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Michael Didas and Jörg Eschmeier

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**Michael Didas**

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
didas@math.uni-sb.de

**Jörg Eschmeier**

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
eschmei@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

# On the essential commutant of analytic Toeplitz operators associated with spherical isometries

Michael Didas and Jörg Eschmeier

Let $T \in B(H)^n$ be an essentially normal spherical isometry with empty point spectrum on a separable complex Hilbert space $H$, and let $\mathcal{A}_T \subset B(H)$ be the unital dual operator algebra generated by $T$. In this note we show that every operator $S \in B(H)$ in the essential commutant of $\mathcal{A}_T$ has the form $S = X + K$ with a $T$-Toeplitz operator $X$ and a compact operator $K$. Our proof actually covers a larger class of subnormal operator tuples, called $A$-isometries, which includes for example the tuple $T = (M_{z_1}, \ldots, M_{z_n}) \in B(H^2(\sigma))^n$ consisting of the multiplication operators with the coordinate functions on the Hardy space $H^2(\sigma)$ associated with the normalized surface measure $\sigma$ on the boundary $\partial D$ of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$. As an application we determine the essential commutant of the set of all analytic Toeplitz operators on $H^2(\sigma)$ and thus extend results proved by Davidson [6] for the unit disc and Ding-Sun [11] for the unit ball.

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## 1 Introduction

Let $m$ denote the linear Lebesgue measure on the unit circle $\partial\mathbb{D}$. A classical theorem of Davidson from 1977 (Theorem 1 in [6]) asserts that an operator $S$ on the Hardy space $H^2(m)$ commutes modulo compact operators with all analytic Toeplitz operators if and only if $S$ is a compact perturbation of a Toeplitz operator $T_f$ with symbol $f \in H^\infty(m) + C(\partial\mathbb{D})$, where $H^\infty(m)$ refers to the space of all bounded holomorphic functions on $\mathbb{D}$ regarded as a subspace of $L^\infty(m)$ by passing to non-tangential boundary values.

In a paper [11] by Ding and Sun from 1997 an analogue of this result is obtained for the Hardy space on the open Euclidean unit ball $\mathbb{B}_n \subset \mathbb{C}^n$. More precisely, if $\sigma$ denotes the normalized surface measure on $\partial\mathbb{B}_n$, then by Theorem 2 in [11], an operator $S \in B(H^2(\sigma))$ essentially commutes with all analytic Toeplitz operators if and only if $S = T_f + K$, where $K$ is compact and $f \in L^\infty(\sigma)$ has the property that the associated Hankel operator $H_f = P_{H^2(\sigma)^\perp} M_f | H^2(\sigma)$ is compact. For $n > 1$, this class of symbols strictly contains the space $H^\infty(\sigma) + C(\partial\mathbb{B}_n)$ (see [7]), while equality holds in the case $n = 1$.

The aim of this paper is to establish variants of the cited results for Toeplitz operators associated with spherical isometries or, more general, with $A$-isometries. Recall that a spherical isometry on a complex Hilbert space $H$ is a commuting tuple $T \in B(H)^n$ satisfying

$$\sum_{i=1}^{n} T_i^* T_i = 1_H.$$

Given a spherical isometry $T$, there is an abstract theory of $T$-Toeplitz operators $X \in B(H)$ defined by Prunaru [20] as the solutions of the operator equation $\sum_{i=1}^{n} T_i^* X T_i = X$. From this point of view, the result of Ding and Sun cited above describes the essential commutant of the dual algebra

$$\mathcal{A}_T = \overline{\mathbb{C}[T_1, \ldots, T_n]}^{w^*} \subset B(H)$$

generated by the special spherical isometry $T = (T_1, \ldots, T_n) \in B(H)^n$ consisting of the multiplication operators $T_i = M_{z_i}$ with the coordinate functions on the Hardy space $H = H^2(\sigma)$. Formulated in the setting of general spherical isometries, the main result of the paper is the following (cf. Theorem 4.6):

*If $T \in B(H)^n$ is an essentially normal spherical isometry with empty point spectrum, then every operator $S \in B(H)$ in the essential commutant of $\mathcal{A}_T$ has the form $S = X + K$ with a $T$-Toeplitz operator $X$ and a compact operator $K$ on $H$.*

As an application we deduce concrete analogues of the above-mentioned results of Davidson and Ding-Sun for multiplication tuples on Hardy-type function spaces. To be more specific, let $\mu$ denote a regular Borel probability measure on $\partial\mathbb{B}_n$ with the property that all one-point sets have $\mu$-measure zero. Then, the multiplication tuple $T_z = (M_{z_1}, \ldots, M_{z_n}) \in B(H^2(\mu))^n$ on the associated Hardy space

$$H^2(\mu) = \overline{\mathbb{C}[z_1, \ldots, z_n]}^{\|\cdot\|_{2,\mu}} \subset L^2(\mu)$$

is a spherical isometry whose $T_z$-Toeplitz operators are precisely the compressions

$$T_f = P_{H^2(\mu)} M_f | H^2(\mu)$$

of multiplication operators $M_f : L^2(\mu) \to L^2(\mu)$ with symbols $f \in L^\infty(\mu)$. In this context, the analytic Toeplitz operators are those with a symbol belonging to the space

$$H^\infty(\mu) = \overline{\mathbb{C}[z_1, \ldots, z_n]}^{w^*} \subset L^\infty(\mu).$$

Our main theorem then takes the following form (see Corollary 4.7):

*If $T_z \in B(H^2(\mu))^n$ is essentially normal, then an operator essentially commutes with all analytic Toeplitz operators if and only if it has the form $S = T_f + K$ with a compact operator $K$ and a symbol $f \in L^\infty(\mu)$ for which the associated Hankel operator $H_f = P_{H^2(\mu)^\perp} M_f | H^2(\mu)$ is compact.*

We actually prove stronger versions of the above results for so-called regular $A$-isometries. The precise definition will be given in Section 2. Let us just mention at the moment that this class is general enough to cover multiplication tuples with the coordinate functions on strictly pseudoconvex domains. For example, we obtain the following exact analogue of the above-mentioned theorem of Ding and Sun in the strictly pseudoconvex situation (see Corollary 4.8).

*If $\sigma$ denotes the normalized surface measure on the boundary $\partial D$ of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with $C^2$-boundary, then an operator $S$ in $B(H^2(\sigma))$ essentially commutes with all analytic Toeplitz operators on $H^2(\sigma)$ if and only if it has the form $S = T_f + K$ with a compact operator $K$ and a symbol $f \in L^\infty(\sigma)$ for which the associated Hankel operator $H_f$ is compact.*

Along the way we extend Prunaru's theory [20] on the existence of short exact Toeplitz sequences from the case of spherical isometries to the class of $A$-isometries and refine his results in the essentially normal case. To illustrate this for a spherical isometry $T \in B(H)^n$, let us write $\mathcal{T}_C(T) = C^*(T_f : f \in C(\partial\mathbb{B}_n))$ for the $C^*$-algebra generated by all $T$-Toeplitz operators with continuous symbols (for the definition of $T_f$, see Section 3). Then Proposition 3.9 says the following:

*Let $T \in B(H)^n$ be an essentially normal, non-normal spherical isometry. If $\mathcal{T}_C(T)$ is irreducible, then there is a short exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{K}(H) \xrightarrow{\subset} \mathcal{T}_C(T) \xrightarrow{\sigma} C(\sigma_n(T)) \longrightarrow 0,$$

*where $\sigma$ maps the Toeplitz operator $T_f$ to $f|\sigma_n(T)$ for every $f \in C(\partial\mathbb{B}_n)$.*

As the above examples show (see also Theorem 3.5 in [10]), many interesting aspects of the theory of Toeplitz operators on classical Hardy spaces can be rediscovered in the context of multi-variable subnormal isometries. The role of the surface measure in the classical theory will then be played by a scalar spectral measure of the minimal normal extension for the underlying subnormal tuple. In general, this measure is far from being explicitly known. So one cannot hope to find as detailed results as in the classical case. Nevertheless, it seems worthwile to pursue this connection further. An interesting question arises from a recent result of Xia (Theorem 1 in [25]) who answered a longstanding problem for Toeplitz operators on the unit disc. By the cited theorem, the condition that $T_{\overline{\theta}} X T_\theta - X$ is compact for every inner function $\theta \in H^\infty(m)$ implies that $X \in B(H^2(m))$ is a compact perturbation of a Toeplitz operator. In the context of spherical isometries $T \in B(H)^n$, the $T$-Toeplitz operators with inner symbols naturally correspond to isometries in the dual operator algebra $\mathcal{A}_T$ (see Lemma 2.3). So we may ask:

*If $T \in B(H)^n$ is an essentially normal spherical isometry with empty point spectrum, and $X \in B(H)$ has the property that $J^*XJ - X$ is compact for every isometry $J \in \mathcal{A}_T$. Must $X$ then necessarily be a compact perturbation of a $T$-Toeplitz operator?*

Xia's proof depends on a special sequence of inner functions $(\theta_k)_{k\geq 0}$ consisting of finite Blaschke products, for which a multi-variable substitute is out of sight at the moment. So it seems that more sophisticated methods are needed to solve this problem.

## 2 $A$-isometries and inner functions

Let $H$ be a separable complex Hilbert space. A commuting tuple $T \in B(H)^n$ is called a *spherical isometry* if it satisfies the relation

$$\sum_{i=1}^{n} T_i^* T_i = 1_H.$$

A result of Athavale [2] from 1990 saying that each spherical isometry is subnormal marks the starting point of the structure theory for this class of multi-operators. Since our approach to spherical isometries and their generalizations is based on the property of subnormality, we briefly recall some central facts about subnormal operator tuples.

By definition, a *subnormal tuple* $T \in B(H)^n$ possesses an extension to a tuple $U \in B(\widehat{H})^n$ consisting of commuting normal operators on some Hilbert space $\widehat{H}$ containing $H$. If the only reducing subspace for $U$ that contains $H$ is the space $\widehat{H}$ itself, then the tuple $U \in B(\widehat{H})^n$ is called *a minimal normal extension* of $T$. Given any normal extension $U$ of $T$, one can always obtain a minimal one by restricting $U$ to the space $\bigvee_{\alpha \in \mathbb{N}_0^n} (U^*)^\alpha H$. It is well known that any two minimal normal extensions of $T$ are unitarily equivalent. In particular, the *normal spectrum* of $T$, which is defined by $\sigma_n(T) = \sigma(U)$ for some minimal normal extension $U$ of $T$, does not depend on the choice of $U$. A result of Putinar [21] guarantees that $\sigma_n(T)$ is always contained in $\sigma(T)$.

Now, fix a subnormal tuple $T \in B(H)^n$ together with a minimal normal extension $U \in B(\widehat{H})^n$. Then one can choose a *separating vector* $z \in H$ for $U$, which means that the projection-valued spectral measure $E(\cdot)$ for $U$ and the scalar-valued measure $\mu = \langle E(\cdot)z, z\rangle$ are mutually absolutely continuous. The measure $\mu$ obtained in this way is a finite regular positive Borel measure supported by $\sigma_n(T) = \sigma(U)$, and will be called a *scalar spectral measure* for $U$. From the identity $\mu(\sigma_n(T)) = \|z\|^2$ it follows that $\mu$ is a probability measure if the underlying separating vector $z \in H$ is a unit vector. Since, up to mutual absolute continuity, the measure $\mu$ does not depend on the special choice of $U$, we may speak of $\mu$ as a *scalar spectral measure associated with* $T$. By the spectral theorem for normal tuples, there exists an isomorphism of von Neumann algebras

$$\Psi_U : L^\infty(\mu) \to W^*(U) \subset B(\widehat{H}),$$

mapping the coordinate functions to the corresponding components of $U$. Defining

$$\mathcal{R}_T = \{f \in L^\infty(\mu) : \Psi_U(f)H \subset H\}$$

one obtains a weak* closed subalgebra of $L^\infty(\mu)$ called the *restriction algebra*. The induced mapping

$$\gamma_T : \mathcal{R}_T \to B(H), \quad f \mapsto \Psi_U(f)|H$$

is known to be isometric again (see Conway [5]). Thus $\gamma_T$ defines a weak* continuous isometric algebra homomorphism mapping $z_i$ to $T_i$ for $i = 1, \ldots, n$. It should be mentioned that the restriction algebra $\mathcal{R}_T$ is independent of the choice of the minimal normal extension $U$ and the concrete spectral measure $\mu$.

From these general considerations about subnormal tuples we now return to the special case of a spherical isometry $T \in B(H)^n$. According to Athavale [2], $T$ is subnormal and the spectral inclusion $\sigma_n(T) \subset \partial\mathbb{B}_n$ holds. An obvious density argument for the polynomials implies that the restriction algebra always contains the ball algebra $A(\mathbb{B}_n) = \{f \in C(\overline{\mathbb{B}}_n) : f|\mathbb{B}_n \text{ is holomorphic}\}$. The rich function-theoretic structure of $A(\mathbb{B}_n)$ and suitable weak* closures then leads to interesting structure theorems for spherical isometries such as the reflexivity [9] of the dual operator algebra generated by $T$ or factorization properties of type $\mathbb{A}_1$ and $\mathbb{A}_{1,\aleph_0}$ (see [14]). Replacing $A(\mathbb{B}_n)$ by an arbitrary function algebra $A$ containing the polynomials one obtains the following very general notion of an isometric operator tuple introduced by the second author in [13].

**2.1 Definition.** *Let $K \subset \mathbb{C}^n$ be a compact set and let $A \subset C(K)$ be a closed subalgebra containing the restrictions of the polynomials $\mathbb{C}[z]$ in $n$ complex variables*

$z = (z_1, \dots, z_n)$. *A subnormal tuple $T \in B(H)^n$ is called an $A$-*isometry *if $A \subset \mathcal{R}_T$ and $\sigma_n(T)$ is contained in the Shilov boundary $\partial_A$ of $A$.*

By definition the Shilov boundary $\partial_A \subset K$ is the smallest closed set such that $\|f\|_{\infty,K} = \|f\|_{\infty,\partial_A}$ holds for every $f \in A$. Since the Shilov boundary of $A(\mathbb{B}_n)$ coincides with the topological boundary $\partial\mathbb{B}_n$, the remarks preceding the definition show that spherical isometries are precisely the $A(\mathbb{B}_n)$-isometries.

Other natural examples of $A$-isometries can be found in the context of generalized Hardy spaces. Fix a compact set $K \subset \mathbb{C}^n$, a closed subalgebra $A \subset C(K)$ containing the polynomials $\mathbb{C}[z_1, \dots, z_n]|K$ and a positive measure $\mu \in M^+(\partial_A)$. The multiplication tuple $M_z = (M_{z_1}, \dots, M_{z_n}) \in B(L^2(\mu))^n$ is normal with scalar spectral measure $\mu$ and Taylor spectrum $\sigma(M_z) = \text{supp}(\mu) \subset \partial_A$. The associated functional calculus is given by the map $\Psi_{M_z} : L^\infty(\mu) \to B(L^2(\mu))$, $f \mapsto M_f$. A Stone-Weiertrass argument shows that the restriction $T_z$ of $M_z$ to the invariant subspace

$$H^2_A(\mu) = \overline{A}^{\|\cdot\|_{2,\mu}} \subset L^2(\mu)$$

has $M_z$ as minimal normal extension. Since $H^2_A(\mu)$ is invariant under each multiplication operator $M_f$ with symbol $f \in A$, it follows that $\mathcal{R}_{T_z} \supset A$. Thus, the tuple

$$T_z = (M_{z_1}, \dots, M_{z_n}) \in B(H^2_A(\mu))^n$$

is an $A$-isometry. Note that the multiplication tuples with the coordinate functions on the classical Hardy spaces over strictly pseudoconvex or bounded symmetric domains in $\mathbb{C}^n$ all fit into this context. This justifies the following terminology.

**2.2 Definition.** *A multiplication tuple of the form $T_z \in B(H^2_A(\mu))^n$ described above will be called a Hardy-space $A$-isometry.*

Let us now return from these concrete examples to the study of a general $A$-isometry $T \in B(H)^n$. Fix a minimal normal extension $U \in B(\widehat{H})^n$ and a scalar spectral measure $\mu$ of $T$. Writing $M^+(C)$ for the set of all finite regular positive Borel measures on a compact set $C \subset \mathbb{C}^n$, we may consider $\mu$ as an element of $M^+(\partial_A)$ in the sequel.

Since the restriction algebra is weak* closed and contains $A$, it also contains the dual algebra

$$H^\infty_A(\mu) = \overline{A}^{w^*} \subset L^\infty(\mu).$$

If we denote the image of $H^\infty_A(\mu)$ under the canonical map $\gamma_T$ introduced above by

$$\mathscr{H}_T = \gamma_T(H^\infty_A(\mu)) \subset B(H),$$

which is a weak* closed subalgebra of $B(H)$, then we obtain a dual algebra isomorphism, that is, a weak* homeomorphism and isometric isomorphism

$$\gamma_T : H^\infty_A(\mu) \to \mathscr{H}_T, \quad f \mapsto \Psi_U(f)|H,$$

extending the polynomially functional calculus of $T$. This map will be referred to as the canonical functional calculus for $T$. Via $\gamma_T$ one can analyze the operator algebra $\mathscr{H}_T$ by studying the function algebra $H^\infty_A(\mu)$. A special role in this context is played by the family

$$I_\mu = \{\theta \in H^\infty_A(\mu) : |\theta| = 1 \quad \mu - \text{a.e. on } \partial_A\},$$

whose elements are called *$\mu$-inner functions*. As in the case of spherical isometries, there is a one-to-one correspondece between $I_\mu$ and the operator family

$$\mathcal{I}_T = \{J \in \mathscr{H}_T : J \text{ is isometric}\}.$$

More precisely, a word-by-word repetition of the proof of Lemma 1.1 in [10] yields the following result.

**2.3 Lemma.** *Let $T \in B(H)^n$ be an $A$-isometry with associated scalar spectral meassure $\mu \in M^+(\partial_A)$. Then $\mathcal{I}_T = \gamma_T(I_\mu)$, where $\gamma_T$ is the canonical functional calculus of $T$.* □

In his celebrated work [1], Aleksandrov gives a sufficient condition ensuring $H_A^\infty(\mu)$ to have a rich supply of inner functions. More explicitly, a triple $(A, K, \mu)$ consisting of a compact set $K \subset \mathbb{C}^n$, a closed subalgebra $A \subset C(K)$ and a measure $\mu \in M^+(K)$, is called *regular* in the sense of Aleksandrov if the following approximation problem is solvable: *For every $\varphi \in C(K)$ with $\varphi > 0$, there exists a sequence of functions $(\varphi_k)$ in $A$ with $|\varphi_k| < \varphi$ on $K$ and $\lim_{k\to\infty} \varphi_k = \varphi$ $\mu$-almost everywhere on $K$.* One of the main results in [1] says that, if the measure $\mu$ in a regular triple is *continuous* in the sense that one-point sets have $\mu$-measure zero, then the set of all $\mu$-inner functions is rich in the following sense (see Corollary 29 in [1]).

**2.4 Theorem. (Aleksandrov)** *Let $(A, K, \mu)$ be a regular triple with a continuous measure $\mu \in M^+(K)$. Then the weak* sequential closure of the set $I_\mu$ contains all $L^\infty(\mu)$-equivalence classes of functions $f \in A$ with $|f| \leq 1$ on $K$.* □

In [10] (Proposition 2.4 and Corollary 2.5) it was observed that the following weaker version of this density assertion is valid without any continuity assumption on the measure.

**2.5 Proposition.** *For every regular triple $(A, K, \mu)$, we have*

$$H_A^\infty(\mu) = \overline{LH}^{w^*}(I_\mu) \qquad \text{and} \qquad L^\infty(\mu) = \overline{LH}^{w^*}(\{\overline{\eta} \cdot \theta : \eta, \theta \in I_\mu\}).$$

□

Now we introduce a regularity criterion for $A$-isometries which guarantees that the above density results hold for the associated scalar spectral measures.

**2.6 Definition.** *An $A$-isometry $T \in B(H)^n$ is called regular if, for some or equivalently every scalar spectral measure $\mu \in M^+(\partial_A)$ associated with $T$, the triple $(A|\partial_A, \partial_A, \mu)$ is regular.*

In general, the regularity condition is hard to check. Nevertheless there are examples of function algebras $A$ for which every $A$-isometry is regular. For example, if $D \subset \mathbb{C}^n$ is a relatively compact stricly pseudoconvex open set and

$$A(D) = \{f \in C(\overline{D}) : f|D \text{ is holomorphic}\}$$

is the generalized ball-algebra, then $\partial_{A(D)} = \partial D$ and the triple $(A(D)|\partial D, \partial D, \mu)$ is regular for every measure $\mu \in M^+(\partial D)$ (see Aleksandrov [1] or, for a more detailed explanation, [8]).

**2.7 Proposition.** *Every $A(D)$-isometry on a relatively compact strictly pseudoconvex open set $D \subset \mathbb{C}^n$ (in partucluar, every spherical isometry) is regular.* $\square$

As another example, take $A = C(K)$. Then $\partial_A = K$ and $(C(K), K, \mu)$ is regular for every measure $\mu \in M^+(K)$. Now, a look at Definition 2.1 shows that the regular $C(K)$-isometries are precisely the normal tuples $T \in B(H)^n$ with Taylor spectrum contained in $K$.

The regularity of an $A$-isometry $T \in B(H)^n$ has immediate and far-reaching consequences for the structure of the dual operator algebras associated with $T$ and its minimal normal extension $U$. For later reference, we collect some of them in the following proposition. Recall from Lemma 2.3 that the family of all isometries in $\mathscr{H}_T$ is $\mathcal{I}_T = \gamma_T(I_\mu) \subset B(H)$. Considering the normal tuple $U \in B(\widehat{H})^n$ also as an $A$-isometry, the corresponding set of all isometries contained in $\mathscr{H}_U$ is $\mathcal{I}_U = \Psi_U(I_\mu) \subset B(\widehat{H})$. Having in mind that the point spectrum

$$\sigma_p(T) = \{\zeta \in \mathbb{C}^n : \cap_{i=1}^n \ker(\zeta_i - T_i) \neq \emptyset\}$$

coincides with the set $\Delta_\mu = \{\zeta \in \partial_A : \mu(\{\zeta\}) > 0\}$ of all one-point atoms of one (equivalently any) scalar spectral measure $\mu$ (cp. the remarks following Proposition 3.1 in [10] for the case of spherical isometries), the following approximation results are immediate consequences of Lemma 2.3, Theorem 2.4 and Proposition 2.5.

**2.8 Proposition.** *Let $T \in B(H)^n$ be a regular $A$-isometry with minimal normal extension $U \in B(\widehat{H})^n$. Then the following assertions hold:*

*(a) The families of isometries $\mathcal{I}_T$ and $\mathcal{I}_U$ defined above satisfy*

$$\mathscr{H}_T = \overline{LH}^{w^*}(\mathcal{I}_T) \quad \text{and} \quad W^*(U) = \overline{LH}^{w^*}(\{J_1^* J_2 : J_1, J_2 \in \mathcal{I}_U\}).$$

*(b) If $T$ has empty point spectrum, then the dual operator algebra $\mathscr{H}_T$ contains a weak* zero sequence of isometries $J_k = \gamma_T(\theta_k)$ with $\theta_k \in I_\mu$ for $k \geq 1$.* $\square$

It seems that a profound theory of Toeplitz operators for $A$-isometries can only be established under the assumption that the associated families of isometries $\mathcal{I}_T$ and $\mathcal{I}_U$ are sufficiently rich (in the sense of part (a) above). This is the reason why we mostly consider regular $A$-isometries from now on.

# 3 Toeplitz operators

Recall that Toeplitz operators associated with a spherical isometry $T \in B(H)^n$ have been introduced by Prunaru in [20] as the solutions $X \in B(H)$ of the operator equation $\sum_{i=1}^n T_i^* X T_i = X$. This relation is modelled after the classical Brown-Halmos condition characterizing Hardy-space Toeplitz operators on the unit disc. A recent result of the authors (Proposition 3.1 in [10]) shows that the following definition for general $A$-isometries is consistent with Prunaru's definition for spherical isometries.

**3.1 Definition.** *Let $T \in B(H)^n$ be an $A$-isometry. Then an operator $X \in B(H)$ is called a $T$-Toeplitz operator if*

$$J^* X J = X \quad \textit{holds for every isometry } J \in \mathscr{H}_T.$$

*We write $\mathcal{T}(T)$ for the set of all $T$-Toeplitz operators on $H$.*

To give an alternative characterization of $T$-Toeplitz operators, fix a minimal normal extension $U \in B(\widehat{H})^n$ and write $(U)'$ for the commutant of $U$ in $B(\widehat{H})$, $P_H \in B(\widehat{H})$ for the orthogonal projection onto $H$. Then every operator $X \in B(H)$ of the form

$$X = P_H A|H \qquad \text{with } A \in (U)'$$

belongs to $\mathcal{T}(T)$. Indeed, if $J = \gamma_T(\theta)$ is an isometry in $\mathscr{H}_T$ and $h, k$ are arbitrary elements of $H$, then the fact that $\theta \in H_A^\infty(\mu)$ is inner immediately implies that $\langle J^* X J h, k\rangle = \langle A\Psi_U(\theta)h, \Psi_U(\theta)k\rangle = \langle Ah, \Psi_U(|\theta|^2)k\rangle = \langle Ah, k\rangle = \langle Xh, k\rangle$. In particular, for every $f \in L^\infty(\mu)$, we obtain an element $T_f \in \mathcal{T}(T)$ by setting

$$T_f = P_H \Psi_U(f)|H \in B(H),$$

called the *$T$-Toeplitz operator with symbol $f$*. The corresponding *Hankel operator with symbol $f$* is defined to be

$$H_f = (1 - P_H)\Psi_U(f)|H \in B(H, H^\perp).$$

In case of a *regular* $A$-isometry the different types of Toeplitz operators considered above are related as follows.

**3.2 Proposition.** *Given a regular $A$-isometry $T \in B(H)^n$ with minimal normal extension $U \in B(\widehat{H})^n$, the following assertions hold:*

(a) *The $T$-Toeplitz operators possess the representation $\mathcal{T}(T) = P_H(U)'|H$.*

(b) *If $W^*(U)$ is a maximal abelian $W^*$-algebra, then $\mathcal{T}(T) = \{T_\varphi : \varphi \in L^\infty(\mu)\}$.*

**Proof.** Note that the $T$-Toeplitz operators in the sense of Definition 3.1 are just the operators $X \in B(H)$ that are T-Toeplitz with respect to the commuting family of isometries $(\gamma_T(\theta))_{\theta \in I_\mu}$ in the sense of Prunaru (Definition 1.1 in [20]) . The representation

$$W^*(U) = \overline{LH}^{w^*}(\{J_1^* J_2 : J_1, J_2 \in \mathcal{I}_U\})$$

obtained in Proposition 2.8 shows that the commutant of the family $(\Psi_U(\theta))_{\theta \in I_\mu}$ coincides with $(W^*(U))' = (U)'$. But then the minimality of $U$ as a normal extension of $T$ implies that $(\Psi_U(\theta))_{\theta \in I_\mu}$ is the minimal normal extension of the commuting family $(\gamma_T(\theta))_{\theta \in I_\mu}$ of isometries. Using Theorem 1.2 in Prunaru [20] for commuting families of isometries, we obtain that

$$\mathcal{T}(T) = P_H(\Psi_U(\theta))'_{\theta \in I_\mu}|H = P_H(U)'|H.$$

To prove part (b), observe that if $W^*(U)$ is a maximal abelian $W^*$-algebra, then $W^*(U) = (W^*(U))' = (U)'$ by Proposition 4.62 in Douglas [12]. Therefore $\mathcal{T}(T) = P_H(U)'|H = P_H\Psi_U(L^\infty(\mu))|H$, as desired. $\square$

**3.3 Corollary.** *For every regular Hardy-space $A$-isometry $T = T_z \in H_A^2(\mu)^n$ associated with a probability measure $\mu \in M_1^+(\partial_A)$, we obtain the identity*

$$\mathcal{T}(T) = \{X \in B(H_A^2(\mu)) : T_{\overline{\theta}} X T_\theta = X \text{ for every } \theta \in I_\mu\} = \{T_\varphi : \varphi \in L^\infty(\mu)\}.$$

**Proof.** Remember that the minimal normal extension of $T$ is $U = M_z \in B(L^2(\mu))^n$. Proposition 4.50 in Douglas [12] says that $W^*(U) = \{M_\varphi : \varphi \in L^\infty(\mu)\} \subset B(L^2(\mu))$

is a maximal abelian $W^*$-algebra. Hence the assertion follows from Lemma 2.3 and part (b) of the above proposition. $\square$

Let us add a simple lemma with two elementary properties of Toeplitz operators that will often be used without a comment throughout the subsequent section. For abbreviation, we say that a map $\Gamma : L^\infty(\mu) \to B(H)$ is *pointwise boundedly SOT-continuous* if, for every bounded sequence $(f_k)_{k\geq 1}$ in $L^\infty(\mu)$ converging pointwise $\mu$-almost everywhere to some $f \in L^\infty(\mu)$ (at the level of representatives), we have $\Gamma(f) = SOT - \lim_{k\to\infty} \Gamma(f_k)$.

**3.4 Lemma.** *Let $T \in B(H)^n$ be an $A$-isometry with minimal normal extension $U \in B(\widehat{H})^n$. Then the following assertions hold:*

*(a) For every $Y \in B(\widehat{H})$, the maps $\Gamma : L^\infty(\mu) \to B(H)$, $f \mapsto P_H(\Psi_U(f)Y)|H$ and $\Gamma^* : L^\infty(\mu) \to B(H)$, $\Gamma^*(f) = \Gamma(f)^*$ are pointwise boundedly SOT-continuous.*

*(b) Given $Y \in (U)'$, $f \in L^\infty(\mu)$ and $g, h \in H^\infty_A(\mu)$, we have*

$$P_H(\Psi_U(\overline{g}fh)Y)|H = T_{\overline{g}}\big(P_H(\Psi_U(f)Y)|H\big)T_h$$

*and in particular $T_{\overline{g}fh} = T_{\overline{g}}T_fT_h$.*

**Proof.** Fix an arbitrary vector $x \in H$ and set $y = Yx$. Then the desired continuity property for $\Gamma$ follows from the dominated convergence theorem and the estimate

$$\|\Gamma(f)x\|^2 \leq \|\Psi_U(f)y\|^2 = \int_{\partial_A} |f|^2 d\langle E(\cdot)y, y\rangle \qquad (f \in L^\infty(\mu)).$$

An analogous argument applies to $\|\Gamma(f)^*x\|^2 \leq \|Y^*\|^2\|\Psi_U(f)x\|^2$. This proves part (a). In order to verify part (b), note that, for $x, y \in H$, the scalar product $\langle P_H\Psi_U(\overline{g}fh)Yx, y\rangle$ can be rewritten as

$$\langle \Psi_U(g)^*\Psi_U(f)Y\Psi_U(h)x, y\rangle = \langle \Psi_U(f)YT_hx, T_gy\rangle = \langle T_{\overline{g}}P_H\Psi_U(f)YT_hx, y\rangle,$$

as desired. $\square$

Now we take a closer look at the identification $\mathcal{T}(T) = P_H(U)'|H$ from part (a) of Proposition 3.2. Having the details from the corresponding proof in mind, Theorem 1.2 in [20] actually yields the following detailed analysis of this identity:

**3.5 Proposition. (Prunaru)** *For a regular $A$-isometry $T \in B(H)^n$, the following assertions hold:*

*(a) The compression map $\rho : (U)' \to B(H)$, $Y \mapsto P_HY|H$ is a complete isometry with range $\mathrm{ran}(\rho) = \mathcal{T}(T)$.*

*(b) There is a surjective unital $*$-representation $\pi : C^*(\mathcal{T}(T)) \to (U)' \subset B(\widehat{H})$ satisfying the identity $\pi(\rho(Y)) = Y$ for every $Y \in (U)'$.*

*(c) There exists a completely positive and unital projection $\Phi : B(H) \to B(H)$ onto $\mathrm{ran}(\Phi) = \mathcal{T}(T)$ such that $\Phi(X) = P_H\pi(X)|H$ holds for every $X \in C^*(\mathcal{T}(T))$.*

*(d) The kernels $\ker(\Phi|C^*(\mathcal{T}(T)))$ and $\ker(\pi)$ are equal and coincide with the two-sided closed ideal in $C^*(\mathcal{T}(T))$ generated by all operators of the form $XY - \Phi(XY)$ with $X, Y \in \mathcal{T}(T)$.* $\square$

As an immediate consequence we obtain the existence of a generalized Toeplitz sequence which, in some sense, justifies the definition of Toeplitz operators via the condition $J^*XJ = X$ for every $J \in \mathcal{I}_T$.

**3.6 Corollary.** *For every regular A-isometry $T \in B(H)^n$, there is a short exact sequence*

$$0 \longrightarrow \mathcal{SC}(T) \xrightarrow{\subset} C^*(\mathcal{T}(T)) \xrightarrow{\pi} (U)' \longrightarrow 0,$$

*where $\mathcal{SC}(T)$ stands for the two-sided closed ideal in $C^*(\mathcal{T}(T))$ generated by all operators of the form $XY - \Phi(XY)$ with $X, Y \in \mathcal{T}(T)$.* □

Restricting the map $\pi$ from the full Toeplitz $C^*$-algebra $C^*(\mathcal{T}(T))$ to the $C^*$-algebra

$$\mathcal{T}_C(T) = C^*(\{T_f : f \in C(\partial_A)\}) \subset B(H)$$

generated by all Toeplitz operators with continuous symbols, we obtain the next result. Let $\mathcal{SC}_C(T) \subset \mathcal{T}_C(T)$ be the closed two-sided ideal generated by all semi-commutators $T_fT_g - T_{fg}$ with $f, g \in C(\partial_A)$.

**3.7 Corollary.** *For every regular A-isometry $T \in B(H)^n$, there is a short exact sequence*

$$0 \longrightarrow \mathcal{SC}_C(T) \xrightarrow{\subset} \mathcal{T}_C(T) \xrightarrow{\sigma} C(\sigma_n(T)) \longrightarrow 0$$

*with a $*$-homomorphism $\sigma$ satisfying $\sigma(T_f) = f|\sigma_n(T)$ for every $f \in C(\partial_A)$.*

**Proof.** With the notations from Proposition 3.5, we have $\rho(\Psi_U(f)) = P_H\Psi_U(f)|H = T_f$ for $f \in C(\partial A)$. Hence by part (b) of Proposition 3.5, the restriction of the map $\pi : C^*(\mathcal{T}(T)) \to (U)'$ to $\mathcal{T}_C(T)$ yields a surjective $C^*$-algebra homomorphism $\tilde{\pi} : \mathcal{T}_C(T) \to C^*(U)$ with $\tilde{\pi}(T_f) = \Psi_U(f)$ for all $f \in C(\partial_A)$. We want to determine the kernel of $\tilde{\pi}$, which is a closed two-sided ideal in $\mathcal{T}_C(T)$. First observe that part (c) of Propositon 3.5 yields the identity

$$\begin{aligned} \Phi(T_{f_1} \cdots T_{f_k}) &= P_H\pi(T_{f_1} \cdots T_{f_k})|H \\ &= P_H\tilde{\pi}(T_{f_1}) \cdots \tilde{\pi}(T_{f_k})|H \\ &= P_H\Psi_U(f_1 \cdots f_k)|H \\ &= T_{f_1 \cdots f_k}, \end{aligned}$$

valid for all $k \geq 1$ and all $f_1, \ldots, f_k \in C(\partial_A)$. The case $k = 2$, together with part (d) of Proposition 3.5, implies that the closed two-sided ideal $\mathcal{SC}_C(T) \subset \mathcal{T}_C(T)$ generated by the semi-commutators $T_fT_g - T_{fg} = T_fT_g - \Phi(T_fT_g)$ with $f, g \in C(\partial_A)$ satisfies the inclusion

$$\mathcal{SC}_C(T) \subset \ker(\tilde{\pi}).$$

To finish the proof, we have to settle the reverse inclusion. Towards this end, first observe that the map $\Phi$ leaves $\mathcal{T}_C(T)$ invariant, as can be shown using the identity $\Phi(T_{f_1} \cdots T_{f_k}) = T_{f_1 \cdots f_k}$ and the fact that $\mathcal{T}_C(T)$ is the closed linear hull of all operators of the form $T_{f_1} \cdots T_{f_k}$ with $k \in \mathbb{N}$ and $f_1, \ldots, f_k \in C(\partial_A)$. Hence $\tilde{\Phi} = \Phi|\mathcal{T}_C(T)$ is a continuous linear map with $\tilde{\Phi}^2 = \tilde{\Phi}$. By part (d) of Proposition 3.5,

$$\operatorname{ran}(1 - \tilde{\Phi}) = \ker(\tilde{\Phi}) = \ker(\Phi) \cap \mathcal{T}_C(T) = \ker(\pi) \cap \mathcal{T}_C(T) = \ker(\tilde{\pi}).$$

So it remains to check that

$$(1 - \Phi)(T_{f_1} \cdots T_{f_k}) = T_{f_1} \cdots T_{f_k} - T_{f_1 \cdots f_k} \in \mathcal{SC}_C(T).$$

for all $k \in \mathbb{N}$ and $f_1, \ldots, f_k \in C(\partial_A)$. But this follows easily from the decomposition

$$\begin{aligned} T_{f_1} \cdots T_{f_k} - T_{f_1 \cdots f_k} &= T_{f_1} \cdots T_{f_k} - T_{f_1 \cdots f_{k-1}} T_{f_k} + T_{f_1 \cdots f_{k-1}} T_{f_k} - T_{f_1 \cdots f_k} \\ &\in (T_{f_1} \cdots T_{f_{k-1}} - T_{f_1 \cdots f_{k-1}}) T_{f_k} + \mathcal{SC}_C(T) \end{aligned}$$

using an elementary induction. To complete the proof, we define the symbol map $\sigma$ as the composition $\sigma = \Gamma \circ \tilde{\pi}$ of $\tilde{\pi}$ and the Gelfand map $\Gamma : C^*(U) \to C(\sigma_n(T))$. $\square$

In the classical theory of Hardy space Toeplitz tuples on the unit ball or, more general, on strictly pseudoconvex domains in $\mathbb{C}^n$ (see [23]), the first space in the above short exact sequence coincides with the commutator ideal of $\mathcal{T}_C(T)$. While this fails to be true for arbitrary $A$-isometries, it holds under some natural additional assumptions on $T$ including essential normality. Recall that a commuting tuple $T \in B(H)^n$ is said to be *essentially normal* if its self-commutators are compact, that is, if

$$[T_i, T_i^*] = T_i T_i^* - T_i^* T_i \in \mathcal{K}(H) \qquad (i = 1, \ldots, n).$$

In other words, the images $\pi(T_i)$ of the components of $T$ under the canonical map

$$\pi : B(H) \to \mathcal{C}(H) = B(H)/\mathcal{K}(H), \qquad X \mapsto X + \mathcal{K}(H)$$

into the Calkin-algebra form a commuting tuple $\pi(T) = (\pi(T_1), \ldots, \pi(T_n))$ of normal elements in $\mathcal{C}(H)$. Some useful characterizations of essentially normal $A$-isometries are collected in the following lemma.

**3.8 Lemma.** *For an $A$-isometry $T \in B(H)^n$, the following assertions are equivalent:*

(a) *The tuple $T$ is essentially normal.*

(b) *All Hankel operators $H_f$ with continuous symbol $f \in C(\partial_A)$ are compact.*

(c) *For every $f \in C(\partial_A)$ and every $Y \in B(\widehat{H})$, the semi-commutators*

$$\big(P_H Y|H\big)T_f - P_H(Y\Psi_U(f))|H \quad \text{and} \quad T_f\big(P_H Y|H\big) - P_H(\Psi_U(f)Y)|H$$

*are compact.*

(d) *The semi-commutators $T_f T_g - T_{fg}$ are compact whenever $f \in C(\partial_A)$ and $g \in L^\infty(\mu)$ (or, equivalently, whenever $f, g \in C(\partial_A)$).*

**Proof.** It is well known that a subnormal tuple $T \in B(H)^n$ with minimal normal extension $U \in B(\widehat{H})^n$ is essentially normal if and only if

$$[U_i, P_H] \in \mathcal{K}(\widehat{H}) \qquad (i = 1, \ldots, n),$$

or equivalently, if $\pi(P_H) \in \mathcal{C}(\widehat{H})$ belongs to the commutant of the $C^*$-algebra generated by the commuting normal elements $\pi(U_i)$ $(i = 1, \ldots, n)$. In the setting of the lemma, this immediately implies the compactness of all commutators $[\Psi_U(f), P_H]$ with $f \in C(\partial_A)$, and thus of all Hankel operators

$$H_f = (1 - P_H)\Psi_U(f)P_H|H = (1 - P_H)[\Psi_U(f), P_H]|H \qquad (f \in C(\partial_A)).$$

This settles the implication $(a) \Rightarrow (b)$. Now, fix arbitrary elements $Y \in B(\widehat{H})$ and $f \in C(\partial_A)$. A look at the algebraic identities

$$\begin{aligned} \big(P_H Y|H\big)T_f - P_H(Y\Psi_U(f))|H &= P_H Y\big(P_H \Psi_U(f) - \Psi_U(f)\big)|H \\ &= P_H Y(P_H - 1)\Psi_U(f)|H = -P_H Y H_f \end{aligned}$$

and $T_f\big(P_H Y|H\big) - P_H(\Psi_U(f)Y)|H = \Big(\big(P_H Y^*|H\big)T_{\overline{f}} - P_H(Y^*\Psi_U(\overline{f}))|H\big)\Big)^*$

shows that $(b)$ implies $(c)$. Setting $Y = \Psi_U(g)$ with $g \in L^\infty(\mu)$ in the last part, we obtain $(d)$ as special case. Using the decompostion

$$[T_i, T_i^*] = T_{z_i}T_{\overline{z}_i} - T_{\overline{z}_i}T_{z_i} = (T_{z_i}T_{\overline{z}_i} - T_{z_i\overline{z}_i}) + (T_{\overline{z}_i z_i} - T_{\overline{z}_i}T_{z_i}) \qquad (i = 1, \dots, n)$$

we get back to condition $(a)$, as desired. □

Part (d) of the preceding lemma can be used to calculate the *commutator ideal* of the Toeplitz algebra $\mathcal{T}_C(T)$, that is, the closed two-sided ideal of $\mathcal{T}_C(T)$ generated by all commutators $[A, B] = AB - BA$ of operators $A, B \in \mathcal{T}_C(T)$. Recall that a subset $\mathcal{S} \subset B(H)$ is called irreducible, if there is no non-zero proper closed subspace $M \subset H$ which is reducing for $H$. It is well known that the classical Toeplitz tuples $T_z$ on the Hardy space $H^2(\sigma)$ with respect to the surface measure of the unit sphere or the boundary of a strictly pseudoconvex domain in $\mathbb{C}^n$ are essentially normal and generate an irreducible Toeplitz algebra $\mathcal{T}_C(T_z)$ (see Upmeier [23]).

**3.9 Proposition.** *Let $T \in B(H)^n$ be an essentially normal, non-normal regular $A$-isometry. If the Toeplitz $C^*$-algebra $\mathcal{T}_C(T)$ is irreducible, then the commutator ideal of $\mathcal{T}_C(T)$ is $\mathcal{K}(H)$, and there is a short exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{K}(H) \stackrel{\subset}{\longrightarrow} \mathcal{T}_C(T) \stackrel{\sigma}{\longrightarrow} C(\sigma_n(T)) \longrightarrow 0,$$

*where the symbol homomorphism $\sigma$ satisfies $\sigma(T_f) = f|\sigma_n(T)$ for every $f \in C(\partial_A)$.*

**Proof.** Let $\mathcal{C} \subset \mathcal{T}_C(T)$ denote the commutator ideal. In view of the simple identity $[T_f, T_g] = T_f T_g - T_{gf} - (T_g T_f - T_{gf})$ with $f, g \in C(\partial_A)$, the assumption on $T$ to be not normal, and part (d) of the previous lemma, we conclude that

$$0 \neq \mathcal{C} \subset \mathcal{SC}_C(T) \subset \mathcal{K}(H).$$

In particular, it follows that $\mathcal{T}_C(T) \cap \mathcal{K}(H) \neq 0$. Hence $\mathcal{T}_C(T) \supset \mathcal{K}(H)$ by the assumed irreducibility (see, e.g., Theorem 5.39 in [12]). So both $\mathcal{C}$ and $\mathcal{SC}_C(T)$ are non-zero closed ideals of $\mathcal{K}(H)$. Since $\mathcal{K}(H)$ is known to contain no proper closed ideals, we conclude that $\mathcal{C} = \mathcal{SC}_C(T) = \mathcal{K}(H)$. Hence the asserted short exact sequence is just the one established in Corollary 3.7. □

# 4 The essential commutant of $\mathscr{H}_T$

The *essential commutant* of an arbitrary subset $\mathcal{F} \subset B(H)$ is defined as

$$\mathrm{EssCom}(\mathcal{F}) = \{C \in B(H) : CY - YC \in \mathcal{K}(H) \quad \text{for all } Y \in \mathcal{F}\}.$$

In other words, an operator $C$ belongs to $\mathrm{EssCom}(\mathcal{F})$ if and only if its image $\pi(C)$ in the Calkin algebra belongs to the commutant $(\pi(\mathcal{F}))'$. Obviously, $\mathrm{EssCom}(\mathcal{F})$ is always a norm-closed subalgebra of $B(H)$. This section is devoted to a detailed study of the essential commutant of the dual algebra $\mathscr{H}_T$ associated with a regular essentially normal $A$-isometry. The following two simple observations show how the assumption on $T$ to be essentially normal influences the structure of $\mathrm{EssCom}(T)$.

**4.1 Lemma.** *If $T \in B(H)^n$ is an essentially normal regular $A$-isometry, then we have $\mathrm{EssCom}(T) = \mathrm{EssCom}(\mathcal{T}_C(T))$, and this is a $C^*$-algebra.*

**Proof.** To prove the non-trivial inclusion, fix an element $R \in \text{EssCom}(T)$. Since $\pi(R)$ commutes with the commuting normal elements $\pi(T_i)$ $(i = 1, \dots, n)$, it commutes with $C^*(\pi(T))$. By Lemma 3.8 the map $C(\partial_A) \to \mathcal{C}(H)$, $f \mapsto \pi(T_f)$ is a $C^*$-algebra homomorphism. The theorem of Stone-Weierstrass implies that $\pi(T_f) \in C^*(\pi(T))$ for all $f \in C(\partial_A)$ and hence that $\pi(\mathcal{T}_C(T)) \subset C^*(\pi(T))$. Therefore $R \in \text{EssCom}(\mathcal{T}_C(T))$. Since $\mathcal{T}_C(T)) \subset B(H)$ is a self-adjoint subset, its essential commutant is a $C^*$-algebra. $\square$

For an arbitrary element $f \in L^\infty(\mu)$, we define the support $\text{supp}(f)$ to be the support of the measure $\mu_f$ induced by $f$ via the formula $\mu_f(\omega) = \int_\omega |f| d\mu$ for every Borel subset $\omega \subset \partial_A$. By definition, $\text{supp}(f) \subset \partial_A$ is closed and $\text{supp}(f)^c$ is the largest open set $G \subset \partial_A$ with the property that $f = 0$ $\mu$-almost everywhere on $G$. Morover, if $g \in C(\partial_A)$ is a function with $g = 1$ on $\text{supp}(f)$, then $(1 - g) \cdot f = 0$ and $gf = f$ $\mu$-almost everywhere on $\partial_A$.

**4.2 Lemma.** *Suppose that $T \in B(H)^n$ is an essentially normal $A$-isometry and that $R \in \text{EssCom}(T)$. Then, for every choice of operators $Y_1, Y_2 \in (U)'$ and every pair of elements $f_1, f_2 \in L^\infty(\mu)$ with disjoint supports, we have*

$$\Big(P_H(\Psi_U(f_1)Y_1)|H\Big) \; R \; \Big(P_H(\Psi_U(f_2)Y_2)|H\Big) \; \in \; \mathcal{K}(H).$$

**Proof.** Let us abbreviate the factors on both sides of $R$ by $X_1 = P_H(\Psi_U(f_1)Y_1|H$ and $X_2 = P_H(\Psi_U(f_2)Y_2)|H$. By Urysohn's lemma, we can choose a continuous function $h : \partial_A \to [0, 1]$ with $h = 1$ on $\text{supp}(f_1)$ and $h = 0$ on $\text{supp}(f_2)$. With this choice of $h$, an application of Lemma 3.8 (c) guarantees that

$$\pi(X_1) = \pi(X_1 T_h) \quad \text{and} \quad \pi(T_h X_2) = 0.$$

Since $R \in \text{EssCom}(T) = \text{EssCom}(\mathcal{T}_C(T))$ (see Lemma 4.1), we obtain that

$$\pi(X_1 R X_2) = \pi(X_1 T_h R X_2) = \pi(X_1 R T_h X_2) = 0,$$

as desired. $\square$

As most ideas occurring in this section, the previous lemma goes back in its original form to Davidson [6]. Our study of $\text{EssCom}(\mathscr{H}_T)$ has been inspired by corresponding results of Le [18] and Ding-Sun [11] who developed Davidson's technique further in the several-variable case.

For the remainder of this section, we fix a regular $A$-isometry $T \in B(H)^n$ with $\sigma_p(T) = \emptyset$ and denote its minimal normal extension as before by $U \in B(\widehat{H})^n$.

**4.3 Lemma.** *For every element $S \in \text{EssCom}(\mathscr{H}_T)$, there are a weak$^*$ zero sequence of isometries $(J_k)_{k \geq 1}$ in $\mathscr{H}_T$ and an operator $Y_S \in (U)'$ such that the limit*

$$X_S = w^* - \lim_{k \to \infty} J_k^* S J_k$$

*exists and satisfies $X_S = P_H Y_S | H$.*

**Proof.** Let $S \in \text{EssCom}(\mathscr{H}_T)$ be given. According to Proposition 2.8 there is a weak$^*$ zero sequence $(J_k)_{k \geq 1}$ of isometries in $\mathscr{H}_T$. By passing to a subsequence we can achieve that the limit $X_S = w^* - \lim_{k \to \infty} J_k^* S J_k \in B(H)$ exists. For every isometry $V \in \mathscr{H}_T$, we obtain that

$$V^* X_S V = w^* - \lim_{k \to \infty} J_k^* V^* S V J_k = w^* - \lim_{k \to \infty} J_k^* S J_k = X_S.$$

Here we have used that $[S,V] \in \mathcal{K}(H)$ and that $w^* - \lim_{k\to\infty} J_k^* K J_k = 0$ for every compact operator $K$ on $H$. Thus $X_S$ is a $T$-Toeplitz operator. By Proposition 3.2 there is an operator $Y_S \in (U)'$ with $X_S = P_H Y_S|H$. □

Before we continue, we need an elementary topological lemma ensuring the existence of suitable open covers of compact sets $Q \subset \mathbb{C}^n$. Since the real dimension is involved, we formulate it for compact sets in $\mathbb{R}^m$. Given a subset $F \subset \mathbb{R}^m$, we denote its diameter with respect to the Euclidean norm by $|F| = \sup_{x,y\in F} |x-y|$.

**4.4 Lemma.** *Let $Q \subset \mathbb{R}^m$ be compact and let $\varepsilon > 0$ be given. Then there exists a finite open cover $Q = \bigcup_{j\in J} U_j$ consisting of relatively open sets $U_j \subset Q$ with $|U_j| < \varepsilon$ and such that the index set $J$ admits a decomposition $J = J_1 \cup \cdots \cup J_{2^m}$ with the property that each of the families $(U_i)_{i\in J_l}$ $(l = 1, \ldots, 2^m)$ consists of pairwise disjoint sets.*

**Proof.** For the convenience of the reader, we indicate the elementary ideas. Clearly it suffices to prove the assertion for every compact rectangle $Q \subset \mathbb{R}^m$. For $m = 1$, the result obviously holds. Suppose that the assertion is true for some $m \geq 1$, and let $Q = Q^1 \times Q^2$ be a compact rectangle with $Q^1 \subset \mathbb{R}, Q^2 \subset \mathbb{R}^m$. Choose open covers $(U_j^1)_{j\in J^1}$ for $Q^1$ and $(U_k^2)_{k\in J^2}$ for $Q^2$ as in the assertion. Let

$$J^1 = J_1^1 \cup J_2^1 \quad \text{and} \quad J^2 = J_1^2 \cup \cdots \cup J_{2^m}^2$$

be the corresponding decompositions of the index sets. Define open sets

$$U_{(j,k)} = U_j^1 \times U_k^2 \subset Q \quad (j \in J^1, k \in J^2)$$

and index sets

$$J = J^1 \times J^2 \quad \text{and} \quad J_{(a,b)} = J_a^1 \times J_b^2 \quad (a \in \{1,2\},\, b \in \{1,\ldots,2^m\}).$$

Then $(U_{(j,k)})_{(j,k)\in J}$ is a cover of $Q$ by open sets of diameter $|U_{(j,k)}| \leq |U_j| + |U_k| < 2\varepsilon$, $J$ is the disjoint union of all $J_{(a,b)}$ and the families $(U_{(j,k)})_{(j,k)\in J_{(a,b)}}$ consist of pairwise disjoint sets. □

Let $Y \in (U)'$ and $S \in \mathrm{EssCom}(T)$ be given operators. By Lemma 3.4 the map

$$F : L^\infty(\mu) \to B(H) \quad \text{by} \quad F(f) = T_f S - P_H(\Psi_U(f)Y)|H.$$

is pointwise boundedly SOT-continuous. A straightforward application of Lemma 4.2 (and Lemma 4.1) yields that, for any pair of functions $f, g \in L^\infty(\mu)$ with disjoint supports, each of the products

$$F(f)F(g), \quad F(f)^*F(g), \quad F(f)F(g)^* \in B(H)$$

is compact.

Our main result will follow by applying the following general observation to functions of the above type.

**4.5 Proposition.** *Let $F : L^\infty(\mu) \to B(H)$ be a linear map such that*

*(P1) $F$ is pointwise boundedly SOT-continuous;*

*(P2) $F(\chi)$ is not compact for a characteristic function $\chi$ of some Borel set in $\partial_A$;*

*(P3) if $f, g \in L^\infty(\mu)$ have disjoint supports, then each of the products $F(f)F(g)$, $F(f)F(g)^*$, $F(f)^*F(g)$ is compact.*

*Then there are a positive real number $\rho > 0$ and a sequence $(f_k)_{k\geq 1}$ of continuous functions $f_k : \partial_A \to [0,1]$ with disjoint supports satisfying $\|F(f_k)\| > \rho$ for all $k \geq 1$.*

**Proof.** Let $\alpha = \|\pi(F(\chi))\|/2 > 0$ and define

$$\mathcal{E} = \{f \in C(\partial_A) : 0 \leq f \leq 1 \text{ and } \|F(f\chi)\| > \alpha/(2N)\}$$

with $N = 2^{2n}$. We obtain a decreasing sequence $(E_k)_{k\geq 1}$ of closed subsets of $\partial_A$ by defining each $E_k$ as the closure of the set

$$\bigcup \left(\mathrm{supp}(f) : f \in \mathcal{E} \text{ with } |\mathrm{supp}(f)| \leq \frac{1}{k}\right).$$

We first prove that the intersection $E = \bigcap_{k\geq 1} E_k$ is non-empty. Let us assume the converse. Then $\|F(f\chi)\| \leq \alpha/(2N)$ for every $f \in C(\partial_A)$ with $0 \leq f \leq 1$ and $|\mathrm{supp}(f)| \leq 1/k$.

According to Lemma 4.4 we can choose an open cover $\partial_A = U_1 \cup \ldots \cup U_r$ such that $|U_j| \leq 1/k$ $(j = 1, \ldots, r)$ and such that the set $\{1, \ldots, r\}$ is the disjoint union of sets $J_1, \ldots, J_N$ with the property that each of the families $(U_i)_{i\in J_l}$ $(l = 1, \ldots, N)$ consists of pairwise disjoint sets. Let $(h_j)_{j=1,\ldots,r}$ be a continuous partition of unity relative to the open cover $(U_j)_{j=1,\ldots,r}$. In view of the decomposition

$$\pi(F(\chi)) = \frac{\pi(F(\chi) + F(\chi)^*)}{2} + i\frac{\pi(F(\chi) - F(\chi)^*)}{2i},$$

we can choose an $\varepsilon \in \{-1, +1\}$ such that $\|\pi(F(\chi) + \varepsilon F(\chi)^*)\| > \alpha$. Then

$$A_j = \pi(F(h_j\chi) + \varepsilon F(h_j\chi)^*) \qquad (j = 1, \ldots, r)$$

defines a family $(A_j)_{j=1,\ldots,r}$ of normal elements in the Calkin algebra such that $A_\mu A_\nu = 0$ whenever $\mu, \nu$ are different indices in one of the sets $J_l$ $(l = 1, \ldots, N)$.

A simple spectral radius argument then yields the estimates

$$\|\sum_{j\in J_l} A_j\| \leq \max_{j\in J_l} \|A_j\| \leq 2 \cdot \max_{j\in J_l} \|F(h_j\chi)\| \leq \alpha/N \qquad (l = 1, \ldots, N)$$

which leads to the contradiction

$$\alpha < \|\sum_{j=1}^{r} A_j\| \leq \sum_{l=1}^{N} \|\sum_{j\in J_l} A_j\| \leq \alpha.$$

Thus we have shown that $E = \bigcap_{k\geq 1} E_k \neq \emptyset$ .

Define $\rho = \alpha/(2N)$. In the second step we prove the existence of a sequence $(g_k)_{k\geq 1}$ in $C(\partial_A)$ with with $0 \leq g_k \leq 1$ and pairwise disjoint supports such that $\|F(g_k\chi)\| > \rho$ for all $k \geq 1$.

To this end, let us fix a point $z_0 \in E$. Suppose that $g_1, \ldots g_k \in \mathcal{E}$ are functions with pairwise disjoint supports such that

$$d = \mathrm{dist}\Big(z_0, \bigcup_{j=1}^{k} \mathrm{supp}(g_j)\Big) > 0.$$

Since $z_0 \in E$, there is a function $f \in \mathcal{E}$ with $|\text{supp}(f)| < d/3$ and $\text{dist}(z_0, \text{supp}(f)) < d/3$. If $z_0 \notin \text{supp}(f)$, we define $g_{k+1} = f$. Otherwise we choose a sequence of functions $(\kappa_j)_{j\geq 1}$ in $C(\partial_A)$ with $0 \leq \kappa_j \leq 1$, $z_0 \notin \text{supp}(\kappa_j)$ for all $j \geq 1$ and

$$\kappa_j(z) \xrightarrow{j} 1 \quad (z \in \partial_A \setminus \{z_0\}).$$

By hypothesis $\sigma_p(T) = \emptyset$ and hence $\mu$ has no one-point atoms. Therefore $(\kappa_j f \chi)_j$ is a bounded sequence in $L^\infty(\mu)$ which converges pointwise $\mu$-almost everywhere to the function $f\chi$. Using condition (P1) we find that

$$F(f\chi) = \text{SOT}-\lim_{j\to\infty} F(\kappa_j f \chi)$$

Since $f \in \mathcal{E}$, we can choose a natural number $j \geq 1$ with $\|F(\kappa_j f \chi)\| > \alpha/(2N)$. In this case we define $g_{k+1} = \kappa_j f$. In both cases we obtain a family $(g_j)_{j=1,\ldots,k+1}$ of functions in $\mathcal{E}$ with pairwise disjoint supports not containing $z_0$.

Inductively, one finds a sequence $(g_k)_{k\geq 1}$ in $\mathcal{E}$ with pairwise disjoint supports and $\|F(g_k\chi)\| > \rho$ for all $k \geq 1$.

A standard application of Lusin's theorem (Theorem 7.4.3 and Proposition 3.1.2 in [4]) shows that there is a sequence of continuous functions $h_j : \partial_A \to [0,1]$ such that $(h_j) \xrightarrow{j} \chi$ $\mu$-almost everywhere. Again using hypothesis (P1) we find that

$$F(g_k\chi) = \text{SOT}-\lim_{j\to\infty} F(g_k h_j)$$

for every $k \geq 1$. Hence, for every $k \geq 1$, there is a natural number $j_k$ such that $\|F(g_k h_{j_k})\| > \rho$. The observation that the resulting functions $f_k = g_k h_{j_k}$ have all required properties completes the proof. □

Now we are able to prove the main theorem of this section. Recall that, by Proposition 2.7, every spherical isometry is a regular $A(\mathbb{B}_n)$-isometry and therefore fits into this context.

**4.6 Theorem.** *Let $T \in B(H)^n$ be an essentially normal regular $A$-isometry with $\sigma_p(T) = \emptyset$, and let $S \in B(H)$ be an operator that essentially commutes with $\mathscr{H}_T$. Then there are a $T$-Toeplitz operator $X \in B(H)$ and a compact operator $K \in \mathcal{K}(H)$ with*

$$S = X + K.$$

**Proof.** According to Lemma 4.3 there is a sequence $(J_k)_{k\geq 1}$ of isometries in $\mathscr{H}_T$ in such that the limit

$$X_S = w^* - \lim_{k\to\infty} J_k^* S J_k$$

defines a $T$-Toeplitz operator. By Proposition 3.2 there is an operator $Y_S \in (U)'$ with $X_S = P_H Y_S|H$. By Lemma 2.3 we can choose a sequence $(\theta_i)_{i\geq 1}$ of bounded measurable functions $\theta_i : \partial_A \to \mathbb{C}$ with $|\theta_i| = 1$ on $\partial_A$ such that $\theta_i$, or better its equivalence class in $L^\infty(\mu)$, belongs to $H_A^\infty(\mu)$ and satisfies $J_i = \gamma_T(\theta_i)$ for every $i \geq 1$. As seen before, the continuous map $F : L^\infty(\mu) \to B(H)$ defined by

$$F(f) = T_f S - P_H(\Psi_U(f)Y_S)|H$$

satisfies the hypotheses (P1) and (P3) of Proposition 4.5. To complete the proof it suffices to show that $F(1)$ is a compact operator. We even show that

$$F(L^\infty(\mu)) \subset \mathcal{K}(H).$$

Let us assume that the inclusion does not hold. Since every bounded measurable function can be approximated uniformly by finite linear combinations of characteristic functions of Borel sets, there is a characteristic function $\chi$ of some Borel set in $\partial_A$ such that $F(\chi)$ is not compact. As an application of Proposition 4.5 we find that there are a real number $\rho > 0$ and a sequence $(f_k)_{k\geq 1}$ of continuous functions $f_k : \partial_A \to [0,1]$ with pairwise disjoint supports $A_k = \mathrm{supp}(f_k)$ and $\|F(f_k)\| > \rho$ for all $k \geq 1$.

Let us fix an index $k \geq 1$. Choose a real number $t$ with $0 < t < 4^{-k}$ and $t \cdot \|F(1)\| < \rho/2$. Then the function $\varphi = f_k + t \in C(\partial_A)$ is strictly positive on $\partial_A$ and satisfies the estimates

$$\|\varphi\|_{\infty,\partial_A} \leq 2, \qquad \|\varphi\|_{\infty,\partial_A\setminus A_k} < 4^{-k}, \qquad \|F(\varphi)\| > \rho/2.$$

Since $(A|\partial_A, \partial_A, \mu)$ is regular, there is a sequence $(\varphi_j)_{j\geq 1}$ in $A$ with $|\varphi_j| < \sqrt{\varphi}$ on $\partial_A$ and $|\varphi_j| \xrightarrow{j} \sqrt{\varphi}$ $\mu$-almost everywhere on $\partial_A$. Property (P1) implies that

$$F(\varphi) = \mathrm{SOT} - \lim_{j\to\infty} F(|\varphi_j|^2).$$

Choose a natural number $j$ with $\|F(|\varphi_j|^2)\| > \rho/2$ and set $g_k = \varphi_j$. Then $g_k \in A$ satisfies the estimates $\|g_k\|_{\infty,\partial_A} \leq 2$ and $\|g_k\|_{\infty,\partial_A\setminus A_k} < 2^{-k}$. The identity

$$\begin{aligned} F(|g_k|^2) &= T_{\overline{g}_k g_k} S - P_H(\Psi_U(\overline{g}_k g_k)Y_S)|H \\ &= T_{\overline{g}_k}\big(T_{g_k}S - P_H(\Psi_U(g_k)Y_S)|H\big) \\ &= T_{\overline{g}_k} F(g_k) \end{aligned}$$

implies that $\|F(g_k)\| > \rho/4$. The observation that

$$\begin{aligned} w^* - \lim_{i\to\infty} J_i^*\big(T_{g_k}J_iS - ST_{g_k}J_i\big) &= w^* - \lim_{i\to\infty}\big(T_{g_k}S - J_i^*SJ_iT_{g_k}\big) \\ &= T_{g_k}S - (P_HY_S|H)T_{g_k} \\ &= T_{g_k}S - P_H(\Psi_U(g_k)Y_S)|H \\ &= F(g_k) \end{aligned}$$

allows us to choose a natural number $i$ such that

$$\|T_{g_k}J_iS - ST_{g_k}J_i\| > \rho/4.$$

The functions $h_k = g_k\theta_i$, where for every given $k \geq 1$ the index $i$ is chosen as explained above, satisfy the estimates

$$\|h_k\|_{\infty,\partial_A} \leq 2 \text{ and } \|h_k\|_{\infty,\partial_A\setminus A_k} < 2^{-k}.$$

Furthermore by construction the functions $h_k$, or better their equivalence classes in $L^\infty(\mu)$, belong to $H_A^\infty(\mu)$ and the commutators $B_k = [T_{h_k}, S] \in B(H)$ are compact operators with $\|B_k\| \geq \rho/4$.

By passing to a subsequence, we can achieve that the limit

$$c = \lim_{k\to\infty} \|B_k\| \in [\rho/4, \infty)$$

exists. Since the sequence $(h_k)_{k\geq 1}$ is uniformly bounded on $\partial_A$ and converges to zero pointwise on $\partial_A$, it follows that both sequences $(B_k)_{k\geq 1}$ and $(B_k^*)_{k\geq 1}$ converge to zero in the strong operator topology (see Lemma 3.4). A result proved by Muhly and Xia in [19] (Lemma 2.1) shows that, by passing to a subsequence again, we can achieve that the series

$$B = \sum_{k=0}^{\infty} B_k$$

converges in the strong operator topology and satisfies $\|\pi(B)\| = c \geq \rho/4$. Since every point $z \in \partial_A$ belongs to at most one of the sets $A_k$, the partial sums of the series $\sum_{k=0}^{\infty} h_k$ are uniformly bounded on $\partial_A$ and converge pointwise to a function $h : \partial_A \to \mathbb{C}$. Clearly, (the equivalence class of) h belongs to $H_A^\infty(\mu)$ and the identities

$$T_h = \sum_{k=1}^{\infty} T_{h_k} \quad \text{and} \quad [T_h, S] = \sum_{k=1}^{\infty} [T_{h_k}, S] = B$$

hold in the strong operator topology. But then $T_h \in \mathscr{H}_T$ would be an operator with non-compact commutator $[S, T_h]$. This contradicts the hypothesis and thus completes the proof. □

**4.7 Corollary.** *Let $T \in B(H)^n$ be an essentially normal regular A-isometry with $\sigma_p(T) = \emptyset$. Denote by $U \in B(\widehat{H})^n$ the minimal normal extension of $T$. Suppose that $W^*(U) \subset B(\widehat{H})$ is a maximal abelian von Neumann algebra. Then a given operator $S \in B(H)$ essentially commutes with $\mathscr{H}_T$ if and only if $S$ has the form $S = T_f + K$ with a compact operator $K \in \mathcal{K}(H)$ and a symbol $f \in L^\infty(\mu)$ having the property that the associated Hankel operator $H_f$ is compact.*

**Proof.** Suppose that $S \in B(H)$ essentially commutes with $\mathscr{H}_T$. Fix a weak* zero sequence of isometries $(J_k)_{k \geq 1}$ in $\mathscr{H}_T$ such that the weak* limit

$$X = w^* - \lim_{k \to \infty} J_k^* S J_k \in B(H)$$

defines a T-Toeplitz operator (see Lemma 4.3). By Proposition 3.2 (b) there is a function $f \in L^\infty(\mu)$ such that $X = P_H \Psi_U(f)|H = T_f$. The proof of the preceding theorem shows that the image of the map

$$F : L^\infty(\mu) \to B(H), \quad F(h) = T_h S - P_H(\Psi_U(hf))|H$$

is contained in $\mathcal{K}(H)$. In particular, the operator $K = F(1) = S - T_f$ is compact. Because of

$$\begin{aligned}
F(\overline{f}) &= T_{\overline{f}} S - T_{|f|^2} \\
&= T_{\overline{f}} T_f - T_{|f|^2} + T_{\overline{f}} K \\
&= P_H \Psi_U(\overline{f}) P_H \Psi_U(f)|H - P_H \Psi_U(\overline{f}) \Psi_U(f)|H + T_{\overline{f}} K \\
&= -P_H \Psi_U(\overline{f}) P_{H^\perp} \psi_U(f)|H + T_{\overline{f}} K \\
&= -H_f^* H_f + T_{\overline{f}} K
\end{aligned}$$

we find that $H_f^* H_f$ and hence also $H_f$ is compact.

Conversely, suppose that $f \in L^\infty(\mu)$ is a function such that $H_f$ is compact. Then, for every $g \in H_A^\infty(\mu)$, it follows that

$$\begin{aligned}
T_f T_g &= P_H \Psi_U(f) \Psi_U(g)|H \;=\; P_H \Psi_U(g) P_H \Psi_U(f)|H + P_H \Psi_U(g) H_f \\
&= T_g T_f + P_H \Psi_U(g) H_f.
\end{aligned}$$

Thus $T_f$ essentially commutes with $\mathscr{H}_T$. □

The preceding corollary in particular applies to each essentially normal regular Hardy-space $A$-isometry $T = T_z \in B(H^2_A(\mu))^n$ (cf. Definition 2.2) with empty point spectrum. To give a concrete example, let $D \subset \mathbb{C}^n$ be a relatively compact stricly pseudoconvex open set with $C^2$-boundary $\partial D$. The normalized surface measure $\sigma$ on the boundary $\partial D$ has no one-point atoms. The associated Toeplitz tuple $T_z = (T_{z_1}, \ldots, T_{z_n}) \in B(H^2_{A(D)}(\sigma))^n$ is a regular Hardy-space $A(D)$-isometry (Proposition 2.7). The space $H^2_{A(D)}(\sigma)$ coincides in this case with the Hardy space $H^2(\sigma) \subset L^2(\sigma)$ on the boundary $\partial D$ (see Section 7 in [22]). In order to identify $H^\infty_{A(D)}(\sigma)$ we use the fact that every function $f \in H^\infty(D)$ possesses non-tangential boundary values $f^* \in L^\infty(\sigma)$ (Theorem 8.4.1 in [17]). The map $r : H^\infty(D) \to L^\infty(\sigma)$, $f \mapsto f^*$ is a weak$^*$ continuous isometry. To see this, fix $f \in H^\infty(D)$. Since $f$ belongs to the Hardy space $H^2(D)$, there exists a function $\tilde{f} \in L^2(\sigma)$ such that

$$f(z) = (\mathscr{P}\tilde{f})(z) = \int_{\partial D} P(z,w)\tilde{f}(w)d\sigma(w) \qquad (z \in D),$$

where $P$ denotes the Poisson kernel of $D$ (Theorem 8.3.6 in [17]). In the proof of Theorem 8.4.1 in [17] it is shown that $r(f) = \tilde{f}$. In particular, it follows that $\tilde{f}$ belongs to $L^\infty(\sigma)$ and satisfies $\|\tilde{f}\|_{\infty,\sigma} \leq \|f\|_{\infty,D}$. Estimating the above Poisson integral of $\tilde{f}$ we obtain the reverse inequality $\|f\|_{\infty,D} \leq \|\tilde{f}\|_{\infty,\sigma}$. Thus the map $r : H^\infty(D) \to L^\infty(\sigma)$ is isometric. As usual we denote its range by $H^\infty(\sigma)$. Since $\mathscr{P}(r(f)) = f$ for every $f \in H^\infty(D)$, the inverse of $r$ is the Poisson transformation $\mathscr{P} : H^\infty(\sigma) \to H^\infty(D)$.

Standard arguments show that $H^\infty(\sigma) \subset L^\infty(\sigma)$ is weak$^*$ closed. We briefly indicate a possible proof. Let $(f_k)$ be a sequence in the closed unit ball of $H^\infty(D)$ such that $g = w^* - \lim_k r(f_k)$ exists in $L^\infty(\sigma)$. By Krein-Smulian's theorem and the separability of $L^1(\sigma)$ it suffices to show that $g \in H^\infty(\sigma)$. By Montel's theorem we may suppose that $(f_k)$ converges to some function $f \in H^\infty(D)$ uniformly on every compact subset of $D$. Since $r(f)$ and $g$ are functions in $L^\infty(\sigma)$ such that

$$\mathscr{P}(r(f))(z) = f(z) = \lim_k f_k(z) = \lim_k \int_{\partial D} P(z,w)(rf_k)(w)d\sigma(w) = \mathscr{P}(g)(z)$$

for all $z \in D$, it follows that $g = r(f)$ (cf. the proof of Theorem 8.4.1 in [17]).

As an application one obtains the weak$^*$ continuity of the map $r : H^\infty(D) \to H^\infty(\sigma)$. Since $H^\infty(D) = (L^1(D)/^\perp H^\infty(D))'$ has a separable predual, it suffices to show that $(r(f_k))$ is a weak$^*$ zero sequence in $L^\infty(\sigma)$ for each weak$^*$ zero sequence $(f_k)$ in $H^\infty(D)$. But this follows from the observation that

$$\langle [P(z,\cdot)], r(f_k)\rangle = \int_{\partial D} P(z,w)r(f_k)(w)d\sigma(w) = f_k(z) \xrightarrow{k} 0$$

for all $z \in D$ and the fact that the predual $L^1(\sigma)/^\perp H^\infty(\sigma)$ of $H^\infty(\sigma)$ is the closed linear span of all equivalence classes $[P(z,\cdot)]$ $(z \in D)$.

Since $H^\infty(\sigma) \subset L^\infty(\sigma)$ is weak$^*$ closed, the inclusion $H^\infty_{A(D)}(\sigma) = \overline{(A(D)|\partial D)}^{w^*} \subset r(H^\infty(D)) = H^\infty(\sigma)$ holds. The reverse inclusion $H^\infty(\sigma) \subset H^\infty_{A(D)}(\sigma)$ follows from the weak$^*$ continuity of $r$ and the fact that there is an open neighbourhood $U$ of $\overline{D}$ in $\mathbb{C}^n$ such that $\mathscr{O}(U)|D$ is sequentially weak$^*$ dense in $H^\infty(D)$ (Proposition 2.1.6 in [8]).

Since $\gamma_{T_z}(f) = \Psi_U(f)|H^2(\sigma) = T_f$ for $f \in H^\infty(\sigma)$, the dual algebra $\mathscr{H}_{T_z}$ coincides with the set of all Toeplitz operators $T_\varphi$ with symbol $\varphi$ in $H^\infty(\sigma)$. By Theorem 4.2.17 in Upmeier [23] the tuple $T_z$ is essentially normal. So the last corollary applies to this case and yields a description of the essential commutant of the set of all analytic Toeplitz operators, which extends Theorem 2 of Ding-Sun [11].

**4.8 Corollary.** *If $\sigma$ denotes the normalized surface measure on the boundary $\partial D$ of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with $C^2$-boundary, then an operator $S \in B(H^2(\sigma))$ essentially commutes with all analytic Toeplitz operators on $H^2(\sigma)$ if and only if it has the form $S = T_f + K$ with a compact operator $K$ and a symbol $f \in L^\infty(\sigma)$, for which the associated Hankel operator $H_f$ is compact.* $\square$

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M. Didas, J. Eschmeier
Fachrichtung Mathematik
Universität des Saarlandes
Postfach 151150
D-66041 Saarbrücken
Germany

didas@math.uni-sb.de, eschmei@math.uni-sb.de