Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 275

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Saarbrücken 2010

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 275
submitted: 20^{th} of September, 2010

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Abstract. In this paper we derive a posteriori error estimates for the Hencky plasticity problem. These estimates are formulated in terms of the stresses and present guaranteed and computable bounds of the difference between the exact stress field and any approximation of it from the energy space of the dual variational problem. They consist of quantities that can be considered as penalties for the violations of the equilibrium equations, the yield condition and the constitutive relations that must hold for the exact stresses and strains. It is proved that the upper bound tends to zero for any sequence of stresses that tends to the exact solution of the Haar-Karman variational problem. An important ingredient of our analysis is a collection of Poincare-type inequalities involving the L^1 norms of the tensors of small deformation. Estimates of this form are not new, however we will present computable upper bounds for the constants being involved even for rather complicated domains.

1. The Hencky plasticity problem

Let Ω be an open bounded set in $\mathbb{R}^n$ (n=2 or 3) with Lipschitz boundary $\partial\Omega$, which is the union of two nonempty measurable nonintersecting parts Γ_D and Γ_N. By $\mathbb{R}^n$ and $\mathbb{M}_s^{n\times n}$ we denote the spaces of real n-vectors and the space of symmetric $(n \times n)$ matrices (tensors), respectively. In what follows, Latin letters are typically used for vectors and the Greek ones for matrices. Tensors associated with stresses are marked by stars. By $\mathbb{I}$ we denote the identity matrix, and the symbols $\operatorname{tr}\tau$ and $\tau^D = \tau - \frac{1}{n}\operatorname{tr}\tau\mathbb{I}$ are used for the trace and the deviator of a matrix τ. The scalar product of vectors is denoted by $\cdot$ (e.g., $u \cdot v$) and the product of tensors by : (e.g., $\tau:\sigma$). For the corresponding Euclidean norms we just write $|\cdot|$.

The classical statement of the boundary value problem for the Hencky plasticity model is as follows: find a stress tensor σ^* and a displacement field u satisfying the following system of equations and inequalities (compare, e.g., [19, 21, 9]):

$$\operatorname{Div}\sigma^* + f = 0, \qquad \text{in } \Omega; \tag{1.1}$$

$$\sigma^* = F, \qquad \text{on } \Gamma_N; \tag{1.2}$$

$$u = u_0, \qquad \text{on } \Gamma_D; \tag{1.3}$$

$$\varepsilon(u) = A\sigma^* + \varepsilon_p \qquad \text{in } \Omega; \tag{1.4}$$

$$|\sigma^{*D}| \leq \sqrt{2}k_*, \qquad \text{in } \Omega; \tag{1.5}$$

$$\varepsilon_p : (\tau - \sigma^*) \leq 0 \qquad \forall \tau \in \mathbb{M}_s^{n\times n},\ |\tau^D| \leq \sqrt{2}k_* \tag{1.6}$$

Here $\operatorname{Div}\sigma^*$ denotes the divergence of σ^*, ν is the unit outward normal to the boundary, u_0 is a given vector valued function determining Dirichlet boundary conditions on Γ_D, f and F are the volume and the surface loads, and $A : \mathbb{M}_s^{n\times n} \to \mathbb{M}_s^{n\times n}$ is the elasticity operator. The quantity $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the tensor of small deformations, where the superscript T denotes matrix transposition. The relation (1.1) is the *equilibrium equation* for the stress tensor and (1.4) – (1.6)

1991 *Mathematics Subject Classification.* Primary 65N30.

Key words and phrases. perfect plasticity, a posteriori error estimates of functional type, variational problems with linear growth, duality theory.

are the *constitutive relations* of the elasto–plastic medium. In (1.4) ε_p stands for the plastic part of the strain tensor, and in (1.5) k_* is a positive constant determining the yield surface.
For isotropic media the elasticity operator has the form

$$A\sigma^* \;=\; \frac{\operatorname{tr}\sigma^*}{n^2K_0}\mathbb{I} + \frac{1}{2\mu}\sigma^{*D}, \tag{1.7}$$

where K_0 and μ are positive constants characterizing the elastic properties of the deformed body.
To give a functional formulation for the classical problem (1.1) – (1.5) we introduce the affine manifold

$$V_0 + u_0 \;:= \left\{ v \in V := W^{1,2}(\Omega, \mathbb{R}^n) \;\;\| \;\; v = w + u_0,\; w \in V_0 \right\},$$

where V_0 is the subspace of V that contains functions vanishing on the boundary part Γ_D in the usual sense of traces. As it follows from (1.1) and (1.4) the admissible stress tensors (which we denote by Greek letters with stars) should belong to the set of tensor valued functions satisfying the pointwise yield condition

$$K^* := \left\{ \tau^* \in Y^* := L_2(\Omega, \mathbb{M}_s^{n\times n}) \;\;\| \mid \tau^{*D}(x) \mid \leq \sqrt{2}k_* \text{ for a.e. } x \;\in \Omega \right\},$$

and to the set

$$Q^*_\ell = \left\{ \tau^* \in Y^* \;\;\| \;\; \int_\Omega \tau^* : \varepsilon(v)dx = \int_\Omega f \cdot v\, dx + \int\limits_{\Gamma_N} F \cdot v d\gamma \;\; \forall v \in V_0 \right\},$$

which contains tensors satisfying (in a generalized sense) the equilibrium equations.
The standard norms in the Lebesgue and Sobolev spaces $L_p(\Omega;\mathbb{R}^m)$, $L_p(\Omega;\mathbb{M}_s^{n\times n})$, $W^l_p(\Omega,\mathbb{R}^m)$ are denoted by $\|\cdot\|_{p,\Omega}$, $\|\cdot\|_{l,p,\Omega}$. For L_2 norms we also use the simplified notation $\|\cdot\|$. Another pair of equivalent norms in Y^* is defined by the relations

$$\| \;\varepsilon\; \|^2 := \int_\Omega A\varepsilon : \varepsilon\, dx, \quad \text{and} \quad \| \;\eta^*\; \|^2_* := \int_\Omega A^{-1}\eta^* : \eta^*\, dx,$$

which are equivalent to the norm $\|\cdot\|_\Omega$.
A natural functional formulation of the classical problem (1.1) – (1.5) is presented by two variational problems (for displacements and stresses), which we denote $\mathcal{P}$ and $\mathcal{P}^*$.
Problem $\mathcal{P}^*$ (Haar–Karman variational principle). *Find a tensor-function $\sigma^* \in K^* \cap Q^*_\ell$ such that*

$$I^*(\sigma^*) = \sup_{\tau^* \in K^* \cap Q^*_\ell} I^*(\tau^*),$$

where

$$I^*(\tau^*) := \int_\Omega \Big(\varepsilon(u_0) : \tau^* - \frac{1}{2}A\tau^* : \tau^* \Big) dx - \ell(u_0),$$

is the complementary energy functional and

$$\ell(v) := \int_\Omega f \cdot v\, dx + \int_{\partial_2\Omega} F \cdot v\, ds.$$

In the practically important case of isotropic media, we have

$$I^*(\tau^*)=\int_\Omega\Big(\frac{1}{n}\operatorname{div}u_0\ \operatorname{tr}(\tau^*)-\frac{1}{2n^2K_0}(\operatorname{tr}\tau^*)^2+\varepsilon^D(u_0):\tau^{*D}-g_0^*(\tau^{*D})\Big)dx-\ell(u_0),$$

where

$$g_0^*(\eta^*)=\begin{cases}\frac{1}{4\mu}|\eta^*|^2 & \text{if } |\eta^*|\le\sqrt{2}k_*\\ +\infty & \text{if } |\eta^*|>\sqrt{2}k_*.\end{cases}$$

Henceforth we assume that

$$f\in L_\infty(\Omega;\mathbb{R}^n),\qquad F\in L_\infty(\Gamma_N;\mathbb{R}^n),\tag{1.8}$$

$$u_0\in W_2^1(\Omega;\mathbb{R}^n).\tag{1.9}$$

We note that one can take $f\in L_n(\Omega;\mathbb{R}^n)$, however, to simplify some estimates (related to the quantity $R(\tau^*)$ defined in (2.15)), we select f from a smaller set.
For the Haar-Karman problem, existence and uniqueness of the solution is not difficult to prove provided that an additional assumption holds. Indeed, the convex and continuous functional $(-I^*)$ is coercive on Y^*, and the set $K^*\cap Q^*_\ell$ is convex and closed in Y^*. If we assume that

$$K^*\cap Q^*_\ell\neq\emptyset,\tag{1.10}$$

then existence and uniqueness of a maximizer σ^* readily follows from theorems of convex analysis (cf.,[10]). Further we suppose that the set $K^*\cap Q^*_\ell$ has interior points in the following sense:

$$\exists\,\epsilon>0 \text{ and } \exists\,\sigma_0^*\in Q^*_\ell\ :\ \text{such that }\ |\,\sigma_0^{*D}\,|\le\sqrt{2}k_*-\epsilon \text{ a.e. in } \Omega.\tag{1.11}$$

Let

$$\mathcal{E}(\tau^*,v):=\int_\Omega\tau^*:\varepsilon(v)\,dx.$$

In view of (1.11) it holds

$$\ell(v)=\mathcal{E}(\sigma_0^*,v)=\int_\Omega\left(\sigma_0^{*D}:\varepsilon^D(v)+\frac{1}{n}\operatorname{tr}\sigma_0^*\operatorname{div}v\right)dx\le\tag{1.12}$$

$$\le(\sqrt{2}k_*-\epsilon)\|\varepsilon^D(v)\|_{1,\Omega}+\frac{1}{n}\int_\Omega\operatorname{tr}\sigma_0^*\operatorname{div}v\,dx,$$

and we conclude that for any solenoidal field $\overset{\circ}{w}$,

$$\ell(\overset{\circ}{w})\le\frac{1}{\lambda}\sqrt{2}k_*\|\varepsilon(\overset{\circ}{w})\|_{1,\Omega},\tag{1.13}$$

where $\lambda:=\frac{\sqrt{2}k_*}{\sqrt{2}k_*-\epsilon}>1$.

Remark 1.1. *A direct verification of the safe load condition in the form (1.11) may be difficult. For a wide class of plasticity models (which includes Hencky plasticity* [8, 25]*), the computation of a number λ such that (1.13) is true can be performed in a different way, namely one can prove that*

$$\lambda:=\inf_{w\in\widehat{V}_0}\ \sup_{\tau^*\in K^*}\ \mathcal{E}(\tau^*,w)>1,\tag{1.14}$$

where

$$\widehat{V}_0(\Omega) := \{w \in V_0 \mid \mathcal{E}(w) = 1\}.$$

Letting

$$j_0(w) := \sup\{\tau^* : \eta \mid \tau^* \in \mathbb{M}_s^{n\times n},\ |\tau^*| \leq \sqrt{2}k_* \},$$

(1.14) reads

$$\inf_{w\in\widehat{V}_0} \int_\Omega j_0(\varepsilon(w))dx = \lambda > 1. \tag{1.15}$$

For our purposes it is convenient to rewrite (1.15) in form of the inequality

$$\ell(w) \ \leq\ \frac{1}{\lambda} \int_\Omega j_0(\varepsilon(w))\, dx, \qquad \forall\, w \in V_0, \tag{1.16}$$

which plays an important role in our analysis. It is easy to see that the functional j_0 has the explicit form

$$j_0(\eta) = \begin{cases} \sqrt{2}k_*|\eta^D| & \textit{if} \ \ \mathrm{tr}\,\eta = 0, \\ \\ +\infty & \textit{if} \ \ \mathrm{tr}\,\eta \neq 0, \end{cases}$$

and it is also easy to verify that (1.16) is equivalent to (1.13) with λ being defined in (1.14).

Problem $\mathcal{P}^*$ has a dual counterpart, which is a variational problem for displacements. We call it **Problem** $\mathcal{P}$: *find $u \in V_0 + u_0$ such that*

$$J(u) = \inf_{v\in V_0+u_0} J(v), \qquad J(v) = \int_\Omega \tfrac{K_0}{2}(\mathrm{div} v)^2 + g_0(\varepsilon^D(v))dx - \ell(v),$$

where

$$g_0(\eta) = \begin{cases} \mu|\eta|^2 & \text{if} \ \ |\eta| \leq t_0 := \frac{k_*}{\sqrt{2}\mu} \\ \\ \mu(2|\eta|t_0 - t_0^2) = \sqrt{2}k_*|\eta| - \frac{k_*^2}{2\mu} & \text{if} \ \ \ |\eta| > t_0. \end{cases}$$

We note that

$$g_0'(\eta) = \begin{cases} 2\mu\eta & \text{if} \ \ |\eta| \leq t_0 \\ \\ 2\mu\eta\frac{t_0}{|\eta|} = \sqrt{2}k_*\frac{\eta}{|\eta|} & \text{if} \ \ \ |\eta| > t_0, \end{cases}$$

and these relations can be rewritten as

$$g_0(\eta) = \mu|\eta|^2 - \mu\,(|\eta| - t_0)_\oplus^2\,, \quad g_0'(\eta) = 2\mu\eta - 2\mu\frac{\eta}{|\eta|}\,(|\eta| - t_0)_\oplus\,,$$

$(\quad)_\oplus$ being the positive part of the function under consideration.

Remark 1.2. *Since*

$$g_0(\varepsilon^D(v)) \geq j_0(\varepsilon^D(v)) - \frac{k_*^2}{2\mu},$$

we find that

$$J(v) \geq \int_\Omega \big((\tfrac{K_0}{2}\mathrm{div} v)^2 + j_0(\varepsilon^D(v)\big) dx - \ell(v) - |\Omega| \frac{k_*^2}{2\mu},$$

and in view of (1.16) this implies

$$J(v) \geq \int_\Omega \big((\tfrac{K_0}{2}\mathrm{div} v)^2 + \left(1 - \frac{1}{\lambda}\right) j_0(\varepsilon^D(v)\big) dx - |\Omega| \frac{k_*^2}{2\mu}.$$

Hence the condition (1.16) together with $\lambda > 1$ guarantees that the energy functional J is coercive.

It follows from the general principles of convex analysis (cf. [10]) that

$$I^*(\tau^*) \leq J(v), \qquad \forall \tau^* \in K^* \cap Q_f^*,\ v \in V_0 + u_0. \tag{1.17}$$

Unfortunately the functional J, which from the physical point of view represents the energy expressed in terms of strains, has linear growth with respect to the first derivatives of v and it is coercive only on a nonreflexive space. For this reason, Problem $\mathcal{P}$ is ill–posed and may have no solution despite of the facts that J is convex and bounded from below. The mathematical properties of this and other variational problems with functionals of linear growth have been investigated in, e.g., [1, 11, 17, 18, 22, 41, 45, 48, 50], where correct mathematical statements were obtained in terms of variational extensions of Problem $\mathcal{P}$. Minimizers of the relaxed problem belong to spaces like $BV(\Omega)$ or $BD(\Omega)$ of summable vector-valued functions whose deformation tensors are Radon measures. A qualitative analysis of variational problems related to functionals with linear growth is presented in, e.g., [1, 3, 16, 17, 18, 46, 47].

In this paper we derive estimates oriented towards a quantitative study of Hencky's plasticity problem. In this context, the Hencky variational problem is much more complicated than, e.g., problems with quadratic functionals associated with linear elliptic equations and systems. In Hencky plasticity as in other convex variational problems with functionals of linear growth minimizing sequences may converge to a very irregular function. The topological structure and the location of the set, where a solution may have jumps is a priori unknown. This produces essential difficulties in the construction of suitable approximations and requires special methods (see [7, 29, 32, 30, 44]) and the references therein). It is also rather difficult to obtain a priori rate convergence estimates for variational-difference methods for Problem $\mathcal{P}^*$, too. To do this we need additional differentiability of weak solutions such as, e.g., local W_2^1 estimates for the stresses as obtained in [3, 46, 47].

For Hencky plasticity these regularity results were used in [40], where qualified convergence estimates for the equilibrium finite element approximations of the stresses have been established. In [33], a priori rate convergence estimates (in terms of stresses) have been derived with the help of a sequence of regularized problems $\mathcal{P}^\delta$ and $\mathcal{P}^{\delta *}$ and a special type of regularity estimates that depend on the parameter δ explicitly. This gives an opportunity to find a dependence between the parameters of sampling and regularization that provides qualified convergence to the solution of Problem $\mathcal{P}^*$.

However, a priori rate convergence estimates provide only a very general information on the quality of approximate solutions. A posteriori estimates of the difference between exact and the approximate solutions are necessary to verify the accuracy of a particular approximate solution and to provide an information of how a finite dimensional subspace should be improved in order to obtain a much better approximate solution. Therefore, it is not surprising that the subject of a posteriori error estimation has been in the focus of many researches that study linear (and also nonlinear) models based on partial differential equations. There are several different approaches to the a posteriori error analysis of boundary value problems among which we select the so-called functional type a posteriori error estimates (see [37, 27, 38] for a consequent exposition). These estimates (unlike many others) provide guaranteed and computable error bounds for a wide class of approximate solutions without such additional requirements as Galerkin orthogonality or higher regularity of exact solutions or/and approximations. By this method, first results for deformation plasticity models with hardening (which belong to the class of nonlinear, but uniformly elliptic problems) have been derived in [43]. It should be said that the corresponding estimates degenerate as the hardening parameter tends to zero and for this reason they can not be applied to get a posteriori estimates for the Hencky plasticity problem. A posteriori estimates for the elasto-plastic torsion problem (that is a special case of the Hencky problem) have been derived in [6] within the framework of the unified technology that was earlier used for variational inequalities (see [13, 14, 34, 35]). A posteriori estimates for incremental plasticity problems with hardening have been recently derived in [42].

Despite of the fact that all the above mentioned problems are strongly nonlinear, the error estimation methods used there can not be applied to the Hencky problem. We recall that the original variational problem may fail to have a minimizer in a reflexive Sobolev space, so that for the displacements it is not clear how to to correctly define the error estimation problem. Another way is to study the dual problem $\mathcal{P}^*$ and try to measure errors in terms of stresses, which from the physical point of view is quite natural. The dual functional possesses a unique minimizer provided that the safe load condition is satisfied. However, it is finite only for tensors from the rather complicated set $K^* \cap Q^*_\ell$, which involves pointwise restrictions together with differential (equilibrium) equations. Getting guaranteed error bounds makes it necessary to derive computable majorants of the distance to this set. All these difficulties are obstacles for the development of a an posteriori error analysis for the Hencky plasticity problem, and the authors are unaware of publications that contain the above mentioned majorants and present guaranteed and computable a posteriori error estimates.

In the present paper, we deduce guaranteed and computable bounds for the difference between the exact maximizer σ^* and any approximation τ^* in terms of the dual energy norm (Theorem 2.1). The derivation is based upon Lemma 2.5 that evaluates the distance to $K^* \cap Q^*_\ell$. Although the proof of Lemma 2.5 follows an idea close to that which has been used in [36], there are essential differences arising from a much more complicated structure of the Hencky variational functional, which is strongly anisotropic with respect to the spherical and the deviatorical parts of $\varepsilon(v)$. In view of this fact, the safe load inequality (1.16) *de facto* is defined only for solenoidal fields, otherwise it does not provide a finite bound of $\ell(v)$. To avoid this difficulty, we apply Lemma 2.4, which is a consequence of the well known LBB condition. As a result, the estimate derived in Lemma 2.5 contains the corresponding constant c_Ω. In our analysis, Problem $\mathcal{P}$ plays a subsidiary role. We exploit only the existence of a minimizing sequence $\{v_k\}$ and the fact that

$$\lim_{k \to +\infty} J(v_k) = I^*(\sigma^*).$$

A minimizing sequence can be constructed by any method, e.g., by using direct minimization of the functional J or of the corresponding relaxed functional.
A general form of the error majorant is presented in Theorem 2.1. The majorant consists of non-negative terms that can be thought as measures for the failure in the constitutive relations and differential equations of the Hencky problem. It is proved, that the majorant computed with the help of a minimizing sequence vanishes if and only if τ^* coincides with the exact solution σ^* of Problem $\mathcal{P}^*$.

2. The basic error estimate

Assume that $\tau^* \in K^*$ is any approximation of the exact dual solution σ^*. Our final goal is to derive computable upper bound for $\sigma^* - \tau^*$. The corresponding result is formulated in Theorem 2.1, which is based upon the Lemmas 2.1–2.5 below.

Lemma 2.1. *For any $v \in V_0 + u_0$ and any $\eta^* \in K^* \cap Q^*_\ell$ the following estimate holds*

$$\frac{1}{2} \,|\!|\!| \; \eta^* - \sigma^* \; |\!|\!|^2 \leq \; J(v) \; - \; I^*(\eta^*). \tag{2.1}$$

Proof. For any $\eta^* \in K^* \cap Q^*_\ell$, we have

$$\begin{aligned} 0 \quad &\leq I^*(\sigma^*) - I^*(\eta^*) = \\ &= \int_\Omega \left(\varepsilon(u_0) : (\sigma^* - \eta^*) + \frac{1}{2} A\eta^* : \eta^* - \frac{1}{2} A\sigma^* : \sigma^* \right) dx = \\ &= \frac{1}{2} \,|\!|\!| \; \eta^* - \sigma^* \; |\!|\!|^2 + \int_\Omega \left(\varepsilon(u_0) : (\sigma^* - \eta^*) + A\sigma^* : (\eta^* - \sigma^*) \right) dx. \end{aligned} \tag{2.2}$$

Let $\eta^* = \lambda\zeta^* + (1-\lambda)\sigma^*$, where $\zeta^* \in K^* \cap Q^*_\ell$ and $\lambda \in [0,1]$. Then, we obtain the inequality

$$0 \leq \frac{1}{2}\lambda^2 \,|\!|\!| \; \sigma^* - \zeta^* \; |\!|\!|^2 + \lambda \int_\Omega (A\sigma^* - \varepsilon(u_0)) : (\zeta^* - \sigma^*) \, dx \,,$$

which shows that

$$\int_\Omega (A\sigma^* - \varepsilon(u_0)) : (\zeta^* - \sigma^*) \, dx \geq 0 \qquad \forall \zeta^* \in K^* \cap Q^*_\ell. \tag{2.3}$$

Now (2.2) and (2.3) yield the estimate

$$\frac{1}{2} \,|\!|\!| \; \eta^* - \sigma^* \; |\!|\!|^2 \leq I^*(\sigma^*) - I^*(\eta^*) \quad \forall \eta^* \in K^* \cap Q^*_\ell, \tag{2.4}$$

By (1.11) we find that

$$\frac{1}{2} \,|\!|\!| \; \eta^* - \sigma^* \; |\!|\!|^2 \leq \inf \mathcal{P} \; - \; I^*(\eta^*) \leq J(v) \; - \; I^*(\eta^*) \quad \forall \eta^* \in K^* \cap Q^*_\ell, \quad v \in V_0 + u_0, \tag{2.5}$$

thus we have established (2.1). □

Remark 2.1. *Lemma 2.1 generalizes Mikhlin's estimate, which was derived in* [24] *for variational problems with quadratic functionals, to the case of the Hencky plasticity functional. We see that in this case the restrictions imposed on the admissible tensors are much stronger than for problems with quadratic functionals, namely (2.1) is valid only for rather special tensors* η^**, which satisfy simultaneously the pointwise yield condition* $\eta^{*D} \in K^*$ *and the system of* equilibrium equations $\mathrm{Div}\eta^* + f = 0$. *In practical computations, such classes of tensors are difficult to construct, so that the estimate (2.1) has in principle only a theoretical meaning. Below, we will find a way to overcome this drawback and derive estimates that are valid for a much wider class of approximations.*

Henceforth, we consider the case of isotropic media.

Lemma 2.2. *For any* $\tau^* \in K^*$ *and* $\beta > 0$*, the following estimate holds*

$$
(2.6)\quad \frac{1}{2} \| \tau^* - \sigma^* \|^2 \leq \frac{1+\beta}{\beta}\Big(\mathcal{D}(\varepsilon^D(v), \tau^{*D}) + \tfrac{K_0}{2}\left\| \operatorname{div} v - \tfrac{1}{nK_0} \operatorname{tr} \tau^* \right\|^2 + \\ + \inf_{\eta^* \in K^* \cap Q^*_\ell} \Upsilon(v, \tau^*, \eta^*, \beta)\Big),
$$

where $v \in V_0 + u_0$,

$$
\mathcal{D}(\eta, \eta^*) := \int_\Omega (g(\eta) + g^*(\eta^*) - \eta : \eta^*) dx
$$

and

$$
(2.7)\quad \Upsilon(v, \tau^*, \eta^*, \beta) = \\ = \frac{1}{n}\int_\Omega \left(\frac{1}{nK_0} \operatorname{tr} \tau^* - \operatorname{div} v\right) \operatorname{tr}(\eta^* - \tau^*) dx + \frac{1+\beta}{2n^2 K_0} \| \operatorname{tr}(\eta^* - \tau^*) \|^2 + \\ + \frac{1}{2\mu}\int_\Omega \left(\tau^{*D} - 2\mu\varepsilon^D(v)\right) : (\eta^{*D} - \tau^{*D}) dx + \frac{1+\beta}{4\mu} \left\| \eta^{*D} - \tau^{*D} \right\|^2 .
$$

Proof. Let β be a positive number and fix $\eta^* \in K^* \cap Q^*_\ell$. We have

$$
(2.8)\quad \frac{1}{2} \| \tau^* - \sigma^* \|^2 \leq \frac{1+\beta}{2} \| \tau^* - \eta^* \|^2 + \frac{1+\beta}{2\beta} \| \eta^* - \sigma^* \|^2 \leq \\ \frac{1+\beta}{2}\left(\frac{1}{n^2K_0} \| \operatorname{tr}(\tau^* - \eta^*) \|^2 + \frac{1}{2\mu}\left\| \tau^{*D} - \eta^{*D} \right\|^2\right) + \frac{1+\beta}{\beta}\left(J(v) - I^*(\eta^*)\right).
$$

Note that

$$
\begin{aligned}
&J(v) - I^*(\eta^*) = \\
&\quad = \int_\Omega \left(\frac{K_0}{2} |\mathrm{div} v|^2 + g_0(|\varepsilon^D(v)|) + \frac{1}{2} A\eta^* : \eta^* - \varepsilon(u_0) : \eta^* \right) dx + \ell(u_0 - v) = \\
&\qquad \int_\Omega \left(\frac{K_0}{2} |\mathrm{div} v|^2 + g_0(|\varepsilon^D(v)|) + \frac{1}{2} A\eta^* : \eta^* - \varepsilon(v) : \eta^* \right) dx = \\
&\qquad \int_\Omega \left(\frac{K_0}{2} |\mathrm{div} v|^2 + \frac{1}{2n^2 K_0} |\operatorname{tr} \eta^*|^2 - \frac{1}{n} \operatorname{tr}(\varepsilon(v)) \operatorname{tr} \eta^* \right) dx + \\
&\qquad\qquad + \int_\Omega \left(g_0(|\varepsilon^D(v)|) + \frac{1}{4\mu} |\eta^{*D}|^2 - \varepsilon^D(v), \eta^{*D} \right) dx.
\end{aligned}
$$

hence we deduce

$$
J(v) - I^*(\eta^*) = \tfrac{K_0}{2} \left\| \operatorname{div} v - \tfrac{1}{nK_0} \operatorname{tr} \eta^* \right\|^2 + \mathcal{D}(\varepsilon^D(v), \eta^{*D}), \tag{2.9}
$$

where

$$
\mathcal{D}(\eta, \eta^*) := \int_\Omega (g_0(\eta) + g_0^*(\eta^*) - \eta : \eta^*) \, dx
$$

is the compound functional generated by the mutually conjugate functions g_0 and g_0^*. Since $\eta^* \in K^*$, we have $g_0^*(\eta^*) = \frac{1}{4\mu} |\eta^{*D}|^2$ and we find that

$$
\begin{aligned}
&\mathcal{D}(\varepsilon^D(v), \eta^{*D}) - \mathcal{D}(\varepsilon^D(v), \tau^{*D}) = \frac{1}{4\mu} \left(|\eta^{*D}|^2 - |\tau^{*D}|^2 \right) - \varepsilon^D(v) : (\eta^{*D} - \tau^{*D}) = \\
&\quad \frac{1}{4\mu} \left\| \eta^{*D} - \tau^{*D} \right\|^2 - \frac{1}{2\mu} \left\| \tau^{*D} \right\|^2 + \frac{1}{2\mu} \eta^{*D} : \tau^{*D} - \varepsilon^D(v) : (\eta^{*D} - \tau^{*D}) = \\
&\qquad \frac{1}{4\mu} \left\| \eta^{*D} - \tau^{*D} \right\|^2 + \left(\frac{1}{2\mu} \tau^{*D} - \varepsilon^D(v) \right) : (\eta^{*D} - \tau^{*D}),
\end{aligned}
$$

hence (2.9) takes the form

$$
\begin{aligned}
J(v) - I^*(\eta^*) &= \frac{K_0}{2} \left\| \operatorname{div} v - \frac{1}{nK_0} \operatorname{tr} \tau^* \right\|^2 + \mathcal{D}(\varepsilon^D(v), \tau^{*D}) + \\
&\frac{1}{n} \int_\Omega \left(\frac{1}{nK_0} \operatorname{tr} \tau^* - \operatorname{div} v \right) \operatorname{tr}(\eta^* - \tau^*) dx + \frac{1}{2n^2 K_0} \left\| \operatorname{tr}(\eta^* - \tau^*) \right\|^2 + \\
&\quad + \frac{1}{4\mu} \left\| \eta^{*D} - \tau^{*D} \right\|^2 + \frac{1}{2\mu} \int_\Omega \left(2\mu \varepsilon^D(v) - \tau^{*D} \right) : (\tau^{*D} - \eta^{*D}) \, dx.
\end{aligned} \tag{2.10}
$$

Combining (2.8) and (2.10), we obtain

$$\frac{1}{2}\parallel \tau^* - \sigma^* \parallel^2 \leq \frac{1+\beta}{\beta}\mathcal{D}(\varepsilon^D(v), \tau^{*D}) + \\ + \frac{K_0(1+\beta)}{2\beta}\left\| \operatorname{div} v - \frac{1}{nK_0}\operatorname{tr}\tau^* \right\|^2 + \Upsilon(v, \tau^*, \eta^*, \beta). \tag{2.11}$$

where Υ is defined by (2.7). Since η^* is an arbitrary tensor in $K^* \cap Q^*_\ell$, we arrive at (2.6). $\square$

Remark 2.2. *Estimate (2.6) can be written in a different (but equivalent) form by noting that*

$$2\mu\frac{\varepsilon^D(v)}{|\varepsilon^D(v)|}\left(|\varepsilon^D(v)| - t_0\right)_\oplus + g_0'(\varepsilon^D(v)) - \tau^{*D} = 2\mu\varepsilon^D(v) - \tau^{*D}.$$

Remark 2.3. *If $\tau^* \in K^* \cap Q^*_\ell$ then*

$$\inf_{\eta^* \in K^* \cap Q^*_\ell} \Upsilon(v, \tau^*, \eta^*, \beta) = 0$$

and (2.6) implies the estimate

$$\frac{1}{2}\interleave \tau^* - \sigma^* \interleave^2 \leq \inf_{v \in V_0 + u_0}\left\{\mathcal{D}(\varepsilon^D(v), \tau^{*D}) + \frac{K_0}{2}\left\| \operatorname{div} v - \frac{1}{nK_0}\operatorname{tr}\tau^* \right\|^2\right\}, \tag{2.12}$$

which can be considered as a first form of the functional error estimate for the Hencky plasticity problem.

In our subsequent analysis we use the following result, which is a simple consequence of the well known Ladyzhenskaya-Babuska-Brezzi condition written in the form of the Ladyzhenskaya-Solonnikov inequality ([23]). It reads as follows:

Lemma 2.3. *Let Ω be a bounded domain with Lipschitz continuous boundary. Then, for any function $f \in L^2(\Omega)$ with zero mean value, one can find a field $w_f \in V_0$ such that $\operatorname{div} w_f = f$ and*

$$\|\nabla w_f\| \leq c_\Omega \|f\|, \tag{2.13}$$

where c_Ω is a positive constant depending only on n and Ω.

For $n = 2$ this result has been also proved by I. Babuška and A. K. Aziz [2].

Lemma 2.4. *For any function $v \in V_0$, there exists a solenoidal field $v_0 \in V_0$ such that*

$$\|\nabla(v - v_0)\| \leq c_\Omega \|\operatorname{div} v\|. \tag{2.14}$$

Proof. Let $f = \operatorname{div} \widehat{v}$. By Lemma 2.3, we find a field $w_f \in V_0$ such that

$$\operatorname{div} w_f = f$$

and

$$\|\nabla w_f\| \leq c_\Omega \|\operatorname{div} \widehat{v}\|.$$

Since $\operatorname{div}(\widehat{v}-w_f)=0$, the function $v_0:=\widehat{v}-w_f$ is solenoidal. Moreover,

$$\|\nabla(\widehat{v}-v_0)\|=\|\nabla w_f\|\le c_\Omega\|\operatorname{div}\widehat{v}\|,$$

and the estimate (2.14) follows. □

For any $\tau^*\in Y^*$, we define the quantity

$$R(\tau^*):=\sup_{w\in V_0}\frac{\int_\Omega \tau^*:\varepsilon(w)\,dx-\ell(w)}{\|\varepsilon(w)\|_{1,\Omega}},\tag{2.15}$$

which can be thought of as a measure of the residuals associated with the equilibrium equation and the Neumann boundary condition.
If $\tau^*\in K^*$ this quantity is finite since

$$|\ \ell(w)\ |\le\|f\|_\infty\|w\|_{1,\Omega}+\|F\|_\infty\|w\|_{1,\Gamma_N}\tag{2.16}$$

and

$$\|w\|_{1,\Omega}\le C_1(\Omega)\|\varepsilon(w)\|_{1,\Omega},\tag{2.17}$$

$$\|w\|_{1,\Gamma_N}\le C_2(\Omega)\|\varepsilon(w)\|_{1,\Omega}.\tag{2.18}$$

Lemma 2.5. *Let $\phi\in L^2(\Omega)$, $\kappa^*\in L^1(\Omega,\mathbb{M}^{n\times n}_s)$, $\tau^*\in K^*$, and let $\lambda>0$ denote the number defined in (1.14). Then*

$$\begin{aligned}d(\tau^*,\phi,\kappa^*,\lambda,\bar\gamma):=\inf_{\eta^*\in K^*\cap Q^*_\ell}\Big\{\int_\Omega(\phi\ \mathrm{tr}\,(\eta^*-\tau^*)+\kappa^*:(\eta^{*D}-\tau^{*D})\,dx+\\
+c_1\|\ \mathrm{tr}\,(\eta^*-\tau^*)\,\|^2+c_2\left\|\,\eta^{*D}-\tau^{*D}\,\right\|^2\Big\}\le\\
\le\frac{\varrho^2(\tau^*,\phi,\lambda,\bar\gamma)}{4c_1\bar\gamma^2}+\|\kappa^*\|_{1,\Omega}\frac{2\sqrt{2}k_*r_\lambda(\tau^*)}{1+r_\lambda(\tau^*)}+c_2\frac{8k_*^2r_\lambda^2(\tau^*)}{(1+r_\lambda(\tau^*))^2},\end{aligned}\tag{2.19}$$

where $\bar\gamma:=\dfrac{1}{2c_1n}$,

$$\varrho(\tau^*,\phi,\lambda,\gamma):=\gamma\|\phi\|+\left(R(\tau^*)+\frac{2\sqrt{2}k_*r_\lambda(\tau^*)}{1+r_\lambda(\tau^*)}\right)|\Omega|^{1/2}c_\Omega,\tag{2.20}$$

$$r_\lambda(\tau^*):=\frac{\lambda}{\sqrt{2}k_*(\lambda-1)}R(\tau^*).\tag{2.21}$$

Proof. For any $\tau^*\in K^*$ and $w\in V_0+u_0$ we define the Lagrangian $\mathcal{L}_{\phi,\kappa^*,\tau^*}(\eta^*,w):Y^*\times V_0\to\mathbb{R}$ as follows:

$$\begin{aligned}\mathcal{L}_{\phi,\kappa^*,\tau^*}(\eta^*,w):=\int_\Omega(\phi\ \mathrm{tr}\,(\eta^*-\tau^*)+\kappa^*:(\eta^{*D}-\tau^{*D})\,dx+\\
+c_1\|\ \mathrm{tr}\,(\eta^*-\tau^*)\,\|^2+c_2\left\|\,\eta^{*D}-\tau^{*D}\,\right\|^2+\ell(w)-\int_\Omega\eta^*:\varepsilon(w)\,dx.\end{aligned}$$

It is easy to verify that

$$d(\tau^*,\phi,\kappa^*,\lambda,\bar{\gamma})=\inf_{\eta^*\in K^*}\sup_{w\in V_0}\mathcal{L}_{\phi,\kappa^*,\tau^*}(\eta^*,w). \tag{2.22}$$

In addition, $\mathcal{L}_{\phi,\kappa^*,\tau^*}(\eta^*,w)$ possesses the following properties:
a) for any $\eta^*\in K^*$ the Lagrangian $\mathcal{L}_{\phi,\kappa^*,\tau^*}(\eta^*,w)$ is an affine continuous function,
b) for any $w\in V_0$ the mapping $\eta^*\to\mathcal{L}_{\phi,\kappa^*,\tau^*}(\eta^*,w)$ is convex and continuous; moreover, for $w=0$ it is coercive with respect to η^*.
c) the set K^* is a convex, closed and bounded subset of Y^*.
In view of (a)–(c) and known saddle–point theorems (see, e.g., [50], Chapter IV, 2.2), we conclude that

$$d(\tau^*,\phi,\kappa^*,\lambda,\bar{\gamma})=\sup_{w\in V_0}\inf_{\eta^*\in K^*}\mathcal{L}_{\phi,\kappa^*,\tau^*}(\eta^*,w). \tag{2.23}$$

Let w be given. By Lemma 2.4 there exists a solenoidal field $\overset{\circ}{w}\in V_0$ such that

$$\|\varepsilon(w-\overset{\circ}{w})\|\le c_\Omega\|\operatorname{div}w\|.$$

We reorganize the two last terms of the Lagrangian as follows:

$$\ell(w)-\int_\Omega\eta^*:\varepsilon(w)\,dx=\ell(\overset{\circ}{w})-\int_\Omega\eta^*:\varepsilon(\overset{\circ}{w})\,dx+\ell(w-\overset{\circ}{w})-\int_\Omega\eta^*:\varepsilon(w-\overset{\circ}{w})\,dx=$$
$$\ell(\overset{\circ}{w})-\int_\Omega\eta^*:\varepsilon(\overset{\circ}{w})\,dx+\ell(w-\overset{\circ}{w})-\int_\Omega\tau^*:\varepsilon(w-\overset{\circ}{w})\,dx+\int_\Omega(\tau^*-\eta^*):\varepsilon(w-\overset{\circ}{w})\,dx.$$

Then we observe that

$$\begin{aligned}\left|\ell(w-\overset{\circ}{w})-\int_\Omega\tau^*:\varepsilon(w-\overset{\circ}{w})\,dx\right|\le&\\ \le R(\tau^*)\|\varepsilon(w-\overset{\circ}{w})\|_{1,\Omega}\le R(\tau^*)|\Omega|^{1/2}\|\varepsilon(w-\overset{\circ}{w})\|_{2,\Omega}&\le\\ \le R(\tau^*)|\Omega|^{1/2}c_\Omega\|\operatorname{div}w\|_{2,\Omega}&\end{aligned} \tag{2.24}$$

and

$$\int_\Omega(\tau^*-\eta^*):\varepsilon(w-\overset{\circ}{w})\,dx=\int_\Omega((\tau^{*D}-\eta^{*D}):\varepsilon^D(w-\overset{\circ}{w})+\frac{1}{n}\operatorname{tr}(\tau^*-\eta^*)\operatorname{div}w)dx. \tag{2.25}$$

Now we select η^* in a special form. We set

$$\zeta_1^*(w):=(\operatorname{tr}\tau^*+\gamma\operatorname{div}w)\mathbb{I},\ \zeta_2^*(\overset{\circ}{w}):=\begin{cases}\sqrt{2}k_*\dfrac{\varepsilon^D(\overset{\circ}{w})}{|\varepsilon^D(\overset{\circ}{w})|} & \text{if } |\varepsilon^D(\overset{\circ}{w})|>0,\\[2ex] 0 & \text{if } |\varepsilon^D(\overset{\circ}{w})|=0,\end{cases} \tag{2.26}$$

and define $\eta^*_{\alpha\gamma}$ as follows:

$$\eta^*_{\alpha\gamma}:=\frac{1}{n}\zeta_1^*+\frac{1}{1+\alpha}\left(\tau^{*D}+\alpha\zeta_2^*\right), \tag{2.27}$$

where α is a positive real number and $\gamma \in \mathbb{R}$.
Since τ^* and ζ_2^* belong to K^*, we find that

$$
\begin{aligned}
&|\eta_{\alpha\gamma}^{*D}| \leq \frac{1}{1+\alpha}|\tau^{*D}| + \frac{\alpha}{1+\alpha}|\zeta_2^{*D}| \leq \sqrt{2}k_* \;\Rightarrow\; \eta_{\alpha\gamma}^* \in K^*,\\
&\operatorname{tr}(\eta_{\alpha\gamma}^* - \tau^*) = \gamma \operatorname{div} w,\\
&\eta_{\alpha\gamma}^{*D} - \tau^{*D} = \frac{\alpha}{1+\alpha}\left(\zeta_2^{*D} - \tau^{*D}\right),\\
&|\eta_{\alpha\gamma}^{*D} - \tau^{*D}| \leq \frac{2\alpha}{(1+\alpha)}\sqrt{2}k_*.
\end{aligned}
$$

We recall that $\overset{\circ}{w}$ is a solenoidal field, hence

$$
\eta_{\alpha\gamma}^* : \varepsilon(\overset{\circ}{w}) = \frac{\tau^{*D} : \varepsilon^D(\overset{\circ}{w})}{1+\alpha} + \frac{\sqrt{2}k_*\alpha}{1+\alpha}|\varepsilon^D(\overset{\circ}{w})|.
$$

Consider the Lagrangian with the function $\eta_{\alpha\gamma}^*$. It is estimated as follows

$$
\mathcal{L}_{\phi,\kappa^*,\tau^*}(\eta_{\alpha\gamma}^*, w) \leq I_1 + I_2, \tag{2.28}
$$

where

$$
\begin{aligned}
I_1 = &\int_\Omega (\phi \operatorname{tr}(\eta_{\alpha\gamma}^* - \tau^*)\, dx + c_1 \left\| \operatorname{tr}(\eta_{\alpha\gamma}^* - \tau^*) \right\|^2 + \\
&R(\tau^*)|\Omega|^{1/2}c_\Omega\|\operatorname{div} w\| + \int_\Omega \frac{1}{n}\operatorname{tr}(\tau^* - \eta_{\alpha\gamma}^*)\operatorname{div} w\, dx + \int_\Omega (\tau^{*D} - \eta_{\alpha\gamma}^{*D}) : \varepsilon^D(w - \overset{\circ}{w})\, dx
\end{aligned}
$$

and

$$
I_2 = \int_\Omega \kappa^* : (\eta_{\alpha\gamma}^{*D} - \tau^{*D})\, dx + c_2 \left\| \eta_{\alpha\gamma}^{*D} - \tau^{*D} \right\|^2 + \ell(\overset{\circ}{w}) - \int_\Omega \eta_{\alpha\gamma}^* : \varepsilon(\overset{\circ}{w})\, dx.
$$

We have

$$
\begin{aligned}
I_1 = &\int_\Omega \phi\, \gamma \operatorname{div} w\, dx + c_1\gamma^2 \| \operatorname{div} w \|^2 + \\
&R(\tau^*)|\Omega|^{1/2}c_\Omega\|\operatorname{div} w\| - \gamma\frac{1}{n}\|\operatorname{div} w\|^2 + \frac{2\sqrt{2}\alpha k_* c_\Omega|\Omega|^{1/2}}{1+\alpha}\|\operatorname{div} w\| \leq \\
&\leq \varrho(\tau^*, \phi, \lambda, \gamma)\|\operatorname{div} w\| - \left(\frac{\gamma}{n} - c_1\gamma^2\right)\|\operatorname{div} w\|^2,
\end{aligned}
$$

where

$$
\varrho(\tau^*, \phi, \lambda, \gamma) := \gamma\|\phi\| + \left(R(\tau^*) + \frac{2\sqrt{2}\alpha k_*}{1+\alpha}\right)|\Omega|^{1/2}c_\Omega.
$$

Let $\gamma = \bar{\gamma} := \frac{1}{2c_1 n}$. Then the estimate takes the form

$$
(2.29) \quad I_1 \leq \varrho(\tau^*, \phi, \lambda, \bar{\gamma})\|\operatorname{div} w\| - c_1 \bar{\gamma}^2 \|\operatorname{div} w\|^2 \leq
\leq \frac{\varrho^2(\tau^*, \phi, \lambda, \bar{\gamma})}{4c_1 \bar{\gamma}^2} = c_1 n^2 \varrho^2(\tau^*, \phi, \lambda, \bar{\gamma}).
$$

Consider the second term. We have

$$
\begin{aligned}
I_2 = \int_\Omega \kappa^* : (\eta^{*D}_{\alpha\gamma} - \tau^{*D})\, dx + c_2 \left\| \eta^{*D}_{\alpha\gamma} - \tau^{*D} \right\|^2 + \ell(\mathring{w}) - \int_\Omega \eta^*_{\alpha\gamma} : \varepsilon(\mathring{w})\, dx \leq \\
\leq \|\kappa^*\|_{1,\Omega} \frac{2\sqrt{2}k_*\alpha}{1+\alpha} + c_2 \frac{8k_*^2\alpha^2}{(1+\alpha)^2} + \frac{1}{1+\alpha}\left(\ell(\mathring{w}) - \int_\Omega \tau^* : \varepsilon(\mathring{w}) dx\right) + \\
+ \frac{\alpha}{1+\alpha}\left(\ell(\mathring{w}) - \sqrt{2}k_* \int_\Omega |\varepsilon^D(\mathring{w})| dx\right),
\end{aligned}
$$

and by (2.15) it holds

$$
\left| \int_\Omega \tau^* : \varepsilon(\mathring{w})\, dx - \ell(\mathring{w}) \right| \leq R(\tau^*) \int_\Omega |\varepsilon^D(\mathring{w})| dx. \tag{2.30}
$$

In view of (1.16) we have

$$
\ell(\mathring{w}) \leq \frac{1}{\lambda}\sqrt{2}k_* \int_\Omega |\varepsilon^D(\mathring{w})| dx. \tag{2.31}
$$

Hence we deduce an upper bound in the form

$$
\begin{aligned}
I_2 \leq \\
\leq \|\kappa^*\|_{1,\Omega} \frac{2\sqrt{2}k_*\alpha}{1+\alpha} + c_2 \frac{8k_*^2\alpha^2}{(1+\alpha)^2} + \frac{R(\tau^*)}{1+\alpha}\int_\Omega |\varepsilon^D(\mathring{w})| dx + \frac{\alpha\sqrt{2}k_*}{1+\alpha}\frac{1-\lambda}{\lambda}\int_\Omega |\varepsilon^D(\mathring{w})| dx = \\
\|\kappa^*\|_{1,\Omega} \frac{2\sqrt{2}k_*\alpha}{1+\alpha} + c_2 \frac{8k_*^2\alpha^2}{(1+\alpha)^2} + \frac{1}{1+\alpha}\left(R(\tau^*) + \alpha\sqrt{2}k_* \frac{1-\lambda}{\lambda}\right)\int_\Omega |\varepsilon^D(\mathring{w})| dx
\end{aligned}
$$

Let $\lambda' := \frac{\lambda}{\lambda - 1}$ and

$$
\alpha = r_\lambda(\tau^*) := \frac{\lambda'}{\sqrt{2}k_*} R(\tau^*).
$$

Then we obtain

$$
I_2 \leq \|\kappa^*\|_{1,\Omega} \frac{2\sqrt{2}k_* r_\lambda(\tau^*)}{1 + r_\lambda(\tau^*)} + c_2 \frac{8k_*^2 r_\lambda^2(\tau^*)}{(1 + r_\lambda(\tau^*))^2}, \tag{2.32}
$$

and by (2.28), (2.30)–(2.32) we arrive at (2.20). □

Remark 2.4. *It is convenient to introduce the quantity*

$$
R_{k_*,\lambda}(\tau^*) := \frac{2\sqrt{2}k_* r_\lambda(\tau^*)}{1 + r_\lambda(\tau^*)} = \frac{2\sqrt{2}k_*\lambda R(\tau^*)}{\sqrt{2}k_*(\lambda - 1) + \lambda R(\tau^*)}.
$$

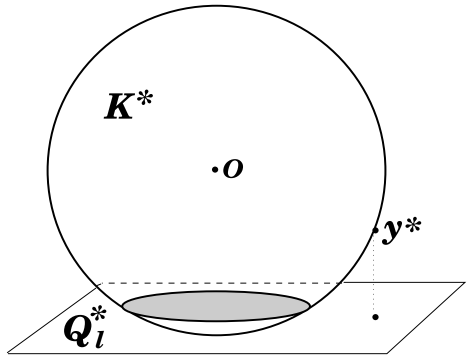

FIGURE 1. Distance to the set $K^* \cap Q^*_\ell$.

Then (2.19) has the shorter form

$$d(\tau^*, \phi, \kappa^*, \lambda, \bar{\gamma}) \le c_1 n^2 \varrho^2(\tau^*, \phi, \lambda, \bar{\gamma}) + \|\kappa^*\|_{1,\Omega} R_{k_*,\lambda}(\tau^*) + c_2 R^2_{k_*,\lambda}(\tau^*), \tag{2.33}$$

where

$$\varrho(\tau^*, \phi, \lambda, \bar{\gamma}) := \bar{\gamma}\|\phi\| + (R(\tau^*) + R_{k_*,\lambda}(\tau^*))\, |\Omega|^{1/2} c_\Omega. \tag{2.34}$$

We note that

$$R_{k_*,\lambda}(\tau^*) \le 2\lambda' R(\tau^*), \quad R(\tau^*) + \frac{2\sqrt{2} k_* r_\lambda(\tau^*)}{1 + r_\lambda(\tau^*)} \le (1 + 2\lambda') R(\tau^*), \tag{2.35}$$

hence, for any $\theta > 0$

$$\varrho^2(\tau^*, \phi, \lambda, \bar{\gamma}) \le (1+\theta)\bar{\gamma}^2 \parallel \phi \parallel^2 + \frac{1+\theta}{\theta}(1+2\lambda')^2 c_\Omega^2 R^2(\tau^*)|\Omega|. \tag{2.36}$$

From (2.34)–(2.36) we deduce another upper bound for the quantity $d(\tau^*, \phi, \kappa^*, \lambda, \bar{\gamma})$:

$$d^2(\tau^*, \phi, \kappa^*, \lambda, \bar{\gamma}) \le c_1 n^2 \left((1+\theta)\bar{\gamma}^2 \parallel \phi \parallel^2 + \left(1 + \frac{1}{\theta}\right)(1+2\lambda')^2 c_\Omega^2 R^2(\tau^*)|\Omega| \right) + 2\lambda' R(\tau^*)\|\kappa^*\|_{1,\Omega} + 4 c_2 \lambda'^2 R^2(\tau^*). \tag{2.37}$$

Remark 2.5. *Set $\phi = 0$, $\kappa = 0$, $c_1 = 1/n$, $c_2 = 1$, and let θ tend to $+\infty$. Then (2.37) implies an estimate of the distance to the set $K^* \cap Q^*_\ell$. If $\tau^* \in K^*$, then*

$$\inf_{\eta^* \in K^* \cap Q^*_\ell} \|\tau - \eta^*\| \le \mathbf{C}(n, \Omega, \lambda, k_*) R(\tau^*), \tag{2.38}$$

where

$$\mathbf{C}(n, \Omega, \lambda, k_*) = \left(n(1+2\lambda')^2 c_\Omega^2 |\Omega| + 4\lambda'^2 \right)^{1/2}.$$

We note that $\mathbf{C}(n,\Omega,\lambda,k_*)$ *tends to* $+\infty$ *as* $\lambda \to 1$, *i.e., the estimate becomes worse if the safe load condition is satisfied only with* λ *close to one.*
It is worth noting, that the problem of finding a computable upper bound of the distance to $K^* \cap Q^*_\ell$ *is quite nontrivial. The geometrical illustration presented in Fig. 1 shows that even if* y^* *belongs to* K^*, *its projection on* Q^*_ℓ *may not belong to* K^*. *Hence, the distance to* $K^* \cap Q^*_\ell$ *(shadowed set on Fig. 2.5) can not be estimated through the distance to* Q^*_l *exclusively. Any estimate that evaluates the distance between* $y^* \in K^*$ *and* $K^* \cap Q^*_\ell$ *in terms of the distance to* Q^*_l *must include a factor essentially depending on the distance of* Q^*_ℓ *from the center* O *of the ball* K^*. *In our relations the latter factor depends on the value of* λ. *If* λ *is close to one, then the distance between* O *and* Q^*_ℓ *is close to one, so that the set* $K^* \cap Q^*_\ell$ *is small and the constant* $\mathbf{C}$ *is large. In the opposite case (if* O *belongs to* Q^*_ℓ*), we can take* $\lambda = +\infty$ *and minimize the value of* $\mathbf{C}$.

Now we will collect all the previous results and deduce a computable upper bound for $\| \tau^* - \sigma^* \|$. It will be expressed in terms of the quantities

$$H_1(v,\tau^*) := \left\| \operatorname{div} v - \tfrac{1}{nK_0} \operatorname{tr} \tau^* \right\|,$$
$$H_2(v,\tau^*) := \left\| \varepsilon^D(v) - \tfrac{1}{2\mu}\tau^{*D} \right\|_{1,\Omega}, \quad \mathcal{D}(\varepsilon(v),\tau^*)$$

that reflects the residuals associated with the constitutive relations, and the quantity $R(\tau^*)$ measuring the violation of the equilibrium equation. From Lemmas 2.2 and 2.5, we obtain the following result:

Theorem 2.1. *Let the assumptions of Lemmas 2.2 and 2.5 hold. Then, for any* $v \in V_0 + u_0$ *and* $\tau^* \in K^*$ *the following estimate holds:*

$$(2.39) \quad \frac{1}{2} \| \tau^* - \sigma^* \|^2 \le M_\oplus(\tau^*, v, \beta, \theta) := \frac{1+\beta}{\beta}\Big(\mathcal{D}(\varepsilon^D(v), \tau^{*D}) + $$
$$+ \tfrac{K_0}{2}\left(1 + \tfrac{1+\theta}{1+\beta}\right) H_1^2(v,\tau^*) + 2\lambda' R(\tau^*) H_2(v,\tau^*) + \vartheta R^2(\tau^*)\Big),$$

where β *and* θ *are arbitrary positive numbers, and*

$$\vartheta = \tfrac{1+\beta}{\mu}\lambda'^2 + \tfrac{1+\beta}{2K_0}\left(1+\tfrac{1}{\theta}\right)(1+2\lambda')^2 c_\Omega^2 |\Omega|$$

with λ *being defined in (1.14) and* $\lambda' = \frac{\lambda}{\lambda - 1}$. *This majorant is consistent, i.e., for any positive* β *and* θ

$$\inf_{v \in V_0 + u_0} M_\oplus(\tau^*, v, \beta, \theta) = 0,$$

if and only if $\tau^* = \sigma^*$.

Proof. In (2.19) we set $c_1 = \frac{1+\beta}{2n^2K_0}$, $c_2 = \frac{1+\beta}{4\mu}$, $\bar\gamma = \frac{nK_0}{1+\beta}$,

$$\kappa^* = \varepsilon^D(v) - \frac{1}{2\mu}\tau^{*D}, \qquad \phi = \frac{1}{n}\left(\frac{1}{nK_0}\operatorname{tr}\tau^* - \operatorname{div} v\right).$$

By (2.6), (2.19), and (2.37) we obtain $M_\oplus$ in the above presented form.

It remains to prove the consistency of $M_\oplus$. Let

$$\inf_{v\in V_0+u_0} M_\oplus(\tau^*, v, \beta, \theta) = 0.$$

Then there exists a sequence $\{v_k\} \in V_0 + u_0$ such that $M_\oplus(\tau^*, v_k, \beta, \theta) \to 0$. Since $\vartheta > 0$, this is possible only if

$$R(\tau^*) = 0, \tag{2.40}$$

$$H_1(v_k, \tau^*) \quad \to\ 0, \tag{2.41}$$

$$\mathcal{D}(\varepsilon^D(v_k), \tau^{*D}) \quad \to 0 \quad \text{as } k \to +\infty. \tag{2.42}$$

In view of (2.40) we have $\tau^* \in Q^*_\ell$ and (2.41) means that $\operatorname{div} v_k \to \frac{1}{nK_0}\tau^*$ in L^2. Using (2.42) recalling $\tau^* \in K^*$ we obtain

$$\begin{aligned}
(2.43)\quad & J(v_k) - I^*(\tau^*) = \\
&= \int_\Omega \big(g_0(\varepsilon^D(v_k)) + g_0^*(\tau^*) - \varepsilon^D(u_0) : \tau^* - \tfrac{1}{n}\operatorname{div} v_k \operatorname{tr}\tau^* - \ell(v_k - u_0)\big)\, dx + \\
&\qquad \tfrac{K_0}{2}\int_\Omega (\operatorname{div}^2 v_k + \frac{1}{n^2K_0^2}\operatorname{tr}^2\tau^*)dx = \\
&= \int_\Omega (g_0(\varepsilon^D(v_k)) + g_0^*(\tau^*) - \varepsilon^D(v_k) : \tau^*)dx + \tfrac{K_0}{2}\int_\Omega (\operatorname{div} v_k - \frac{1}{nK_0}\operatorname{tr}\tau^*)^2 dx = \\
&\qquad = D(\varepsilon^D(v_k), \tau^*) + \tfrac{K_0}{2} H_1^2(v_k, \tau^*) \to 0.
\end{aligned}$$

Therefore, we conclude that τ^* realizes the exact upper bound of the dual functional and, therefore, coincides with the unique solution σ^* of Problem $\mathcal{P}^*$.
Finally, let $\tau^* = \sigma^*$. Again, let $\{v_k\} \in V_0$ be a minimizing sequence in Problem $\mathcal{P}$. By (2.43) we have

$$D(\varepsilon^D(v_k), \sigma^*) + \tfrac{K_0}{2} H_1^2(v_k, \sigma^*) = J(v_k) - I^*(\sigma^*) \to\ 0. \tag{2.44}$$

Since $R(\sigma^*) = 0$, $D(\varepsilon^D(v_k), \sigma^*) \to 0$ and $H_1(v_k, \sigma^*) \to 0$, we conclude that for any positive numbers β and θ

$$\inf_{v\in V_0+u_0} M_\oplus(\sigma^*, v, \beta, \theta) = 0,$$

which completes the proof of Theorem 2. □

Let us discuss the meaning of (2.39). First of all, we note that for $\tau^* \in K^*$ the functional $\mathcal{D}(\tau^{*D}, \varepsilon^D(v))$ has the form

$$\mathcal{D}(\varepsilon^D(v), \tau^{*D}) = \begin{cases} \frac{1}{4\mu} \mid \tau^{*D} - 2\mu\varepsilon^D(v) \mid^2 & \text{if} \quad \mid \varepsilon^D(v) \mid \leq\ t_0 = \frac{\sqrt{2}k_*}{2\mu}, \\ \frac{1}{4\mu} \mid \tau^{*D} \mid^2 + \sqrt{2}k_* \mid \varepsilon^D(v) \mid - \frac{k_*^2}{2\mu} - \varepsilon^D(v) : \tau^{*D} & \text{if} \quad \mid \varepsilon^D(v) \mid >\ t_0. \end{cases}$$

It is easy to see that $\mathcal{D}(\varepsilon^D(v), \tau^{*D}) \geq 0$. If $|\varepsilon^D(v)| \leq t_0$, then $\mathcal{D}(\varepsilon^D(v), \tau^{*D})$ vanishes if $\tau^{*D} = 2\mu\varepsilon^D(v)$.

We represent the second line in the definition of $\mathcal{D}(\varepsilon^D(v),\tau^{*D})$ in the form

$$\frac{1}{4\mu}\left(\tau^{*D}-\sqrt{2}k_*\frac{\varepsilon^D(v)}{|\varepsilon^D(v)|}\right)^2+\left(\mid \varepsilon^D(v)\mid -\frac{\sqrt{2}k_*}{2\mu}\right)\left(\sqrt{2}k_*-\frac{\tau^{*D}\cdot\varepsilon^D(v)}{\mid \varepsilon^D(v)\mid}\right).$$

If $|\varepsilon^D(v)|>t_0$, then $\mathcal{D}(\varepsilon^D(v),\tau^{*D})$ vanishes if and only if $\tau^{*D}=\sqrt{2}k_*\frac{\varepsilon^D(v)}{|\varepsilon^D(v)|}$. Hence, the compound functional vanishes if and only if $\tau^{*D}=g_0'(\varepsilon^D(v))$ and the condition $\mathcal{D}(\varepsilon^D(v),\tau^{*D})=0$ means that $\varepsilon^D(v)$ and τ^{*D} are joined by the Henky constitutive relation, which holds for the exact stress σ^{*D} and $\varepsilon^D(u)$ (if the corresponding u exists). Thus, we conclude that if $M_\oplus(v,\tau^*,\beta,\theta)=0$, then the primal as well as the dual problem are solvable and v and τ^* coincide with the respective exact solutions.

Remark 2.6. *Assume that $\tau^*\notin K^*$. Let*

$$\bar{\tau^*}:=\frac{1}{n}\operatorname{tr}\tau^*\mathbb{I}+\sqrt{2}k_*\frac{\tau^{*D}}{|\tau^{*D}|}.$$

Then it holds

$$\|\!|\ \tau^*-\bar{\tau^*}\ \|\!|^2=\frac{1}{2\mu}\int_\Omega\left(|\tau^{*D}|-\sqrt{2}k_*\right)^2_\oplus dx$$

and

$$\frac{1}{2}\|\!|\ \sigma^*-\bar{\tau^*}\ \|\!|^2\leq M_\oplus(v,\bar{\tau^*},\beta,\theta).$$

Hence we arrive at the estimate

$$\frac{1}{2}\|\!|\ \tau^*-\sigma^*\ \|\!|^2\leq(1+\xi)\frac{1}{4\mu}\int_\Omega\left(|\tau^{*D}|-\sqrt{2}k_*\right)^2_\oplus dx+ \\ +\left(1+\frac{1}{\xi}\right)M_\oplus(v,\bar{\tau^*},\beta,\theta), \tag{2.45}$$

where ξ is a positive constant.

3. Computable bounds for $r_\lambda(\tau^*)$

To have a fully computable estimate, we must suggest a way of computing the quantity $R(\tau^*)$. This subject is strongly related to estimates of the type

$$\|v\|_{1,\Omega}\leq C_1(n,\Omega)\|\varepsilon(v)\|_{1,\Omega}\qquad\forall v\in\overset{\circ}{W}{}^{1,1}(\Omega,\mathbb{R}^n) \tag{3.1}$$

or

$$\inf_{\kappa\in\mathfrak{R}(\Omega)}\|v-\kappa\|_{1,\Omega}\leq C_2(n,\Omega)\|\varepsilon(v)\|_{1,\Omega}\qquad\forall v\in W^{1,1}(\Omega,\mathbb{R}^n), \tag{3.2}$$

where $\mathfrak{R}(\Omega)$ denotes the space of rigid motions (i.e., the kernel of the operator $\varepsilon(v)$). These estimates can be viewed as generalizations of Friedrich's and Poincaré's inequalities for vector valued functions in L_1 type norms.

It should be emphasized, that for our purposes we need more than the fact that the above estimates hold with "some" positive constants. We need also computable and realistic bounds of the constants $C_1(n,\Omega)$ and $C_2(n,\Omega)$.
For (3.1) we refer to [49] but it is possible to prove the following stronger estimate (cf. [15])

Theorem 3.1. *For any* $v \in \overset{\circ}{W}{}^{1,1}(\Omega,\mathbb{R}^n)$ *it holds*

$$\|v\|_{1,\Omega} \leq c_1(n)\,\mathrm{diam}(\Omega)\,\|\varepsilon^D(v)\|_{1,\Omega},$$

where c_1 *is a positive constant depending only on* n.
Moreover, for any $p \in [1, \frac{n}{n-1})$, *there exists* $c_1(n,p)$ *such that*

$$\|v\|_{p,\Omega} \leq c_1(n,p)\ (\mathrm{diam}(\Omega))^{1-n+\frac{n}{p}}\,\|\varepsilon^D(v)\|_{1,\Omega},$$

Theorem 3.1 states that $C_1(n,\Omega) \leq c_1(n)\mathrm{diam}(\Omega)$.

Remark 3.1. *We note that if* $v \in BD(\Omega)$, *then (see* [1]*)*

$$\int\limits_{R^n} |v|dx \leq c_1(n)\mathrm{diam}\,(\mathrm{supp}\,v)\int\limits_{R^n} |\varepsilon(v)|dx.$$

It is not difficult to show that for BD functions with compact support this estimate can be also rewritten in terms of $\varepsilon^D(v)$ *(see* [15]*).*

We can use these Friedrich's type estimates as follows: assume that

$$\tau^* \in H_\infty(\Omega,\mathrm{div}\,) := \{\tau^* \in L^\infty(\Omega,\mathbb{M}^{n\times n}_s)\ \mid\ \mathrm{Div}\,\tau^* \in L^\infty(\Omega,\mathbb{R}^n)\}$$

and $\partial\Omega = \Gamma_D$. Then $V_0 = \overset{\circ}{W}{}^{1,1}(\Omega,\mathbb{M}^{n\times n}_s)$ and we have

$$(3.3)\quad R(\tau^*) := \sup_{w\in V_0}\frac{\int_\Omega(\mathrm{Div}\,\tau^* + f)\cdot w\,dx}{\|\varepsilon(w)\|_{1,\Omega}} \ \leq\ C_1(n,\Omega)\|\mathrm{Div}\,\tau^* + f\|_{\infty,\Omega} \leq$$
$$\leq c_1(n)\mathrm{diam}(\Omega)\|\mathrm{Div}\,\tau^* + f\|_{\infty,\Omega}.$$

Let $\widehat{\Omega}$ be a certain "simple" domain (e.g., square or circle), for which the corresponding constant c_1 can be found analytically or computed with high accuracy. It is easy to see that

$$C_1(n,\Omega) \leq C_1(n,\widehat{\Omega}) \leq \widehat{c}_1(n)\mathrm{diam}(\Omega)$$

provided that $\Omega \subset \widehat{\Omega}$ and $\mathrm{diam}(\Omega) = \mathrm{diam}(\widehat{\Omega})$. Hence we have

$$R(\tau^*) \leq \widehat{c}_1(n)\mathrm{diam}(\Omega)\|\mathrm{Div}\,\tau^* + f\|_{\infty,\Omega}.$$

This estimate and (2.39) yield a guaranteed upper bound for $\|\tau^* - \sigma^*\|$.
However, in more general cases (e.g., for problems with mixed boundary conditions) finding an explicitly computable upper bound of $C_1(n,\Omega)$ may be an uneasy task. For this case, we suggest another way. It is based on the decomposition of Ω into a collection of convex subdomains and using an analog of the Poincaré inequality (see 3.2). We note that estimates of deviations from exact solutions of such a type were earlier derived for linear elliptic problems in [38] and some classes of generalized Newtonian fluids in [14].

Let Ω be decomposed into N elementary subdomains, i.e.,

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j.$$

Assume that $\tau^* \in Q^*$ and

$$\tau^* n = F \quad \text{on } \Gamma_N, \tag{3.4}$$

$$\int_{\Omega_i} (\mathrm{Div}\, \tau^* + f) \cdot \kappa \; dx = 0 \quad \forall \kappa \in \mathfrak{R}(\Omega_i)\,, \; i = 1, ..., N\,, \tag{3.5}$$

which means that the residuals generated by τ^* are orthogonal to the spaces of rigid motions $\mathfrak{R}(\Omega_i)$ related to Ω_i. Obviously (3.5) can be rewritten as

$$\int_{\partial\Omega_i} (\tau^* n) \cdot \kappa \, ds + \int_{\Omega_i} f \cdot \kappa \, dx = 0 \quad \forall \kappa \in \mathfrak{R}(\Omega_i),\; i = 1, ...N\,. \tag{3.6}$$

We note that the condition $\tau^* \in Q^*$ means continuity of $\tau^* n$ on $\Gamma_{ij} := \partial\bar{\Omega}_i \cap \bar{\partial}\Omega_j$. If τ^* is constructed with the help of standard piecewise polynomial approximations then it is usually satisfied. For example, if τ^* is presented on a simplicial mesh by piecewise affine nodal type approximations, then continuity of τ^* is guaranteed. In this case, each Ω_i is a collection of simpleces (whose amount may be very large) and (3.6) imposes $N\frac{n(n+1)}{2}$ algebraic relations on the nodal values that define τ^*. This number does not depend on the amount of elementary simpleces and it is much lesser then the amount of nodal values. Hence, satisfying (3.6) does not lead to serious numerical complications and practically, it means that in the approximation of τ^* some degrees of freedom must be excluded. Assume that τ^* satisfies the above discussed conditions. Then

$$\int_{\Omega} (\mathrm{Div}\;\; \tau^* + f) \cdot w \; dx = \sum_{i=1}^{N} \int_{\Omega_i} (\mathrm{Div}\, \tau^* + f) \cdot (w - \kappa_i) \;\; dx \leq \tag{3.7}$$
$$\leq \sum_{i=1}^{N} C_2(n, \Omega_i) \|\mathrm{Div}\;\; \tau^* + f\|_{\infty,\Omega_i} \|\varepsilon w\|_{1,\Omega_i}\,.$$

Using (3.7), we arrive at the estimate

$$R(\tau^*) \leq \max_{i=1,\ldots,N} C_2(n, \Omega_i) \|\mathrm{Div}\;\; \tau^* + f\|_{\infty,\Omega_i}, \tag{3.8}$$

and computable estimates for the constants $C_2(n, \Omega_i)$ are derived in the next section.

4. Estimates of the constant in a Poincaré type inequality for vector valued functions in L^1 type norms

Consider a bounded domain $\mathfrak{D} \subset \mathbb{R}^n$ with Lipschitz boundary. Below we derive a computable upper bound of $C_2(n, \mathfrak{D})$ under the assumption that $\mathfrak{D}$ *is star-shaped with respect to some subdomain* $\mathfrak{D}_0 \in \mathfrak{D}$, which means that

$$x + \rho(y - x) \in \mathfrak{D} \qquad \forall x \in \mathfrak{D},\; y \in \mathfrak{D}_0,\; \rho \in [0, 1]. \tag{4.1}$$

In our analysis we use the method suggested in [26].

Lemma 4.1. *Let $f \in L^1(\mathfrak{D})$ and $f(x) \geq 0$ for a. a. $x \in \mathfrak{D}$. Then it holds*

$$\int_{\mathfrak{D}} \int_0^1 \int_{\mathfrak{D}_0} f(x+\rho(y-x))dyd\rho dx \ \leq \ \varpi_n \left(|\mathfrak{D}| + |\mathfrak{D}_0|\right) \int_{\mathfrak{D}} f(z)\, dz, \tag{4.2}$$

where $\varpi_n = \dfrac{2^{n-1}-1}{n-1}$.

Proof. By the Fubini theorem, we have

$$\int_{\mathfrak{D}} \int_0^1 \int_{\mathfrak{D}_0} f(x+\rho(y-x))dyd\rho dx = \int_0^{1/2} \int_{\mathfrak{D}_0} \int_{\mathfrak{D}} f(x+\rho(y-x))dxdyd\rho +$$
$$\int_{1/2}^1 \int_{\mathfrak{D}} \int_{\mathfrak{D}_0} f(x+\rho(y-x))dydxd\rho = \imath_1 + \imath_2,$$

and obviously it holds

$$\imath_1 = \int_0^{1/2} \int_{\mathfrak{D}_0} \left(\int_{\mathfrak{D}} f(x+\rho(y-x))(1-\rho)^n dx \right) dy (1-\rho)^{-n} d\rho.$$

Let ρ and y be fixed. We set

$$z = Tx := (1-\rho)x + \rho y.$$

Then we get

$$\int_{\mathfrak{D}} f(x+\rho(y-x))(1-\rho)^n dx = \int_{T(\mathfrak{D})} f(z)dz \leq \int_{\mathfrak{D}} f(x)dx,$$

and therefore

$$\imath_1 \leq |\mathfrak{D}_0| \int_{\mathfrak{D}} f(x)dx \int_0^{1/2} (1-\rho)^{-n} d\rho. \tag{4.3}$$

The second term can be treated analogously:

$$\begin{aligned}\imath_2 = &\int_{1/2}^1 \int_{\mathfrak{D}} \int_{\mathfrak{D}_0} f(x+\rho(y-x))dydxd\rho = \\ &= \int_{1/2}^1 \int_{\mathfrak{D}} \left(\int_{\mathfrak{D}_0} f(x+\rho(y-x))\rho^n dy \right) dx \rho^{-n} d\rho \leq \left(\int_{\mathfrak{D}} f(x)dx \right) \int_{1/2}^1 \int_{\mathfrak{D}} dx \rho^{-n} d\rho = \\ &= |\mathfrak{D}| \left(\int_{\mathfrak{D}} f(x)dx \right) \int_{1/2}^1 \rho^{-n} d\rho = \frac{2^{n-1}-1}{n-1} |\mathfrak{D}| \left(\int_{\mathfrak{D}} f(x)dx \right).\end{aligned}$$

Together with (4.3) we arrive at (4.3). $\square$

Remark 4.1. *If $\mathfrak{D}$ is convex, then it is star-shaped with respect to $\mathfrak{D}_0 = \mathfrak{D}$ and (4.4) holds with the constant $2\varpi_n|\mathfrak{D}|$.*

Assume that $w \in C^\infty(\overline{\mathfrak{D}}, \mathbb{R}^n)$. To this function we can apply an integral representation known as Cesaro's formula (see, e.g., equation (13) in [26]):

$$(4.4) \quad w_i(x) = w_i(y) + \sum_{j=1}^n \omega_{ij}(y)(x_j - y_j) +$$

$$+ \sum_{j,k=1}^n \int_0^1 \left(\varepsilon_{ik}(z) - \sum_{j=1}^n \rho(y_i - x_i) \left\{ \frac{\partial \varepsilon_{ik}}{\partial z_j}(z) - \frac{\partial \varepsilon_{kj}}{\partial z_i}(z) \right\} \right) (x_k - y_k) d\rho$$

Here $\varepsilon_{ij} := \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$ is the deformation tensor associated with w and

$$\omega_{ij} := \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i}\right)$$

is a skew-symmetric tensor. Let $q \in C_0^1(\mathfrak{D}_0)$ be a function satisfying

$$(4.5) \qquad 0 \le q(y) \le 1, \qquad m(\mathfrak{D}_0) := \int_{\mathfrak{D}_0} q(y)dy > 0.$$

It is not difficult to see that the vector valued function $\widehat{w}$ with the components

$$\widehat{w}_i(x) := \frac{1}{m(\mathfrak{D}_0)} \left\{ \int_{\mathfrak{D}_0} w_i q(y) dy + \sum_{j=1}^n \int_{\mathfrak{D}_0} \omega_{ij}(y) q(y)(x_j - y_j) dy \right\}$$

belongs to the space of rigid motions $\mathfrak{R}$. By (4.4), we find that

$$(4.6) \qquad \int_{\mathfrak{D}} |w_i(x) - \widehat{w}_i(x)| dx \le \frac{1}{m(\mathfrak{D}_0)} (T_1 + T_2),$$

where

$$T_1 = \int_{\mathfrak{D}} \int_{\mathfrak{D}_0} \int_0^1 \sum_{k=1}^n |\varepsilon_{ik}(z)||x_k - y_k| q(y) d\rho dy dx,$$

$$T_2 = \sum_{j,k=1}^n \Big| \int_{\mathfrak{D}} \int_{\mathfrak{D}_0} \int_0^1 q(y)\, \rho\, (y_i - x_i)(y_k - x_k) \left\{ \frac{\partial \varepsilon_{ik}}{\partial z_j}(z) - \frac{\partial \varepsilon_{kj}}{\partial z_i}(z) \right\} d\rho dy dx \Big|.$$

It is clear that

$$(4.7) \quad T_1 \le \operatorname{diam} \mathfrak{D} \sum_{k=1}^n \int_{\mathfrak{D}} \int_{\mathfrak{D}_0} \int_0^1 |\varepsilon_{ik}(z)| d\rho dy dx \le$$

$$\le \varpi_n \left(|\mathfrak{D}| + |\mathfrak{D}_0| \right) \operatorname{diam} \mathfrak{D} \sum_{k=1}^n \int_{\mathfrak{D}} |\varepsilon_{ik}(x)| dx.$$

Note that

$$\rho\frac{\partial \varepsilon_{ik}}{\partial z_j}(z)=\frac{\partial}{\partial y_j}\,\varepsilon_{ik}(z),$$
$$\rho\frac{\partial \varepsilon_{kj}}{\partial z_i}(z)=\frac{\partial}{\partial y_i}\,\varepsilon_{kj}(z)$$

and consider one term of the sum T_2 (we denote it by T_{2jk}). Since q has a compact support in $\mathfrak{D}_0$, we have

$$T_{2jk}=\Big|\int_{\mathfrak{D}}\int_{\mathfrak{D}_0}\int_0^1 q(y)\rho\,(y_i-x_i)(y_k-x_k)\left\{\frac{\partial \varepsilon_{ik}}{\partial z_j}(z)-\frac{\partial \varepsilon_{kj}}{\partial z_i}(z)\right\}d\rho dy dx\Big|= \tag{4.8}$$
$$\Big|\int_{\mathfrak{D}}\int_{\mathfrak{D}_0}\int_0^1 q(y)\,(y_i-x_i)(y_k-x_k)\left(\frac{\partial}{\partial y_j}\varepsilon_{ik}(z)-\frac{\partial}{\partial y_i}\varepsilon_{kj}(z)\right)d\rho dy dx\Big|=$$
$$=\Big|\int_{\mathfrak{D}}\int_{\mathfrak{D}_0}\int_0^1\Big\{\frac{\partial}{\partial y_j}\big\{q(y)\,(y_i-x_i)(y_k-x_k)\big\}\varepsilon_{ik}(z)-\frac{\partial}{\partial y_i}\big\{q(y)\,(y_i-x_i)(y_k-x_k)\big\}\varepsilon_{kj}(z)\Big\}d\rho dy dx\Big|.$$

Note also that

$$\left|\frac{\partial q(y)}{\partial y_j}\,(y_i-x_i)(y_k-x_k)\right|,\quad \left|\frac{\partial q(y)}{\partial y_i}(y_i-x_i)(y_k-x_k)\right|\le \operatorname{ess}\sup_{y\in\mathfrak{D}_0}\{|\nabla q\,|(y)\}\;(\operatorname{diam}(\mathfrak{D}))^2$$

and

$$q(y)\;(\delta_{ij}(y_k-x_k)\varepsilon_{ik}(z)+\delta_{kj}(y_i-x_i)\varepsilon_{ik}(z)-(y_k-x_k)\varepsilon_{kj}(z)-\delta_{ik}(y_i-x_i)\varepsilon_{kj}(z))=$$
$$q(y)\;(\delta_{ij}(y_k-x_k)\varepsilon_{ik}(z)+\delta_{kj}(y_i-x_i)\varepsilon_{ik}(z)-2(y_k-x_k)\varepsilon_{kj}(z))=$$
$$q(y)\;((y_k-x_k)\varepsilon_{jk}(z)+\delta_{kj}(y_i-x_i)\varepsilon_{ik}(z)-2(y_k-x_k)\varepsilon_{kj}(z))=$$
$$q(y)\;(\delta_{kj}(y_i-x_i)\varepsilon_{ik}(z)-(y_k-x_k)\varepsilon_{kj}(z))\le \operatorname{diam}(\mathfrak{D})(|\varepsilon_{ik}(z)|+|\varepsilon_{kj}(z)|).$$

Let

$$\Theta(q,\mathfrak{D}):=1+\operatorname{ess}\sup_{y\in\mathfrak{D}}\{|\nabla q\,|(y)\}\operatorname{diam}(\mathfrak{D}).$$

Then we find that

$$T_{2jk}\le\Theta(q,\mathfrak{D})\operatorname{diam}(\mathfrak{D})\int_{\mathfrak{D}}\int_{\mathfrak{D}_0}\int_0^1(|\varepsilon_{ik}(z)|+|\varepsilon_{kj}(z)|)d\rho dy dx.$$

Now we apply Lemma 4.1 and obtain

$$T_{2jk}\;\le\;\varpi_n\,(|\mathfrak{D}|+|\mathfrak{D}_0|)\Theta(q,\mathfrak{D})\operatorname{diam}(\mathfrak{D})\int_{\mathfrak{D}}(|\varepsilon_{ik}|+|\varepsilon_{kj}|)dx \tag{4.9}$$

hence it holds

$$T_2\le\varpi_n\,(|\mathfrak{D}|+|\mathfrak{D}_0|)\Theta(q,\mathfrak{D})\operatorname{diam}(\mathfrak{D})\sum_{j,k=1}^{n}\int_{\mathfrak{D}}(|\varepsilon_{ik}|+|\varepsilon_{kj}|)dx. \tag{4.10}$$

By (4.6), (4.7) and (4.10), we conclude that

$$
(4.11) \qquad \int_{\mathfrak{D}} |w_i(x) - \widehat{w}_i(x)|dx \leq \frac{\varpi_n(|\mathfrak{D}| + |\mathfrak{D}_0|)}{m(\mathfrak{D}_0)} \operatorname{diam}(\mathfrak{D}) \Big(\sum_{k=1}^{n} \int_{\mathfrak{D}} |\varepsilon_{ik}(x)|dx + \Theta(q, \mathfrak{D}) \sum_{j,k=1}^{n} \int_{\mathfrak{D}} (|\varepsilon_{ik}| + |\varepsilon_{kj}|)dx \Big).
$$

Estimate (4.11) implies

$$
(4.12) \qquad \sum_{i=1}^{n} \int_{\mathfrak{D}} |w_i(x) - \widehat{w}_i(x)|dx \leq \leq \frac{\varpi_n(|\mathfrak{D}| + |\mathfrak{D}_0|)}{m(\mathfrak{D}_0)} \operatorname{diam}(\mathfrak{D}) \Big(1 + 2\Theta(q, \mathfrak{D})\Big) \sum_{i,k=1}^{n} \int_{\mathfrak{D}} |\varepsilon_{ik}|dx
$$

and we find that

$$
(4.13) \qquad \|w - \widehat{w}\|_{1,\mathfrak{D}} \leq \frac{n\varpi_n(|\mathfrak{D}| + |\mathfrak{D}_0|)}{m(\mathfrak{D}_0)} \operatorname{diam}(\mathfrak{D}) \Big(1 + 2\Theta(q, \mathfrak{D})\Big) \|\varepsilon(w)\|_{1,\mathfrak{D}}.
$$

If $w \in W^{1,1}(\mathfrak{D}, \mathbb{R}^n)$, then we use an approximating sequence $\{w_m\}$ that consists of smooth functions and obtain (4.13) by density arguments. Thus we find that

$$
(4.14) \qquad \inf_{\kappa \in \mathfrak{R}(\mathfrak{D})} \|w - \widehat{w}\|_{1,\mathfrak{D}} \leq C_2(n, \mathfrak{D}) \|\varepsilon(w)\|_{1,\mathfrak{D}} \qquad \forall v \in W^{1,1}(\mathfrak{D}, \mathbb{R}^n),
$$

where

$$
(4.15) \qquad C_2(n, \mathfrak{D}) \leq \frac{n\varpi_n(|\mathfrak{D}| + |\mathfrak{D}_0|)}{m(\mathfrak{D}_0)} \operatorname{diam}(\mathfrak{D}) \Big(1 + 2\Theta(q, \mathfrak{D})\Big).
$$

Remark 4.2. *In case of $w \in BD(\mathfrak{D})$ we can also derive (4.12), where $\int_{\mathfrak{D}} |\varepsilon_{ik}|$ is understood as the variation of the measure and in the definition of $\widehat{w}$ the second integral in the r.h.s. is integrated by parts in order to put derivatives on $q(y)(x_j - y_j)$.*

Remark 4.3. *In the process of deriving (4.14) we have overestimated the right hand side of (4.6), so that (4.15) gives a general but not very accurate upper bound of $C_2(n, \mathfrak{D})$. If $\mathfrak{D}$ has some special form then a better constant might be derived by using these specifics in the integration formula (4.8).*

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