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Abstract

We introduce the space $BLD(\Omega)$ consisting of all fields $u : \Omega \to \mathbb{R}^n$ defined on a domain $\Omega \subset \mathbb{R}^n$, whose symmetric gradient satisfies $\int_\Omega |\varepsilon(u)| \ell n(1 + |\varepsilon(u)|)\, dx < \infty$. These fields of bounded logarithmic defomation form a proper subspace of the class $BD(\Omega)$ consisting of all functions having bounded deformation. With the help of Reshetnyak's representation formulas we prove that $BLD(\Omega)$ is compactly embedded in $L^p(\Omega; \mathbb{R}^n)$ even for $p = n/n-1$. The space $BLD(\Omega)$ plays in important role in the theory of plasticity with logarithmic hardening as well as in the modelling of Prandtl-Eyring fluids.

1 Introduction

As outlined in the paper of Frehse and Seregin [FrSe] spaces of vector fields $u : \Omega \to \mathbb{R}^n$ defined on a bounded domain $\Omega \subset \mathbb{R}^n$ satisfying ($\varepsilon(u)$ denoting the symmetric gradient)

$$\int_\Omega |\varepsilon(u)| \ell n(1 + |\varepsilon(u)|)\, dx < \infty \tag{1.1}$$

play an important role in the deformation theory of plasticity with logarithmic hardening. Another application of fields with the property (1.1) arises in the setting of Prandtl-Eyring fluids, whose physical background was introduced in Eyring's paper [Ey]. To be more precise, let us define the space of fields of bounded logarithmic deformation

$$BLD(\Omega) := \left\{ u \in L^1(\Omega; \mathbb{R}^n) : |\varepsilon(u)| \in L_h(\Omega) \right\} ,$$

where $h(t) := t\, \ell n(1 + t),\ t \geq 0$. Here $L_h(\Omega)$ denotes the Orlicz class generated by the N–function h equipped with the Luxemburg norm

$$\|\varphi\|_{L_h(\Omega)} := \inf \left\{ k > 0 : \int_\Omega h\left(\frac{1}{k}|\varphi|\right)\, dx \leq 1 \right\} .$$

For further details concerning Orlicz classes and related spaces we refer the reader to the textbooks of Adams [Ad] or Rao and Ren [RR]. We also remark that the space $L_h(\Omega)$ is just the famous class $L \log L(\Omega)$ investigated for example in Stein's deep paper [St]. By definition a field $u \in L^1(\Omega; \mathbb{R}^n)$ belongs to the space $BLD(\Omega)$ if and only if the (distributional) symmetric gradient is generated by a tensor valued function $\varepsilon(u)$ for which (1.1) holds. Letting

$$\|u\|_{BLD(\Omega)} := \|u\|_{L^1(\Omega)} + \|\varepsilon(u)\|_{L_h(\Omega)}$$

$BLD(\Omega)$ turns into a Banach space being a proper subspace of the space $BD(\Omega)$ of functions of bounded defomation introduced by Suquet [Su] and by Matthies, Strang and Christiansen [MSC]. The class $BD(\Omega)$ has been widely considered in the literature in connection with problems from plasticity theory, and we additionally refer to the works of Suquet [Su], Anzellotti and Giaquinta [AG], Teman and Strang [TS] and Teman [Te], a historical overview is given in the monograph [FuSe]. A natural norm on $BD(\Omega)$ is defined through

$$\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega)} + \int_\Omega |\varepsilon(u)| \,,$$

where $\int_\Omega |\varepsilon(u)|$ is the total variation of the tensor measure $\varepsilon(u)$. From the above references it follows that for bounded Lipschitz domains Ω $BD(\Omega)$-functions have a $L^1(\partial\Omega;\mathbb{R}^n)$-trace denoted by $u|_{\partial\Omega}$, and therefore we also have a trace operator in $BLD(\Omega)$. If we define the subspace

$$BLD_0(\Omega) := \text{ closure of } C_0^\infty(\Omega;\mathbb{R}^n) \text{ in } BLD(\Omega) \text{ with respect to } \|\cdot\|_{BLD(\Omega)} \,,$$

then it holds (see, e.g., [FuSe], Lemma 4.1.6)

$$BDL_0(\Omega) = \{u \in BLD(\Omega) : u|_{\partial\Omega} = 0\} \,,$$

which means that for fields from $BLD(\Omega)$ with vanishing trace there exists an approximation through smooth functions with compact support.

The main feature of the present paper is the analysis of the problem for which exponents p the embedding

$$BLD(\Omega) \hookrightarrow L^p(\Omega;\mathbb{R}^n)$$

is compact. This question naturally arises, when investigating variational problems for functionals of the form

$$\int_\Omega h\left(|\varepsilon(u)|\right)\,dx + \int_\Omega f(u)\,dx$$

with potential f being of power growth, since then minimizing sequences $\{u_\nu\}$ under natural assumptions satisfy

$$\sup_\nu \|u_\nu\|_{BLD(\Omega)} < \infty \,, \tag{1.2}$$

so that (see Theorem 1.1 below) $\varepsilon(u_\nu) \rightharpoondown \varepsilon(u)$ in $L^1(\Omega;\mathbb{R}^{n\times n})$ for a subsequence and a function $u \in BLD(\Omega)$, and one likes to conclude that

$$\int_\Omega f(u_\nu)\,dx \to \int_\Omega f(u)\,dx \,.$$

A related problem occurs for stationary Prandtl-Eyring fluids: in this setting a sequence $\{u_\nu\}$ satisfying (1.2) arises for example as solutions of regularized problems, and by interpreting the convective term in the weakest way one now wants to show that

$$\int_\Omega u_\nu \otimes u_\nu : \varepsilon(\varphi)\,dx \to \int_\Omega u \otimes u : \varepsilon(\varphi)\,dx \tag{1.3}$$

as $\nu \to \infty$ for all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^n)$, $\operatorname{div}\varphi = 0$. In order to discuss the question of compactness, let us recall some facts from the BD-case. There it holds (see, e.g., [AG] or [TS]):

Lemma 1.1. *Let Ω denote a bounded Lipschitz domain.*

a) The space $BD(\Omega)$ is continuously embedded into $L^{n/n-1}(\Omega;\mathbb{R}^n)$. If $u|_{\partial\Omega} = 0$, then $\|u\|_{L^{n/n-1}(\Omega)} \le c(n,\Omega)\int_\Omega |\varepsilon(u)|$.

b) For $1 \le p < n/n-1$ the embedding $BD(\Omega) \hookrightarrow L^p(\Omega;\mathbb{R}^n)$ is compact.

If we look at the Prandtl-Eyring example, we first see from Lemma 1.1a) that we need $n = 2$ in order to have $|u \otimes u| \in L^1(\Omega)$ for $u \in BLD(\Omega)$, but even if we assume this, (1.3) does not follow from the BD-compactness, since we are in the limit case "$p = n/n-1$". However it turns out that we have BLD-compactness even for the limit exponent. We will derive this result with the help of representation formulas due to Reshetnyak [Re] valid for special domains.

Definition 1.1. *Let Ω denote a bounded Lipschitz domain. Ω is called a star region with respect to a ball $B_0 \subset \Omega$ if every segment $\overline{x_0 x}$ connecting an arbitrary point x_0 of B_0 with an arbitrary point x of Ω is contained in Ω.*

Then we have

THEOREM 1.1. *Suppose that the bounded Lipschitz domain is a finite union of domains as described in Definition 1.1. Then the embedding*

$$BLD(\Omega) \hookrightarrow L^{n/n-1}(\Omega;\mathbb{R}^n)$$

is compact, more precisely: consider a sequence $\{u_\nu\}$ from $BLD(\Omega)$ such that

$$\sup_\nu \left\{ \int_\Omega |u_\nu|\,dx + \int_\Omega |\varepsilon(u_\nu)|\ell n(1+|\varepsilon(u_\nu)|)\,dx \right\} < \infty\,.$$

Then there exist a subsequence $\{\widetilde{u}_\nu\}$ and a field $u \in BLD(\Omega)$ such that

a) $\|\widetilde{u}_\nu - u\|_{L^{n/n-1}(\Omega)} \to 0$

b) $\varepsilon(\widetilde{u}_\nu) \rightharpoondown \varepsilon(u)$ weakly in $L^1(\Omega;\mathbb{R}^{n\times n})$

as $\nu \to \infty$. Moreover, if all fields u_ν agree on $\partial\Omega$ with some function $\varphi \in L^1(\partial\Omega;\mathbb{R}^n)$, then also $u = \varphi$ on $\partial\Omega$.

REMARK 1.1. *Of course Theorem 1.1 remains valid if $h(t) = t\,\ell n(1+t)$ is replaced by an N-function $\widetilde{h}$ of stronger growth. On the contrary, we do not know, if it is possible to consider N-functions of slower growth.*

Our paper is organized as follows: in Section 2 we present an approximation lemma for functions from $BLD(\Omega)$. In Section 3 we combine this lemma with representation formulas due to Reshetnyak [Re] in order to get a $L^1 - L\log L$ Korn-type inequality, and Section 4 contains the proof of Theorem 1.1. In Section 5 we collect various other Korn-type inequalities which can be deduced from the techniques used in Section 3.

2 Approximation

In order to carry out the calculations in the subsequent sections more easily we need the following approximation lemma:

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain. For any $u \in BLD(\Omega)$ there exists a sequence $\{u_j\}$ in $C^\infty(\overline{\Omega};\mathbb{R}^n)$ such that ($h(t) := t\,\ell n(1+t)$)*

a) $\int_\Omega |u_j - u|\,dx \to 0$ *and*

b) $\int_\Omega h(|\varepsilon(u_j) - \varepsilon(u)|)\,dx \to 0, \quad \|\varepsilon(u_j) - \varepsilon(u)\|_{L_h(\Omega)} \to 0$ *and*

$\int_\Omega h(|\varepsilon(u_j)|)\,dx \to \int_\Omega h(|\varepsilon(u)|)\,dx$ *as* $j \to \infty$.

REMARK 2.1. *From the first convergence stated in b) it is immediate that*

$$\varepsilon(u_j) \to \varepsilon(u) \text{ in } L^1(\Omega;\mathbb{R}^{n\times n})\,, \tag{2.1}$$

but we like to remark that (2.1) already follows from the last statement of b): in fact, since $\lim\limits_{t\to\infty} \frac{h(t)}{t} = \infty$, we deduce the equi-integrability of the tensors $\varepsilon(u_j)$, hence by the Dunford-Pettis theorem (see, e.g. [AFP], Theorem 1.3.8)

$$\varepsilon(u_j) \rightharpoonup \varepsilon(u) \text{ in } L^1(\Omega;\mathbb{R}^{n\times n})\,. \tag{2.2}$$

Next we combine (2.2) with Theorem 2, p.92, of [GMS2], to see

$$\int_\Omega \sqrt{1+|\varepsilon(u_j)|^2}\,dx \to \int_\Omega \sqrt{1+|\varepsilon(u)|^2}\,dx\,, \tag{2.3}$$

where of course the last claim of b) has been exploited again. Finally, (2.1) is a consequence of (2.2) and (2.3) on account of Proposition 1, p.95, from [GMS2].

Proof of Lemma 2.1: The main ideas are due to Frehse and Seregin and outlined in Appendix 2 of their paper [FrSe]. In [FuSe], Lemma 4.1.6a), we gave a formulation of a density result suitable for the situation at hand. So suppose that we are given a field $u \in BLD(\Omega)$. Then according to this lemma from [FuSe] we can find a sequence $\{u_j\}$ in $C^\infty(\overline{\Omega};\mathbb{R}^n)$ such that part a) of Lemma 2.1 is satisfied, moreover it holds

$$|||\varepsilon(u_j) - \varepsilon(u)||| \to 0 \text{ as } j \to \infty\,. \tag{2.4}$$

Here we have abbreviated ($\varphi \in L_h(\Omega)$)

$$|||\varphi||| := \int_{Q_0} M_{Q_0}\widetilde{\varphi}\,dx\,,$$

where Q_0 denotes an open cube with sides parallel to the axes of $\mathbb{R}^n$ containing Ω and $\widetilde{\varphi}$ is the extension of φ to Q_0 with value 0. $M_{Q_0}\widetilde{\varphi}$ stands for the maximal function of $\widetilde{\varphi}$. From Lemma 4.1.1 in [FuSe] we deduce

$$\int_\Omega |\varphi| \ell n \left(2 + \frac{1}{[\varphi]}|\varphi|\right) dx \le c|||\varphi||| \tag{2.5}$$

for a suitable positive constant e and with $[\varphi] := \fint_{Q_0} |\widetilde{\varphi}|\, dx$. Letting $\varphi_j := |\varepsilon(u_j - u)|$ we get from (2.4) and (2.5)

$$\lim_{j\to\infty} \int_\Omega \varphi_j \; \ell n \left(2 + \frac{1}{[\varphi_j]}\varphi_j\right) dx = 0\,. \tag{2.6}$$

(2.6) clearly implies $\int_\Omega \varphi_j\, dx \to 0$ as $j \to \infty$ and in consequence $[\varphi_j] \to 0$. For j sufficiently large it therefore holds

$$\varphi_j \; \ell n(1 + \varphi_j) \le \varphi_j \; \ell n \left(2 + \frac{1}{[\varphi_j]}\varphi_j\right)\,,$$

thus by (2.6) and the definition of φ_j

$$\lim_{j\to\infty} \int_\Omega h\left(|\varepsilon(u_j) - \varepsilon(u)|\right) dx = 0\,. \tag{2.7}$$

Quoting for example Lemma 2.1 b) from [FO] we see that (2.7) implies

$$\|\varepsilon(u_j) - \varepsilon(u)\|_{L_h(\Omega)} \to 0\,, \tag{2.8}$$

which means that we have $\varepsilon(u_j) \to \varepsilon(u)$ w.r.t. the Luxemburg norm. Having established (2.7) and (2.8) it remains to show that

$$\lim_{j\to\infty} \int_\Omega h\left(|\varepsilon(u_j)|\right) dx = \int_\Omega h(|\varepsilon(u)|)\, dx \tag{2.9}$$

is true. From (2.7) we get (2.1), hence

$$\int_\Omega h(|\varepsilon(u)|)\, dx \le \liminf_{j\to\infty} \int_\Omega h(|\varepsilon(u_j)|)\, dx$$

by lower semicontinuity. The convexity of h combined with inequality (16), p.234, from [Ad] implies

$$\begin{aligned}
&\int_\Omega h\left(|\varepsilon(u_j)|\right) dx \\
&\quad\le \int_\Omega h(|\varepsilon(u)|)\, dx + \int_\Omega h'(|\varepsilon(u_j)|)(|\varepsilon(u_j)| - |\varepsilon(u)|)\, dx \\
&\quad\le \int_\Omega h(|\varepsilon(u)|)\, dx + 2\|h'(|\varepsilon(u_j)|)\|_{L_{h^*}(\Omega)} \||\varepsilon(u_j)| - |\varepsilon(u)|\|_{L_h(\Omega)}\,,
\end{aligned}$$

h^* denoting the conjugate function. From (2.8) it follows that $\| |\varepsilon(u_j)| - |\varepsilon(u)| \|_{L_h(\Omega)} \to 0$, hence our claim (2.9) is established as soon as we can show the validity of

$$\sup_j \|h'(|\varepsilon(u_j)|)\|_{L_{h^*}(\Omega)} < \infty \,. \tag{2.10}$$

We have

$$h^*\left(h'(|\varepsilon(u_j)|)\right) = |\varepsilon(u_j)| h'(|\varepsilon(u_j)|) - h(|\varepsilon(u_j)|) \le h(|\varepsilon(u_j)|) \,,$$

and therefore

$$\sup_j \int_\Omega h^*\left(h'(|\varepsilon(u_j)|)\right) dx < \infty \,. \tag{2.11}$$

But then (2.10) is an immediate consequence of (2.11) and the elementary estimate

$$\|\Psi\|_{L_{h^*}(\Omega)} \le \max\left\{1, \int_\Omega h^*(|\Psi|)\, dx\right\}$$

for the Luxemburg norm of a function Ψ from the Orlicz class $L_{h^*}(\Omega)$. This completes the proof of Lemma 2.1. □

3 A Korn-type inequality and consequences

We start with a $L^1 - L\log L$ Korn-type inequality, which might be known (see Remark 5 in the paper [MM] of Mosolov and Mjasnikov), but we could not trace its explicit form in the literature.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain, which is the union of a finite number of domains, each of them being a star region relative to some ball. Then there is a constant $C = C(n, \Omega)$ such that for all $u \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ we have*

$$\int_\Omega |\nabla u|\, dx \le C\left[\int_\Omega |\varepsilon(u)| \ell n(1 + |\varepsilon(u)|)\, dx + \int_\Omega |u|\, dx + 1\right] . \tag{3.1}$$

Proof: Consider $u \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ and assume for simplicity that Ω is a star region with respect to a ball $B_0 \subset \Omega$. From Reshetnyak's work [Re] we get following his notation

$$u(x) = (P_1 u)(x) + (R_1 Q_1 u)(x) \,. \tag{3.2}$$

Here the operator P_1 (= projection on the space of rigid motions) is defined through formulas (2.33) and (2.34) in [Re], $Q_1 u$ is just the symmetric gradient of u, and the definition of R_1 is given in equation (2.37) of [Re]. Abbreviating $U := \varepsilon(u)$, we see from [Re], proof of Lemma 2, that

$$R_1(U) = S(U) + T(U) \,,$$

where $(i = 1, \ldots, n)$

$$S(U)^i(x) = \int_\Omega \frac{\omega_{kli}(x,e)}{|x-y|^{n-1}} U_{kl}(y)\,dy\,, \tag{3.3}$$

$$T(U)^i(x) = \int_\Omega \Theta_{kli}(x,y) U_{kl}(y)\,dy\,. \tag{3.4}$$

In formulas (3.3) and (3.4) the sum is taken from 1 to n with respect to indices repeated twice, and the quantities ω_{kli} and Θ_{kli} are explained after (2.38) in [Re]. Extending U_{kl} to the whole space by setting $U_{kl} = 0$ outside of Ω, we see - after dropping all indices - that the right-hand side of (3.3) is of the form

$$V(x) := \int_{\mathbb{R}^n} K(x-y)U(y)\,dy\,,\ x \in \mathbb{R}^n\,,$$

with K being essentially homogeneous of degree $1-n$ in the sense of Morrey [Mo], Theorem 3.4.2. From part (b) of this theorem and the subsequent remark we deduce that almost everywhere it holds

$$\partial_\alpha V(x) = C_\alpha U(x) + \lim_{\rho\downarrow 0} \int_{\mathbb{R}^n - B_\rho(x)} (\partial_\alpha K)(x-y)U(y)\,dy\,, \tag{3.5}$$

where $\alpha \in \{1, \ldots n\}$ and where C_α denotes a constant. In particular this means, that $\lim\limits_{\rho\downarrow 0} \ldots$ exists for almost all centers $x \in \mathbb{R}^n$. Combining (3.4) and (3.5) with Theorem G (case (ii)) of Calderón's and Zygmund's paper [CZ], in which it is noted that this part of Theorem G was proved by Cotlar [Co], we get

$$\|\nabla S(U)\|_{L^1(\Omega)} \le C\|U\|_{L^1(\Omega)} + A_\Omega \int_{\mathbb{R}^n} |U|\ell n(1+|U|)\,dx + B_\Omega\,,$$

A_Ω, B_Ω denoting constants depending on n and Ω as introduced after Theorem B in [CZ]. We therefore arrive at

$$\|\nabla S(\varepsilon(u))\|_{L^1(\Omega)} \le C\left[\int_\Omega |\varepsilon(u)|\ell n(1+|\varepsilon(u)|)\,dx + 1\right] \tag{3.6}$$

for a suitable constant $C = C(n,\Omega)$. The right-hand side of (3.4) behaves "nicely", i.e. we have

$$\partial_\alpha T(U)^i(x) = \int_\Omega \frac{\partial}{\partial x_\alpha} \Theta_{kli}(x,y) U_{kl}(y)\,dy$$

with bounded derivatives $\frac{\partial}{\partial x_\alpha}\Theta(x,y)$ (see [Re], comments after (2.46)), thus

$$\|\nabla T(U)\|_{L^1(\Omega)} \le C\|\varepsilon(u)\|_{L^1(\Omega)}\,. \tag{3.7}$$

Returning to (3.2) we have a bound for the L^1-norm of $\nabla(R_1Q_1u)$ by combining (3.6) and (3.7). Let us finally look at P_1u: if we calculate $\nabla P_1 u$ with the help of (2.33) from [Re], it is immediate that in order to estimate $\|\nabla P_1 u\|_{L^1(\Omega)}$ we have to bound the quantities $\int_{B_0} |S_j u|\, dx$, $j = 1, \ldots, n$. However, since $S_{ij} u := \frac{1}{2}(\partial_j u^i - \partial_i u^j)$ is formed of first partial derivatives of u, this is not very helpful. We therefore use the representation (2.34) of [Re], where an integration by parts has been performed with the consequence that on the right-hand side of (2.34) only the function u itself occurs, hence we arrive at

$$\|\nabla P_1 u\|_{L^1(\Omega)} \leq C \|u\|_{L^1(B_0)} . \tag{3.8}$$

Note that in (3.8) the constant C also depends on B_0. Putting together our results we have shown (3.1). $\square$

REMARK 3.1. *Let Ω denote a domain as described in Lemma 3.1 and assume in addition that $n \geq 3$. Then we have the following stronger variant of inequality (3.1)*

$$\int_\Omega |\nabla u|\, dx \leq C(n, \Omega) \left[\int_\Omega |\varepsilon^D(u)| \ell n(1 + |\varepsilon^D(u)|)\, dx + \int_\Omega |u|\, dx + 1 \right] \tag{3.1'}$$

valid for $u \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$, where $\varepsilon^D(u)$ is the deviatoric part of $\varepsilon(u)$, i.e.

$$\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}(div\ u)\, \mathbf{1} ,$$

1 *denoting the unit matrix. For the proof we observe that (3.2) can be replaced by the representation*

$$u(x) = (P_2 u)(x) + (R_2 Q_2 u)(x)$$

with operators P_2, R_2 and Q_2 defined in (2.40'), (2.41) and (1.2) of [Re]. Starting from this formula, the ε^D-variant of (3.1) is obtained by repeating the arguments from the proof of Lemma 3.1. It is worth noting that inequalities (3.1) and (3.1)' are in some sense limit cases of Korn-type inequalities for functions from Orlicz-Sobolev spaces. We will discuss this issue in Section 5.

As an application we get

Lemma 3.2. *Under the hypothesis of Lemma 3.1 we have that $w := \ell n(1 + |u|)u$ is in the space $BD(\Omega)$ for any function $u \in BLD(\Omega)$ together with*

$$\int_\Omega |\varepsilon(w)| \leq c(n, \Omega) \left[\int_\Omega h(|u|)\, dx + \int_\Omega h(|\varepsilon(u)|)\, dx + 1 \right] . \tag{3.9}$$

Proof: We consider first the smooth case, i.e. we choose $u \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$. Then it holds for $w := \ell n(1 + |u|)u$

$$\varepsilon(w) = \ell n(1 + |u|)\varepsilon(u) + \frac{1}{2} \left(u^i \partial_j \ell n(1 + |u|) + u^j \partial_i \ell n(1 + |u|) \right)_{1 \leq i,j \leq n} ,$$

hence

$$|\varepsilon(w)| \leq \ell n(1 + |u|)|\varepsilon(u)| + c(n) \frac{|u|}{1 + |u|} |\nabla u| .$$

From Young's inequality for the N-function $h(t) = t\,\ell n(1+t)$ we get for $s,\, t \geq 0$

$$h'(t)s \leq h^*(h'(t)) + h(s)\,,$$

h^* denoting the conjugate function, moreover we have

$$h^*(h'(t)) = t\,h'(t) - h(t) \leq h(t)\,.$$

These inequalities together imply

$$\ell n(1+|u|)|\varepsilon(u)| \leq h'(|u|)|\varepsilon(u)| \leq h(|u|) + h(|\varepsilon(u)|)\,,$$

hence

$$\int_\Omega |\varepsilon(w)|\,dx \leq \int_\Omega h(|u|)\,dx + \int_\Omega h(|\varepsilon(u)|)\,dx + c(n)\int_\Omega |\nabla u|\,dx\,. \tag{3.10}$$

To the last integral on the right-hand side of (3.10) we apply (3.1) and immediately deduce (3.9) from (3.10). Now let $u \in BLD(\Omega)$ and choose a sequence $\{u_j\}$ in $C^\infty(\overline{\Omega};\mathbb{R}^n)$ according to Lemma 2.1. Then $u_j \to u$ strongly in L^p for any $p < n/n-1$, thus

$$\lim_{j\to\infty}\int_\Omega h(|u_j|)\,dx = \int_\Omega h(|u|)\,dx\,,$$

and from (3.9) we infer

$$\sup_j \int_\Omega |\varepsilon(w_j)|\,dx < \infty\,, \tag{3.11}$$

$w_j = \ell n(1+|u_j|)u_j$. By (3.11) $\{w_j\}$ is a bounded sequence in $BD(\Omega)$, and after passing to a subsequence we find $\widetilde{w} \in BD(\Omega)$ such that $w_j \to \widetilde{w}$ in $L^1(\Omega;\mathbb{R}^n)$ and a.e., moreover

$$\int_\Omega |\varepsilon(\widetilde{w})| \leq \liminf_{j\to\infty}\int_\Omega |\varepsilon(w_j)|\,dx\,. \tag{3.12}$$

Since we may also assume that $u_j \to u$ a.e., it follows $\widetilde{w} = w(:= \ell n(1+|u|)u)$, and the desired inequality (3.9) for w follows from the validity of (3.9) for w_j in combination with (3.12). □

REMARK 3.2. *If we combine the approximation procedure from Lemma 2.1 with inequality (3.1), then we get*

$$BLD(\Omega) \subset BV(\Omega;\mathbb{R}^n)\,, \tag{3.13}$$

where $BV(\Omega;\mathbb{R}^n)$ is the space of functions of bounded variation from Ω into $\mathbb{R}^n$ as introduced for example in [Gi].

REMARK 3.3. *From the recent survey paper of Mingione (see [Mi]) it can be deduced that*

$$\|\nabla u\|_{L^1(\mathbb{R}^n)} \le c(n)\|\varepsilon(u)\|_{L_h(\mathbb{R}^n)} \tag{3.14}$$

holds for $u \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$*. In fact, for smooth functions* u *we have the formula*

$$\Delta u^j = 2\partial_i \varepsilon^D(u)_{ij} - 2\left(\frac{1}{2}-\frac{1}{n}\right)\partial_j \ div\ u,\ \ j=1,\dots,n\,,$$

where $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}\ div\ u\ \mathbf{1}$*,* $\mathbf{1}$ *denoting the unit matrix. This can be rewritten as*

$$\Delta u^j = \ div\ F^j, F^j := \left(2\varepsilon^D(u)_{ij} - 2\left(\frac{1}{2}-\frac{1}{n}\right)\ div\ u\, \delta_{ij}\right)_{1\le i\le n}\,,$$

and (2.28) from [Mi] implies (3.14). An immediate consequence of (3.14) is

$$BLD_0(\Omega) \subset \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^n)\,. \tag{3.15}$$

In fact, if u *is in* $BLD_0(\Omega)$*, then by definition there exists a sequence* $\{u_\nu\} \subset C_0^\infty(\Omega;\mathbb{R}^n)$ *such that* $\|\varepsilon(u_\nu - u)\|_{L_h(\Omega)} \to 0$*. Hence* $\{\varepsilon(u_\nu)\}$ *is a Cauchy sequence in the space* $L_h(\Omega;\mathbb{R}^{n\times n})$ *and (3.14) shows that the same is true for the sequence* $\{\nabla u_\nu\}$ *now in the class* $L^1(\Omega;\mathbb{R}^{n\times n})$*. Therefore* $\{u_\nu\}$ *converges strongly in* $\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^n)$ *towards some element* $\widetilde{u}$ *from this class. Since* $u - \widetilde{u}$ *has trace zero and since obviously* $\varepsilon(u) = \varepsilon(\widetilde{u})$*, we obtain* $u = \widetilde{u}$ *on account of Lemma 1.1. This proves the inclusion (3.15).*

REMARK 3.4. *It would be interesting to know, if (3.13) can be replaced by the stronger result* $BLD(\Omega) \subset W_1^1(\Omega;\mathbb{R}^n)$*. We think that this inclusion is true and suggest to prove the claim in two steps:*

i) Given $u \in BLD(\Omega)$ *the construction of the trace* $u|_{\partial\Omega}$ *outlined by Temam and Strang (see [TS], Lemma 1.1) probably gives that* $u|_{\partial\Omega}$ *is in* $L\log L(\partial\Omega)$*.*

ii) Applying the arguments used by Gagliardo [Ga] to $u|_{\partial\Omega}$ *we may end up with* $\overline{u} \in W_1^1(\Omega;\mathbb{R}^n)$ *having trace* $u|_{\partial\Omega}$ *and satisfying in addition* $|\nabla\overline{u}| \in L\log L(\Omega)$*. Therefore* $u - \overline{u}$ *is in* $BLD_0(\Omega)$ *and (3.15) can be applied, hence* u *is in the space* $W_1^1(\Omega;\mathbb{R}^n)$*.*

4 Proof of Theorem 1.1

Suppose that ($h(t) := t\,\ell n(1+t)$)

$$\sup_\nu \int_\Omega [|u_\nu| + h(|\varepsilon(u_\nu)|)]\,dx < \infty \tag{4.1}$$

for the sequence $\{u_\nu\} \subset BLD(\Omega)$. The compactness of the embedding

$$BD(\Omega) \hookrightarrow L^1(\Omega;\mathbb{R}^n)$$

gives the existence of $u \in L^1(\Omega;\mathbb{R}^n)$ such that

$$u_\nu \to u \text{ in } L^1(\Omega;\mathbb{R}^n) \text{ and a.e.}\,, \tag{4.2}$$

where here and in what follows we always pass to subsequences whenever this is necessary. From (4.1) and the Dunford-Pettis theorem (see [AFP], Theorem 1.38) we infer

$$\varepsilon(u_\nu) \rightharpoondown: \sigma \text{ in } L^1(\Omega;\mathbb{R}^{n\times n})\,,$$

which in combination with (4.2) implies

$$\varepsilon(u_\nu) \rightharpoondown \varepsilon(u) \text{ in } L^1(\Omega;\mathbb{R}^{n\times n})\,. \tag{4.3}$$

Now, using standard theorems on lower semicontinuity of convex variational integrals, we get from (4.3)

$$\int_\Omega h(|\varepsilon(u)|)\,dx \le \liminf_{\nu\to\infty} \int_\Omega h(|\varepsilon(u_\nu)|)\,dx\,,$$

so that in particular $u \in BLD(\Omega)$. Next we apply arguments of Frehse and Seregin [FrSe]: on account of (4.3) there exists a sequence $\{\sigma_\mu\}$, σ_μ being an element of the convex hull of $\{\varepsilon(u_\nu) : \nu \ge \mu\}$, such that $\sigma_\mu \to \varepsilon(u)$ in $L^1(\Omega;\mathbb{R}^{n\times n})$. We have

$$\sigma_\mu = \sum_{\nu=\mu}^{N(\mu)} \lambda_\nu^\mu \varepsilon(u_\nu),\ \sum_{\nu=\mu}^{N(\mu)} \lambda_\nu^\mu = 1,\ 0 \le \lambda_\nu^\mu \le 1\,,$$

with suitable coefficients λ_ν^μ and integers $N(\mu) \ge \mu$. Let

$$\overline{u}_\mu := \sum_{\nu=\mu}^{N(\mu)} \lambda_\nu^\mu\, u_\nu\,.$$

These functions belong to $BLD(\Omega)$ and satisfy on account of (4.2)

$$\|\overline{u}_\mu - u\|_{L^1(\Omega)} \le \sum_{\nu=\mu}^{N(\mu)} \lambda_n^\mu u \|u_\nu - u\|_{L^1(\Omega)} \to 0$$

as $\mu \to \infty$. Moreover it holds

$$\int_\Omega |\varepsilon(\overline{u}_\mu)|\,dx = \int_\Omega |\sigma_\mu|\,dx \to \int_\Omega |\varepsilon(u)|\,dx,\ \mu \to \infty\,,$$

and according to [AG], remarks after Theorem 1.4, these two convergences imply the L^1–convergence of the traces of u_μ towards the trace of u. In other words: if the sequence $\{u_\nu\}$ has fixed traces, then also the limit function u has this boundary datum. It remains to show the validity of

$$\lim_{\nu\to\infty} \|u_\nu - u\|_{L^{n/n-1}(\Omega)} = 0\,. \tag{4.4}$$

From (4.1) and (3.9) we first obtain

$$\sup_{\nu\in\mathbb{N}} \int_\Omega |\varepsilon(w_\nu)| < \infty\,, \tag{4.5}$$

$w_\nu := \ell n(1+|u_\nu|)u_\nu$; clearly $\{u_\nu\}$ is a bounded sequence in $BD(\Omega)$, Lemma 1.1 implies

$$\sup_{\nu} \int_\Omega |u_\nu|^{n/n-1}\,dx < \infty\,, \tag{4.6}$$

which yields in combination with (4.5) the $BD(\Omega)$-boundedness of $\{w_\nu\}$. Quoting Lemma 1.1 one more time we arrive at

$$\sup_{\nu} \|w_\nu\|_{L^{n/n-1}(\Omega)} < \infty\,. \tag{4.7}$$

Let $\Gamma(t) := h\left(t^{\frac{n-1}{n}}\right)^{\frac{n}{n-1}}$, $t \geq 0$. Then

$$\frac{\Gamma(t)}{t} = \left\{\frac{h\left(t^{\frac{n-1}{n}}\right)}{t^{\frac{n-1}{n}}}\right\}^{\frac{n}{n-1}} \longrightarrow \infty$$

as $t\to\infty$, and from (4.7) it follows

$$\int_\Omega \Gamma\left(|u_\nu|^{\frac{n}{n-1}}\right)\,dx = \int_\Omega h(|u_\nu|)^{\frac{n}{n-1}}\,dx = \int_\Omega |w_\nu|^{\frac{n}{n-1}}\,dx \leq \text{ const } < \infty\,.$$

The Dunford-Pettis theorem implies

$$|u_\nu|^{\frac{n}{n-1}} \rightharpoondown: g$$

weakly in $L^1(\Omega)$, but $g = |u|^{n/n-1}$ on account of (4.2). This in particular yields

$$\|u_\nu\|_{L^{n/n-1}(\Omega)} \to \|u\|_{L^{n/n-1}(\Omega)},\ \nu\to\infty\,. \tag{4.8}$$

From (4.6) and $u_\nu \to u$ a.e. we get

$$u_\nu \rightharpoondown u \text{ in } L^{n/n-1}(\Omega)\,, \tag{4.9}$$

and from (4.8) and (4.9) our claim (4.4) follows by applying the Radon–Riesz lemma (see, e.g., [GMS1], Proposition 3, p.47). This completes the proof of Theorem 1.1. □

5 Korn-type inequalities in Orlicz-Sobolev spaces

The most important tool for proving Theorem 1.1 evidently is inequality (3.1), which in principle states that the L^1-norm of ∇u is bounded in terms of the $L\log L$-norm of $\varepsilon(u)$ plus the L^1- norm of the field u itself. Unfortunately we do not have a version of (3.1) now with the $L\log L$-norm of ∇u on the left-hand side, and we do not think that this actually holds. One formal objection against the validity of this stronger inequality is the fact that "our" N-function $h(t) = t\,\ell n(1+t)$ has the doubling property ($\Delta 2$), but the ($\nabla 2$)-condition, i.e. the validity of ($\Delta 2$) for the conjugate function h^*, does not hold, and as we will show below stronger results can be obtained for N-functions φ of type $(\Delta 2)\cap(\nabla 2)$. Before we give some details let us remark that Stein [St] shows that a function f (with compact support) is in $L\log L$ if and only if the maximal function $M(f)$ is in L^1. If we take $f := |\varepsilon(u)|$, then the singular integral occurring in (3.5) shows a behaviour as $M(f)(x)$, so that according to Stein's characterization of the class $L\log L$ we can not hope for an improvement of either (3.1) or (3.1)'.

Now we assume that we are given a N-function φ of type $(\Delta 2)\cap(\nabla 2)$, and let Ω denote a bounded Lipschitz domain in $\mathbb{R}^n$. Then it holds ($W^1_\varphi(\Omega;\mathbb{R}^n)$ and $\overset{\circ}{W}{}^1_\varphi(\Omega;\mathbb{R}^n)$ denoting the Orlicz-Sobolev spaces defined in e.g. [Ad])

THEOREM 5.1. *Assume that the N-function $\varphi : [0,\infty) \to [0,\infty)$ is of class C^2 satisfying $(\Delta 2)\cap(\nabla 2)$ and in addition*

(H) $\quad a\frac{\varphi'(t)}{t} \le \varphi''(t)$

for all $t \ge 0$ with a constant $a > 0$. Then we have the following statements:

a) There is a constant $c = c(n,\Omega,\varphi)$ such that

$$\|\nabla u\|_{L_\varphi(\Omega)} \le c\|\varepsilon^D(u)\|_{L_\varphi(\Omega)}$$

holds for all fields $u \in \overset{\circ}{W}{}^1_\varphi(\Omega;\mathbb{R}^n)$, $\|\cdot\|_{L_\varphi(\Omega)}$ denoting the Luxemburg norm.

Suppose in addition that Ω is a region as described in Theorem 1.1. Then, with another constant $c = c(n,\Omega,\varphi)$ it holds for all $u \in W^1_\varphi(\Omega;\mathbb{R}^n)$

*b) $\|\nabla u\|_{L_\varphi(\Omega)} \le c\left[\|\varepsilon(u)\|_{L_\varphi(\Omega)} + \|u\|_{L_\varphi(\Omega)}\right]$
and if $n \ge 3$, then we can replace $\varepsilon(u)$ on the right-hand side by $\varepsilon^D(u)$,*

*c) $\|u - r\|_{L_\varphi(\Omega)} \le c\|\varepsilon(u)\|_{L_\varphi(\Omega)}$,
where $r = r(u)$ is a suitable rigid motion,*

*d) $\|u - \kappa\|_{L_\varphi(\Omega)} \le c\|\varepsilon^D(u)\|_{L_\varphi(\Omega)}$,
$\kappa = \kappa(u)$ being a Killing vector as explained in [Re] and [Da].*

REMARK 5.1. *For the choice $\varphi(t) = t^p$, $1 < p < \infty$, these inequalities are obtained by e.g. Mosolov-Mjasnikov [MM] and Reshetnyak [Re]. Moreover, in [MM], Remark 5, there is a comment how to get variants for N-functions.*

REMARK 5.2. *Inequality a) for $n = 2$ has been proved in [Fu1], the ε-variant of a) is discussed in [Fu2] and a proof of d) for domains $\Omega \subset \mathbb{R}^2$ and $\varphi(t) := t^p$, $1 < p < \infty$, is presented in [Fu3].*

REMARK 5.3. *From (H) it easily follows that*

$$\inf_{t\ge 0} \frac{\varphi'(t)t}{\varphi(t)} \ge 1 + a\,,$$

which by Corollary 4, p.26, of [RR] implies that φ is already of type $(\nabla 2)$*. The reader should note that for $h(t) := t\,\ell n(1+t)$ it is not possible to find $a > 0$ such that (H) holds.*

REMARK 5.4. *Condition (H) is a rather strong hypothesis on the N-function φ implying the validity of the inequalities a) - d) stated in Theorem 5.1. It is motivated through certain recent applications (see, e.g. [BF] and [Fu4]), and as a matter of fact (compare the statements after (5.3)) condition (H) can be replaced by the assumption that φ satisfies (1.11) - (1.16) from [Ko].*

Proof of Theorem 5.1: a) Let B denote the open unit ball in $\mathbb{R}^n$ and consider a field $u \in C_0^\infty(B;\mathbb{R}^n)$. According to Remark 5.2 we may also assume $n \ge 3$. Then we quote Reshetnyak's representation formula (2.43), i.e.

$$u(x) = (P_2 u)(x) + R_2(Q_2 u)(x),\ x \in B\,. \tag{5.1}$$

Here P_2u is a Killing vector, and in order to get more information on P_2u we use formula (2.20) of [Re] and pass to the mean value $\fint_B \quad \ldots dy$ with respect to the variable $y \in B$ on the right-hand side. According to the comment given after (2.22) the i^{th} component of $(P_2u)(x)$ is the remaining expression on the right-hand side, in which no integration with respect to $t \in [0,1]$ is performed. But then it is easy to see (using integration by parts) that P_2u is a constant vector, and (5.1) yields

$$\nabla u(x) = \nabla R_2(Q_2u)(x),\ x \in B\,. \tag{5.2}$$

The operator R_2 can be decomposed in the same way as done after (3.2) (compare the proof of Lemma 2 in [Re]) leading to a discussion of the singular integral from (3.5). This time we quote Theorem 2 of Koizumi [Ko] and deduce from (5.2) the inequality

$$\|\nabla u\|_{L_\varphi(B)} \le c\|\varepsilon^D(u)\|_{L_\varphi(B)}\,. \tag{5.3}$$

In fact, the hypotheses (1.12) and (1.15) of Theorem 2 from [Ko] follow along the lines of Lemma 4.3 in [BF], whereas (1.13) and (1.16) are immediate consequences of inequality (1.5) from [BF] combined with the fact that $t \mapsto \frac{\varphi'(t)}{t^a}$ is increasing, which is implied by (H). For further details we refer to the Appendix. By scaling we obtain (5.3) now for functions $u \in C_0^\infty(B_R(x_0);\mathbb{R}^n)$ with constant c additionaly depending on R. If Ω a bounded Lipschitz domain, we choose a ball $B_R(x_0)$ with radius proportional to diam (Ω) such that $\Omega \subset B_R(x_0)$. This yields the inequality from a) now for $u \in C_0^\infty(\Omega;\mathbb{R}^n)$, and from this the general claim follows by approximation.

b) Let us look at the ε^D-variant, which is only true for $n \ge 3$ (compare e.g. [Da]). For $u \in C^\infty(\overline{\Omega};\mathbb{R}^n)$ we again return to (2.43) of [Re] valid now for $x \in \Omega$, where for simplicity we assume that Ω is a star domain with respect to a ball $B_0 \subset \Omega$. Then it holds

$$\nabla(u - P_2u)(x) = \nabla R_2(Q_2u)(x)$$

and we deduce as in the proof of a)

$$(5.4) \qquad \|\nabla(u - P_2 u)\|_{L_\varphi(\Omega)} \leq c\|\varepsilon^D(u)\|_{L_\varphi(\Omega)} ,$$

thus it remains to discuss $\nabla P_2 u$. This can be done with formula (2.40)' of [Re], since by this identity it is immediate how to calculate $\nabla P_2 u$ and to use (5.4) in order to obtain

$$(5.5) \qquad \|\nabla u\|_{L_\varphi(\Omega)} \leq c\left[\|u\|_{L_\varphi(\Omega)} + \|\varepsilon^D(u)\|_{L_\varphi(\Omega)}\right] .$$

The general case of fields $u \in W^1_\varphi(\Omega;\mathbb{R}^n)$ follows from (5.5) by approximation.

Finally, we look at d), the proof of c) is similar. Let us assume $n \geq 3$ and consider a field $u \in C^\infty(\overline{\Omega};\mathbb{R}^n)$. The discussion from b) yields

$$(5.6) \qquad u - \kappa(u) = R_2(Q_2 u)$$

with Killing vector $\kappa(u) = P_2 u$. This time we do not have to take derivatives, and the estimate

$$(5.7) \qquad \|R_2(Q_2 u)\|_{L_\varphi(\Omega)} \leq c\|\varepsilon^D(u)\|_{L_\varphi(\Omega)}$$

for the non-singular operator R_2 is immediate. Combination of (5.6) and (5.7) gives d) for the smooth case. If we approximate $u \in W^1_\varphi(\Omega;\mathbb{R}^n)$ through a smooth sequence $\{u_m\}$, then it is easy to see that $\|\kappa(u_m)\|_{L_\varphi(\Omega)}$ stays bounded, and since for $n \geq 3$ the space of Killing vectors is of finite dimension, we arrive at d).

□

A Appendix

We like to show that the set of hypotheses imposed on the N-function φ in Theorem 5.1 implies the validity of the conditions (1.11) - (1.16) required in Koizumi's paper [Ko], more precisely we claim

Lemma A.1. *Let φ denote an N-function as described in Theorem 5.1. Then the following statements hold:*

a) There is a positive constant k such that

$$k\,\varphi'(t)t \leq \varphi(t) \leq \varphi'(t)t$$

holds for all $t \geq 0$.

b) The function $t \mapsto \varphi'(t)/t^a$ is increasing.

c) $\displaystyle\int_1^u \frac{\varphi(t)}{t^2}\,dt = O\left(\frac{\varphi(u)}{u}\right) \quad as \quad u \to \infty.$

d) $\displaystyle\int_0^u \frac{\varphi(t)}{t^2}\,dt = O\left(\frac{\varphi(u)}{u}\right) \quad as \quad u \to 0.$

e) $\exists\, r > 1: \quad \displaystyle\int_u^\infty \frac{\varphi(t)}{t^{r+1}}\,dt = O\left(\frac{\varphi(u)}{u^r}\right).$

f) $\exists\, r > 1: \quad \displaystyle\int_u^1 \frac{\varphi(t)}{t^{r+1}}\,dt = O\left(\frac{\varphi(u)}{u^r}\right).$

REMARK A.1. *Since we assume that φ is of type $(\Delta 2)$, we clearly have the validity of (1.11) and (1.14) from [Ko]. This observation together with c) - f) of Lemma A.1 shows that Theorem 2 of [Ko] in fact is applicable.*

Proof of Lemma A.1 : a) Since $\varphi(0) = 0$, convexity of φ immediately implies $\varphi(t) \le t\,\varphi'(t)$. At the same time we have

$$\varphi(t) = \int_0^t \varphi'(u)\,du \ge \int_{t/2}^t \varphi'(u)\,du \ge \frac{t}{2}\varphi'(t/2)\,,$$

hence $u\varphi'(u) \le \varphi(2u)$, and the $(\Delta 2)$ property of φ yields the claim.

b) This is an immediate consequence of our assumption (H).

c) From a) we get $(u > 1)$

$$\begin{aligned} \int_1^u \frac{\varphi(t)}{t^2}\,dt &\le \int_1^u \frac{\varphi'(t)}{t}\,dt \\ &= \int_1^u \frac{\varphi'(t)}{t^a} t^{a-1}\,dt \overset{\text{b)}}{\le} \frac{\varphi'(u)}{u^a} \int_1^u t^{a-1}\,dt = \frac{1}{a}\,\frac{\varphi'(u)}{u^a}(u^a - 1)\,, \end{aligned}$$

and the first inequality from a) applies.

d) As in c) we have for $u > 0$

$$\int_0^u \frac{\varphi(t)}{t^2}\,dt \le \frac{1}{a}\,\varphi'(u)$$

and again we benefit from a).

e) Let p denote a number $\ge 1/k$. Then we get from a)

$$\frac{d}{dt}\left[\frac{\varphi(t)}{t^p}\right] \le 0\,.$$

Let us fix $p := 1/k\,(> 1)$ and select some $r > p$. Then

$$\int_u^\infty \frac{\varphi(t)}{t^{r+1}}\,dt = \int_u^\infty \frac{\varphi(t)}{t^p}\frac{1}{t^{r-p+1}}\,dt \le \frac{\varphi(u)}{u^p}\int_u^\infty t^{p-r-1}\,dt = \frac{1}{r-p}\,\frac{\varphi(u)}{u^r}\,,$$

and e) is established.

f) For $u \in [0,1]$ the claim follows as in e) working with arbitrary exponent $r > 1/k$. In case "$u > 1$" we prove more general:

$$\exists\, s > 1 : \int_0^v \frac{\varphi(t)}{t^{s+1}}\, dt = O\left(\frac{\varphi(v)}{v^s}\right)$$

In fact, we first observe that by (H) it holds

$$\begin{aligned} t\,\varphi'(t) &= \int_0^t \frac{d}{du}\left[u\varphi'(u)\right]\, du \\ &= \int_0^t \left[\varphi'(u) + u\varphi''(u)\right]\, du \geq (1+a)\int_0^t \varphi'(u)\, du = (1+a)\varphi(t)\,, \end{aligned}$$

which gives for exponents $p \leq 1+a$

$$\frac{d}{dt}\left[\frac{\varphi(t)}{t^p}\right] \geq 0\,.$$

Let us fix $s \in (1, 1+a)$. Then the monotonicity of $\frac{\varphi(t)}{t^{1+a}}$ implies

$$\int_0^v \frac{\varphi(t)}{t^{s+1}}\, dt \leq \frac{\varphi(v)}{v^{a+1}}\int_0^v \frac{1}{t^{s-a}}\, dt = \frac{1}{a+1-s}\,\frac{\varphi(v)}{v^s}\,,$$

and the proof is complete.

□

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