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Representation of Quadratic Forms by Integral Quadratic Forms

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1. Introduction.

It is a classical problem to study the solvability and the number of integral solutions of the quadratic diophantine equation

$$\sum_{i,j=1}^{m} a_{ij}x_ix_j = t$$

for an integral symmetric matrix $A = (a_{ij})$ and an integer t. Already Gauß studied more generally systems of such equations of the form

$${}^tXAX = T$$

where now T is another (half-) integral symmetric matrix of size $n \leq m$. If one looks for rational instead of integral solutions, the Hasse-Minkowski theorem states the validity of the local-global principle for this problem, i.e., a rational solution exists if and only if solutions exist over $\mathbb{R}$ and over all p-adic fields $\mathbb{Q}_p$. That the local-global principle fails for integral solutions is already seen in simple examples like $Q(x_1, x_2) = 5x_1^2+11x_2^2$ which represents 1 over $\mathbb{R}$ and over all $\mathbb{Z}_p$ but not over $\mathbb{Z}$. Whereas the integral local-global principle can be saved with some modifications for indefinite A by the theory of spinor genera, the best possible results in the definite case prove that local representability implies global representability for T that are sufficiently large in a suitable sense and yield asymptotic formulas for T which are locally represented. The case of one equation ($n = 1$) is already classical, and considerable effort has been spent in the last sixty years on the case of $n > 1$, using both analytic and purely arithmetic methods. The introduction of ergodic theory as a new tool in [16] by Ellenberg and Venkatesh in 2008 has brought dramatic progress, it builds on the arithmetic approach of Eichler and Kneser and is also inspired by work of Linnik on representation of integers by ternary quadratic forms.

In this survey we sketch all three approaches (arithmetic, analytic, ergodic) and compare their results. At present each of the methods gives results which cannot be achieved by one of the others.

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2. Statement of the problem and notations.

An integral valued quadratic form $Q = Q_A$ on $\mathbb{Z}^m$ is given as $Q_A(\mathbf{x}) = \frac{1}{2}{}^t\mathbf{x}A\mathbf{x}$, where $A \in M_m^{\mathrm{sym}}(\mathbb{Z})$ is an integral symmetric matrix with even diagonal. Associated to it are the symmetric bilinear forms $b(\mathbf{x},\mathbf{y}) = Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}) = {}^t\mathbf{x}A\mathbf{y}$ and $B = \frac{1}{2}b$ with $B(\mathbf{x},\mathbf{x}) = Q(\mathbf{x})$. One says that the symmetric matrix T of size n is represented by Q_A or that Q_{2T} is represented by Q_A over $\mathbb{Z}$ (resp. over $\mathbb{Q}$) if there is $X \in M_{m,n}(\mathbb{Z})$ (resp. $\in M_{m,n}(\mathbb{Q})$) with $Q_A(X) = \frac{1}{2}{}^tXAX = T$.

An integral representation is called primitive if all elementary divisors of X are 1, in particular for $n = 1$ this says that the coefficients $x_1, \dots, x_m$ of the representing vector $\mathbf{x} \in \mathbb{Z}^m$ are relatively prime. If the matrix A is positive definite the matrix equation $Q_A(\mathbf{x}) = T$ has only finitely many solutions over $\mathbb{Z}$ and one calls

$$r(A,T) := \#\{X \in M_{m,n}(\mathbb{Z}) \mid Q_A(X) = T\}$$

the number of representations of T by A. The matrices $A_1, A_2 \in M_m^{\mathrm{sym}}(\mathbb{Z})$ and their associated quadratic forms Q_{A_1}, Q_{A_2} are called rationally resp. integrally equivalent if the equations $A_1 = {}^tX_1A_2X_1$, $A_2 = {}^tX_2A_1X_2$ are solvable with $X_1, X_2 \in \mathrm{GL}_m(\mathbb{Q})$ resp. in $\mathrm{GL}_m(\mathbb{Z})$. Clearly, integrally equivalent forms represent the same numbers and matrices and have the same representation numbers. The forms Q_{A_1}, Q_{A_2} (or their associated symmetric matrices) are said to be locally everywhere integrally (resp. rationally) equivalent, if ${}^tXA_1X = A_2$ is solvable with $X \in \mathrm{GL}_m(\mathbb{Z}_p)$ (resp. $\in \mathrm{GL}_m(\mathbb{Q}_p)$) for all primes p and A_1 and A_2 have the same signature (i.e., ${}^tXA_1X = A_2$ is solvable in $\mathrm{GL}_m(\mathbb{R})$). Forms which are locally everywhere integrally equivalent are said to belong to the same genus. Analogously one defines the notion that T is locally everywhere representable by Q_A (integrally or rationally). The Hasse-Minkowski theorem [17] asserts that rational representation locally everywhere is equivalent to representation (over $\mathbb{Q}$); this is not true for integral representation, where representation over $\mathbb{Z}$ is stronger than representation locally everywhere.

Since it is in principle easy to determine the numbers or matrices which are represented locally everywhere by determining the solvability of finitely many congruences, the problem to determine all T which are represented by Q_A is reduced to

Problem 1. *Given $A \in M_m^{\mathrm{sym}}(\mathbb{Z})$ determine conditions on $T \in M_n^{\mathrm{sym}}(\mathbb{Z})$ (with $n \le m$) such that T meeting these conditions is represented integrally by Q_A if it is represented locally everywhere integrally by Q_A.*

Similarly, determine conditions under which primitive representability locally everywhere implies (primitive) representability over $\mathbb{Z}$.

For many purposes it is convenient to use the equivalent but slightly more flexible language of quadratic spaces and lattices in them which has been introduced by Witt; in particular for generalizations to forms over number fields and their integers it is the more natural framework:

Definition 1. *Let F be a field. A* quadratic space (V, Q) *over F is a finite dimensional vector space V over F equipped with a map $Q : V \to F$ satisfying*

a) $Q(ax) = a^2Q(x)$ *for all* $x \in V$, $a \in F$

b) *$b(x,y) := Q(x+y) - Q(x) - Q(y)$ defines a symmetric bilinear form on V.*

The map Q is called the quadratic form *on V and b is called its* associated bilinear form.

If $\mathcal{B} = (e_1, \dots, e_m)$ is a basis of V we call the matrix $M_{\mathcal{B}}(Q) := (b(e_i, e_j)) \in M_m^{\mathrm{sym}}(F)$ the Gram matrix *of (V, Q) (or just of Q) with respect to $\mathcal{B}$.*

If (V', Q') is another quadratic space over F a linear isomorphism $f : V \to V'$ is called an isometry *if $Q'(f(x)) = Q(x)$ holds for all $x \in V$. If an isometry $f : (V, Q) \to (V', Q')$ exists one says that the spaces are* isometric *or* equivalent *or that they belong to the same* class.
If the mapping f above is just injective but may fail to be surjective it is called an isometric embedding *of (V, Q) into (V', Q') and one says that (V, Q) is* represented by (V', Q').

The geometric formulation of integral quadratic forms is obtained by considering lattices on quadratic spaces. In the most classical case we have:

Definition and Lemma 2. *Let (V, Q) be a quadratic space over the field $\mathbb{Q}$ of rational numbers.*
A $\mathbb{Z}$- lattice *(or simply lattice) on V is a finitely generated $\mathbb{Z}$-submodule L of V which generates V over $\mathbb{Q}$.*
Equivalently, $L = \bigoplus_{i=1}^m \mathbb{Z}e_i$ for some basis $(e_1, \dots, e_m)$ of the space V (which is then also a basis of the $\mathbb{Z}$-module L). We also call (L, Q) a quadratic lattice.
The lattice L is called integral *if $Q(L) \subseteq \mathbb{Z}$.*
If (L, Q) is an integral $\mathbb{Z}$-lattice and $\mathcal{B} = (e_1, \dots, e_m)$ is a basis of L, the quadratic polynomial $P_{Q,\mathcal{B}}(x_1, \dots, x_m) = Q(\sum_{i=1}^m x_i e_i)$ has integral coefficients; this polynomial is then what is usually called an integral valued quadratic form *(see* [9]*). One obtains a* classically integral quadratic form *in the sense of* [9] *if in addition the bilinear form $B = b/2$ assumes integral values.*

Since we will also consider the number field situation we need the following more general definition:

Definition 3. *Let F be a number field and R its ring of integers or let F be the completion of a number field at a non-archimedean place and again R its ring of integers.*
*A finitely generated R-submodule L of V is called an R-*lattice *on V if L generates V over F. We also call (L, Q) a* quadratic lattice over R.
The lattice L is called integral *(with respect to R) if $Q(L) \subseteq R$.*
If the lattice L is free with basis $\mathcal{B} = (e_1, \dots, e_m)$ over R the matrix $A = (b(e_i, e_j)) \in M_m^{\mathrm{sym}}(\mathbb{Z})$ is called its Gram matrix with respect to $\mathcal{B}$.

Lattices L on (V,Q) and L' on (V',Q') are called isometric or equivalent if there is an isometry $f:(V',Q')\to(V,Q)$ with $f(L')=L$; one writes $L'\in \mathrm{cls}(L)$ and also says that L and L' belong to the same class. The lattice L' is said to be represented by L if there is an isometric embedding $f:(V',Q')\to(V,Q)$ with $f(L')\subseteq L$. We write $r(L,L')$ for the number of such representations if this number is finite.
The representation f is called primitive if $f(L')=L\cap f(V')$. For $a\in R$ it is called of imprimitivity bounded by a if $a(L\cap f(V'))\subseteq f(L')$.
If F is totally real and L is (totally) positive definite we denote by $\min(L):=\min\{N^F_{\mathbb{Q}}(Q(x))\mid x\in L, x\neq \mathbf{0}\}$ the minimum of the lattice L. (For the question which lattices have large minimum it does not matter whether we chose this definition or $\min\{Tr^F_{\mathbb{Q}}(Q(x))\mid x\in L, x\neq \mathbf{0}\}$ instead, see the remark in [20, p.139]*.)*

Remark. a) If one wants to use the language of matrices instead of that of lattices and R is no principal ideal domain, one has to consider Gram matrices with respect to linearly dependent generating sets (see [48]); this is one of the reasons why lattices give the more convenient framework.

b) An equivalent characterization of a lattice on V is: L is an R-submodule of F, and for some basis $(x_1,\dots,x_m)$ of V and some $c\in R$ one has $cL\subseteq Rx_1+\ldots+Rx_m\subseteq L$. If R is a principal ideal domain we can instead require $L=Rx_1+\ldots+Rx_m$ for some basis of V as before. If R is no PID one admits non free lattices as well.

c) If R is the ring of integers of the number field F and $S=R_v$ its completion at some place v we write (L_v,Q) for the extension of (L,Q) to R_v and call it the completion of L at v; if v is archimedean we have $R_v=F_v$ and $L_v=V_v$.

In the sequel we let F be a number field with ring of integers R.

The R-lattices Λ, Λ' are in the same genus ($\Lambda'\in\mathrm{gen}(\Lambda)$) if Λ_v is isometric to Λ'_v for all places v of F. The R-lattice N is represented by Λ locally everywhere if N_v is represented by Λ_v for all places v of F. If the lattices in question are free and have Gram matrices A, T with respect to bases $\mathcal{B}$ of Λ and $\mathcal{B}'$ of Λ' resp. N, the notions of equivalence, genus, (primitive) representation (locally everywhere) for lattices given above translate into those for symmetric matrices described earlier in this section.

Problem 1 from above becomes

Problem 1'. Given an R-lattice Λ of rank m describe conditions on an R-lattice N of rank $n\le m$ such that N satisfying these conditions is represented by Λ (primitively) if it is represented by Λ (primitively) locally everywhere. If possible give (approximate) formulas for the numbers or measures of representations.

3. Siegel's theorem.

Although a strict local-global principle is not valid for representation of numbers or forms by integral quadratic forms, the Hasse-Minkowski theorem for quadratic forms over a number field has the consequence that a lattice N which is represented by the lattice Λ locally everywhere (primitively) is represented by some lattice Λ' in the genus of Λ (primitively), see [38, 9]. Siegel's celebrated theorem in fact gives the so-called mass formula (German: Maßformel, verbal translation to English: measure formula) for the average number of representations of K by Λ.

Theorem 4. *Let $\{L_1, \dots, L_h\}$ be a set of representatives of the classes of lattices in the genus of Λ. If Q is definite put $w = \sum_{i=1}^{h} \frac{1}{|O(L_i)|}$ (where $O(L_i)$ is the group of isometries of L onto itself with respect to Q) and write*

$$r(\text{gen } \Lambda, N) = \frac{1}{w} \sum_{i=1}^{h} \frac{r(L_i, N)}{|O(L_i)|}$$

for Siegel's weighted average of the representation numbers of N by the lattices L_i in the genus of Λ.
If Q is not definite and neither the space $F\Lambda$ nor the orthogonal complement of a representation of FM in $F\Lambda$ is a hyperbolic plane the measure (mass) $w = \mu(\Lambda)$ of Λ and the representation measures $\mu(L_i, N)$ of N by the L_i can be defined as in [48] *and one puts*

$$r(\text{gen } \Lambda, N) = \frac{1}{w} \sum_{i=1}^{h} \mu(L_i, N).$$

Then $r(\text{gen } L, N)$ can be expressed as a product of local densities over the non-archimedean places v of F,

$$r(\text{gen } \Lambda, N) = c \cdot (N_{\mathbb{Q}}^{F}(\det N))^{\frac{m-n-1}{2}} (N_{\mathbb{Q}}^{F}(\det \Lambda))^{-\frac{n}{2}} \prod_{v} \alpha_v(\Lambda, N)$$

with some constant c.

Here by $N_{\mathbb{Q}}^{F}(\det(\Lambda))$ resp. $N_{\mathbb{Q}}^{F}(\det(N))$ we denote the norm of the ideal generated by the determinants of the Gram matrices with respect to linearly independent subsets of the respective lattice, the local density $\alpha_v(\Lambda, N)$ is for a non-archimedean place v of F with residue field of order q_v and local prime element $\omega_v \in R_v$ given as

$$\begin{aligned} \alpha_v(\Lambda, N) &= \alpha_v(S_v, T_v) \\ &= q_v^{j \cdot (\frac{n \cdot (n+1)}{2} - mn)} \# \mathcal{A}_j(S_v, T_v), \end{aligned}$$

for sufficiently large j with an additional factor $\frac{1}{2}$ if $m = n$, where S_v, T_v denote Gram matrices of the local lattices Λ_v, N_v and where we *write*

$$\mathcal{A}_j(S_v, T_v) = \{X = (x_{ij}) \in M_{m,n}(R_v)/\omega_v^j M_{m,n}(R_v) \mid {}^tXS_vX \equiv T \bmod \omega_v^j\}$$

with the congruence being required modulo integral symmetric matrices with even diagonal.

An analogous formula holds for averaged primitive representation numbers resp. measures and primitive local densities $\alpha_v^*(\Lambda, N)$ counting congruence solutions as above but with the representing matrix X being primitive.

The (primitive) local densities in Siegel's theorem are nonzero if N is represented (primitively) locally by Λ and their product converges, so the theorem implies that (as mentioned above) such an N is represented (primitively) by at least one class of lattices in the genus of Λ; it gives a quantitative version of this qualitative result.

If Λ happens to be in a genus of one class, as is the case e.g. for the lattice over $\mathbb{Z}$ corresponding to the sum of k integral squares with $k \leq 8$, Siegel's theorem gives an exact formula for $r(\Lambda, N)$ resp. the measure $\mu(\Lambda, N)$. Since one can give closed formulae for $\alpha_v(\Lambda, N)$ for almost all v (where the exceptional set depends on Λ) the average representation numbers or measures can be determined explicitly for given Λ by determining the numbers of solutions of finitely many congruence systems.

In the asymptotic formulas to be discussed later the average representation number $r(\text{gen }\Lambda, N)$ will be the main term.

4. The indefinite case.

For the rest of this article we restrict attention to quadratic spaces and lattices with non-degenerate quadratic form Q, as usual we will often suppress the quadratic form Q in the notation.

The case that Λ_v is isotropic (i.e., represents zero nontrivially) for at least one archimedean place v of F has been solved as completely as possible:

Theorem 5 (Eichler [15], Kneser [35], Weil [51], Hsia [19])**.** *Let Λ be a non-degenerate quadratic R-lattice of rank m such that Λ_v is isotropic for at least one archimedean place v of F.*

a) *If the non-degenerate quadratic R-lattice N of rank $n \leq m - 3$ is represented by Λ locally everywhere it is represented by Λ, and the measure of representations (Darstellungsmaß) of N by Λ' is the same for all lattices Λ' in the genus of Λ.*
b) *If N is as above with $n = m - 2$ then either the measure of representations of N by a lattice Λ' is the same for all Λ' in the genus of Λ or the genus of Λ splits into two half genera consisting*

of equally many classes such that the measure of representations of N by Λ', Λ'' is the same if Λ', Λ'' belong to the same half genus.

The latter case occurs only for N for which the discriminant of the space FN (i.e., the determinant of a Gram matrix of that quadratic space) belongs to one of finitely many square classes depending on Λ which can be explicitly determined.

Remark. a) The proof uses the theory of spinor genera, a modified version of it plays a role in the arithmetic and the ergodic approach to problem 1' for definite lattices, see Lemma 13 below.

b) The measure of representations has been defined by Siegel in [47]; an equivalent definition using measures on adelic orthogonal groups is given e.g. in [35].

c) In the case $n = 1$, $m = 3$, the difference between the representation measures of the half genera occurring in part b) of the theorem has been calculated in [43]. The integers represented (primitively) locally everywhere but not globally by all classes in the genus have been determined explicitly in [42] without the primitivity condition and in [14] for the primitive case; they are called (primitive) spinor exceptions.

d) Some further results for the case $n = m - 2$ have been obtained in [52, 53].
The cases $n = m - 1$ and $m = n$ do not admit clean solutions; what can be done is shown in [52, 53].

e) The determination of the square classes in part b) of the theorem is achieved by computing the spinor norms of the local orthogonal groups of the lattices N_v. This computation is given in [34], [18, 13].

f) An analytic proof of the result for $n = 1$, $m = 4$ has been given by Siegel in [49].

5. Representation of integers ($n = 1$).

By the results of the previous section we can restrict attention from now on to the case that F is totally real and Λ is totally (positive) definite. In this case the representation numbers

$$r(\Lambda, N) = \#\{\varphi : N \longrightarrow \Lambda \mid \varphi \text{ linear isometry}\}$$

are finite. The first general result here is the following theorem. It has been proven with the help of the Hardy Littlewood circle method in [32] by Kloosterman for diagonal forms and in [33] for general forms using both modular forms and the Hardy Littlewood method in 1927; Kloosterman's first proof has been generalized in [50] by Tartakovskii to general forms in 1929.

Theorem 6 (Kloosterman, Tartakovskii)**.** *Let Λ be a positive definite $\mathbb{Z}$-lattice of rank $m \geq 5$. Then Λ represents all sufficiently large numbers t which are represented by it locally everywhere. The same is true for $m = 4$ if one restricts attention to t which are represented locally everywhere primitively or which satisfy at least for some fixed a that for each p there is $x \in \Lambda_p$ with $Q(x) = t$ and $a^{-1}x \notin \Lambda_p$ (one also says that t is represented locally everywhere with imprimitivity bounded by a).*

In both cases one has an asymptotic formula

$$r(\Lambda, t) = r(\operatorname{gen}\Lambda, t) + \mathrm{O}(t^{\frac{m}{4}-\delta})$$

for any $\delta < \frac{5}{16}$ for odd m and $\delta < \frac{1}{2}$ for even m, where the main term $r(\operatorname{gen}\Lambda, t)$ grows at least like $t^{\frac{m}{2}-1-\epsilon}$ for all $\epsilon > 0$ for t satisfying the conditions given.

Remark. a) The condition on bounded imprimitivity for the local representations is automatically satisfied for all primes p for which Λ_p is isotropic (represents zero nontrivially), hence in particular for all p not dividing the determinant of Λ.

b) The exponents in the error terms are better than the original ones; the bound for even m is the Ramanujan-Petersson bound (proven by Deligne), the bound for odd m is the bound from [4, (1.3)].

c) In the Hardy Littlewood method the main term appears as the singular series.

This result already contains some essential features of the general situation (i.e., arbitrary n):

- Instead of an exact formula one has an asymptotic formula whose main term is determined by the local arithmetic of N and Λ.
- The asymptotic formula is unconditional for $m = 5 = 2 \cdot n + 3$ (with $n = 1$) and needs an additional primitivity assumption for $m = 4 = 2 \cdot n + 2 = n + 3$.
- Results for representation of sufficiently large integers follow directly from the asymptotic formula and can be made explicit.

The result has been generalized to the number field case and (as far as possible) to $m = 3$, for details see the survey [44] and notice that the bound in the error term for odd m has meanwhile been improved in [5]. All these results can be generalized to representations with congruence conditions and to statements about the equidistribution of lattice points on (higher dimensional) ellipsoid surfaces, see [12].

Remark. There are several results about representation of numbers by an integral quadratic form that don't fit well into this survey but should at least be mentioned:

- The 15-theorem of Conway and Schneeberger [2] states that a classically integral quadratic form represents all natural numbers if it represents all natural numbers up to 15. A modification of this result is the 290-conjecture, stating that an integral valued quadratic form represents all natural numbers if it represents all natural numbers up to 290; a proof of this conjecture has been announced by Bhargava and Hanke in 2008. A generalization of both these results to representation of quadratic forms has been proven by B. M. Kim, M.-H. Kim and Oh in [23].

- In several recent articles the representation of numbers by a quadratic form with restricted variables is investigated, e.g. [3] treats the number of representations of t as a sum of four squares whose largest prime factor is bounded ("smooth squares").

6. Representation of forms ($n > 1$), analytic method.

We continue to assume Λ to be positive definite (and F totally real), so that the representation number $r(\Lambda, N)$ is finite.

All results obtained for $r(\Lambda, N)$ with $n = \mathrm{rk}(N) > 1$ obtained so far by analytic methods are for the case $R = \mathbb{Z}$, $F = \mathbb{Q}$ and use the fact that the theta series of degree n of Λ is a Siegel modular form with respect to a suitable congruence subgroup of the modular group $\mathrm{Sp}_n(\mathbb{Z})$.

To fix some notation let

$$\mathrm{Sp}_n(\mathbb{R}) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}) \mid {}^t g \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} g = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}\}$$

be the real symplectic group of rank (degree, genus) n and $\mathcal{H}_n$ the Siegel upper half space of degree (genus) n, with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R})$ acting by

$$Z \longmapsto g\langle Z \rangle := gZ := (AZ + B)(CZ + D)^{-1}.$$

A Siegel modular form of weight k for the congruence subgroup $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$ is a holomorphic function $F : \mathcal{H}_n \longrightarrow \mathbb{C}$ satisfying $F(g\langle Z \rangle) = \det(CZ + D)^k F(Z)$ for all $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$. If $\chi : \Gamma \longrightarrow \mathbb{C}^\times$ is a character we will also use Siegel modular forms with character (nebentype) χ, where one has $F(g\langle Z \rangle) = \chi(g) \det(CZ + D)^k F(Z)$ for $g \in \Gamma$. (If $n = 1$, one has to add a holomorphy condition at the cusps.) The theta series of degree (genus) n of the positive definite lattice Λ is given as

$$\vartheta^{(n)}(\Lambda, Z) = \sum_{\mathbf{x}=(x_1,\dots,x_n)\in\Lambda^n} \exp(2\pi i \mathrm{tr}(Q(\mathbf{x})Z))$$

with $Q(\mathbf{x}) = (B(x_i, x_j)) \in M_n^{\mathrm{sym}}(\frac{1}{2}\mathbb{Z})$, if S is a Gram matrix of Λ we can also write

$$\begin{aligned}\vartheta^{(n)}(\Lambda, Z) &= \vartheta^{(n)}(S, Z) = \sum_{X \in M_{m,n}(\mathbb{Z})} \exp(\pi i \operatorname{tr}({}^tXSXZ)) \\ &= \sum_T r(S,T) \exp(2\pi i \operatorname{tr}(TZ))\end{aligned}$$

where T runs over half integral positive semidefinite symmetric matrices with integral diagonal.

Proposition 7. *Let Λ as above have even rank and Gram matrix S, let M be an integer such that MS^{-1} is integral with even diagonal and write*

$$\chi \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\frac{(-1)^{\frac{m}{2}} \det S}{\det D} \right) \quad \textit{(generalized Jacobi symbol)}$$

for

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(M) = \{g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n\mathbb{Z}) \mid C \equiv 0 \bmod M\}.$$

Then $\vartheta^{(n)}(\Lambda, \cdot)$ is a Siegel modular form of weight $k = \frac{m}{2}$ with character χ for the group $\Gamma_0^{(n)}(M)$.

In particular, if $\det S = 1$ (Λ and S are then called even unimodular), $\vartheta^{(n)}(\Lambda, \cdot)$ is a Siegel modular form for the full modular group $\mathrm{Sp}_n(\mathbb{Z})$.

Proof. A proof can e.g. be found in [1], where also a similar formula is given for the case of odd rank m. □

For a Siegel modular form F of degree n for some congruence subgroup Γ the ϕ-operator is defined by

$$(F|\phi)(t) = (\phi F)(Z) = \lim_{t \to \infty} F \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix} \quad (Z \in \mathcal{H}_{n-1}),$$

the function $F|\phi$ is then a Siegel modular form of degree $n-1$ for the group

$$\{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid \begin{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \in \Gamma\}.$$

F is a cusp form if $F|\gamma|\phi = 0$ for all $\gamma \in \mathrm{Sp}_n(\mathbb{Z})$, it is said to vanish in all zero-dimensional cusps if

$$(F|\gamma) \mid \Phi^{n-1} = 0 \quad \text{for all } \gamma \in \mathrm{Sp}_n(\mathbb{Z})$$

(i.e., if the constant term in the Fourier expansion of $F|\gamma$ vanishes for all $\gamma \in \mathrm{Sp}_n(\mathbb{Z})$). A well-known fact is:

Proposition 8. *If Λ, Λ' are lattices in the same genus, the function $\vartheta^{(n)}(\Lambda,\cdot) - \vartheta^{(n)}(\Lambda',\cdot)$ vanishes in all zero dimensional cusps. If we define*

$$\vartheta^{(n)}(\mathrm{gen}(\Lambda), Z) = \sum_T r(\mathrm{gen}\,\Lambda, N_T)\exp(\pi i\,\mathrm{tr}(TZ))$$

with N_T denoting a lattice with Gram matrix T, then also

$$\vartheta^{(n)}(\Lambda,\cdot) - \vartheta^{(n)}(\mathrm{gen}(\Lambda),\cdot)$$

vanishes in all zero dimensional cusps.

Proof. In the case $n = 1$ this has been noticed in [46, p. 376], the general case is an immediate consequence. The reason is that $\mathrm{Sp}_n(\mathbb{Z})$ is generated by $\begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ and matrices not changing the Fourier expansion of $\vartheta^{(n)}(\Lambda,\cdot)$ and that $F|\begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ can be expressed with the help of the Poisson summation formula as a sum of terms whose constant term in the Fourier expansion depends only on the congruence properties of S. ☐

The analytic approach to Problem 1' can now be formulated as follows.

Write $r(\Lambda, N) = r(\mathrm{gen}\,\Lambda, N) + (r(\Lambda, N) - r(\mathrm{gen}\ \Lambda, N))$ and try to estimate the main term $r(\mathrm{gen}\,\Lambda, N)$ from below and the error term $r(\Lambda, N) - r(\mathrm{gen}\,\Lambda, N)$ from above, using the fact that the latter expression is a Fourier coefficient of a Siegel modular form which vanishes in all zero dimensional cusps. In the case $n = 1$ we have to estimate the Fourier coefficients of a cusp form, which allows to use Deligne's theorem (i.e., the truth of the Ramanujan-Petersson conjecture) if m is even. For $n > 1$, the difference $\vartheta(\Lambda,\cdot) - \vartheta(\mathrm{gen}\,\Lambda,\cdot)$ will in general not be a cusp form; this makes the estimation of the error term from above considerably more difficult.

The first result for our problem in the case $n > 1$ is due to Raghavan:

Theorem 9 ([39], 1959)**.** *Let N run through positive definite integral lattices of rank n with $2n + 3 \le m$ and $(\det N) \to \infty$ satisfying one of the equivalent conditions*

a) $\min(N) \cdot \min(N^{\#}) < c_1$ *for some fixed* $c_1 > 0$
b) $\min(N^{\#}) \ge c_1(\det N)^{-\frac{1}{n}}$ *for some fixed* $c_2 > 0$
c) $\min(N) \ge c_3(\det N)^{\frac{1}{n}}$ *for some fixed* $c_3 > 0$

Then one has

$$r(\Lambda, N) = r(\mathrm{gen}\,\Lambda, N) + \mathrm{O}((\det N)^{\frac{m-n-1}{2}} \cdot (\min(N))^{(n+1-\frac{m}{2})/2}).$$

Proof. The idea of the proof is to compute the Fourier coefficient at T of $g(Z) := \vartheta^{(n)}(\Lambda, Z) - \vartheta^{(n)}(\operatorname{gen}\Lambda, Z)$ as

$$\int_{\mathfrak{E}} g(Z) \exp(-2\pi i \operatorname{tr}(TZ)) dX,$$

where the variable $Z = X + iT^{-1}$ runs over a cube $\mathfrak{E}$ of side length 1 with one corner in T^{-1}, using a generalized Farey dissection of this cube which has been introduced by Siegel in [49].

Raghavan proves in fact more generally an estimate for the Fourier coefficient of $F-\varphi$, where F is a Siegel modular form of weight $k > n+1$ and φ the associated Eisenstein series; the analytic version of Siegel's theorem states that $\vartheta^{(n)}(\operatorname{gen}(\Lambda), \cdot)$ is the Eisenstein series associated to $\vartheta^{(n)}(\Lambda, \cdot)$.

Raghavan shows that the formula given above is indeed an asymptotic formula for the number of representations of N by Λ in the case $n = 2$; this is achieved by estimating the local densities (and hence the main term $r(\operatorname{gen}(\Lambda), N)$) from below; we will see a more general version of that below. $\square$

Remark. Minkowski's reduction theory of positive definite quadratic forms implies that the minimum of N grows at most like a constant multiple of $(\det(N))^{\frac{1}{n}}$, so the condition of the theorem roughly says that the minimum of N grows as fast as it can.

Since Λ represents (by the case $n = 1$) only all sufficiently large numbers that it represents locally everywhere, there are in general some small numbers which are represented by Λ locally everywhere but not globally. It is then easy to construct a sequence of lattices N of growing determinant and one of these exceptional numbers as minimum which are represented locally everywhere but not globally (since their minimum is not represented globally). The most common example of this type is to take the Leech lattice for Λ and the number 1 as minimum of N. It is therefore clear that an asymptotic formula for $r(\Lambda, N)$ can not rely on the growth of $\det(N)$ alone.

Raghavan's result was further improved by Kitaoka in [24, 25].

Theorem 10 (Kitaoka, 1982)**.** a) *If $m \geq 2n + 3$ holds, the product $\prod_p \alpha_p(\Lambda, N)$ is bounded from below and above by constants depending only on Λ for all N which are represented locally everywhere by Λ.*

In particular, one has

$$r(\operatorname{gen}\Lambda, N) > c_4(\det N)^{\frac{m-n-1}{2}}$$

for all N which are represented locally everywhere by Λ , with c_4 depending only on Λ.

b) *The formula in Raghavan's theorem is an asymptotic formula for* N *which are represented locally everywhere by* Λ *and for which* $\min(N) \geq c_3(\det N)^{\frac{1}{n}}$ *with some constant* $c_3 > 0$ *independent of* N.
If N *runs through lattices of rank* n *with* $\min(N) > c_3(\det N)^{\frac{1}{n}}$, *any* N *of sufficiently large minimum which is represented locally everywhere by* Λ *is represented globally by* Λ.

c) *If* $n = 2$, $m = 7$, *there is a constant* c_5 *depending only on* Λ *such that the condition* $\min(N) \geq c_3(\det N)^{\frac{1}{2}}$ *above can be replaced by* $\min(N) \geq c_5$.
In particular, any N *which is represented locally everywhere by* Λ *and has minimum* $\geq c_5$ *is represented globally by* Λ.

The method of proof here is essentially the same as for Raghavan's theorem, but considerably refined.

Another interesting result of Kitaoka is proved in [26].

Theorem 11 (Kitaoka, 1988)**.** *Let* $n = 2$, $m = 6$. *Let* N_0 *have Gram matrix* T_0 *and be represented by* Λ *locally everywhere. Then for* $t \to \infty$ *with* $\gcd(t, \det(\Lambda)) = 1$ *and such that* tT_0 *is represented by* Λ_p *for all* p *dividing* $\det(\Lambda)$ *one has*

$$r(\Lambda, tT_0) = r(\mathrm{gen}(\Lambda), tT_0) + O(t^{\frac{5}{2}+\delta}) \qquad \textit{for all } \delta > 0,$$

where the main term $r(\mathrm{gen}(\Lambda), tT_0)$ *grows at least like* $t^{3-\epsilon}$ *for all* $\epsilon > 0$.

In particular, for t *large enough and satisfying the conditions above the matrix* tT_0 *is represented by* Λ.

For the case that Λ is even unimodular (i.e., the Gram matrix S of Λ has even diagonal and determinant 1), the estimation of the error term has also been investigated by Kitaoka using instead of the circle method as above the decomposition of $\vartheta^{(n)}(\Lambda, \cdot) - \vartheta^{(n)}(\mathrm{gen}\,\Lambda, \cdot)$ into a cusp form and Klingen-Eisenstein series associated to cusp forms of degree $r < n$. The result is a similar asymptotic formula as above for the range $m \geq 4n + 4$, namely

$$r(\Lambda, N) = r(\mathrm{gen}\,\Lambda, N) + \mathrm{O}((\min(N))^{1-\frac{m}{4}}(\det N)^{\frac{m-n-1}{2}}.$$

In particular, in exchange for the restriction on Λ and the stronger condition $m \geq 4n + 4$ one gets rid of the condition $\min(N) > c_3(\det N)^{\frac{1}{n}}$.

In all of the above results one can deduce global representability from local representability only for lattices N of large minimum, a condition which excludes many cases of interest in which the determinant of N grows but the minimum remains small.

The examples which show its necessity are lattices N of large determinant which have small minimum or more generally a sublattice N' of small determinant, so that it can happen that N' is not represented globally by Λ. On the other hand, a lattice N' of rank $n' < n$ and small determinant of which one already knows that it is represented globally by Λ may have extensions to lattices N of rank n which are represented locally everywhere. A result towards the global representation of such N by Λ has been obtained in [8]

Theorem 12 (Böcherer, Raghavan 1988)**.** *Let T_1 be positive definite symmetric Minkowski reduced of rank n_1 and Λ be even unimodular. Then for Minkowski reduced symmetric $n \times n$-matrices $T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix}$ with $m > 4n$ and sufficiently large $\min(T_4)$ the primitive representation number $r^*(\Lambda, T)$ satisfies*

$$\begin{aligned} r^*(\Lambda, T) &= c_6 r^*(\Lambda, T_1) \cdot (\det T_4)^{\frac{m-n-1}{2}} \\ &\quad + O((\det T_4)^{\frac{m-n-1}{2}} \min(T_4)^{-\frac{m}{4}+v(n_1)} \end{aligned}$$

for some constant $c_6 \neq 0$, with $v(n_1) < \frac{m}{4}$.

The proof uses again the decomposition of $\vartheta^{(n)}(\Lambda) - \vartheta^{(n)}(\operatorname{gen}\Lambda)$ into a sum of Klingen-Eisenstein series associated to cusp forms of degrees $\leq n$. Notice that for even unimodular Λ and $m \geq 2n+3$ the condition of local representability is satisfied automatically.

In the case $n_1 = 1$ Böcherer has shown in [6] that for a square free integer $t_1 = T_1$ this problem can also be treated using the theory of Jacobi forms; a generalization of that result to general t_1 and to not necessarily unimodular Λ will be the subject of the PhD thesis of T. Paul in Saarbrücken.

7. Representation of forms, arithmetic method.

In order to present the arithmetic method we need some terminology. We denote by $O_V(F)$ the group of isometries of V with respect to Q (the orthogonal group of the quadratic space (V, Q)), by $O_V(\mathbb{A})$ its adelization, and by $SO_V(F)$ resp. $SO_V(\mathbb{A})$ their subgroups of elements of determinant 1. For a lattice Λ on V we denote its automorphism group (or unit group) $\{\sigma \in O_V(F) \mid \sigma(\Lambda) = \Lambda\}$ by $O_\Lambda(R)$ and similarly for the local or adelic analogues. $\mathrm{Spin}_V(\mathbb{A})$ is the adelic spin group and $O'_V(\mathbb{A})$ its image in $O_V(\mathbb{A})$, i.e., the subgroup of adelic transformations of determinant and spinor norm 1.

The orbit of a fixed lattice Λ under $O_V(\mathbb{A})$ consists then of all lattices on V in the genus of Λ, the lattices on V in the spinor genus of Λ comprise the orbit under $O_V(F)O'_V(\mathbb{A})$.

The proof of the theorem of section 4 on representation by indefinite lattices rests on the strong approximation theorem for the spin group with respect to an archimedean place of F at which Q is indefinite.

In the case of a definite lattice one can, following Eichler [15], consider it as "arithmetically indefinite" if there is an archimedean place w of F for which Λ_w is isotropic (i.e., represents zero nontrivially). The strong approximation theorem gives then

Lemma 13. *Let w be a non-archimedean place of F for which Λ_w is isotropic.*

a) *Each class in the spinor genus of Λ has a representative Λ' such that $\Lambda'_v = \Lambda_v$ for all places $v \neq w$.*
b) *If the genus of Λ consists of only one spinor genus there is an integer s such that Λ represents every R-lattice N for which N_v is represented by $\mathfrak{p}_w^s \Lambda_v$ for all finite places v of F (where $\mathfrak{p}_w$ is the ideal of R corresponding to w).*
c) *If $m \geq n+3$ and N is represented (primitively) locally everywhere by Λ there is a lattice Λ' in the spinor genus of Λ with $\Lambda'_v = \Lambda_v$ for all places $v \neq w$ and Λ'_w in the* $\mathrm{Spin}(F_w)$*-orbit of Λ_w, such that N is represented (primitively) by Λ.*

The lemma alone is not sufficient to deduce global representability of N by Λ from representability locally everywhere since in the definite situation the spinor genus consists in general of more than one class. We will see in the next section that it provides the starting point for the ergodic method of Ellenberg and Venkatesh. It is also basic for the purely arithmetic method of Hsia, Kitaoka and Kneser.

For N which is represented by Λ locally everywhere they construct in [20], using the local arithmetic of lattices, a finite set of sublattices $K(J)$ of rank n and $L(J)$ of rank $m - n \geq n + 3$ of Λ which are orthogonal to each other and such that for each finite place v of F the lattice N_v is represented either by $K(J)_v$ or by $\mathfrak{p}_v^s L(J)_v$. With the help of b) of the Lemma and some additional rather tricky approximation arguments they can then deduce that N is represented by one of the $K(J) + L(J)$ and hence by Λ if the minimum of N is large enough. The final result is

Theorem 14 (Hsia, Kitaoka, Kneser, 1978)**.** *There is a constant $c_7 = c_7(\Lambda)$ such that for $m \geq 2n + 3$ every lattice N which is represented locally everywhere by Λ and has minimum $\geq c_7$ is represented by Λ.*

The constant in the theorem can in principle be made effective; such an effective version (with a rather large constant) has been given by Chan and Icaza in [10] for $m \geq 3n + 3$ and for $n = 2, m = 7$. It has been shown by Jöchner and Kitaoka in [22] that the proof of the theorem can be modified to give the same result for representations with additional congruence and primitivity conditions and by Hsia and Prieto-Cox in [21] that it can also be generalized to hermitian forms.

Kitaoka has further noticed (see [8][p. 95]) that a version of the result on extensions of representations in Theorem 12 can also be obtained by the arithmetic method, the result given in [8] has been further improved by Chan, B. M. Kim, M.-H. Kim, and Oh in [11] to give

Theorem 15 (Chan, B. M. Kim, M.-H. Kim, Oh 2008)**.** *Let $F = \mathbb{Q}$, let K be a lattice of rank k on the space of Λ, let $\sigma : K \longrightarrow \Lambda$ be a representation. Then there is a constant $c_8 > 0$ such that one has:*

If $N \supseteq K$ is a lattice of rank n with $m \geq k+2(n-k)+3$ on the space of Λ such that for all primes p the local representation $\sigma_p : K_p \longrightarrow \Lambda_p$ can be extended to a representation $\rho_p : N_p \longrightarrow \Lambda_p$ and such that the minimum of the orthogonal projection $\pi(N)$ on the orthogonal complement of $\mathbb{Q}K$ in $\mathbb{Q}\Lambda$ is larger than c_8, the representation σ can be extended to a representation $\rho : N \longrightarrow \Lambda$. One can in addition specify congruence conditions modulo an integer prime to $2\det(K)\det(\Lambda)$.

8. Representation of forms, ergodic method.

The result of c) of Lemma 13 can be rephrased group theoretically:

There exists an isometric embedding of N into $\Lambda' = u\Lambda$ with

$$u \in O_V(F)(\prod_{v \neq w} O_\Lambda(R_v))\mathrm{Spin}_V(F_w).$$

Representability of N by Λ is equivalent to being able to choose $u \in O_V(F)O_\Lambda(\mathbb{A})$ instead with $O_\Lambda(\mathbb{A}) = \prod_v O_\Lambda(R_v)$.

If we consider N as a sublattice of Λ' we can clearly modify u by a suitable element of $O_{W_1}(F_w)$, where $W_1 = (FN)^\perp$.

Ellenberg and Venkatesh show in [16] that the necessary modification of u is indeed possible for N of sufficiently large minimum if one has $m \geq n+3$, the lattice N has square free determinant, and N satisfies some additional conditions; their proof uses ergodic theory, in particular results of Ratner and Margulis/Tomanov (see [40, 37]). In view of the fact that before their work it was generally considered to be possible that $m = 2n+2$ is the natural barrier for the validity of a representability result this represented a dramatic breakthrough. That their conditions on the lattice N (but not on its dimension) can be further relaxed has been shown in [45], where also the arithmetic parts of their proof were reformulated in a way closer to previous work in the arithmetic theory of quadratic forms.

The final result is:

Theorem 16. *Let $(V, Q), \Lambda$ be as before, fix a finite place w of F and $j \in \mathbb{N}, a \in R$.*

Then there exists a constant $c_9 := c_9(\Lambda, j, w, a)$ such that Λ represents all R - lattices N of rank $n \le m-3$ satisfying

a) *N is represented by Λ locally everywhere with imprimitivity bounded by a and with isotropic orthogonal complement at the place w.*
b) $\mathrm{ord}_w(\det(N_w)) \le j$
c) *The minimum of N is $\ge c_9$.*

The representation may be taken to be of imprimitivity bounded by a. The isotropy condition is satisfied automatically if $n \le m-5$ or if w is such that the determinants of the local lattices Λ_w and N_w are units in R_w.

It is not difficult to adapt the method in order to obtain a version for extensions of representations:

Corollary 17. *Let $(V, Q), \Lambda$ be as before, fix a finite place w of F and $j \in \mathbb{N}, a \in R$.*
Let $K \subseteq \Lambda$ be a fixed R-lattice of rank k, $\sigma : K \longrightarrow \Lambda$ a representation of K by Λ and assume that K_w is unimodular.
Then there exists a constant $c_{10} := c_{10}(\Lambda, R, j, w, a)$ such that one has: If $N \supseteq K$ is an R-lattice of rank $n \le m-3$ and

a) *For each place v of F there is a representation $\tau_v : N_v \longrightarrow \Lambda_v$ with $\tau_v|_{K_v} = \sigma_v$ with imprimitivity bounded by a and with isotropic orthogonal complement in Λ at the place w*
b) *For the w-adic order $\mathrm{ord}_w(\det(N_w))$ of the determinant of a Gram matrix of N_w one has $\mathrm{ord}_w(\det(N_w)) \le j$*
c) *The minimum of $N \cap (FK)^\perp$ is $\ge c_{10}$,*

then there exists a representation $\tau : N \longrightarrow \Lambda$ with $\tau|_K = \sigma$.
The representation may be taken to be of imprimitivity bounded by a. The isotropy condition is satisfied automatically if $n \le m-5$ or if w is such that the local lattices Λ_w and N_w are unimodular.

Ellenberg and Venkatesh prove the theorem in [16] under the stronger restriction that the determinant of N is square free; the version of it given here and the corollary are proven in [45].
For the reader's convenience we add a matrix version of the main result for the case $F = \mathbb{Q}$:

Theorem 18. *Let $S \in M_m^{\mathrm{sym}}(\mathbb{Z})$ be a positive definite integral symmetric $m \times m$-matrix, fix a prime q and positive integers j, a.*
Then there is a constant c_{11} such that a positive definite matrix $T \in M_n^{\mathrm{sym}}(\mathbb{Z})$ with $n \le m-3$ is represented by S (i.e., $T = {}^tXSX$ with $X \in M_{mn}(\mathbb{Z})$) provided it satisfies:

a) *For each prime p there exists a matrix $X_p \in M_{mn}(\mathbb{Z}_p)$ with ${}^tX_pSX_p = T$ such that the elementary divisors of X_p divide a and such that the equations ${}^tX_qS\mathbf{y} = \mathbf{0}$ and ${}^t\mathbf{y}S\mathbf{y} = 0$ have a nontrivial common solution $\mathbf{y} \in \mathbb{Z}_q^m$*

b) $q^j \nmid \det(T)$
c) $\min\{{}^t\mathbf{y}T\mathbf{y} \mid \mathbf{0} \neq \mathbf{y} \in \mathbb{Z}^n\} > c_{11}$

The matrix X may be chosen to have elementary divisors dividing a.

As remarked earlier the primitivity (or bounded imprimitivity) condition is satisfied automatically in the range $m \geq 2n+3$ covered by the analytic and arithmetic results. Kitaoka has proved in [27, 28, 30, 31] some lemmas which imply that one can drop or weaken the primitivity conditions in some lower dimensional cases; the original purpose of those lemmas was to obtain improved estimates for the main term in the analytic method which could be used once the analytic estimate for the error term in the asymptotic formula could be improved sufficiently much. This leads to the following corollaries, proven in [45]:

Corollary 19. *Let $F = \mathbb{Q}$, let $(V, Q), \Lambda$ be as before and fix a prime q and $j \in \mathbb{N}$.*

a) *Let $n \geq 6$ and $m = \dim(V) \geq 2n$. Then there exists a constant $c_{12} := c_{12}(\Lambda, j, q)$ such that Λ represents all $\mathbb{Z}$ - lattices N of rank n which are represented by Λ locally everywhere, have minimum $\geq c_{12}$ and satisfy $\mathrm{ord}_q(\det(N)) \leq j$.*

b) *Let $n \geq 3$ and $m = \dim(V) \geq 2n+1$. Then there exists a constant $c_{13} := c_{13}(\Lambda, j, q)$ such that Λ represents all $\mathbb{Z}$ - lattices N of rank n which are represented by Λ locally everywhere, have minimum $\geq c_{13}$, satisfy $\mathrm{ord}_q(\det(N)) \leq j$ and which are in the case $n = 3$ such that the orthogonal complement of N_q in Λ_q is isotropic.*

c) *Let $n = 2$ and $m = \dim(V) \geq 6$. Then there exists a constant $c_{14} := c_{14}(\Lambda, j, q)$ such that Λ represents all $\mathbb{Z}$ - lattices N of rank n which are represented by Λ locally everywhere, have minimum $\geq c_{14}$, satisfy $\mathrm{ord}_q(\det(N)) \leq j$ and which are such that the orthogonal complement of N_q in Λ_q is isotropic.*

d) *Let a positive definite $\mathbb{Z}$-lattice N_0 of rank $n_0 \leq m-3$ with Gram matrix T_0 be given. Let S be a finite set of primes with $q \in S$ such that one has*
   i) *Λ_p and $(N_0)_p$ are unimodular for all primes $p \notin S$ and for $p = q$.*
   ii) *Each isometry class in the genus of Λ has a representative Λ' on V such that $\Lambda'_p = \Lambda_p$ for all primes $p \notin S$.*

   Then there exists a constant $c_{15} := c_{15}(\Lambda, T_0, S)$ such that for all sufficiently large integers $t \in \mathbb{Z}$ which are not divisible by a prime in S, the $\mathbb{Z}$-lattice N with Gram matrix tT_0 is represented by Λ if it is represented by all completions Λ_p.

9. Comparison of results.

Concerning dimension bounds the theorem of Ellenberg and Venkatesh is clearly superior to the results obtained by other methods, and it should not be difficult to show that it is best possible in this respect. The method makes it necessary to impose a bound on the power to which some fixed prime is allowed to divide the determinant of the lattice N; this is not necessary for the arithmetic and the analytic results in the dimension range where they are valid. At least at present the ergodic method gives neither an effective bound on the "sufficiently large" minimum of the lattice N nor an asymptotic formula for the number of representations. This may of course change with further refinements of the results from ergodic theory which make the proof possible.

Results of Kitaoka (see [27, 25, 29]) on estimates of local densities show that at least the main term $r(\text{gen}\Lambda, N)$ is still growing like $(\det\ N)^{\frac{m-n-1}{2}}$ in the range $n+3 < m \leq 2n+2$ if one puts suitable restrictions on N, e.g., if one supposes a Gram matrix of N to have square free determinant.

On the other hand, even for a Siegel cusp form of weight k the best known estimtes for the Fourier coefficient $a(F,T)$ at T bound it by a term of the type $(\det\ T)^r$ where r is not much smaller than $\frac{k}{2}$, see [7] for some results in that direction. The famous conjecture of Resnikoff and Saldaña [41] (for which meanwhile counterexamples are known, see [36]) predicts an estimate

$$|a(F,T)| = O((\det(T))^{\frac{k}{2}-\frac{n+1}{4}+\epsilon}),$$

hence (with $m = 2k$) an exponent $\frac{m-n-1}{4} + \epsilon$ at $\det(T)$, which, like the exponent in the main term, depends only on the difference $m - n$ but not on m itself.
An asymptotic formula for $r(\Lambda, N)$ valid in a range $m \geq n + n_0$ for some fixed n_0 would have a main term growing like $(\det(T))^{\frac{n_0-1}{2}}$ (with some restrictions on T), in particular the exponent would be independent of the weight of the theta series. Its validity would therefore in particular require that the Fourier coefficients of the modular form $\vartheta^{(n)}(\Lambda) - \vartheta^{(n)}(\text{gen}(\Lambda))$, which in general is not cuspidal, satisfy an estimate similar to that of the Resnikoff-Saldaña conjecture for Fourier coefficients of cusp forms.

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