+\alpha$.*

**Proof of Lemma 2.1:** We first observe that

$$\frac{h'(t)}{t} = g(t) \leq 1 + \int_0^\infty g'(s)\,ds = 1 + \int_0^\infty \Theta(s)\,ds = 1 + \sum_{i=1}^{\infty} \varepsilon_i a_i^{\alpha-1}\,,$$

hence by (2.3) we find a number $g_\infty \in (0,\infty)$ such that

$$1 = g(0) \leq g(t) \leq g_\infty,\ t \geq 0\,. \tag{2.4}$$

The properties of $h$ stated in front of a) are immediate. For a) we use (2.4), since this inequality implies $g(2t) \leq g_\infty g(t)$, hence $h(2t) = \int_0^{2t} sg(s)\,ds = 4\int_0^t sg(2s)\,ds \leq 4g_\infty \int_0^t sg(s)\,ds = 4g_\infty h(t)$.
b) is elemantary: $h''(t) = \frac{d}{dt}(tg(t)) = g(t) + tg'(t) \geq g(t) = \frac{h'(t)}{t}$.
c) is a consequence of a): we have $h(2t) = \int_0^{2t} h'(s)\,ds \geq \int_t^{2t} h'(s)\,ds \geq th'(t)$, hence $th'(t) \leq c_1 h(t)$.

The validity of d) is equivalent to the existence of a constant $c_6$ such that

$$tg'(t) \leq c_6 t^\alpha g(t) \tag{2.5}$$

holds for all large $t$. The left–hand side of (2.5) equals $t\Theta(t)$, and according to (2.4) the right–hand side of (2.5) behaves like $t^\alpha$, thus our claim is immediate by the definition of $\Theta$ and we also see that $\alpha$ can not be replaced by a smaller exponent. Finally, the validity of e) is obvious. $\square$

Now we can state our examples:

**Lemma 2.2.** *Let $B : \mathbb{R}^{2M} \times \mathbb{R}^{2M} \to \mathbb{R}$ denote any symmetric, strictly positive bilinear function. With $h$ from Lemma 2.1 we let*

$$\begin{aligned} H_1(Z) &:= B(Z,Z) + h(|Z|)\,, \\ H_2(Z) &:= B(Z,Z) + h\left(\sqrt{B(Z,Z)}\right),\ Z \in \mathbb{R}^{2M}\,. \end{aligned}$$

*Then $H_1$ and $H_2$ satisfy (A1-4) with $q = \alpha + 2$.*

With Lemma 2.1 the proof of Lemma 2.2 is immediate. The reader should note that in these examples condition (1.5) is violated, provided we choose $\alpha \geq 2$. In this case we can not refer to the paper [BF6], however - according to Theorem 1.1 and 1.2 - we still have some regularity results for local minima.

# 3 Proof of Theorem 1.1 and 1.3

Let the assumptions of Theorem 1.1 hold and consider a local minimizer $u \in \mathcal{C}$. The following calculations can be made precise by replacing derivatives through difference quotients. If in addition we have (A4), then we can work alternatively with a local regularization with exponent $q$ (see, e.g. [BF1]) having a sufficient degree of regularity. For the particular examples involving "the function $h$" from Section 2 a quadratic regularization from below can be applied (compare [BF7]). Let $\eta \in C_0^\infty(\Omega)$. With " $:$ " denoting the scalar product of matrices and using "$\otimes$" as symbol for the tensor product of vectors from $\mathbb{R}^M$ we obtain from Euler's equation valid for $u$ (from now on we use the convention of summation with respect to indices repeated twice)

$$
\begin{aligned}
0 &= \int_\Omega \partial_\gamma \left[DH(\nabla u)\right] : \nabla(\eta^2 \partial_\gamma u)\, dx \\
&= \int_\Omega \eta^2 D^2 H(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma \nabla u)\, dx - \int_\Omega DH(\nabla u) : \partial_\gamma \left[\nabla \eta^2 \otimes \partial_\gamma u\right]\, dx
\end{aligned}
$$

and in conclusion

$$
\begin{aligned}
(3.1) \qquad & \int_\Omega D^2 H(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma \nabla u)\eta^2\, dx \\
&= \int_\Omega DH(\nabla u) : (\partial_\gamma \nabla \eta^2 \otimes \partial_\gamma u)\, dx + \int_\Omega DH(\nabla u) : (\nabla \eta^2 \otimes \Delta u)\, dx \\
&=: T_1 + T_2\,.
\end{aligned}
$$

From (A1) we get

$$
\begin{aligned}
T_1 &\le 2 \int_\Omega |DH(\nabla u)||\nabla u| \left\{|\nabla \eta|^2 + |\nabla^2 \eta|\right\}\, dx \\
&\le 2a \left\{\|\nabla \eta\|^2_{L^\infty(\Omega)} + \|\nabla^2 \eta\|_{L^\infty(\Omega)}\right\} \int_{\operatorname{spt}\eta} H(\nabla u)\, dx \\
&=: c(\eta) \int_{\operatorname{spt}\eta} H(\nabla u)\, dx\,,
\end{aligned}
$$

and for $T_2$ we observe

$$
\begin{aligned}
T_2 &\le 2 \int_\Omega \eta |DH(\nabla u)||\nabla^2 u||\nabla \eta|\, dx \\
&= 2 \int_\Omega \eta \left(\frac{|DH(\nabla u)|}{|\nabla u|}\right)^{1/2} |\nabla^2 u||\nabla \eta| \left(|DH(\nabla u)||\nabla u|\right)^{1/2}\, dx \\
&\le \varepsilon \int_\Omega \eta^2 \frac{|DH(\nabla u)|}{|\nabla u|} |\nabla^2 u|^2\, dx + c(\varepsilon) \int_\Omega |\nabla \eta|^2 |DH(\nabla u)||\nabla u|\, dx\,,
\end{aligned}
$$

where $\varepsilon > 0$ is arbitrary. Recalling (A3), choosing $\varepsilon$ sufficiently small and applying (A1) one more time, we deduce from (3.1) and the estimates from above

$$
(3.2) \qquad \int_\Omega \eta^2 D^2 H(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma \nabla u)\, dx \le c(\eta) \int_{\operatorname{spt}\eta} H(\nabla u)\, dx\,.
$$

This proves part a) of Theorem 1.1, since by (A2) and (A3)

$$
D^2 H(\nabla u)\,(\partial_\gamma \nabla u, \partial_\gamma \nabla u) \ge \frac{\lambda_0}{A} |\nabla^2 u|^2\,.
$$

Moreover we note that the right-hand side of (3.2) is finite. We emphasize that up to now we did not make use of our assumption that $\Omega$ is a domain in $\mathbb{R}^2$. For b) we apply

Sobolev's inequality and get

$$
\begin{aligned}
\int_\Omega (\eta H(\nabla u))^2 \, dx &\le c \left[ \int_\Omega |\nabla(\eta H(\nabla u))| \, dx \right]^2 \\
&\le \; c \left[ \int_\Omega |\nabla\eta| H(\nabla u) \, dx + \int_\Omega \eta |DH(\nabla u)||\nabla^2 u| \, dx \right]^2 \\
&\le \; c(\eta) \left[ \int_{\operatorname{spt}\eta} H(\nabla u) \, dx \right]^2 + c \left[ \int_\Omega \eta |DH(\nabla u)||\nabla^2 u| \, dx \right]^2 .
\end{aligned}
$$

The last integral on the right-hand side is handled with the help of (A3) and Hölder's inequality:

$$
\begin{aligned}
\left[ \int_\Omega |DH(\nabla u)||\nabla^2 u| \eta \, dx \right]^2 &\le c(\eta) \left( \int_{\operatorname{spt}\eta} |DH(\nabla u)||\nabla u| \, dx \right) \\
&\quad \cdot \int_{\operatorname{spt}\eta} \frac{|DH(\nabla u)|}{|\nabla u|} |\nabla^2 u|^2 \, dx \\
&\le \; c(\eta) \left( \int_{\operatorname{spt}\eta} |DH(\nabla u)||\nabla u| \, dx \right) \int_{\operatorname{spt}\eta} D^2 H(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma \nabla u) \, dx \, .
\end{aligned}
$$

Applying (3.2) (with appropriate choice of $\eta$) and using (A1), we arrive at our claim

$$
\int_{\Omega_1} H(\nabla u)^2 \, dx \le c(\Omega_1, \Omega_2) \left( \int_{\Omega_2} H(\nabla u) \, dx \right)^2
$$

for arbitrary subdomains $\Omega_1 \Subset \Omega_2 \Subset \Omega$. □

For proving Theorem 1.3 we first observe that (3.2) immediately implies (for all dimensions $n \ge 2$)

$$
\int_{B_R(0)} D^2 H(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma \nabla u) \, dx \le c R^{-2} \int_{B_{2R}(0)} H(\nabla u) \, dx \tag{3.3}
$$

for any radius $R > 0$. Therefore, if we assume that

$$
\int_{\mathbb{R}^n} H(\nabla u) \, dx < \infty \, , \tag{3.4}
$$

we get from (3.3) by passing to the limit $R \to \infty$ that $\nabla^2 u = 0$. At the same time it is easy to see that the estimates for the $2D$-case stated after (3.2) yield

$$
\int_{B_R(0)} H(\nabla u)^2 \, dx \le c R^{-2} \left( \int_{B_{2R}(0)} H(\nabla u) \, dx \right)^2
$$

thus, under the hypothesis (3.4), we must have

$$
\int_{\mathbb{R}^2} H(\nabla u)^2 \, dx = 0
$$

and therefore the Jacobian matrix of the affine linear function $u$ actually vanishes. Assume now that in place of (3.4) we have

$$L := \|u\|_{L^\infty(\mathbb{R}^2)} < \infty \, . \tag{3.5}$$

We first claim that (3.5) implies

$$\lim_{R\to\infty} R^{-2} \int_{B_R(0)} H(\nabla u)\, dx = 0 \, , \tag{3.6}$$

so that $\nabla^2 u = 0$ will follow from (3.3) and (3.6). For proving (3.6) let us fix $a$ number $\tau > 1$ to be specified later. For $\xi \in \mathbb{R}^{2M}$, $|\xi| \geq 1$, it holds on account of (A1) and (1.7)

$$\begin{aligned} |DH(\xi)|^\tau &\leq a\, H(\xi)^\tau |\xi|^{-\tau} = a\, H(\xi) H(\xi)^{\tau-1}|\xi|^{-\tau} \\ &\leq \ c\, H(\xi)|\xi|^{a(\tau-1)-\tau} \leq c\, H(\xi) \, , \end{aligned}$$

provided we choose $\tau$ such that $a(\tau-1)-\tau \leq 0$. In this case we arrive at

$$|DH(\xi)| \leq c \left( H(\xi)^{1/\tau} + 1 \right) , \ \xi \in \mathbb{R}^{2M} \, . \tag{3.7}$$

Next consider $R > 0$, let $k \in \mathbb{N}$ and choose $\eta \in C_0^\infty(B_{2R}(0))$, $\eta = 1$ on $B_R(0)$, $0 \leq \eta \leq 1$, $|\nabla \eta| \leq c/R$.

Starting from

$$0 = \int_{B_{2R}(0)} DH(\nabla u) : \nabla \left( \eta^{2k} u \right) \, dx \, ,$$

observing $H(\xi) \leq \xi : DH(\xi)$ (see Remark 1.3) and using (3.5) we find

$$\int_{B_{2R}(0)} \eta^{2k} H(\nabla u)\, dx \leq c \int_{B_{2R}(0)} \eta^{2k-1} |\nabla \eta| |DH(\nabla u)|\, dx \tag{3.8}$$

with constant $c$ depending on $L$ and $k$. On the right-hand side of (3.8) we apply Young's inequality in combination with (3.7) and get for all $\varepsilon > 0$

$$\begin{aligned} &\int_{B_{2R}(0)} \eta^{2k-1} |\nabla\eta| |DH(\nabla u)|\, dx \\ &\leq \ c \int_{B_{2R}(0)} \eta^{2k-1} H(\nabla u)^{1/\tau} |\nabla \eta|\, dx + c \int_{B_{2R}(0)} |\nabla\eta|\, dx \\ &\leq \ \varepsilon \int_{B_{2R}(0)} \eta^{(2k-1)\tau} H(\nabla u)\, dx + c(\varepsilon) \int_{B_{2R}(0)} |\nabla \eta|^{\frac{\tau}{\tau-1}}\, dx \\ &\quad + c \int_{B_{2R}(0)} |\nabla \eta|\, dx \, . \end{aligned}$$

Thus, if $\varepsilon$ is small enough and if we choose $k$ so large that $(2k-1)\tau \geq 2k$, then the $\varepsilon$-term can be absorbed into the left-hand side of (3.8). Observing that $\tau/\tau - 1 > 1$, we deduce at least for $R \geq 1$

$$\int_{B_R(0)} H(\nabla u)\, dx \leq c \int_{B_{2R}(0)} |\nabla \eta|\, dx \leq cR \, ,$$

and (3.6) follows. From the above inequality and the estimate stated after (3.4) we infer $\int_{\mathbb{R}^2} H(\nabla u)^2\,dx < \infty$, but since $\nabla u$ is a constant matrix, this is only possible in the case that $\nabla u$ vanishes. $\square$

# 4 Proof of Theorem 1.2

Suppose now that in addition to (A1-3) the hypothesis (A4) with $q > 2$ is valid and fix a local minimizer $u \in \mathcal{C}$. By Theorem 1.1 and Sobolev's embedding theorem $|\nabla u|$ belongs to the space $\bigcap\limits_{1 \le s < \infty} L^s_{\mathrm{loc}}(\Omega)$, thus the excess function

$$E(x,r) =: \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2\; dy + \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^q\; dy\,,$$

where $(g)_{x,r}$ denotes the mean value of a function $g$ with respect to a disc $B_r(x) \Subset \Omega$, is well defined. We claim:

**Lemma 4.1.** *Fix $L > 0$. Then there exists a constant $C_*(L)$ such that for every $\tau \in (0, 1/4)$ there is an $\varepsilon = \varepsilon(L,\tau)$ satisfying: if $B_r(x) \Subset \Omega$ and if we have*

$$|(\nabla u)_{x,r}| \le L,\; E(x,r) \le \varepsilon(L,\tau)\,,$$

*then $E(x,\tau r) \le C_*(L) E(x,r)\tau^2$.*

**Proof:** We argue by contradiction following the ideas of [BF1] assuming that $L > 0$ is fixed. The constant $C_*(L)$ will be specified below. If the lemma is wrong, then for some $\tau$ there are discs $B_{r_n}(x_n) \Subset \Omega$ such that

$$|(\nabla u)_{x_n,r_n}| \le L,\; E(x_n,r_n) =: \lambda_n^2 \to 0,\; n \to \infty\,, \tag{4.1}$$

$$E(x_n, r_n\tau) > C_*\tau^2\lambda_n^2\,. \tag{4.2}$$

Letting $a_n := (u)_{x_n,r_n}$, $A_n := (\nabla u)_{x_n,r_n}$ and

$$u_n(z) := \frac{1}{\lambda_n r_n}\left[u(x_n + r_n z) - a_n - r_n A_n z\right],\; |z| < 1\,,$$

we obtain from (4.1) and (4.2)

$$|A_n| \le L,\; \fint_{B_1} |\nabla u_n|^2\, dz + \lambda_n^{q-2} \fint_{B_1} |\nabla u_n|^q\, dz = 1\,. \tag{4.3}$$

$$\fint_{B_\tau} |\nabla u_n - (\nabla u_n)_{0,\tau}|^2\, dx + \lambda_n^{q-2} \fint_{B_\tau} |\nabla u_n - (\nabla u_n)_{0,\tau}|^q\, dz > C_*\tau^2\,. \tag{4.4}$$

From (4.3) we get after passing to subsequences

$$
\begin{array}{llll}
(4.5) \qquad & u_n \rightharpoonup: \overline{u} & \text{in} & W^1_2(B_1;\mathbb{R}^M),\ A_n \to: \overline{A}\,, \\
& \lambda_n \nabla u_n \to 0 & \text{in} & L^2(B_1;\mathbb{R}^{2M}) \text{ and a.e.}\,, \\
& \lambda_n^{1-2/q} \nabla u_n \rightharpoonup 0 & \text{in} & L^q(B_1;\mathbb{R}^{2M})\,.
\end{array}
$$

It is easy to verify that $\overline{u}$ satisfies

$$\fint_{B_1} D^2H(\overline{A})(\nabla \overline{u}, \nabla \varphi)\, dx = 0,\ \varphi \in C^\infty_0(B_1;\mathbb{R}^M)\,,$$

and thereby is a smooth function for which the Campanato estimate

$$\fint_{B_\tau} |\nabla \overline{u} - (\nabla \overline{u})_{0,\tau}|^2\, dx \le C^* \tau^2 \tag{4.6}$$

holds with a suitable constant $C^* = C^*(L)$. Let us choose $C_* := 2C^*$. Then (4.4) and (4.6) are in contradiction, if we can improve the weak convergences from (4.5) to

$$\nabla u_n \to \nabla \overline{u} \text{ in } L^2_{\mathrm{loc}}(B_1;\mathbb{R}^{2M})\,, \tag{4.7}$$

$$\lambda_n^{1-2/q} \nabla u_n \to 0 \text{ in } L^q_{\mathrm{loc}}(B_1;\mathbb{R}^{2M})\,. \tag{4.8}$$

The claim (4.7) follows exactly as (4.16) i) in [BF1] by quoting Proposition 4.3 from this reference (letting $\mu = 0$ there). For verifying (4.8) we let

$$\Psi_n := \lambda_n^{-1} \left[ (1 + |A_n + \lambda_n \nabla u_n|^2)^{1/2} - (1 + |A_n|^2)^{1/2} \right]$$

and observe that from ($Q \in \mathbb{R}^{2M}$, $\eta \in C^\infty_0(\Omega)$)

$$0 = \int_\Omega \partial_\gamma [DH(\nabla u)] : \nabla(\eta^2 \partial_\gamma [u - Qx])\, dx$$

we obtain the Caccioppoli inequality

$$\int_\Omega \eta^2 D^2H(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma \nabla u)\, dx \le c \int_\Omega |\nabla \eta|^2 |D^2H(\nabla u)| |\nabla u - Q|^2\, dx\,. \tag{4.9}$$

Combining (4.9) with (A2) and (A3) we get after scaling for $\rho \in (0,1)$ (choosing $Q = A_n$)

$$\int_{B_\rho} |\nabla \Psi_n|^2\, dz \le c(\rho) \int_{B_1} |D^2H(A_n + \lambda_n \nabla u_n)| |\nabla u_n|^2\, dx\,. \tag{4.10}$$

Now, by (4.3), the right-hand side of (4.10) is bounded through a finite constant so that

$$\sup_n \int_{B_\rho} |\nabla \Psi_n|^2\, dz \le c(\rho) < \infty,\ 0 < \rho < 1\,. \tag{4.11}$$

At the same time we have

$$|\Psi_n| \le c |\nabla u_n|$$

and therefore in addition to (4.11) (recall (4.3))

$$\sup_n \int_{B_\rho} |\Psi_n|^2 \, dz \le c(\rho) < \infty, \ 0 < \rho < 1 \,. \tag{4.12}$$

With (4.11) and (4.12) it is shown that $\{\Psi_n\}$ is a bounded sequence in each space $W^1_2(B_\rho)$, $0 < \rho < 1$, and this will imply (4.8): to this purpose we fix a number $K \gg 1$. Letting

$$U_n := U_n(K, \rho) := \{ z \in B_\rho : \lambda_n |\nabla u_n(z)| \le K \}$$

we obtain from $q > 2$ and (4.7) (using the smoothness of $\overline{u}$)

$$\begin{aligned} \int_{U_n} \lambda_n^{q-2} |\nabla u_n|^q \, dz &\le c \left\{ \lambda_n^{q-2} \int_{U_n} |\nabla u_n - \nabla \overline{u}|^q \, dz + \lambda_n^{q-2} \int_{U_n} |\nabla \overline{u}|^q \, dz \right\} \\ &\le c \left\{ \int_{U_n} \lambda_n^{q-2} \left( |\nabla u_n|^{q-2} + |\nabla \overline{u}|^{q-2} \right) |\nabla u_n - \nabla \overline{u}|^2 \, dz + \int_{U_n} \lambda_n^{q-2} |\nabla \overline{u}|^q \, dz \right\} \to 0 \\ &\text{as} \quad n \to \infty \,. \end{aligned}$$

On the other hand, for $K$ large enough and $z \in B_\rho - U_n$ it holds $\Psi_n(z) \ge c|\nabla u_n(z)|$, hence

$$\lambda_n^{q-2} \Psi_n^q(z) \ge c \, \lambda_n^{q-2} |\nabla u_n(z)|^q \,.$$

By Sobolev's embedding theorem we have

$$\sup_n \int_{B_\rho} |\Psi_n|^q \, dz < \infty \,,$$

and since we assume $q > 2$, it follows from the above estimates

$$\int_{B_\rho - U_n} \lambda_n^{q-2} |\nabla u_n|^q \, dz \to 0, \ n \to \infty \,.$$

This proves (4.8) and thereby Lemma 4.1. □

Let us define the set

$$\Omega_0 := \{ x \in \Omega : \ \sup_{r > 0} |(\nabla u)_{x,r}| < \infty \text{ and } \liminf_{r \searrow 0} E(x, r) = 0 \} \,.$$

Then, according to Lemma 4.1, $u$ is of class $C^{1,\mu}$ in a neighborhood of each point $x \in \Omega_0$, and it remains to check more precisely, which points $x \in \Omega$ belong to the set $\Omega_0$. Since $u$ is in $W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^M)$, we deduce from Poincaré's inequality

$$\begin{aligned} \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2 \, dy &\le c r^2 \fint_{B_r(x)} |\nabla^2 u|^2 \, dy \\ &= c \int_{B_r(x)} |\nabla^2 u|^2 \, dy \to 0, \ r \to 0 \,, \end{aligned}$$

for all points $x \in \Omega$. Next let $s := \frac{2q}{2+q}$. The Sobolev-Poincaré estimate gives

$$\left(\fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^q \, dy\right)^{1/q} \le cr \left(\fint_{B_r(x)} |\nabla^2 u|^s \, dy\right)^{1/s},$$

hence we find after applying Hölder's inequality

$$\begin{aligned}\fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^q \, dy &\le cr^q \, r^{-\frac{2}{s}q} \left(\int_{B_r(x)} |\nabla^2 u|^s \, dy\right)^{q/s} \\ & c\, r^{q-\frac{2}{s}q} r^{2\frac{q}{s}-q} \left(\int_{B_r(x)} |\nabla^2 u|^2 \, dy\right)^{q/2} \to 0,\ r \to 0\,,\end{aligned}$$

so that $\lim_{r\to 0} E(x,r) = 0$ for all $x \in \Omega$. Therefore $\Omega_0 = \{x \in \Omega : \sup_{r>0} |(\nabla u)_{x,r}| < \infty\}$ and the complement of this set is of Hausdorff-dimension zero. This finishes the proof of Theorem 1.2. □

# References


[AF] Acerbi, E., Fusco, N., Partial regularity under anisotropic $(p, q)$ growth conditions. J. Differ. Equ. 107(1), 46–67 (1994).

[Ad] Adams, R. A., Sobolev spaces. Academic Press, New York (1975).

[ABF] Apushkinskaya, D., Bildhauer, M., Fuchs, M., Interior gradient bounds for local minimizers of variational integrals under nonstandard growth conditions. Problems in Math. Anal.43, 35–50 (2009).

[BF1] Bildhauer, M., Fuchs, M., Partial regularity for variational integrals with $(s, \mu, q)$-growth. Calc. Var.13, 537–560 (2001).

[BF2] Bildhauer, M., Fuchs, M., $C^{1,\alpha}$–solutions to non–autonomous anisotropic variational problems. Calc. Var.24, 309–340 (2005).

[BF3] Bildhauer, M., Fuchs, M., Higher integrability of the gradient for vectorial minimizers of decomposable variational integrals. Manus. Math.123, 269–283 (2007).

[BF4] Bildhauer, M., Fuchs, M., Partial regularity for local minimizers of splitting-type variational integrals. Asymp. Anal.55, 33–47 (2007).

[BF5] Bildhauer, M., Fuchs, M., Variational integrals of splitting type: higher integrability under general growth conditions. Ann. Pura Appl.188(3), 467–496 (2009).

[BF6] Bildhauer, M., Fuchs, M., Twodimensional anisotropic variational problems. Calc. Var.16, 177–186 (2003).

[BF7] Bildhauer, M., Fuchs, M., Differentiability and higher integrability results for local minimizers of splitting type variational integrals in 2D with applications to nonlinear Hencky materials. Calc. Var.37, 167–186 (2010).

[BFZ] Bildhauer, M., Fuchs, M., Zhong, X., A regularity theory for scalar local minimizers of splitting type variational integrals. Ann. SNS Pisa Serie V VI(Fasc.3), 385–404 (2007).

[Ch] Choe, H. J., Interior behaviour of minimizers for certain functionals with nonstandard growth. Nonlinear Anal. Theory Methods Appl.19.10, 933–945 (1992).

[CGM] Cupini, G., Guidorzi, M., Mascolo, E., Regularity of minimizers of vectorial integrals with $p-q$ growth. Nonlinear Anal.54, 591–616(2003).

[ELM1] Esposito, L., Leonetti, F., Mingione, G., Regularity results for minimizers of irregular integrals with $(p-q)$– growth. Forum Math.14, 245–272 (2002).

[ELM2] Esposito, L., Leonetti, F., Mingione, G., Sharp regularity for functionals with $(p, q)$ growth. J. Differ. Eq.204, 5–55 (2004).

[Fu] Fuchs, M., Local Lipschitz regularity of vector valued local minimizers of variational integrals depending on the modulus of the gradient. to appear in Math. Nachrichten.

[FS] Fusco, N., Sbordone, C., Some remarks on the regularity of minima of anisotropic integrals. Comm. P.D.E.18, 153–167 (1993).

[Gi] Giaquinta, M., Growth conditions and regularity, a counterexample. Manus. Math.59, 245–248 (1987).

[Ma1] Marcellini, P., Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. Arch. Rat. Mech. Anal. 105, 267–284 (1989).

[Ma2] Marcellini, P., Everywhere regularity for a class of elliptic systems without growth conditions. Ann. Scuola Norm. Sup. Pisa CI. Sci.(4)23, 1–25 (1996).

[Ma3] Marcellini, P., Regularity and existence of solutions of elliptic equations with $(p, q)$–growth conditions. J. Diff. Equ.90, 1–30 (1991).

[Ma4] Marcellini, P., Regularity for elliptic equations with general growth conditions. J. Diff. Equ.105, 296–333 (1993).

[MP] Marcellini, P., Papi, G., Nonlinear elliptic systems with general growth. J. Diff. Equ.221, 412–443 (2006).

[MS] Mingione, G., Siepe, F., Full $C^{1,\alpha}$-regularity for minimizers of integral functionals with L log L-growth. Z. Anal. Anw.18, 1083–1100 (1999).

[PS] Passarelli Di Napoli, A., Siepe, F., A regularity result for a class of anisotropic systems. Rend. Ist. Mat. Univ. Trieste 28(1-2), 13–31 (1996).