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# ANDERSON'S DOUBLE COMPLEX AND GAMMA MONOMIALS FOR RATIONAL FUNCTION FIELDS

Sunghan Bae*, Ernst-Ulrich Gekeler, Pyung-Lyun Kang
Linsheng Yin**

Abstract. We investigate algebraic $\Gamma$-monomials of Thakur's positive characteristic $\Gamma$-function, by using Anderson-Das' double complex method of computing the sign-cohomology of the universal ordinary distribution. We prove that the $\Gamma$-monomial associated to an element of the second sign-cohomology of the universal ordinary distribution of $\mathbb{F}_q(T)$ generates a Kummer extension of the Carlitz cyclotomic function field, which is also a Galois extension of the base field $\mathbb{F}_q(T)$. These results are characteristic-$p$ analogues of those of Deligne on classical $\Gamma$-monomials, proofs of which were given by Das using the double complex method. In this paper, we also obtain some results on $e$-monomials of Carlitz's exponential function.

## 0. Introduction.

Motivated by the purpose of calculating the indices of circular units and the Stickelberger ideal of a cyclotomic number field, Sinnott [S] computed the sign-cohomology of the universal ordinary distribution of $\mathbb{Q}$ by an inductive method. Galovich and Rosen [GR1] gave an analogous unit index formula in a cyclotomic function field by using a result parallel to Sinnott's on the sign-cohomology of the universal ordinary distribution of $\mathbb{F}_q(T)$. To extend Galovich-Rosen's formula for the unit index to a cyclotomic extension of a general global function field, Yin attempted to compute the sign-cohomology of the universal ordinary distribution of such a field. He determined the Galois module structure of the sign-cohomology conditionally by Sinnott's inductive method, and thus obtained a conditional unit-index formula, see [Y1]. Soon later, Anderson [An] found a new method of computing in an identical way the sign cohomology of the universal ordinary distributions, both for the rational number field and a global function field. He introduced a certain double complex which is a resolution of the universal ordinary distribution. This double complex enabled him to construct canonical basis classes of the sign cohomology. Das [Da] used this double complex in the rational number field case for the study of classical $\Gamma$-monomials and got a series of results, which greatly illuminated the power of Anderson's method.

In this paper, using Anderson's double complex and following Das' way, we study $\Gamma$-monomials for rational function fields. Thakur [Th] defined the $\Gamma$-function in characteristic $p$ and showed that it has many interesting properties analogous to the classical $\Gamma$-function. Especially, it satisfies a reflection formula and a multiplication formula. Sinha [Si] used Anderson's soliton theory to develop an analogue of Deligne's reciprocity for function fields. In the course he found that certain $\Gamma$-monomials generate Kummer extensions of cyclotomic function fields, a result which will be reproved below with the aid of the double complex. In the paper [BGY], necessary and sufficient conditions for elements of the value group of the universal distribution to describe first or second sign-cohomology classes are derived using distributions. In the present article we get only the necessity of these criteria, but a little more information about $\Gamma$-monomials and $e$-monomials, which seem out of reach for the method employed in [BGY]. Using $\Gamma$-monomials we also find extensions of cyclotomic function fields, and these happen to be Galois even over the basic rational function field.

We would like to emphasize the following technical points: Besides the double complex, there are three main ingredients in computing the $\Gamma$-monomials in Das' paper, and these are used frequently. The first one is the vertical shift operator. In the case of a rational function field there are more roots of unity, which causes the definition of the vertical shift operator to be more complicated. The second one is the $\Gamma$-function itself, of which the reflection formula and the multiplication formula play important roles. These formulae in the function field case have some extra factors, and thus one has to be more careful

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in applying them. The third one is a certain " canonical lifting", which fails to exist in the function field case for $q > 3$. This is due to the fact that here the definition of the double complex depends on the non-canonical choice of a generator of the sign group. Thus we have to content with a "semi-canonical lifting", which still has nice properties, and is good enough for our purposes.

**1.The Double Complex for $\mathbb{F}_q(T)$.**

Let $K = \mathbb{F}_q(T)$ and $A = \mathbb{F}_q[T]$, the rational function field and polynomial ring, respectively, over the finite field $\mathbb{F}_q$. We choose a fixed generator $\gamma$ of $J = \mathbb{F}_q^*$. Let $\mathcal{A}$ be the free abelian group generated by symbols of the form $[a]$, where $a \in K/A$. Let $\mathbb{U}$ be the quotient of $\mathcal{A}$ by the subgroup generated by all elements of the form $[a] - \sum_{\mathfrak{n}b=a}[b]$, where $\mathfrak{n}$ is a monic polynomial in $A$, and $\mathbb{U}^-$ (resp. $\mathbb{U}^+$) the quotient of $\mathcal{A}$ by the subgroup generated by all elements of the form $[a] - \sum_{\mathfrak{n}b=a}[b]$, along with all those of the form $\sum_{\theta\in J}[\theta a]$ (resp. $[a] - [\gamma a]$). The map $a \mapsto [a]$ is the universal ordinary distribution on $K/A$. By abuse of notation, we also call the group $\mathbb{U}$ itself the universal ordinary distribution on $K/A$. Further, $J$ acts on $\mathbb{U}$ in the natural way. Let $H^*(J, \mathbb{U})$ denote the sign cohomology group for $\mathbb{U}$. It is known that $tor(\mathbb{U}^+) \simeq H^1(J, \mathbb{U})$ and $tor(\mathbb{U}^-) \simeq H^2(J, \mathbb{U})$([BGY], Proposition 2.4). If $\mathbf{a} = \sum m_i[a_i] \in \mathcal{A}$ represents an element in $H^*(J, \mathbb{U})$, we often write $\mathbf{a} \in H^*(J, \mathbb{U})$. It is clear from the context whether elements of $\mathcal{A}$, $\mathbb{U}$, $H^1(J, \mathbb{U})$, or $H^2(J, \mathbb{U})$ are intended. We use gothic letters to denote elements of $A$.

Define

$$\langle \frac{\mathfrak{a}}{\mathfrak{f}} \rangle = \begin{cases} 1, & \text{if } \mathfrak{a} \text{ is monic} \\ 0, & \text{otherwise,} \end{cases}$$

assuming that $\deg \mathfrak{a} < \deg \mathfrak{f}$ and $\mathfrak{f}$ monic. For $\mathbf{a} = \sum m_i[a_i] \in \mathcal{A}$ we define the *degree* $m(\mathbf{a})$ of $\mathbf{a}$ by $\sum m_i$ and the *internal sum* $IS(\mathbf{a})$ of $\mathbf{a}$ by

$$IS(\mathbf{a}) = \sum m_i \langle a_i \rangle.$$

Let $\mathfrak{f}$ be the least common multiple of the denominators of the $a_i$ and let $\mathfrak{t} \in (A/\mathfrak{f})^*$. We define $\mathbf{a}^{\mathfrak{t}}$ by

$$\mathbf{a}^{\mathfrak{t}} = \sum m_i[\mathfrak{t}a_i].$$

Let $\mathcal{P}$ be the set of all monic irreducible polynomials in $A$. We fix a linear order '$<$' on $\mathcal{P}$. Let

$$\mathcal{S} = \{[a, \mathfrak{g}, n] : a \in K/A, \mathfrak{g}\,\text{squarefree monic polynomial}, n\,\text{an integer}\}.$$

We write $|\mathfrak{g}|$ to denote the number of monic irreducible polynomials dividing $\mathfrak{g}$. We will define a double complex $\mathbb{SK}$ as follows:

$\mathbb{SK}_{m,n}$ = the free abelian group generated by the symbols $[a, \mathfrak{g}, n] \in \mathcal{S}$ with $m = |\mathfrak{g}|$. The chain maps $\partial$ and $\delta$ of bidegree $(-1, 0)$ and $(0, -1)$, respectively, are:

$$\partial[a, \mathfrak{g}, n] = \sum_{i=1}^{|\mathfrak{g}|}(-1)^{i-1}([a, \mathfrak{g}/\mathfrak{p}_i, n] - \sum_{\mathfrak{p}_i b = a}[b, \mathfrak{g}/\mathfrak{p}_i, n]),$$

where $\mathfrak{g} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$ with $\mathfrak{p}_i < \mathfrak{p}_j$ for $i < j$, and

$$\delta[a, \mathfrak{g}, n] = \begin{cases} (-1)^m \sum_{i=0}^{q-2}[\gamma^i a, \mathfrak{g}, n-1], & \text{for } n \text{ odd} \\ (-1)^m([a, \mathfrak{g}, n-1] - [\gamma a, \mathfrak{g}, n-1]), & \text{for } n \text{ even.} \end{cases}$$

Then it is easy to see that

$$\partial^2 = 0, \quad \delta^2 = 0, \quad \text{and} \quad \delta\partial + \partial\delta = 0.$$

Let $(T(\mathbb{SK}), \partial + \delta)$ be the total complex of $\mathbb{SK}$. We use the same notation $\mathbb{SK}$ for the total complex when the meaning is evident.

Let $\mathbb{SK}'$ be the subcomplex of $\mathbb{SK}$ generated by the elements $\beta(a, n)[a, \mathfrak{g}, n]$, where

$$\beta(a, n) = \begin{cases} q-1, & \text{if } a = 0 \text{ and } n \text{ is even} \\ 1, & \text{otherwise.} \end{cases}$$

Then following the method employed by Ouyang in [Ou],§2.3, §3.4, we have:

**Proposition 1.** *Let $\mathbb{U}$ be the universal ordinary distribution on $K/A$. There exist canonical isomorphisms*

$$H^2(J,\mathbb{U}) = H_0(H_0(\mathbb{SK},\partial),\delta) = H_0(\mathbb{SK},\partial+\delta) = H_0(\mathbb{SK}/\mathbb{SK}',\partial+\delta),$$

*and*

$$H^1(J,\mathbb{U}) = H_{-1}(H_0(\mathbb{SK},\partial),\delta) = H_{-1}(\mathbb{SK},\partial+\delta) = H_{-1}(\mathbb{SK}/\mathbb{SK}',\partial+\delta).$$

Since

$$\delta([0,\mathfrak{g},n]) = \begin{cases} (-1)^n(q-1)[0,\mathfrak{g},n-1], & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even,} \end{cases}$$

and $\partial([0,\mathfrak{g},n])$ lies in $\mathbb{SK}'$, we have:

**Proposition 2.** *Given a squarefree monic polynomial $\mathfrak{g}$ with $|\mathfrak{g}| = i$, we define*

$$k_{\mathfrak{g}} = \begin{cases} [0,\mathfrak{g},-i] \in \mathbb{SK}_{i,-i}/\mathbb{SK}'_{i,-i}, & \text{for } i \text{ even} \\ [0,\mathfrak{g},-i-1] \in \mathbb{SK}_{i,-i-1}/\mathbb{SK}'_{i,-i-1}, & \text{for } i \text{ odd.} \end{cases}$$

*Then the collection $\{k_{\mathfrak{g}}; |\mathfrak{g}| \text{ even}\}$ (resp. $\{k_{\mathfrak{g}}; |\mathfrak{g}| \text{ odd}\}$) forms a $\mathbb{Z}/(q-1)$-basis for $H_0(\mathbb{SK}/\mathbb{SK}',\partial+\delta)$ (resp. $H_{-1}(\mathbb{SK}/\mathbb{SK}',\partial+\delta)$). The $k_{\mathfrak{g}}$ are referred to as canonical basis classes.*

Define the *vertical shift operator* $S : \mathbb{SK}_{m,n} \longrightarrow \mathbb{SK}_{m,n+1}$ by the rule

$$S([a,\mathfrak{g},n]) = (-1)^{|\mathfrak{g}|}\sum_{i=0}^{q-2} N_i[\gamma^i a,\mathfrak{g},n+1],$$

where $N_i = \frac{(q-1)(q-2)+2}{2} - i(q-1)$.

**Remark.** Let $d = g.c.d.(\frac{(q-1)(q-2)+2}{2}, q-1)$. Then we could define a vertical shift operator $S'$ by

$$S'([a,\mathfrak{g},n]) = (-1)^{|\mathfrak{g}|}\sum_{i=0}^{q-2} N'_i[\gamma^i a,\mathfrak{g},n+1],$$

where $N'_i = \frac{N_i}{d}$. In the case when $q = 3$, $S'$ is just the same as that in [Da].

Define the *diagonal shift operator* $\Delta_{\mathfrak{p}} : \mathbb{SK}_{m,n} \longrightarrow \mathbb{SK}_{m-1,n+2}$ associated with a prime $\mathfrak{p}$ by the rule:

$$\Delta_{\mathfrak{p}}([a,\mathfrak{g},n]) = 0, \text{if } \mathfrak{p} \nmid \mathfrak{g}.$$

If $\mathfrak{g} = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_m$ with $\mathfrak{p}_1 < \mathfrak{p}_2 < \cdots < \mathfrak{p}_m$,

$$\Delta_{\mathfrak{p}_r}([a,\mathfrak{g},n]) = (-1)^{m+n-r}[a,\mathfrak{g}/\mathfrak{p}_r,n+2].$$

Then we get the following lemma, whose proof is exactly the same as in the classical case([Da], Theorem 4, Theorem 5).

**Lemma 3.** i) $S\delta + \delta S = (q-1)^2$ *and* $\partial S + S\partial = 0$. *Thus* $(\partial+\delta)S + S(\partial+\delta) = (q-1)^2$.

ii) $\partial\Delta_{\mathfrak{p}} = \Delta_{\mathfrak{p}}\partial$ *and* $\delta\Delta_{\mathfrak{p}} = \Delta_{\mathfrak{p}}\delta$.

Given a canonical basis class $[0,\mathfrak{g},-n]$ with $|\mathfrak{g}| = n$ even, one can construct a representing cycle

$$C = \oplus_{i=0}^n C_{i,-i}, \quad C_{n,-n} = [0,\mathfrak{g},-n],$$

such that $C_{i,-i} = \sum n_j[a_j,\mathfrak{g}_j,-i]$ with $sgn(a_j) = 1$ for $i$ odd, $sgn(a_j) \neq \gamma^{q-2}$ for $i$ even, and no term of the form $[0,\mathfrak{h},-m]$ except $[0,\mathfrak{g},-n]$ occurs, as follows:

Suppose that one has constructed $C_{i,-i}$. If

$$\partial C_{i,-i} = \sum m_j[a_j,\mathfrak{g}_j,-i],$$

then

$$C_{i-1,1-i} = \begin{cases} (-1)^{i-1}\sum\langle a_j\rangle[a_j,\mathfrak{g},1-i], & \text{if } i \text{ is even} \\ (-1)^{i-1}\sum\sum_{k\geq 0}^{\kappa(a_j)-1}[\gamma^{k-\kappa(a_j)}a_j,\mathfrak{g},1-i], & \text{if } i \text{ is odd} \end{cases}$$

where $sgn(a_j) = \gamma^{\kappa(a_j)}$.

We call $C$ a *semi-canonical lifting*. Such a construction also works for canonical basis classes of $H^1$ and for the boundary elements of $\mathbb{SK}$.

For an element $C$ of $\mathbb{SK}$ and a squarefree monic polynomial $\mathfrak{g}$, we let $C^{\{\mathfrak{g}\}}$ be the $\mathfrak{g}$-component, i.e., the part that includes those of the form $[*,\mathfrak{g},*]$. Following the same lines as Proposition 7 of [Da], we have:

**Proposition 4.** *Let $C = \oplus_{i+j=\ell} C_{i,j}$ be a cycle in $\mathbb{SK}$. For a fixed monic squarefree polynomial $\mathfrak{g}$, write $C^{\{\mathfrak{g}\}}_{k,\ell-k} = \sum n_i[a_i, \mathfrak{g}, \ell-k]$. Then if $\ell - k$ is odd, we have*

$$\sum n_i[a_i] \in H^1(J, \mathbb{U}).$$

*If $\ell - k$ is even, then*

$$\sum n_i[a_i] \in H^2(J, \mathbb{U}).$$

It is shown in [BGY] that $\mathbf{a} = \sum m_i[a_i] \in H^2(J, \mathbb{U})$ if and only if $\sum m_i\langle a_i\rangle = \sum m_i\langle \mathfrak{t}a_i\rangle$ for all $\mathfrak{t} \in (A/\mathfrak{f})^*$. We give another proof of the necessity of this using our double complex.

**Proposition 5.** *If $\mathbf{a} = \sum m_i[a_i] \in H^2(J, \mathbb{U})$, then $\sum m_i\langle a_i\rangle = \sum m_i\langle \mathfrak{t}a_i\rangle$ for all $\mathfrak{t} \in (A/\mathfrak{f})^*$.*

*Proof.* Let $C$ be a cycle in $\mathbb{SK}$ such that $C = \oplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i[a_i, 1, 0]$ and $(\partial + \delta)C = 0$. Write $C_{1,-1} = \sum n_i[b_i, \mathfrak{p}_i, -1]$. Then

$$IS(\partial C_{1,-1}) = IS(\partial C^{\mathfrak{t}}_{1,-1}) = \sum n_i \frac{q^{\deg \mathfrak{p}_i} - 1}{q-1}.$$

Also

$$IS(\delta C_{0,0}) = \sum m_i\langle a_i\rangle - \sum m_i\langle \gamma a_i\rangle = IS(\partial C_{1,-1}),$$

and

$$IS(\delta C^{\mathfrak{t}}_{0,0}) = \sum m_i\langle \mathfrak{t}a_i\rangle - \sum m_i\langle \gamma\mathfrak{t}a_i\rangle = IS(\partial C^{\mathfrak{t}}_{1,-1}).$$

Summing over $\mathfrak{t} = 1, \gamma, \dots, \gamma^{q-2}$, we get 0, which implies that $\sum n_i(q^{\deg \mathfrak{p}_i} - 1) = 0$. This in turn implies that

$$\sum m_i\langle \mathfrak{t}a_i\rangle = \sum m_i\langle \gamma\mathfrak{t}a_i\rangle.$$

Then $(q-1)\sum m_i\langle a_i\rangle = \sum_{a_i \neq 0} m_i = (q-1)\sum m_i\langle \mathfrak{t}a_i\rangle$. Hence

$$\sum m_i\langle a_i\rangle = \sum m_i\langle \mathfrak{t}a_i\rangle.$$

This finishes the proof. □

**Corollary 1.** *Let $\mathbf{a} = \sum m_i[a_i] \in H^2(J, \mathbb{U})$. Assume that $a_i \neq 0$ for every $i$. Then $q-1$ divides $\sum m_i$. More generally, if $m$ is the coefficient of $[0]$ in $\mathbf{a}$, then $q-1$ divides $\sum m_i - m$.*

**Corollary 2.** *Let $\mathbf{a} = \sum m_i[a_i] \in H^2(J, \mathbb{U})$. Let $C$ be a cycle in $\mathbb{SK}$ such that $C = \oplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i[a_i, 1, 0]$ and $(\partial + \delta)C = 0$. Let*

$$C_{1,-1} = \sum n_j[b_j, \mathfrak{p}_j, -1].$$

*Then $\sum n_j \deg \mathfrak{p}_j \equiv 0 \mod (q-1)$.*

*Proof.* It is shown in the proof of Proposition 5 that $IS(\delta C_{0,0}) = 0$ and

$$IS(\partial C_{1,-1}) = \sum n_j \frac{q^{\deg \mathfrak{p}_j} - 1}{q-1}.$$

But $\frac{q^d-1}{q-1} \equiv d \mod (q-1)$. Hence we get the result. □

**2. Algebraic Gamma Monomials.**

Thakur([Th]) defined some $\Gamma-$function in characteristic $p$. We change Thakur's definition slightly by the formula

$$\Gamma(z) = \prod_{\mathfrak{a} \in A_+} \left(1 + \frac{z}{\mathfrak{a}}\right)^{-1},$$

where $A_+$ is the set of all monic polynomials in $A$. Then our $\Gamma(z)$ is just $\Pi(z)$ of Thakur. Let $\tilde{\pi}$ denote the fundamental period of the Carlitz module, which is unique up to a factor of $\mathbb{F}_q^*$, and $e = e_C$ the Carlitz exponential. This $\Gamma$-function has the following nice properties.

**Theorem 6.** ([Th] Theorem 6.1.1, Theorem 6.2.1)
Reflection formula:

$$\prod_{\theta\in J}\Gamma(\theta z)=\frac{\tilde{\pi}z}{e(\tilde{\pi}z)}.$$

Multiplication formula: *For $\mathfrak{f}\in A_+$ of degree $d$ we have*

$$\prod_{\substack{\mathfrak{a}\in A\\ \deg\mathfrak{a}<d}}\Gamma(\frac{z+\mathfrak{a}}{\mathfrak{f}})=\tilde{\pi}^{(q^d-1)/(q-1)}((-1)^d\mathfrak{f})^{q^d/(1-q)}R_d(z)\Gamma(z),$$

*where* $R_d(z)=\prod_{\substack{\deg\mathfrak{a}<d\\ \mathfrak{a}\text{ monic}}}(z+\mathfrak{a})$.

Let $Q_d(z)=\prod_{\substack{\deg\mathfrak{a}<d\\ \mathfrak{a}\neq 0}}(z+\mathfrak{a})$. Then $Q_d(\gamma z)=Q_d(z)$ and $\prod_{1=0}^{q-2}R_d(\gamma^i z)=-Q_d(z)$. We also wirte $Q_\mathfrak{f}=Q_{\deg\mathfrak{f}}$ and $R_\mathfrak{f}=R_{\deg\mathfrak{f}}$.

For $a\in K/A$ we write $\{a\}$ to denote the representative of $a$ such that $|a|_\infty<1$, where $|\,|_\infty$ is the absolute value at $\infty=(\frac{1}{T})$. For each $\mathbf{a}=\sum m_i[a_i]\in\mathcal{A}$, we define the $\Gamma$-monomial, $e$-monomial, and $r$-monomial, respectively, by

$$\Gamma(\mathbf{a})=\tilde{\pi}^{\frac{m(\mathbf{a})}{q-1}}\prod\Gamma(\{a_i\})^{-m_i},$$

$$e(\mathbf{a})=\prod_{a_i\neq 0}e(\tilde{\pi}a_i)^{m_i},$$

and

$$r(\mathbf{a})=\prod_{a_i\neq 0}\{a_i\}^{m_i}.$$

Here $e(\mathbf{a})$ can be thought of as the analogue of a classical sine-monomial. By abuse of notation, we also write $\Gamma(\sum n_i[a_i,*,*])$ to mean $\Gamma(\sum n_i[a_i])$. This notation will be also applied to $e$- and $r$-monomials. In what follows $a\in K/A$ always means that $a=\{a\}$ unless otherwise stated. It is known that $\Gamma(\mathbf{a})$ is algebraic over $K$ if $\mathbf{a}\in H^2(J,\mathbb{U})$(see [Th]). Moreover, Sinha([Si]) has shown that $\Gamma(\mathbf{a})$ generates a Kummer extension of $K_\mathfrak{f}$ for $\mathbf{a}\in H^2(J,\mathbb{U})$. We will give another proof of this fact in §4, using the double complex. The first step is the following weaker version.

**Theorem 7.** *Let* $\mathbf{a}=\sum m_i[a_i]\in H^2(J,\mathbb{U})$. *Then*

$$\Gamma(\mathbf{a})^{(q-1)^2}\equiv e(\mathbf{a})^{q-1}\mod K^*.$$

*In fact,* $\Gamma(\mathbf{a})^{q-1}=\sqrt[q-1]{r}e(\mathbf{a})$ *for some* $r\in K^*$.

*Proof.* It is easy to see from the definitions that

$$\partial S[b,\mathfrak{p},-1]=\sum_{i=0}^{q-2}N_i(-[\gamma^i b,1,0]+\sum_{\deg\mathfrak{u}<\deg\mathfrak{p}}[\frac{\gamma^i b+\mathfrak{u}}{\mathfrak{p}},1,0]),$$

and

$$\delta S[a,1,0]=(q-1)\sum_{i=0}^{q-2}[\gamma^i a,1,0],$$

since $\sum_{i=0}^{q-2}N_i=q-1$. Then from Theorem 6 we can see that

$$\Gamma(\partial S[b,\mathfrak{p},-1])=(-1)^d\mathfrak{p}^{-q^d}Q_\mathfrak{p}(\{b\})^{-\frac{(q-1)(q-2)+2}{2}}\cdot\left(\prod_{i=0}^{q-2}R_\mathfrak{p}(\gamma^i\{b\})^i\right)^{q-1},\tag{*}$$

where $d=\deg\mathfrak{p}$ and

$$\Gamma(\delta S[a,1,0])=\left(\frac{e(\tilde{\pi}a)}{\tilde{\pi}\{a\}}\right)^{q-1}.\tag{**}$$

Now follow the proof of Theorem 6 of [Da], using the relation $(q-1)^2C=(\partial+\delta)SC$ for a cycle $C$. □

**Remark.** Let the notations be the same as in the proof of Proposition 5. Since $\sum n_j\deg\mathfrak{p}_j\equiv 0$ mod $(q-1)$, it is easy to see that $\sqrt[q-1]{\prod\mathfrak{p}_j^{n_j}}$ lies in $K_\mathfrak{f}$, the $\mathfrak{f}$-th cyclotomic function field, where $\mathfrak{f}$ is the least common multiple of the denominators of the $a_i$'s. Thus

$$\Gamma(\mathbf{a})^{q-1}\equiv\prod\left(Q_{\mathfrak{p}_j}(b_j)\right)^{n_j\frac{(q-1)(q-2)+2}{2(q-1)}}\mod K_\mathfrak{f}^*.$$

But it is shown in [Si] that $\Gamma(\mathbf{a})^{q-1}$ itself lies in $K_\mathfrak{f}$. We will come to this later.

**Proposition 8.** *Let* $\mathbf{a} = \sum m_i[a_i] \in H^2(J, \mathbb{U})$*. Then, with notations as in Proposition 5, Thus we have*

$$\frac{e(\mathbf{a})}{e(\mathbf{a}^{\mathfrak{t}})} = \theta_{\mathfrak{t}} e(\mathbf{b})^{(q-1)},$$

*for some* $\mathbf{b} \in \mathbb{U}$ *and* $\theta_{\mathfrak{t}} = \pm 1$.

*Proof.* Let $C = \oplus_{i+j=0} C_{i,j}$ be a cycle in $\mathbb{SK}$ such that $C_{0,0} = \sum m_i[a_i, 1, 0]$. Then $C - C^{\mathfrak{t}}$ is a boundary. Let $B = \oplus_{i+j=1} B_{i,j}$ be a chain in $\mathbb{SK}$ such that $(\partial + \delta)B = C - C^{\mathfrak{t}}$. Note that

$$e(\partial[0, \mathfrak{p}, 0]) = \mathfrak{p} = \prod_{\substack{\deg \mathfrak{u} < \deg \mathfrak{p} \\ monic}} (-1)^{\frac{q^{\deg p} - 1}{q-1}} e(\frac{\mathfrak{u}}{\mathfrak{p}})^{q-1},$$

and

$$e(\partial[a, \mathfrak{p}, 0]) = 1, \quad \text{if } a \neq 0,$$

since $e$ is a punctured even distribution, and that $e(\delta[a, 1, 1]) = -e(a)^{q-1}$. Thus we get the result from $\frac{e(\mathbf{a})}{e(\mathbf{a}^{\mathfrak{t}})} = e(\partial B_{1,0}) e(\delta B_{0,1})$. □

**Remark.** 1. Proposition 8 is the analog of the fact that

$$\frac{\sin \mathbf{a}}{\sin \mathbf{a}^t} = (\sin \mathbf{b})^2,$$

in the corresponding classical situation ([Da], Theorem 11). But the constant $\theta_{\mathfrak{t}}$ may be different from 1 as the following example shows. This $\theta_{\mathfrak{t}}$ will play a crucial role in constructing Galois extensions of $K_{\mathfrak{f}}$ using $\Gamma$-monomials. It is not difficult to see that $\theta_{\mathfrak{t}}$ is only dependent of the cohomology class of $\mathbf{a}$, and that $\theta_{\mathfrak{t}_1 \mathfrak{t}_2} = \theta_{\mathfrak{t}_1} \theta_{\mathfrak{t}_2}$.

2. It is known that $\sin : \mathbb{Q}/\mathbb{Z} \to \mathbb{C}^*$, $a \mapsto 2 \sin \pi \langle a \rangle$ is an even punctured distribution. If one uses this fact, then the proof of Theorem 11 of [Da] can be simplified as the proof of Proposition 8 above.

**Example.** Let $q = 3$, $\mathbf{a} = [\frac{1}{T+1}] - [\frac{T-1}{T(T+1)}]$, and $\mathfrak{t} = -T + 1$. Then

$$\frac{e(\mathbf{a})}{e(\mathbf{a}^{\mathfrak{t}})} = \frac{e(\tilde{\pi} \frac{1}{T(T+1)})}{e(\tilde{\pi} \frac{T-1}{T(T+1)})} = -e(\tilde{\pi} \frac{1}{T(T+1)})^2,$$

since $\lambda = e(\tilde{\pi} \frac{1}{T(T+1)})$ satisfies the relation $\lambda^4 + (T-1)\lambda^2 + 1 = 0$.
If $\mathfrak{t} = -1$, then

$$\frac{e(\mathbf{a})}{e(\mathbf{a}^{\mathfrak{t}})} = 1 = e(\mathbf{0})^2.$$

Thus $\theta_{\mathfrak{t}}$ changes as $\mathfrak{t}$ varies.

**Theorem 9.** *Let* $\mathbf{a} = \sum m_i[a_i] \in H^2(J, \mathbb{U})$*. Let* $\mathfrak{f}$ *be the least common multiple of the denominators of the* $a_i$ *and let* $\mathfrak{t} \in (A/\mathfrak{f})^*$*. Then*

$$\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}^{\mathfrak{t}})} \in K_{\mathfrak{f}}.$$

*Proof.* Let the notations be as in the proof of Proposition 8. Let $B_{1,0} = \sum \ell_j [c_j, \mathfrak{p}_j, 0]$. We may assume that $B$ is a semi-canonically lifted chain. Then the denominators of $c_j$ divide $\mathfrak{f}$. From the proof of Propositon 9, it suffices to show that $\Gamma(\partial B_{1,0}) \in K_{\mathfrak{f}}$. It can be easily checked that

$$\Gamma(\partial B_{1,0}) \equiv (-1)^{\frac{\sum \ell_j \deg \mathfrak{p}_j}{q-1}} \prod \mathfrak{p}_j^{\frac{\ell_j}{q-1}} \mod K_{\mathfrak{f}}^*.$$

The extension $K(\sqrt[q-1]{\prod((-1)^{\deg \mathfrak{p}_j} \mathfrak{p}_j)^{\ell_j}})$ is abelian over $K$, and ramified only at the primes over $\mathfrak{p}_j$ and $\infty$. But since $\sqrt[q-1]{(-T)^d} \in \mathbb{F}_q(\sqrt[q-1]{-\frac{1}{T}})$, we see that $\sqrt[q-1]{\prod((-1)^{\deg \mathfrak{p}_j} \mathfrak{p}_j)^{\ell_j}} \in K_{\mathfrak{f}}$. □

Similarly, if $\mathbf{a}$ and $\mathbf{a}'$ represent the same class in $H^2(J, \mathbb{U})$, then $\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}')} \in K_{\mathfrak{f}}$.

## 3. Criterion for $H^1(J,\mathbb{U})$.

One can follow the proof of Proposition 12 of [Da] to get:

**Lemma 10.** *Let* $\mathbf{b}=\sum m_i[b_i]\in H^1(J,\mathbb{U})$. *Then for all* $i$, $b_i\neq 0$.

**Lemma 11.** *Let* $\mathbf{b}=\sum m_i[b_i]\in H^1(J,\mathbb{U})$, *representing any canonical basis class of* $H^1(J,\mathbb{U})$ *indexed by a monic squarefree polynomial divisible by at least three primes. Let* $C=\oplus_{i+j=1}C_{i,j}$ *be a cycle such that* $C_{0,1}=\sum m_i[b_i,1,1]$. *Assume that no term of the form* $[0,\mathfrak{p},0]$ *appears in* $C_{1,0}$. *Then*

$$\sum m_i\equiv 0 \mod q-1.$$

*Proof.* Note that $IS(\partial[b,p,0])=\frac{q^{\deg p}-1}{q-1}\equiv \deg\mathfrak{p} \mod (q-1)$ for $b\notin A$. Now follow the proof of Proposition 13 of [Da]. ☐

It is shown in [BGY] that $\mathbf{b}\in H^1(J,\mathbb{U})$ if and only if $|r(\mathbf{b})|_\infty=|r(\mathbf{b}^{\mathfrak{t}})|_\infty$ for any $\mathfrak{t}\in(A/\mathfrak{f})^*$, where $\mathfrak{f}$ is the least common multiple of the denominators of the $b_i$. Here we give another proof of the necessity of this using the double complex. In this way one can get some more information about the $e$-monomials.

**Theorem 12.** *Let* $\mathbf{b}=\sum m_i[b_i]\in H^1(J,\mathbb{U})$ *and let* $\mathfrak{f}$ *be the least common multiple of the denominators of the* $b_i$. *Then*

$$|r(\mathbf{b})|_\infty=|r(\mathbf{b}^{\mathfrak{t}})|_\infty,$$

*for all* $\mathfrak{t}\in(A/\mathfrak{f})^*$. *Furthermore, we have more information about e-monomials in the following special cases:*

*First case: If* $\mathbf{b}$ *represents a canonical basis class of* $H^1(J,\mathbb{U})$, *indexed by a single irreducible polynomial, then* $e(\mathbf{b})^{2(q-1)}=e(\mathbf{b})^{2(q-1)}=1$ *and* $e(\mathbf{b})e(\mathbf{b}^{\mathfrak{t}})^{-1}\in\mathbb{F}_q^*$.

*Second case: Let* $\mathbf{b}$ *represent a canonical basis class of* $H^1(J,\mathbb{U})$ *indexed by a monic squarefree polynomial divisible by at least three primes. Let* $C=\oplus C_{i,-i+1}$ *be a cycle such that* $C_{0,1}=\sum m_i[b_i,1,1]$. *Assume that no term of the form* $[0,\mathfrak{p},0]$ *appears in* $C_{1,0}$. *Then* $e(\mathbf{b}^{\mathfrak{t}})\in\mathbb{F}_q^*$ *for any* $\mathfrak{t}\in(A/\mathfrak{f})^*$.

*Proof.* The first statement follows by linearity from the two special cases, since $|r(\mathbf{b})|_\infty=|e(\mathbf{b})|_\infty$. Let $C=\oplus C_{i,-i+1}$ a cycle such that $C_{0,1}=\sum m_i[b_i,1,1]$.
*First Case:* Let $C_{1,0}=[0,\mathfrak{p},0]$. Then we know that $e(-\partial C_{1,0})=e(-\partial C_{1,0}^{\mathfrak{t}})=\mathfrak{p}$. Thus

$$e(\delta C_{0,1})=e(\delta C_{0,1}^{\mathfrak{t}})=\mathfrak{p}.$$

However, $e(\delta C_{0,1})=(-1)^{\sum m_i}e(\mathbf{b})^{q-1}$ and $e(\delta C_{0,1}^{\mathfrak{t}})=(-1)^{\sum m_i}e(\mathbf{b}^{\mathfrak{t}})^{q-1}$. Then the result follows from the fact that $\sigma_{\mathfrak{t}}e(\mathbf{b})=e(\mathbf{b}^{\mathfrak{t}})$.
*Second Case:* In this case $e(\partial C_{1,0})=1$. Then the result follows in the same way as the first case, by using Lemma 11. ☐

**Remark.** Theorem 12 is the analogue of Theorem 14 of [Da]. Again if one uses the fact that sin is an even punctured distribution, then he gets that $w=\sqrt{p}$ in the first case of Theorem 14 of [Da] too.

## 4. Kummer Properties.

In this section, we use the semi-canonical lifting to show that $\Gamma(\mathbf{a})^{q-1}\in K_{\mathfrak{f}}$, where $\mathbf{a}=\sum n_i[a_i]\in H^2(J,\mathbb{U})$ and $\mathfrak{f}$ is the least common multiple of the denominators of the $a_i$. For this we need the following Lemma.

**Lemma 13.** *Let* $\mathfrak{p}$*(resp.* $\mathfrak{q}$*) be a monic irreducible polynomial of degree* $d$ *(resp.* $e$*). Then we have*

i) $$\Big(\prod_{\substack{\deg\mathfrak{a}<d\\ monic}}Q_{\mathfrak{q}}(\tfrac{\mathfrak{a}}{\mathfrak{p}})\Big)^{q-1}=\mathfrak{p}^{q^d(1-q^e)}\frac{\prod_{\substack{\deg\mathfrak{a}<d+e\\ \mathfrak{a}\neq 0}}\mathfrak{a}}{\prod_{\substack{\deg\mathfrak{b}<d\\ \mathfrak{b}\neq 0}}\mathfrak{b}\prod_{\substack{\deg\mathfrak{c}<e\\ \mathfrak{c}\neq 0}}\mathfrak{c}}$$

ii) *For* $b\in K\setminus A$, *we have*

$$\prod_{\substack{\deg\mathfrak{a}<d\\ monic}}\prod_{\theta\in J}Q_{\mathfrak{q}}(\frac{\theta b+\mathfrak{a}}{\mathfrak{p}})=\prod_{\substack{\deg\mathfrak{a}<d\\ \mathfrak{a}\neq 0}}Q_{\mathfrak{q}}(\frac{\theta b+\mathfrak{a}}{\mathfrak{p}})$$
$$=\mathfrak{p}^{q^e(1-q^d)}\frac{Q_{d+e}(b)}{Q_d(b)Q_e(\frac{b}{\mathfrak{p}})}$$

*Hence*
$$\frac{Q_{\mathfrak p}(b)Q_{\mathfrak q}(\frac{b}{\mathfrak p})\prod_{\substack{\deg\mathfrak a<d\\ monic}}\prod_{\theta\in J}Q_{\mathfrak q}(\frac{\theta b+\mathfrak a}{\mathfrak p})}{Q_{\mathfrak q}(b)Q_{\mathfrak p}(\frac{b}{\mathfrak q})\prod_{\substack{\deg\mathfrak b<e\\ monic}}\prod_{\theta\in J}Q_{\mathfrak p}(\frac{\theta b+\mathfrak b}{\mathfrak q})}=\frac{\mathfrak q^{q^d(q^e-1)}}{\mathfrak p^{q^e(q^d-1)}}.$$

*Proof.* For i) we have
$$\begin{aligned}\left(\prod_{\substack{\deg\mathfrak a<d\\ monic}}Q_e(\frac{\mathfrak a}{\mathfrak p})\right)^{q-1} &= \prod_{\substack{\deg\mathfrak a<d\\ \mathfrak a\neq 0}}Q_e(\frac{\mathfrak a}{\mathfrak p})\\ &= \prod_{\substack{\deg\mathfrak a<d\\ \mathfrak a\neq 0}}\prod_{\substack{\deg\mathfrak b<e\\ \mathfrak b\neq 0}}\frac{\mathfrak a+\mathfrak b\mathfrak p}{\mathfrak p}\end{aligned}$$

For ii) we have
$$\prod_{\substack{\deg\mathfrak a<d\\ monic}}\prod_{\theta\in J}Q_e(\frac{\theta b+\mathfrak a}{\mathfrak p})=\prod_{\substack{\deg\mathfrak a<d\\ \mathfrak a\neq 0}}Q_e(\frac{b+\mathfrak a}{\mathfrak p}).$$

Then a similar computation as in i) will give the result. □

We come back to $\mathbf a\in H^2(J,\mathbb U)$, for which we want to calculate $\Gamma(\mathbf a)^{q-1}$.

**First case:** Let $\mathbf a=\mathbf a_{\mathfrak p\mathfrak q}$ be semi-canonically lifted from the basis class $[0,\mathfrak p\mathfrak q,-2]$. Then it is easy to see that
$$C_{1,-1}=\sum_{\substack{\deg\mathfrak a<\deg\mathfrak p\\ monic}}[\frac{\mathfrak a}{\mathfrak p},\mathfrak q,-1]-\sum_{\substack{\deg\mathfrak b<\deg\mathfrak q\\ monic}}[\frac{\mathfrak b}{\mathfrak q},\mathfrak p,-1].$$

We need to compute $\Gamma(\partial SC_{1,-1})$. Let $d=\deg\mathfrak p$ and $e=\deg\mathfrak q$. Then
$$\begin{aligned}&\Gamma(\partial SC_{1,-1})\\ &\equiv\frac{\prod_{\substack{\deg\mathfrak a<d\\ monic}}(-1)^e\mathfrak q^{q^e}Q_e(\frac{\mathfrak a}{\mathfrak p})^{-\frac{(q-1)(q-2)+2}{2}}}{\prod_{\substack{\deg\mathfrak b<e\\ monic}}(-1)^d\mathfrak p^{q^d}Q_d(\frac{\mathfrak b}{\mathfrak q})^{-\frac{(q-1)(q-2)+2}{2}}}\mod (K^*)^{q-1}\\ &\equiv\theta\frac{\mathfrak p^{q^d\frac{q^e-1}{q-1}(\frac{(q-1)(q-2)+2}{2}-1)}}{\mathfrak q^{q^e\frac{q^d-1}{q-1}(\frac{(q-1)(q-2)+2}{2}-1)}}\mod (K^*)^{q-1}\\ &\equiv\theta\frac{\mathfrak p^{\frac{q^d(q^e-1)(q-2)}{2}}}{\mathfrak q^{\frac{q^e(q^d-1)(q-2)}{2}}}\mod (K^*)^{q-1},\end{aligned}$$

for some $\theta\in J$ by Lemma 13, i). But comparing the signs of second and third lines, we must have $\theta=1$. Thus
$$\Gamma(\mathbf a_{\mathfrak p\mathfrak q})^{q-1}\equiv\sqrt{\frac{\mathfrak p^e}{\mathfrak q^d}}e(\mathbf a_{\mathfrak p\mathfrak q})\mod K^*.$$

We read off that $\Gamma(\mathbf a_{\mathfrak p\mathfrak q})^{q-1}\in K_{\mathfrak p\mathfrak q}$.

**Second Case:** Let $n\geq 4$ be even. Let $C=\oplus C_{i,-i}$ be the semi-canonically lifted cycle of the basis class $[0,\mathfrak g,-n]$, where $\mathfrak g$ is a squarefree monic polynomial divisible by $n$ irreducible polynomials. Let $C_{0,0}=\sum m_i[a_i,1,0]$ and $\mathbf a=\sum m_i[a_i]$. Express $C_{2,-2}=\sum C_{2,-2}^{\{\mathfrak p\mathfrak q\}}$ and $C_{2,-2}^{\{\mathfrak p\mathfrak q\}}=\sum n_i[b_i,\mathfrak p\mathfrak q,-2]$. Note that $b_i\notin A$. We have
$$\begin{aligned}\partial C_{2,-2}^{\{\mathfrak p\mathfrak q\}}=&\sum n_i\left([b_i,\mathfrak q,-2]-\sum_{\deg\mathfrak a<\deg\mathfrak p}[\frac{b_i+\mathfrak a}{\mathfrak p},\mathfrak q,-2]\right)\\ &-\sum n_i\left([b_i,\mathfrak p,-2]-\sum_{\deg\mathfrak b<\deg\mathfrak q}[\frac{b_i+\mathfrak b}{\mathfrak q},\mathfrak p,-2]\right).\end{aligned}$$

Then the contribution to $C_{1,-1}$ from $C_{2,-2}^{\{\mathfrak{p}\mathfrak{q}\}}$ is

$$\sum n_i \left( \langle b_i \rangle [b_i, \mathfrak{q}, -1] - \sum_{\substack{\deg \mathfrak{a} < \deg \mathfrak{p} \\ monic}} [\frac{b_i + \mathfrak{a}}{\mathfrak{p}}, \mathfrak{q}, -1] - \langle b_i \rangle [\frac{b_i}{\mathfrak{p}}, \mathfrak{q}, -1] \right)$$
$$- \sum n_i \left( \langle b_i \rangle [b_i, \mathfrak{p}, -1] - \sum_{\substack{\deg \mathfrak{b} < \deg \mathfrak{q} \\ monic}} [\frac{b_i + \mathfrak{b}}{\mathfrak{q}}, \mathfrak{p}, -1] - \langle b_i \rangle [\frac{b_i}{\mathfrak{q}}, \mathfrak{p}, -1] \right).$$

Now let $b = \frac{b_i + \mathfrak{a}}{\mathfrak{q}}$, with $\mathfrak{a}$ monic, $\deg \mathfrak{a} < \deg \mathfrak{q}$. First we need to find the term $[\frac{\theta b_i + \mathfrak{a}}{\mathfrak{q}}, \mathfrak{p}, -1]$ from $[\frac{b_i + \mathfrak{a}}{\mathfrak{q}}, \mathfrak{p}, -1]$ for any $\theta \in J$. Note that $[\theta b, \mathfrak{p}, -2]$ lives in $\partial C_{2,-2}$. So if $[\theta b_i, \mathfrak{p}\mathfrak{q}, -2]$ lives in $C_{2,-2}$, we are done. If $[\theta b_i, \mathfrak{p}\mathfrak{q}, -2]$ does not appear in $C_{2,-2}$, then $\theta b = \frac{c + \mathfrak{c}}{\mathfrak{r}}$ and $[c, \mathfrak{p}, -2]$ lives in $C_{2,-2}$. Then $c = \frac{c_1 + \mathfrak{v}}{\mathfrak{q}}$, and $b_i = \frac{a_1 + \mathfrak{u}}{\mathfrak{r}}$. Thus

$$c_1 + \mathfrak{v} + \mathfrak{q}\mathfrak{c} = \theta a_1 + \theta \mathfrak{u} + \theta \mathfrak{r}\mathfrak{a}.$$

Then we must have $c_1 = \theta a_1$, since they are the only terms with absolute value less than 1. Because we can find $\mathfrak{v}'$ and $\mathfrak{c}'$ such that

$$\mathfrak{v}' + \mathfrak{q}\mathfrak{c}' = \theta \mathfrak{u} + \mathfrak{r}\mathfrak{b},$$

the term $[\frac{c_1 + \mathfrak{v}'}{\mathfrak{q}}, \mathfrak{p}\mathfrak{r}, -2]$ can be obtained from $[c_1, \mathfrak{p}\mathfrak{q}\mathfrak{r}*, *]$, and this will give the term $[\frac{\theta b_i + \mathfrak{a}}{\mathfrak{q}}, \mathfrak{p}, -1]$ in $C_{1,-1}$.

In conclusion, we see that

$$D = \left( [b_i, \mathfrak{q}, -1] - \sum_{\substack{\deg \mathfrak{a} < \deg \mathfrak{p} \\ monic}} [\frac{b_i + \mathfrak{a}}{\mathfrak{p}}, \mathfrak{q}, -1] - [\frac{b_i}{\mathfrak{p}}, \mathfrak{q}, -1] \right)$$
$$- \left( [b_i, \mathfrak{p}, -1] - \sum_{\substack{\deg \mathfrak{b} < \deg \mathfrak{q} \\ monic}} [\frac{b_i + \mathfrak{b}}{\mathfrak{q}}, \mathfrak{p}, -1] - [\frac{b_i}{\mathfrak{q}}, \mathfrak{p}, -1] \right)$$

appears in $C_{1,-1}$ for $b_i$ with $\langle b_i \rangle = 1$. Now applying Lemma 14, ii), we have

$$\Gamma(\partial S C_{1,-1}) \in (K^*)^{q-1}.$$

Hence $\Gamma(\mathbf{a})^{q-1} \equiv e(\mathbf{a}) \mod K^*$. We summarize these calculations in the following Theorem, which is a refined version of Theorem 7.

**Theorem 14.** *Let $n$ be an even positive integer. Let $C = \oplus C_{i,-i}$ be the semi-canonically lifted cycle from the basis class $[0, \mathfrak{g}, -n]$, where $\mathfrak{g}$ is a squarefree monic polynomial divisible by $n$ irreducible polynomials. Let $\mathbf{a} = \sum m_i [a_i]$, where $C_{0,0} = \sum m_i [a_i, 1, 0]$. Then $\Gamma(\mathbf{a}) \in K_{\mathfrak{g}}$. Furthermore,*

i) *If $\mathfrak{g} = \mathfrak{p}\mathfrak{q}$, then*

$$\Gamma(\mathbf{a})^{q-1} \equiv \sqrt{\frac{\mathfrak{q}^d}{\mathfrak{p}^e}} e(\mathbf{a}) \mod K^*.$$

ii) *If $n \geq 4$, then*

$$\Gamma(\partial S C_{1,-1}) \in (K^*)^{q-1} \quad \textit{and} \quad \Gamma(\mathbf{a})^{q-1} \equiv e(\mathbf{a}) \mod K^*.$$

**Remark.** In fact, one can show that in Theorem 15, ii), even $\Gamma(\mathbf{a})^{q-1} = \theta e(\mathbf{a})$, for some $\theta \in J$ as follows:

The contribution to $C_{0,0}$ from $D$ is

$$D_{0,0} = -\sum_{\substack{\deg \mathfrak{u}<\deg \mathfrak{p}\\ monic}} \sum_{k=1}^{q-2}\sum_{\ell\geq 0}^{k-1}[\frac{\gamma^{\ell-k}b+\gamma^{\ell}\mathfrak{u}}{\mathfrak{p}},1,0] + \sum_{\substack{\deg \mathfrak{u}<\deg \mathfrak{p}\\ monic}} \sum_{k=1}^{q-2}\sum_{\ell\geq 0}^{k-1}[\frac{\gamma^{\ell-k}b+\gamma^{\ell}\mathfrak{q}\mathfrak{u}}{\mathfrak{p}\mathfrak{q}},1,0]$$
$$+\sum_{\theta\in J}\sum_{\substack{\deg \mathfrak{u}<\deg \mathfrak{p}\\ monic}}\sum_{\substack{\deg v<\deg q\\ monic}} \sum_{k=1}^{q-2}\sum_{\ell\geq 0}^{k-1}[\frac{\gamma^{\ell-k}(\theta b+\mathfrak{v})+\gamma^{\ell}\mathfrak{q}\mathfrak{u}}{\mathfrak{p}\mathfrak{q}},1,0]$$
$$+\sum_{\substack{\deg \mathfrak{v}<\deg \mathfrak{q}\\ monic}} \sum_{k=1}^{q-2}\sum_{\ell\geq 0}^{k-1}[\frac{\gamma^{\ell-k}b+\gamma^{\ell}\mathfrak{v}}{\mathfrak{q}},1,0] - -\sum_{\substack{\deg \mathfrak{v}<\deg \mathfrak{q}\\ monic}} \sum_{k=1}^{q-2}\sum_{\ell\geq 0}^{k-1}[\frac{\gamma^{\ell-k}b+\gamma^{\ell}\mathfrak{p}\mathfrak{v}}{\mathfrak{p}\mathfrak{q}},1,0]$$
$$-\sum_{\theta\in J}\sum_{\substack{\deg \mathfrak{u}<\deg \mathfrak{p}\\ monic}}\sum_{\substack{\deg v<\deg q\\ monic}} \sum_{k=1}^{q-2}\sum_{\ell\geq 0}^{k-1}[\frac{\gamma^{\ell-k}(\theta b+\mathfrak{u})+\gamma^{\ell}\mathfrak{p}\mathfrak{v}}{\mathfrak{p}\mathfrak{q}},1,0].$$

By a tedious computation one can see that

$$r(D_{0,0})^{q-1} = \Gamma(\partial SD).$$

Then

$$\Gamma(\mathbf{a})^{(q-1)^2} = \Gamma(\delta SC_{0,0})\Gamma(\partial SC_{1,-1}) = \Gamma(\partial SC_{1,-1})\left(\frac{e(\mathbf{a})}{r(\mathbf{a})}\right)^{q-1} = e(\mathbf{a})^{q-1}.$$

**Corollary 1.** *Let $n\geq 4$ be an even integer. Let $\mathbf{a}=\sum m_i[a_i]$ represent the basis class $[0,\mathfrak{g},-n]$ with $|\mathfrak{g}|=n$, not necessarily a semi-canonical representative. Then*

$$\Gamma(\mathbf{a})^{q-1}\equiv e(\mathbf{a}) \mod K^*.$$

*Proof.* Let $C=\oplus C_{i,-i}$ be the cycle such that $C_{0,0}=\sum m_i[a_i,1,0]$, and $C'=\oplus C'_{i,-i}$ be the semi-canonically lifted cycle of the class $[0,\mathfrak{g},-n]$. Then there exists a chain $B=\oplus B_{i,-i+1}$ such that $(\partial+\delta)B = C-C'$. Then

$$\partial S(C_{1,-1}-C'_{1,-1}) = \partial S(\partial B_{2,-1}+\delta B_{1,0}) = \partial S\delta B_{1,0},$$

since $\partial S=-S\partial$ and $\partial^2=0$. Now it is easy to see from (*) that $\Gamma(\partial S\delta[b,\mathfrak{p},0])\in (K^*)^{q-1}$. Hence $\Gamma(\partial SC_{1,-1})\in(K^*)^{q-1}$, since we know from the proof of Theorem 15 that $\Gamma(\partial SC'_{1,-1})\in (K^*)^{q-1}$. Then the result follows from the relation $\Gamma(\mathbf{a})^{(q-1)^2}=\Gamma(\partial SC_{1,-1})\left(\frac{e(\mathbf{a})}{r(\mathbf{a})}\right)^{q-1}$. □

It is not hard to show that in case i) above, we have $\sqrt{\frac{\mathfrak{p}^e}{\mathfrak{q}^d}}\in K_{\mathfrak{p}\mathfrak{q}}$. Thus the following holds.

**Corollary 2.** *If $\mathbf{a}=\sum m_i[a_i]\in H^2(J,\mathbb{U})$, then $\Gamma(\mathbf{a})^{q-1}\in K_{\mathfrak{f}}$, where $\mathfrak{f}$ is the least common multiple of the denominators of the $a_i$.*

## 5. Galois Properties of $K_{\mathfrak{f}}(\Gamma(\mathbf{a}))/K$.

Let $\mathbf{a}=\sum m_i[a_i]\in H^2(J,\mathbb{U})$, and let $\mathfrak{f}$ be the least common multiple of the denominators of the $a_i$. In this section we consider the extension $K_{\mathfrak{f}}(\Gamma(\mathbf{a}))$ over $K$. Let $C=\oplus C_{i,-i}$ be a cycle in $\mathbb{SK}$ such that $C_{0,0}=\sum m_i[a_i,1,0]$. Let $v$ be an element of $\bar{K}$ such that

$$v^{q-1}=\Gamma(\partial SC_{1,-1}) \quad\text{and}\quad \Gamma(\mathbf{a})^{q-1}=v\frac{e(\mathbf{a})}{r(\mathbf{a})}.$$

Let $\sigma$ be an element of $Gal(\bar{K}/K)$ whose restriction to $K_{\mathfrak{f}}$ is $\sigma_{\mathfrak{t}}$, where $\bar{K}$ is the separable closure of $K$. Then

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right)^{q-1} = \frac{v}{\sigma v}\theta_{\mathfrak{t}}e(\mathbf{b})^{q-1},$$

where $\theta_{\mathfrak{t}}$ and $\mathbf{b}$ are given in Proposition 9. Hence $\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right)\in K_{\mathfrak{f}}$ if and only if $\frac{v}{\sigma v}\theta_{\mathfrak{t}}=1$.

**Example.** We keep the example after Proposition 8. Let $\sigma \in Gal(\bar{K}/K)$ restrict to $\sigma_{-T+1}$ on $K_{T(T+1)}$. Then $\frac{v}{\sigma v} = -1$ and $\theta_{\mathfrak{t}} = -1$ as we saw in the example in §2.
On the other hand, let $\tau$ be an element of $Gal(\bar{K}/K)$ which restricts to $\sigma_{-1}$ on $K_{T(T+1)}$. Then $\frac{v}{\tau v} = 1$ and $\theta_{-1} = 1$ as we saw in the example in §2. Thus $K_{T(T+1)}(\Gamma(\mathbf{a}_{T(T+1)}))$ is Galois over $K$.

We know from Proposition 5 that $e(\mathbf{a})$ and $e(\mathbf{a}^{\mathfrak{t}})$ lie in $K_{\mathfrak{f}}^{+}$, the maximal real subfield of $K_{\mathfrak{f}}$. Hence the signs of $e(\mathbf{a})$ and $e(\mathbf{a}^{\mathfrak{t}})$ make sense. Then

$$sgn(e(\mathbf{a})) = \prod_{\theta \in J} \theta^{\sum m_i \langle \theta^{-1} a_i \rangle} = (-1)^{\sum m_i \langle a_i \rangle}.$$

Similarly we have

$$sgn(e(\mathbf{a}^{\mathfrak{t}})) = (-1)^{\sum m_i \langle a_i \rangle}.$$

Therefore

$$sgn(\frac{e(\mathbf{a})}{e(\mathbf{a}^{\mathfrak{t}})}) = 1.$$

Now let $B = \oplus B_{i,-i+1}$ be a chain such that $(\partial + \delta)B = C - C^{\mathfrak{t}}$, and assume that $B$ is semi-canonically lifted. Let $B_{0,1} = \sum n_j [b_j, 1, 1]$. Then it is easy to see that $sgn(e(\mathbf{b})^{q-1}) = (-1)^{\sum n_j}$, where $\mathbf{b} = \sum n_j [b_j]$. Since $B$ is semi-canonically lifted, we know that $\frac{e(\mathbf{a})}{e(\mathbf{a}^{\mathfrak{t}})} = \theta_{\mathfrak{t}} e(\mathbf{b})^{q-1}$. Hence $\theta_{\mathfrak{t}} = (-1)^{\sum n_j}$.

**Lemma 15.** $\theta_{\mathfrak{t}} = sgn(\Gamma(\partial B_{1,0})^{q-1})$.

*Proof.* It is easy to see that $sgn(\Gamma(\partial[b, \mathfrak{p}, 0])^{q-1}) = (-1)^{\deg \mathfrak{p}}$. Write $B_{1,0} = \sum \ell_k [c_k, \mathfrak{p}_k, 0]$. Then

$$sgn(\Gamma(\partial B_{1,0})^{q-1}) = (-1)^{\sum \ell_k \deg \mathfrak{p}_k}.$$

Since $B$ is canonically lifted, $IS(\delta B_{0,1}) = IS(B_{0,1}) = \sum n_j$. Since $C$ represents an element of $H^2(J, \mathbb{U})$, $IS(C_{0,0} - C_{0,0}^{\mathfrak{t}}) = 0$. Also we see that $IS(\partial B_{1,0}) = \sum \frac{q^{\deg \mathfrak{p}_k} - 1}{q-1} \ell_k$. Since $\partial B_{1,0} + \delta B_{0,1} = C_{0,0} - C_{0,0}^{\mathfrak{t}}$, $\sum \frac{q^{\deg \mathfrak{p}_k} - 1}{q-1} \ell_k + \sum n_j = 0$. But

$$\frac{q^{\deg \mathfrak{p}_k} - 1}{q-1} \equiv \deg \mathfrak{p}_k \mod (q-1).$$

Thus

$$\sum \ell_k \deg \mathfrak{p}_k \equiv \sum n_j \mod 2,$$

which implies the result. □

**Remark.** As we saw in the proof of Theorem 14 that $v^2 \in K$, $\frac{v}{\sigma v} = \pm 1$, which makes sense since we know that $\theta_{\mathfrak{t}} = \pm 1$.

In what follows, we assume that $q$ is odd. We have the following analog of the classical Gauss lemma, whose proof is exactly the same.

**Lemma 16.** *For $\mathfrak{p} \in A$ irreducible and $\mathfrak{a}$ prime to $\mathfrak{p}$, let $(\frac{\mathfrak{a}}{\mathfrak{p}})$ be the quadratic symbol, i.e., $(\frac{\mathfrak{a}}{\mathfrak{p}}) \equiv \mathfrak{a}^{\frac{q^{\deg \mathfrak{p}} - 1}{2}}$. Then for $\mathfrak{t}$ prime to $\mathfrak{p}$, we have*

$$\left(\frac{\mathfrak{t}}{\mathfrak{p}}\right) = (-1)^n,$$

*where $n = \#\{\mathfrak{a} : monic, \deg \mathfrak{a} < \deg \mathfrak{p}, sgn(\{\frac{\mathfrak{t}\mathfrak{a}}{\mathfrak{p}}\}) \notin \mathbb{F}_q^2\}$.*

As in [Da], to show that $\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})} \in K_{\mathfrak{f}}$ it suffices to consider the two cases that $\mathbf{a}$ represents $[0, \mathfrak{p}\mathfrak{q}, -2]$ and $\mathbf{a}$ represents $[0, \mathfrak{g}, -n]$, where $\mathfrak{g}$ is a product of $n$ distinct irreducible polynomials and $n \geq 4$ is an even integer. Also it is easy to see that if $\mathbf{a}$ and $\mathbf{a}'$ represent the same canonical basis class of $H^2(J, \mathbb{U})$, then

$$\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})} \in K_{\mathfrak{f}} \quad \text{if and only if} \quad \frac{\Gamma(\mathbf{a}')}{\sigma\Gamma(\mathbf{a}')} \in K_{\mathfrak{f}}.$$

**First Case:** $\mathbf{a}$ represents $[0, \mathfrak{p}\mathfrak{q}, -2]$. Suppose that $C = C_{0,0} + C_{1,-1} + C_{2,-2}$ is a cycle such that $\mathbf{a} = \sum n_i[a_i]$, where $C_{0,0} = \sum n_i[a_i, 1, 0]$. We may also assume that $C$ is canonically lifted. Let $d = \deg \mathfrak{p}$ and $e = \deg \mathfrak{q}$. Then from the proof of Theorem 14, $v \equiv \sqrt{\frac{\mathfrak{q}^d}{\mathfrak{p}^e}} \mod K^*$. By Lemma 16 we have

$$\frac{v}{\sigma v} = \left(\frac{\mathfrak{t}}{\mathfrak{p}}\right)^e \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^d = (-1)^{dN_2 + eN_1},$$

where

$$N_1 = \#\{\mathfrak{a} : \text{monic}, \deg \mathfrak{a} < \deg \mathfrak{p}, sgn(\{\frac{\mathfrak{t}\mathfrak{a}}{\mathfrak{p}}\}) \notin \mathbb{F}_q^2\},$$

and

$$N_2 = \#\{\mathfrak{a} : \text{monic}, \deg \mathfrak{a} < \deg \mathfrak{q}, sgn(\{\frac{\mathfrak{t}\mathfrak{a}}{\mathfrak{q}}\}) \notin \mathbb{F}_q^2\}.$$

Now $B_{2,-1} = 0$ and

$$C_{1,-1} = \sum_{\substack{\mathfrak{a},\text{monic} \\ \deg \mathfrak{a} < \deg \mathfrak{p}}} [\frac{\mathfrak{a}}{\mathfrak{p}}, \mathfrak{q}, -1] - - \sum_{\substack{\mathfrak{b},\text{monic} \\ \deg \mathfrak{b} < \deg \mathfrak{q}}} [\frac{\mathfrak{b}}{\mathfrak{q}}, \mathfrak{p}, -1].$$

For each $\mathfrak{a}$ (resp. $\mathfrak{b}$) in the sum, there exists a unique $\mathfrak{a}'$ (resp. $\mathfrak{b}'$) with $\deg \mathfrak{a}' < \deg \mathfrak{p}$ (resp. $\deg \mathfrak{b}' < \deg \mathfrak{q}$) and $i(\mathfrak{a})$ (resp. $j(\mathfrak{b})$) such that

$$\{\frac{\mathfrak{a}'\mathfrak{t}}{\mathfrak{p}}\} = \gamma^{i(\mathfrak{a})} \frac{\mathfrak{a}}{\mathfrak{p}} \quad (\text{resp. } \{\frac{\mathfrak{b}'\mathfrak{t}}{\mathfrak{q}}\} = \gamma^{j(\mathfrak{b})} \frac{\mathfrak{b}}{\mathfrak{q}}).$$

It is easy to verify that

$$i(\mathfrak{a}) \equiv \begin{cases} 0 \mod 2, & \text{if } sgn(\{\frac{\mathfrak{a}'\mathfrak{t}}{\mathfrak{p}}\}) \in \mathbb{F}_q^2 \\ 1 \mod 2, & \text{otherwise} \end{cases}$$

A similar formula holds for $j(\mathfrak{b})$. Thus

$$\sum i(\mathfrak{a}) \equiv N_1 \mod 2 \qquad \text{and} \qquad \sum j(\mathfrak{b}) \equiv N_2 \mod 2.$$

The semi-canonical lifting gives

$$B_{1,0} = \sum_{\substack{\mathfrak{a}\text{monic} \\ \deg a < \deg p}} \sum_{k=0}^{i(\mathfrak{a})-1} [\gamma^k \frac{\mathfrak{a}'}{\mathfrak{p}}, \mathfrak{q}, 0] - \sum_{\substack{\mathfrak{b}\text{monic} \\ \deg \mathfrak{b} < \deg \mathfrak{q}}} \sum_{k=0}^{j(\mathfrak{b})-1} [\gamma^k \frac{\mathfrak{b}'}{\mathfrak{q}}, \mathfrak{p}, 0].$$

Now we easily see that

$$sgn(\Gamma(\partial B_{1,0})^{q-1}) = (-1)^{dN_2 + eN_1},$$

which implies the wanted result $\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})} \in K_{\mathfrak{f}}$ in the first case.

**Second Case:** $\mathbf{a}$ represents a canonical basis class indexed by a squarefree polynomial divisible by at least four distinct irreducibles.

The following result can be regarded as an analogue of Proposition 15 in [Da].

**Proposition 17.** *Let $\mathbf{b} = \sum m_i[b_i] \in H^1(J, \mathbb{U})$. Assume that $\mathbf{b}$ represents a canonical basis class of $H^1(J, \mathbb{U})$, indexed by monic squarefree polynomials divisible by at least three monic irreducibles. Let $C$ be a cycle in $\mathbb{SK}$ such that $C = \oplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i[b_i, 1, 0]$ and $(\partial + \delta)C = 0$. Assume that no term of the form $[0, \mathfrak{p}, -1]$ appears in $C_{1,-1}$. Then for each $\mathfrak{t}$ coprime to the least common multiple of the denominators of the $b_i$, we have*

$$\prod_i sgn(\{b_i\mathfrak{t}\})^{m_i} = \prod_i sgn(\{b_i\})^{m_i}.$$

*Proof.* From Theorem 12, we know that $e(\mathbf{b}) \in K^*$. Thus $e(\mathbf{b}^{\mathfrak{t}}) = \sigma_{\mathfrak{t}} e(\mathbf{b}) = e(\mathbf{b})$ and further $sgn(e(\mathbf{b})) = sgn(e(\mathbf{b}^{\mathfrak{t}}))$. Since $\sum m_i \equiv 0 \mod (q-1)$ from Lemma 11, we get the result. $\square$

Given $C=\sum n_i[*,*,*]\in\mathbb{SK}$, the total sum is defined by $TS(C)=\sum n_i$. We also define some operators as follows.

$$\begin{aligned}
&\mathfrak{t}:[a,*,*]\mapsto[a^{\mathfrak{t}},*,*],\\
&I:[a,*,k]\mapsto\langle a\rangle[a,*,k-1]\quad\text{for } k \text{ odd},\\
&J:[a,*,k]\mapsto\sum_{l=0}^{\kappa(a)-1}[\gamma^{l-\kappa(a)}a,*,k-1]\quad\text{for } k \text{ even},
\end{aligned}$$

where $\kappa(a)$ is defined by $\operatorname{sgn}(a)=\gamma^{\kappa(a)}$. Note that $JI=0$.

Let $\mathbf{a}\in H^2(J,\mathbb{U})$ represent a canonical basis class $k_{\mathfrak{g}}$ indexed by a monic squarefree polynomial divisible by at least four irreducible polynomials. Let $C$ be a cycle obtained by the semi-canonical lifting of $k_{\mathfrak{g}}$. Let $\mathfrak{t}$ be prime to $\mathfrak{g}$. Then $C-C^{\mathfrak{t}}=(\partial+\delta)B$ for some chain $B$. Again assume that $B$ is a semi-canonically lifted. Our aim is to compute $\operatorname{sgn}(\Gamma(\partial B_{1,0})^{q-1})=(-1)^{\sum n_i\deg\mathfrak{p}_i}$, where $B_{1,0}=\sum n_i[a_i,\mathfrak{p}_i,0]$. So we only need to consider the parities of the total sum of $B_{1,0}^{\{\mathfrak{p}_i\}}$. Let $C_{n,-n}=[0,\mathfrak{g},-n]$. Then we have

$$C_{n-1,1-n}=I\partial C_{n,-n}$$

$$C_{n-2,2-n}=J\partial I\partial C_{n,-n}$$

$$C_{n-3,3-n}=I\partial J\partial I\partial C_{n,-n}=I\partial J\partial C_{n-1,1-n}=I\partial C_{n-2,2-n}$$

$$B_{n-1,2-n}=J(I\partial-tI\partial)C_{n,-n}$$

$$\begin{aligned}
B_{n-2,3-n}&=I(J\partial I\partial-tJ\partial I\partial-\partial JI\partial+\partial JtI\partial)C_{n,-n}\\
&=I(J\partial-tJ\partial-\partial J+\partial Jt)C_{n-1,1-n}
\end{aligned}$$

$$\begin{aligned}
B_{n-3,4-n}&=J(I\partial J\partial-tI\partial J\partial-\partial IJ\partial+\partial ItJ\partial+\partial I\partial J-\partial I\partial Jt)C_{n-1,1-n}\\
&=(J\partial It-JtI\partial)C_{n-2,2-n}+J\partial IJ\partial E+J\partial I\partial JF
\end{aligned}$$

$$\vdots$$

$$B_{1,0}=(J\partial It-JtI\partial)C_{2,,-3}+J\partial IJ\partial E'+J\partial I\partial JF',$$

for some chains $E,F,E',F'\in\mathbb{SK}$. To show the $sgn(\Gamma(\partial B_{1,0})^{q-1})=1$, it suffices to prove that $TS(B_{1,0})\equiv 0 \mod 2$.

**Lemma 18.** *We have*

$$TS((J\partial IJ\partial[a,\mathfrak{pqr},-3])^{\{\mathfrak{p}\}})=0\quad\textit{and}\quad TS((J\partial I\partial J[a,\mathfrak{pqr},-3])^{\{\mathfrak{p}\}})=0.$$

*Proof.* We define $\kappa(b)$ by $\operatorname{sgn}(b)=\gamma^{\kappa(b)}$. Then up to $\pm1$ we have

$$\begin{aligned}
(J\partial[a,\mathfrak{pq},-2])^{\{\mathfrak{p}\}}=&\sum_{k\ge0}^{\kappa(a)-1}\Big([\gamma^{-\kappa(a)+k}a,\mathfrak{p},-1]-[\gamma^{-\kappa(a)+k}a/\mathfrak{q},\mathfrak{p},-1]\Big)\\
&-\sum_{\substack{\deg v<\deg q\\ \text{monic}}}\sum_{k=0}^{q-2}\sum_{l\ge0}^{k-1}[\gamma^{-k+l}(a+\gamma^k v)/\mathfrak{q},\mathfrak{p},-1].
\end{aligned}$$

$$\begin{aligned}
(\partial J[a,\mathfrak{pq},-2])^{\{\mathfrak{p}\}}=&\sum_{k\ge0}^{\kappa(a)-1}\Big([\gamma^{-\kappa(a)+k}a,\mathfrak{p},-1]-[\gamma^{-\kappa(a)+k}a/\mathfrak{q},\mathfrak{p},-1]\Big)\\
&-\sum_{\substack{\deg\mathfrak{v}<\deg\mathfrak{q}\\ \mathfrak{v}\neq0}}[\gamma^{-\kappa(a)+k}a+\mathfrak{v})/\mathfrak{q},\mathfrak{p},-1].
\end{aligned}$$

$$\begin{aligned}
(IJ\partial[a,\mathfrak{pqr},-3])^{\{\mathfrak{pq}\}}=&\kappa(a)\Big([\gamma^{-\kappa(a)}a,\mathfrak{pq},-2]-[\gamma^{-\kappa(a)}a/\mathfrak{r},\mathfrak{pq},-2]\Big)\\
&-\sum_{\substack{\deg\mathfrak{v}<\deg\mathfrak{r}\\ \text{monic}}}\sum_{k=1}^{q-2}[(\gamma^{-k}a+\mathfrak{v})/\mathfrak{r},\mathfrak{pq},-2].
\end{aligned}$$

$$(I\partial J[a,\mathfrak{pqr},-3])^{\{\mathfrak{pq}\}} = \kappa(a)\left([\gamma^{-\kappa(a)}a,\mathfrak{pq},-2]-[\gamma^{-\kappa(a)}a/\mathfrak{r},\mathfrak{pq},-2]\right) - \sum_{\substack{\deg\mathfrak{v}<\deg\mathfrak{r}\\ \text{monic}}}\sum_{k\geq 0}^{\kappa(a)-1}[(\gamma^{-\kappa(a)+k}a+\mathfrak{v})/\mathfrak{r},\mathfrak{pq},-2].$$

We thus get

$$\begin{aligned}
TS(J\partial[a,\mathfrak{pq},-2]^{\{\mathfrak{p}\}}) &= \pm\left(\frac{(q-1)(q-2)(q^{\deg\mathfrak{q}}-1)}{2(q-1)}\right)\\
TS(I\partial J[a,\mathfrak{pqr},-3]^{\{\mathfrak{pq}\}}) &= \pm\left(\kappa(a)\frac{q^{\deg\mathfrak{r}}-1}{q-1}\right),\\
TS(IJ\partial[a,\mathfrak{pqr},-3]^{\{\mathfrak{pq}\}}) &= \pm\left(\frac{(q-2)(q^{\deg\mathfrak{r}}-1)}{q-1}\right)\\
TS(I\partial J[a,\mathfrak{pqr},-3]^{\{\mathfrak{pr}\}}) &= \mp\left(\kappa(a)\frac{q^{\deg\mathfrak{q}}-1}{q-1}\right),\\
TS(IJ\partial[a,\mathfrak{pqr},-3]^{\{\mathfrak{pr}\}}) &= \mp\left(\frac{(q-2)(q^{\deg\mathfrak{q}}-1)}{q-1}\right)
\end{aligned}$$

Therefore $TS(J\partial IJ\partial[a,\mathfrak{pqr},-3]^{\{\mathfrak{p}\}})=0$ and $TS(J\partial I\partial J[a,\mathfrak{pqr},-3]^{\{\mathfrak{p}\}})=0$ □

Now we need to consider $J\partial I\mathfrak{t}C_{2,-2}$ and $J\mathfrak{t}I\partial C_{2,-2}$. We first consider $J\mathfrak{t}I\partial C_{2,-2}$. We know $I\partial C_{2,-2}=C_{1,-1}$. By Lemma 11, $\sum n_i\equiv 0 \mod (q-1)$ if $C_{1,-1}^{\{\mathfrak{p}\}}=\sum n_i[a_i,\mathfrak{p},-1]$. Since $C_{1,-1}$ is semi-canonically lifted, $\langle a_i\rangle = 1$ for every $i$. Then by Proposition 17, $\prod_i sgn(\{\mathfrak{t}a_i\})^{n_i}=\prod_i sgn(a_i)^{n_i}=1$. Thus $\sum n_i\kappa(\{\mathfrak{t}a_i\})\equiv 0 \mod (q-1)$. Therefore

$$TS(J\mathfrak{t}C_{1,-1}^{\{\mathfrak{p}\}})=\sum n_i(\kappa(\{\mathfrak{t}a_i\})-1)\equiv 0 \mod (q-1).$$

Next we consider $J\partial I\mathfrak{t}C_{2,-2}$. Write $C_{2,-2}^{\{\mathfrak{pq}\}}=\sum m_j[a_j,\mathfrak{pq},-2]$. Then $\sum m_j[a_j]\in H^2(J,U)$. Thus $\sum m_j\langle\{\mathfrak{t}a_j\}\rangle=\sum m_j\langle a_j\rangle=\sum m_j\langle\gamma a_j\rangle$. But since $C_{2,-2}$ is semi-canonically lifted, there is no term like $[a,\mathfrak{pq},-2]$ with $\mathrm{sgn}(a)=\gamma^{q-2}$. Hence $\sum m_j\langle\gamma a_j\rangle=0$. Now $I\mathfrak{t}C_{2,-2}^{\{\mathfrak{pq}\}}=\sum m_j\langle\{\mathfrak{t}a_j\}\rangle[\{\mathfrak{t}a_j\},\mathfrak{pq},-2]$. As in the proof of the Lemma 18, we have

$$TS(J\partial[b,\mathfrak{pq},-1])=\frac{(q-1)(q-2)(q^{\deg\mathfrak{q}}-1)}{2(q-1)}-\frac{(q-1)(q-2)(q^{\deg\mathfrak{p}}-1)}{2(q-1)}.$$

Thus

$$TS(J\partial I\mathfrak{t}C_{2,-2}^{\{\mathfrak{pq}\}})=(\sum m_j\langle\{\mathfrak{t}a_j\}\rangle)\frac{(q-1)(q-2)(q^{\deg\mathfrak{p}}-q^{\deg\mathfrak{q}})}{2(q-1)}=0$$

We conclude that $\mathrm{sgn}(\Gamma(\partial B_{1,0})^{q-1})=1$, which concludes the second case before Proposition 17.

Combining the two cases we finally get:

**Theorem 19.** *Let $q$ be odd, let $\mathbf{a}=\sum m_i[a_i]\in H^2(J,\mathbb{U})$, and let $\mathfrak{f}$ be the least common multiple of the denominators of the $a_i$. Then $K_{\mathfrak{f}}(\Gamma(\mathbf{a}))$ is a Galois extension of $K$.*

We know from Proposition 8 that $\mathbb{F}K_{\mathfrak{f}}(\sqrt[q-1]{e(\mathbf{a})})$ is a Galois extension of $K$, where $\mathbb{F}$ is the $(q-1)$-th extension of $\mathbb{F}_q$. But in the second case we get more.

**Theorem 20.** *Let $\mathbf{a}\in H^2(J,\mathbb{U})$ represent a canonical basis class indexed by a squarefree polynomial $\mathfrak{f}$ divisible by at least four distinct irreducibles. Then $K_{\mathfrak{f}}(\sqrt[q-1]{e(\mathbf{a})})$ is Galois over $K$.*

*Proof.* Since $\frac{e(\mathbf{a})}{\sigma e(\mathbf{a})}=\frac{e(\mathbf{a})}{e(\mathbf{a}^{\mathfrak{t}})}=\theta_{\mathfrak{t}}e(\mathbf{b})^{q-1}$, for some $\mathbf{b}$, $K_{\mathfrak{f}}(\sqrt[q-1]{e(\mathbf{a})})$ is Galois over $K$ if and only if $\theta_{\mathfrak{t}}=1$. Thus the result follows from the proof of Theorem 19. □

Suppose that $n\geq 4$ is even and that $\mathfrak{g}$ is a monic squarefree polynomial divisible by $n$ primes. Let $\mathbf{a}\in H^2(J,\mathbb{U})$ be the semi-canonically lifted representative of $[0,\mathfrak{g},-n]$. We keep the same notations as in the proof of Theorem 19. Then for any $\mathfrak{t}$ prime to $\mathfrak{g}$, we have

$$\begin{aligned}
\left(\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}^{\mathfrak{t}})}\right)^{q-1} &= \Gamma(\partial B_{1,0})^{q-1}\Gamma(\delta B_{0,1})^{q-1}\\
&= \Gamma(\partial B_{1,0})^{q-1}r(B_{0,1})^{1-q}\theta_{\mathfrak{t}}\frac{e(\mathbf{a})}{e(\mathbf{a}^{\mathfrak{t}})}
\end{aligned}$$

But we know from the proof of Theorem 19 that $\theta_{\mathfrak{t}} = 1$ and $\Gamma(\partial B_{1,0}) \in K$. From the Remark after Theorem 14, we further know that $\Gamma(\mathbf{a})^{q-1} = \theta e(\mathbf{a})$. Thus we see that

$$\Gamma(\mathbf{a}^{\mathfrak{t}})^{q-1} = \theta s^{q-1} e(\mathbf{a}^{\mathfrak{t}}),$$

for some $s \in K^*$. Combining with the Remark after Theorem 14, we have the following stronger version of Theorem 14.

**Theorem 14'.** *Let $n \geq 4$ be an even positive integer. Let $C = \oplus C_{i,-i}$ be the semi-canonically lifted cycle from the basis class $[0, \mathfrak{g}, -n]$, where $\mathfrak{g}$ is a squarefree monic polynomial divisible by $n$ irreducible polynomials. Let $\mathbf{a} = \sum m_i[a_i]$, where $C_{0,0} = \sum m_i[a_i, 1, 0]$. Then*

$$\Gamma(\mathbf{a})^{q-1} = \theta e(\mathbf{a}),$$

*and*

$$\Gamma(\mathbf{a}^{\mathfrak{t}})^{q-1} = \theta s^{q-1} e(\mathbf{a}^{\mathfrak{t}}),$$

*for some $s \in K^*$ and $\theta \in \mathbb{F}_q^*$.*

Now we will give an example where $K_{\mathfrak{f}}(\Gamma(\mathbf{a}))$ is not abelian over $K$.

**Example.** We assume that $q = 3$. We can easily compute that the cycle $C = C_{0,0} \oplus C_{1,-1} \oplus C_{2,-2}$ is the semi-canonically lifted cycle of the canonical basis class $[0, T(T+1), -2]$, where

$$C_{2,-2} = [0, T(T+1), -2],$$

$$C_{1,-1} = [\frac{1}{T}, T+1, -1] - [\frac{1}{T+1}, T, -1],$$

$$C_{0,0} = [\frac{1}{T+1}, 1, 0] - [\frac{T-1}{T(T+1)}, 1, 0].$$

Thus

$$\mathbf{a}_{T(T+1)} = [\frac{1}{T+1}] - [\frac{T-1}{T(T+1)}].$$

A simple computation gives

$$\Gamma(\mathbf{a}_{T(T+1)})^2 = \sqrt{\frac{T}{T+1}} \frac{e(\frac{\tilde{\pi}}{T+1})}{e(\frac{(T-1)\tilde{\pi}}{T(T+1)})} = u,$$

using the relation $\Gamma(\mathbf{a}_{T(T+1)})^2 = \Gamma(\delta S C_{0,0}) \Gamma(\partial S C_{1,-1})$. Here we used the vertical shift operator $S'$ as in the Remark after Proposition 2. Let $\sigma = \sigma_{T-1}$ and $\tau = \sigma_{-T+1}$. Let $\lambda = e(\frac{\tilde{\pi}}{T(T+1)})$. Then we can check that

$$\frac{u}{\sigma u} = \lambda^2, \quad \frac{u}{\sigma u} = \lambda^2, \quad \text{and} \quad \frac{u}{\sigma\tau u} = 1.$$

Let

$$v_\sigma = \lambda \quad v_\tau = \lambda \quad \text{and} \quad v_{\sigma\tau} = 1.$$

Let $\tilde{\eta} \in Gal(K_{\mathfrak{f}}(\Gamma(\mathbf{a}))/K)$ be the lifting of $\eta \in Gal(K_{\mathfrak{f}}/K)$ such that $v_\eta \eta \sqrt{u} = \sqrt{u}$. Then as in the proof of Proposition 19 of [Da], using the fact that $\lambda^4 + (T-1)\lambda^2 + 1 = 0$, we get $\tilde{\sigma}\tilde{\tau} = - - \tilde{\tau}\tilde{\sigma}$ on $\sqrt{u}$.

**Remark.** It would be interesting to know whether $K_{\mathfrak{f}}(\Gamma(\mathbf{a}))$ is abelian over $K$ if $\mathbf{a}$ represents a canonical basis class indexed by a monic squarefree polynomial divisible by at least four irreducibles. In the classical case this is verified by Das, [Da] Theorem 21, with the aid of a theorem of Deligne(Theorem 7.18(b) of [De],Theorem 19 of [Da]), which is beyond the reach of this paper. If one disposes of an analogue of this theorem, then one can easily how, with the aid of Theorem 14', that $K_{\mathfrak{f}}(\Gamma(\mathbf{a}))$ is abelian over $K$ if $\mathbf{a}$ satisfies the above conditions.

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