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Abstract We prove variants of Korn's inequality involving the trace-free part of the symmetric gradient of vector fields $v : \Omega \to \mathbb{R}^n$ ($\Omega \subset \mathbb{R}^n$), that is,

$$\int_\Omega h(|\nabla v|)\,\mathrm{d}x \leqslant c \int_\Omega h(|\mathcal{E}^D v|)\,\mathrm{d}x$$

for functions with zero trace as well as some further variants of this inequality. Here, h is an N-function of rather general type. As an application we prove partial $C^{1,\alpha}$-regularity of minimizers of energies of the type

$$\int_\Omega h(|\mathcal{E}^D v|)\,\mathrm{d}x,$$

occurring, for example, in general relativity.

Keywords Generalized Korn inequalities in Orlicz-Sobolev spaces · Variational problems · Nonstandard growth · Regularity

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1 Introduction and formulation of the main results

A crucial tool in the mathematical approach for the behavior of Newtonian fluids is Korn's inequality: Given a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) with Lipschitz boundary $\partial\Omega$, we have for all $v \in \mathring{W}^{1,2}(\Omega;\mathbb{R}^n)$

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x \leqslant 2 \int_\Omega |\mathcal{E} v|^2\,\mathrm{d}x, \tag{1.1}$$

wherein $\mathcal{E} v := \frac{1}{2}(\nabla v + \nabla^{\mathrm{T}} v)$ denotes the symmetric part of the gradient ∇v of v. For smooth vector fields v with compact support (1.1) follows by integration by parts, whereas in the general case (1.1) is proved by approximation. We note that L^2-variants of Korn's inequality are due to Courant and Hilbert [10], Friedrichs [23], Èidus [15] and Mihlin [34].
Many problems in mathematical theory of generalized Newtonian fluids and mechanics of solids lead to the following question (compare, for example, the monographs of Málek, Nečas, Rokyta and Růžička [36], Duvaut and Lions [11] as well as Zeidler [47]): Is it possible to control a certain energy depending on ∇v by the corresponding one depending just on $\mathcal{E} v$, that is, does

$$\int_\Omega |\nabla v|^p\,\mathrm{d}x \leqslant c(p,\Omega) \int_\Omega |\mathcal{E} v|^p\,\mathrm{d}x \tag{1.2}$$

hold for functions $v \in \mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$?
As shown by Gobert [27]-[28], Nečas [38], Mosolov and Mjasnikov [35], Temam [45], and later by Fuchs [19] the inequality (1.1) is true for all $1 < p < \infty$. (It should be emphasized that inequality (1.1) does not hold in case $p = 1$; see [39], or [9].) We also like to remark that the case of Sobolev spaces $\mathring{W}^{1,p(\cdot)}(\Omega;\mathbb{R}^n)$ with variable exponents, which are the natural spaces for the study of electro-rheological fluids (compare [40]), is considered in [12].

In order to generalize (1.2), we replace $t \mapsto t^p$ by an N-function h (see, for example, [1] for a definition) of rather general type and consider the inequality

$$\int_\Omega h(|\nabla v|)\,\mathrm{d}x \leqslant c(\varphi,\Omega)\int_\Omega h(|\mathcal{E}v|)\,\mathrm{d}x \tag{1.3}$$

for functions $v \in \overset{\circ}{W}{}^{1,h}(\Omega;\mathbb{R}^n)$. A first step is mentioned in [2]: Acerbi and Mingione prove a variant of (1.3) (in the Luxemburg norm, and not in the integral version) for the N-function $h(t) = (1+t^2)^{\frac{p-2}{2}}t^2$ with $p > 1$. Although, they just consider a special case, they provide the necessary tools to deal with much more general situation. Moreover, the more general result (1.3) is proved in [13], [21], [8], and [7] with the result: (1.3) holds if h satisfies Δ_2- and ∇_2-conditions (a precise definition is given below).

In this note we prove Korn inequalities of the above type, where $\mathcal{E}v$ is replaced by its trace-free part $\mathcal{E}^D v := \mathcal{E}v - \frac{1}{n}(\operatorname{div} v)I$, that is, we prove an $\mathcal{E}^D v$-version of (1.3) (and variants of this). Korn inequalities involving the trace-free part of the symmetric gradient have applications in general relativity, Cosserat elasticity, and geometry; compare [26], [17], [43], and the references therein. On the other hand, since the kernel of the operator

$$W^{1,p}(\Omega;\mathbb{R}^n) \to L^p(\Omega;\mathbb{M}^n), \quad v \mapsto \mathcal{E}^D v$$

($\mathbb{M}^n$ denoting the space of trace-free matrices of order n) is much larger than the kernel of the operator

$$W^{1,p}(\Omega;\mathbb{R}^n) \to L^p(\Omega;\mathbb{R}^{n\times n}), \quad v \mapsto \mathcal{E}v$$

such Korn-type inequalities are also of great interest from the mathematical point of view. Contributions to these issues can be found, for example, in [42], [43] as well as [44] with the result:

$$\int_\Omega |\nabla v|^p\,\mathrm{d}x \leqslant c(p,\Omega)\int_\Omega |\mathcal{E}^D v|^p\,\mathrm{d}x$$

for all $v \in \overset{\circ}{W}{}^{1,p}(\Omega;\mathbb{R}^n)$ and all $1 < p < \infty$.

It should be emphasized that even the case $p = 2$ requires hard mathematical arguments being totally different from those needed in situation of (1.1). Moreover, the ideas from the Orlicz setting used to prove (1.3) are not applicable here. Specifically, the proof of (1.3) presented in [13] (Theorems 6.13 and 5.17) is based on the inequality (and a generalization of Nečas' lemma [38])

$$|\nabla^2 v| \leqslant c|\nabla\mathcal{E}v|,$$

which does not hold for $n \geqslant 3$ if we replace $\mathcal{E}v$ by $\mathcal{E}^D v$ on the right-hand side. In [21] the main tool in the proof of (1.3) is a regularity theorem for elliptic equations in Orlicz spaces (see [31]) and the representation

$$Lv = \operatorname{div} V$$

with an elliptic differential operator L of second order (here the Laplace operator) and a suitable vector field V depending on elements of $\mathcal{E}v$. In case $n \geqslant 3$ this technique is also not applicable to the situation, where $\mathcal{E}v$ is replaced by $\mathcal{E}^D v$.

As an application of our new Korn-type inequalities we discuss the regularity of local minimizers of functionals of the form

$$\int_{\Omega} h(|\mathcal{E}^D v|)\,\mathrm{d}x,$$

defined on an appropriate Orlicz-Sobolev class, wherein h is an N-function of rather general type. Corresponding results are shown by the first author and Fuchs [8] in the context of the nonlinear Stokes problem, where the density of the functional depends on the symmetric gradient.

Let us give a detailed formulation of our results: Assume that $h : [0,\infty) \to [0,\infty)$ is a function of class C^2 that satisfies the conditions

$$\begin{aligned}
&\text{(H1)} \quad h \text{ is strictly increasing and convex}\\
&\text{(H2)} \quad h''(0) > 0 \quad \text{and} \quad \lim_{t \searrow 0} \frac{h(t)}{t} = 0\\
&\text{(H3)} \quad \frac{h'(t)}{t} \leqslant h''(t) \leqslant A(1+t^2)^{\omega/2}\frac{h'(t)}{t}\\
&\text{(H4)} \quad h(2t) \leqslant K h(t)
\end{aligned}$$

for all $t \geqslant 0$ with constants $A, K > 0$ and an exponent $\omega \geqslant 0$. Let us give some remarks on the above conditions; the details can be found in [3] and [8]. Examples of functions h satisfying (H1)–(H4) are given in [22].

Remark 1.1 i) Conditions (H1)–(H3) imply that h is an N-function (according to the definition of Adams [1], Section 8.2). In particular, $h(0) = 0 = h'(0)$ and $h'(t) > 0$ for all $t > 0$. Note also $h''(0) = \lim_{t\searrow 0} h'(t)/t$.

ii) Condition (H4) states that h fulfills a global Δ_2-condition. In particular,

$$h(s+t) \leqslant \frac{K}{2}\big(h(s)+h(t)\big), \quad h(\lambda t) \leqslant K\lambda^{\beta} h(t) \tag{1.4}$$

for all $\lambda > 1$ and $s, t \geqslant 0$ with $\beta \coloneqq \log K/\log 2$. Note that for $\lambda \leqslant 1$ we clearly have $h(\lambda t) \leqslant \lambda h(t)$. Moreover, (H3) implies that h fulfills a global ∇_2-condition, that is,

$$h(t) \leqslant \frac{1}{2L} h(Lt)$$

for all $t \geqslant 0$ with some $L > 1$; compare [41], Section 2.3.

iii) From the lower bound in (H3) we deduce that the function $t \mapsto h'(t)/t$ is increasing and

$$h(t) \geqslant \frac{h''(0)}{2} t^2, \quad h'(t) \geqslant h''(0)t \quad (t \geqslant 0). \tag{1.5}$$

Moreover, from (H4) and the convexity of h it follows

$$\frac{h'(t)t}{K} \leqslant h(t) \leqslant h'(t)t \quad (t \geqslant 0). \tag{1.6}$$

iv) There is an exponent $\overline{q} \geqslant 2$ such that

$$h(t) \leqslant c(1+t^2)^{\overline{q}/2}, \quad \frac{h'(t)}{t} \leqslant c(1+t^2)^{\overline{q}/2-1} \tag{1.7}$$

for all $t \geqslant 0$.

Let us state our main results.

Theorem 1.2 *Let (H1)–(H4) be fulfilled. For each* $v \in \overset{\circ}{W}{}^{1,h}(\Omega;\mathbb{R}^n)$ *we have*

$$\int_\Omega h(|\nabla v|)\,\mathrm{d}x \leqslant c(n,h,\Omega)\int_\Omega h(|\mathcal{E}^D v|)\,\mathrm{d}x.$$

For further variants of this inequality and some comments we refer the reader to Theorem 2.1 and Remark 2.2 in the next section.
Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ denote a bounded Lipschitz domain, and let $H : \mathbb{M}^n \to [0,\infty)$ be a function on the space $\mathbb{M}^n$ of trace-free matrices of order n. Assume that H has the special structure $H(\sigma) = h(|\sigma|)$ with a function h as above. From (H3) we deduce the ellipticity condition

$$\frac{h'(|\sigma|)}{|\sigma|}|\tau|^2 \leqslant D^2H(\sigma)(\tau,\tau) \leqslant A(1+|\sigma|^2)^{\omega/2}\frac{h'(|\sigma|)}{|\sigma|}|\tau|^2 \tag{1.8}$$

for all $\sigma,\tau \in \mathbb{M}^n$. Using (1.7) we conclude with $q \coloneqq \overline{q}+\omega$ (recall (1.5))

$$h''(0)|\tau|^2 \leqslant D^2H(\sigma)(\tau,\tau) \leqslant \Lambda(1+|\sigma|^2)^{q/2-1}|\tau|^2 \tag{1.9}$$

for all $\sigma,\tau \in \mathbb{M}^n$ with a positive number Λ, which means that H is of anisotropic $(2,q)$-growth.
We consider the functional

$$J[v] = J[v;\Omega] \coloneqq \int_\Omega H(\mathcal{E}^D v)\,\mathrm{d}x \tag{1.10}$$

among vector fields v from the class $\mathbb{K} \coloneqq u_0 + \overset{\circ}{W}{}^{1,h}(\Omega;\mathbb{R}^n)$ with prescribed Dirichlet boundary data u_0 from the Orlicz-Sobolev space $W^{1,h}(\Omega;\mathbb{R}^n)$ generated by h; see [1] for a definition. Then we have the following existence and regularity theorem, which in the two-dimensional is already proved by Fuchs [19].

Theorem 1.3 *Let (H1)–(H4) hold.*

a) The minimization problem $J \to \min$ *in* $\mathbb{K}$ *admits a unique solution* u.
b) If $n \geqslant 3$ *and* $\omega < 4/n$, *there is an open set of full Lebesgue measure such that* $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^n)$ *for each* $\alpha \in (0,1)$.
c) Let $n = 2$ *and* $\omega < 2$. *Then* $u \in C^{1,\alpha}(\Omega;\mathbb{R}^2)$ *for each* $\alpha \in (0,1)$.

Remark 1.4 i) In the proof of part b) of the above theorem we use a blow-up argument, which generalizes the approach used in [8], where an $\mathcal{E}$-version of the above theorem is proved, but only in case $n = 3$. We are able to extend this result to arbitrary dimensions owing to our Korn-type inequalities.

ii) Regularity results for functionals of this type for $n \geqslant 3$ are only known if the density H behaves like a power of $\mathcal{E}^D u$ [43], or in the anisotropic case under restrictive assumptions concerning the growth rates [44]. In our approach the range of anisotropy can be arbitrary high (remember (1.9)).

Corollary 1.5 *Let (H1)–(H4) hold and suppose that u is a local J-minimizer, that is, $u \in W^{1,h}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ fulfills for each subdomain $\Omega' \Subset \Omega$ the conditions*

$$J[u;\Omega'] < \infty \quad \text{and} \quad J[u;\Omega'] \leqslant J[v;\Omega']$$

for all $v \in W^{1,h}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ such that $\operatorname{spt}(u-v) \Subset \Omega'$. Then the statements b) and c) of Theorem 1.3 continue to hold.

2 Generalized Korn-type inequalities in Orlicz-Sobolev spaces

In this section we collect variants of Korn's inequality in Orlicz-Sobolev spaces involving the trace-free part of the symmetric gradient. Corresponding versions of these Korn-type inequalities for Sobolev functions are shown by the second author in [44] and by Fuchs and the second author in [26].

We denote by $\mathcal{K}_\Omega$ the kernel of the operator

$$W^{1,h}(\Omega;\mathbb{R}^n) \to L^h(\Omega;\mathbb{M}^n), \quad v \mapsto \mathcal{E}^D v,$$

which for $n \geqslant 3$ is finite-dimensional and coincides with the space of the so-called conformal Killing vectors (Möbius transformations). For a proof and a precise characterization we refer to [44]; compare also [42]. In the two-dimensional case $\mathcal{K}_\Omega$ is infinite-dimensional and coincides with the space of holomorphic functions on Ω.

Theorem 2.1 *Let (H1)–(H4) be fulfilled.*

a) For each $v \in \mathring{W}^{1,h}(\Omega;\mathbb{R}^n)$ we have

$$\int_\Omega h(|\nabla v|)\,\mathrm{d}x \leqslant c(n,h,\Omega) \int_\Omega h(|\mathcal{E}^D v|)\,\mathrm{d}x. \tag{2.1}$$

b) Let $\partial\Omega$ be Lipschitz. For each $v \in W^{1,h}(\Omega;\mathbb{R}^n)$ there exists $\chi \in \mathcal{K}_\Omega$ such that

$$\int_\Omega h(|v-\chi|)\,\mathrm{d}x \leqslant c(n,h,\Omega) \int_\Omega h(|\mathcal{E}^D v|)\,\mathrm{d}x. \tag{2.2}$$

c) Let $\partial\Omega$ be Lipschitz and suppose $n \geqslant 3$. Then for each $v \in W^{1,h}(\Omega;\mathbb{R}^n)$ it holds

$$\int_\Omega h(|\nabla v|)\,\mathrm{d}x \leqslant c(n,h,\Omega)\left(\int_\Omega h(|v|)\,\mathrm{d}x + \int_\Omega h(|\mathcal{E}^D v|)\,\mathrm{d}x\right). \tag{2.3}$$

Remark 2.2 i) When Ω is a ball $B_R = B_R(x_0) \subset \mathbb{R}^n$ the constant c in part a) of the above theorem is independent of R and x_0, which follows by a standard scaling argument; compare [43]. Moreover, the same argument together with (1.4) shows that the inequalities in b) and c) take the form

$$\int_{B_R} h(|v-\chi|)\,\mathrm{d}x \leqslant cR^\gamma \int_{B_R} h(|\mathcal{E}^D v|)\,\mathrm{d}x, \tag{2.4}$$

$$\int_{B_R} h(|\nabla v|)\,\mathrm{d}x \leqslant c\left(\frac{1}{R^\gamma}\int_{B_R} h(|v|)\,\mathrm{d}x + \int_{B_R} h(|\mathcal{E}^D v|)\,\mathrm{d}x\right), \tag{2.5}$$

where $\gamma := \max(1,\beta)$ and $c = c(n,h,K)$ with β as in (1.4).

ii) The last statement of the above theorem does not hold in the two-dimensional case since the corresponding Korn-type inequality in Sobolev spaces is not valid in this case; see [44]. However, we have the following variant of (2.5) for $n = 2$: Let $v \in W^{1,h}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. Then for balls $B_r = B_r(x_0)$ and $B_R = B_R(x_0)$ with $B_r \Subset B_R \Subset \Omega$ it holds

$$\int_{B_r} h(|\nabla v|)\,\mathrm{d}x \leqslant c\left(\frac{1}{(R-r)^\gamma}\int_{B_R} h(|v|)\,\mathrm{d}x + \int_{B_R} h(|\mathcal{E}^D v|)\,\mathrm{d}x\right) \tag{2.6}$$

with $c = c(n,h,K)$.

iii) In the two-dimensional case the proof of part b) requires different methods since in this case $\mathcal{K}_\Omega$ is not finite-dimensional so that the representation formula of Reshetnyak [42] used in case $n \geqslant 3$ is not applicable in case $n = 2$. Here, we can argue as in the proof of the corresponding Korn-type inequality in Sobolev spaces, that is, we combine the Cauchy-Pompeiu formula with a well-known estimate for the Riesz potential; compare [19], or [44].

iv) Part a) of Theorem 2.1 holds for arbitrary bounded domains, whereas the statements b) and c) hold if Ω is a bounded domain allowing a decomposition of the form

$$\Omega = \bigcup_{\ell=1}^{L} \Omega_\ell \quad (L \in \mathbb{N})$$

with domains Ω_ℓ being star-shaped with respect to a ball $B_\ell \Subset \Omega_\ell$. In particular, bounded domains satisfying the cone condition allow such a decomposition; see [33]. Moreover, bounded domains with the cone property are decomposable in finitely many Lipschitz domains.

v) Since $|\mathcal{E}^D v| \leqslant |\mathcal{E}v|$, the statements a) and c) are also valid with $\mathcal{E}^D v$ replaced by $\mathcal{E}v$. The corresponding versions of the Korn inequalities, which are already proved in [7], [8] and [13], are the essential tools in the study of variational problems for generalized Newtonian fluids.

vi) Part b) is also true with $\mathcal{E}^D v$ replaced by $\mathcal{E}v$ if χ is a suitable rigid motion. To the best of our knowledge, this Korn-type inequality (in the version with $\mathcal{E}v$) is new. Note that [42] contains a representation formula for $\mathcal{E}v$, which is also valid in case $n = 2$, so that similar arguments as in the proof of b) yield the corresponding inequality with $\mathcal{E}^D v$ replaced by $\mathcal{E}v$.

vii) From our proof of b) we see that the Killing vector χ is independet of h, which means that (2.2) is true with the same function χ for each N-function h satisfying the conditions (H1)–(H4).

The main tool in the proof of Theorem 2.1 is an interpolation argument due to Koizumi [32]; see Lemma 2.4 below. If we use instead the theory of Torchinsky [46], we can control the constant in a better way; compare [8] (Appendix) for details. The constant now only depends on the constant K from condition (H4). But the argument only works in the Luxembourg norm and not in the integral version.

Corollary 2.3 *Let (H1)–(H4) be fulfilled.*

a) For each $v \in \mathring{W}^{1,h}(\Omega;\mathbb{R}^n)$ *we have*

$$\|\nabla v\|_h \leqslant c(n,K,\Omega)\,\|\mathcal{E}^D v\|_h. \tag{2.7}$$

b) For each $v \in W^{1,h}(\Omega;\mathbb{R}^n)$ *there exists* $\chi \in \mathcal{K}_\Omega$ *such that*

$$\|v-\chi\|_h \leqslant c(n,K,\Omega)\,\|\mathcal{E}^D v\|_h. \tag{2.8}$$

c) Suppose $n \geqslant 3$. *Then for each* $v \in W^{1,h}(\Omega;\mathbb{R}^n)$ *it holds*

$$\|\nabla v\|_h \leqslant c(n,K,\Omega)\big(\|v\|_h + \|\mathcal{E}^D v\|_h\big). \tag{2.9}$$

We begin with the proof of Theorem 2.1 now. The main tool in the proof is the following lemma, which follows from an interpolation argument due to Koizumi [32].

Lemma 2.4 *Let (H1)–(H4) be fulfilled, and let* $\mathcal{T}$ *be a linear operator, which is continuous from* $L^p(\Omega) \to L^p(\Omega)$ *for every* $p \in (1,\infty)$. *Then* $\mathcal{T}$ *is continuous from* $L^h(\Omega) \to L^h(\Omega)$. *Moreover,*

$$\int_\Omega h(|\mathcal{T}v|)\,\mathrm{d}x \leqslant c(n,h,\Omega)\int_\Omega h(|v|)\,\mathrm{d}x$$

for each $v \in L^h(\Omega)$.

Proof According to Theorem 4 in [32] it suffices to show that there are numbers $1 < a < b < \infty$ such that

$$h(2t) = O(h(t)), \tag{2.10}$$

$$\int_t^\infty \frac{h(s)}{s^{b+1}}\,\mathrm{d}s = O\bigg(\frac{h(t)}{t^b}\bigg), \tag{2.11}$$

$$\int_1^t \frac{h(s)}{s^{a+1}}\,\mathrm{d}s = O\bigg(\frac{h(t)}{t^a}\bigg) \tag{2.12}$$

as $t \to \infty$ and

$$h(2t) = O(h(t)), \tag{2.13}$$

$$\int_t^1 \frac{h(s)}{s^{b+1}}\,\mathrm{d}s = O\Big(\frac{h(t)}{t^b}\Big), \tag{2.14}$$

$$\int_0^t \frac{h(s)}{s^{a+1}}\,\mathrm{d}s = O\Big(\frac{h(t)}{t^a}\Big) \tag{2.15}$$

as $t \to 0$. Clearly, (2.10) and (2.13) follow immediately from (H4). For the other conditions, we choose $a \in (1,2)$, $b > \max(a, K)$ and observe (recall (1.6))

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{h(t)}{t^b}\Big) = \frac{h'(t)t - bh(t)}{t^{b+1}} \leqslant 0.$$

We deduce

$$\int_t^\infty \frac{h(s)}{s^{b+1}}\,\mathrm{d}s = \int_t^\infty \frac{h(s)}{s^K}\, s^{K-b-1}\,\mathrm{d}s \leqslant \frac{h(t)}{t^K}\int_t^\infty s^{K-b-1}\,\mathrm{d}s = c\,\frac{h(t)}{t^b},$$
$$\int_t^\infty \frac{h(s)}{s^{b+1}}\,\mathrm{d}s \geqslant \int_t^{2t} \frac{h(s)}{s^K}\, s^{K-b-1}\,\mathrm{d}s \geqslant \frac{h(t)}{t^K}\int_t^{2t} s^{K-b-1}\,\mathrm{d}s \geqslant c\,\frac{h(t)}{t^b}.$$

On the other hand, using (1.6) and (H4), we find

$$\int_1^t \frac{h(s)}{s^{a+1}}\,\mathrm{d}s \leqslant \int_1^t \frac{h'(s)}{s}\, s^{1-a}\,\mathrm{d}s \leqslant \frac{h'(t)}{t}\int_1^t s^{1-a}\,\mathrm{d}s \leqslant c\,\frac{h(t)}{t^a},$$
$$\int_1^t \frac{h(s)}{s^{a+1}}\,\mathrm{d}s \geqslant c\,\frac{h'(t)}{t}\int_{t/2}^t s^{1-a}\,\mathrm{d}s \geqslant c\,\frac{h(t)}{t^a}.$$

This proves (2.11) and (2.12). The remaining conditions (2.14) and (2.15) follow by similar calculations.

Proof (of Theorem 2.1) Assume $n \geqslant 3$ and that Ω is star-shaped with respect to a ball $B \subset \Omega$. Then, according to formula (2.43) in [42] each $v \in C^\infty(\Omega;\mathbb{R}^n)$ can be represented as

$$v(x) = \chi(x) + \mathcal{R}(\mathcal{E}^D v)(x), \tag{2.16}$$

where $\chi = \chi(v)$ is a suitable element of $\mathcal{K}_\Omega$ (compare (2.40) in [42]) and $\mathcal{R}$ is a singular integral operator (compare (2.41) in [42]) given by

$$\mathcal{R}(\varphi) := \mathcal{S}(\varphi) + \mathcal{T}(\varphi) \quad (\varphi \in C^\infty(\Omega;\mathbb{R}^{n\times n}))$$

with ($i \in \{1,\dots,n\}$)

$$\mathcal{S}^i(\varphi)(x) := \int_\Omega \frac{\omega^i_{kl}(x,e)}{|x-z|^{n-1}}\,\varphi^{kl}(z)\,\mathrm{d}z \tag{2.17}$$
$$\mathcal{T}^i(\varphi)(x) := \int_\Omega \theta^i_{kl}(x,z)\varphi^{kl}(z)\,\mathrm{d}z$$

for $x \in \Omega$ with summation with respect to $k, l \in \{1, \ldots, n\}$. Here, $\omega^i_{kl}(x, e)$ are smooth functions ($e := (x-z)/|x-z|$), and $\theta^i_{kl}(x, z)$ are bounded continuous functions; see [42] after (2.38). (Note that the representation formulas from [42] are also used in the paper [5].)
Now, assume $v \in C^\infty_0(\Omega; \mathbb{R}^n)$. Then we have $\chi \equiv \alpha \fint_B v \, \mathrm{d}z$ (compare (10) in [25]) with a certain constant $\alpha = \alpha(n)$ so that from (2.16) we deduce

$$\nabla v(x) = \nabla \mathcal{R}(\mathcal{E}^D v)(x) = \nabla \mathcal{S}(\mathcal{E}^D v)(x) + \nabla \mathcal{T}(\mathcal{E}^D v)(x). \tag{2.18}$$

After dropping all indices for notational simplicity we see that the right-hand side of (2.17) is of the form (note that we can extend v to the hole space by setting $v = 0$ outside Ω)

$$V(x) := \int_{\mathbb{R}^n} K(x-z)\varphi(z) \, \mathrm{d}z \quad (x \in \mathbb{R}^n)$$

with K being essentially homogeneous of degree $1-n$ in the sense of Morrey [37]. From part b) of Theorem 3.4.2 in [37] and the subsequent remark we deduce

$$\partial_j V(x) = c(j)\varphi(x) + \lim_{\rho \searrow 0} \int_{\mathbb{R}^n - B_\rho(x)} (\partial_\alpha K)(x-z)\varphi(z) \, \mathrm{d}z \tag{2.19}$$

for each $j \in \{1, \ldots, n\}$ and almost every $x \in \mathbb{R}^n$. If we consider the right-hand side of (2.19) as a function of φ, it is continuous from $L^p(\Omega) \to L^p(\Omega)$ for each $p \in (1, \infty)$ according to the Calderon-Zygmund theory (compare [37], Theorem 3.4.2 b)) so that from Lemma 2.4 we infer

$$\int_\Omega h(|\nabla \mathcal{S}(\mathcal{E}^D v)|) \, \mathrm{d}x \leqslant c(n, h, \Omega) \int_\Omega h(|\mathcal{E}^D v|) \, \mathrm{d}x. \tag{2.20}$$

On the other hand, we have (compare [42], p. 325)

$$\partial_j \mathcal{T}^i(\varphi)(x) = \int_\Omega (\partial_j \theta^i_{kl})(x, z)\varphi^{kl} v(z) \, \mathrm{d}z \tag{2.21}$$

with $\partial_j \theta^i_{kl}(x, z)$ being bounded and continuous when $x \neq z$. Therefore, the right-hand side of (2.21) is also continuous from $L^p(\Omega) \to L^p(\Omega)$ for each $p \in (1, \infty)$ so that from Lemma 2.4 we obtain

$$\int_\Omega h(|\nabla \mathcal{T}(\mathcal{E}^D v)|) \, \mathrm{d}x \leqslant c(n, h, \Omega) \int_\Omega h(|\mathcal{E}^D v|) \, \mathrm{d}x. \tag{2.22}$$

Hence, returning to (2.18), the latter estimate together with (2.20) shows

$$\int_\Omega h(|\nabla v|) \, \mathrm{d}x \leqslant c(n, h, \Omega) \int_\Omega h(|\mathcal{E}^D v|) \, \mathrm{d}x$$

for every $w \in C^\infty_0(\Omega, \mathbb{R}^n)$, which by approximation gives us part a) of the theorem.
To prove part b), we assume $v \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$. On account of (2.16) we have

$$\int_\Omega h(|v - \chi|) \, \mathrm{d}x = \int_\Omega h(|\mathcal{R}(\mathcal{E}^D v)|) \, \mathrm{d}x$$

and since $\mathcal{R}$ is continuous from $L^p(\Omega) \to L^p(\Omega)$ for each $p \in (1,\infty)$ we obtain by Lemma 2.4

$$\int_\Omega h(|v-\chi|)\,\mathrm{d}x \leqslant c(n,h,\Omega)\int_\Omega h(|\mathcal{E}^D v|)\,\mathrm{d}x.$$

The continuity of $\mathcal{R}$ follows since the coefficients of $\mathcal{T}$ are smooth and bounded (compare [42] after (3.38)), whereas for $\mathcal{S}$ one can argue by the Calderon-Zygmund theory [37] (Theorem 3.4.2). Hence, b) is valid for smooth functions; in the general case it follows from an approximation argument stated in [25] after (13).

So far, we have established the first inequalities in a) and b) in case $n \geqslant 3$. We remark that for $n = 2$ the proof of a) is outlined in [21]. To prove b) for $n = 2$, we argue as in the proof of Lemma A.1 in [19] (compare also [44]): Assume $v \in C^\infty(\overline{\Omega};\mathbb{C})$. Then there exists a holomorphic function $\chi : \Omega \to \mathbb{C}$ such that

$$|v(z) - \chi(z)| \leqslant \frac{1}{\pi}\int_{B_r}\frac{\partial_{\overline{z}} v(\zeta)}{|\zeta - z|}\,\mathrm{d}\mathcal{L}^2(\zeta),$$

wherein $\partial_{\overline{z}} v$ is the Wirtinger derivative $\frac{1}{2}(\partial_x v + \mathrm{i}\partial_y v)$ of $v = v(z) = v(x,y)$, and $\int_{B_r}$ has to be calculated with respect to the two-dimensional Lebesgue measure $\mathcal{L}^2$. Using $|\mathcal{E}^D v| = \sqrt{2}\,|\partial_{\overline{z}} v|$ the right-hand side is bounded from above by

$$\frac{1}{\pi\sqrt{2}}\int_{B_r}\frac{|\mathcal{E}^D v(\zeta)|}{|\zeta - z|}\,d\mathcal{L}^2(\zeta) =: \frac{1}{\pi\sqrt{2}}\,V_{1/2}(|\mathcal{E}^D v|)(z).$$

Here, $V_{1/2}(|\mathcal{E}^D v|)(z)$ is the Riesz potential of $|\mathcal{E}^D v|$ defined in [30], formula (7.31), with the choices $\mu = 1/2$ and $n = 2$. Since the Riesz potential is continuous from $L^p(\Omega) \to L^p(\Omega)$ for each $p \in (1,\infty)$ and since

$$\|V_{1/2}(|\mathcal{E}^D v|)\|_p \leqslant 2\sqrt{|B_1||\Omega|}\,\|\mathcal{E}^D v\|_p$$

(see Lemma 7.12 in [30]), the claim follows from Lemma 2.4 and a standard approximation argument.

For c), we assume as before $v \in C^\infty(\overline{\Omega};\mathbb{R}^n)$ and observe that according to (2.40') in [42] we have

$$\chi^i(x) = \sum_{0\leqslant|\alpha|\leqslant 2} x^\alpha \int_\Omega H^i_{\alpha k}(z) v^k(z)\,\mathrm{d}z$$

with smooth functions $H^i_{\alpha k}$. Hence, $\|\nabla\chi\|_p \leqslant c\|v\|_p$ for all $p \in (1,\infty)$ so that Lemma 2.4 gives us

$$\int_\Omega h(|\nabla\chi|)\,\mathrm{d}x \leqslant c(n,h,\Omega)\int_\Omega h(|v|)\,\mathrm{d}x.$$

By combining (2.16) with the estimates (2.20) and (2.22), we end up with

$$\int_\Omega h(|\nabla v|)\,\mathrm{d}x \leqslant c(n,\varphi,\Omega)\left(\int_\Omega h(|v|)\,\mathrm{d}x + \int_\Omega h(|\mathcal{E}^D v|)\,\mathrm{d}x\right)$$

valid for all $v \in C^\infty(\overline{\Omega};\mathbb{R}^n)$, from which c) follows by an approximation argument.

3 Existence of minimizers: proof of Theorem 1.3 a)

Let $(u_m) \subset \mathbb{K}$ be a J-minimizing sequence, that is,

$$J[u_m] \xrightarrow{m} \inf_{\mathbb{K}} J.$$

Since $u_m - u_0 \in \overset{\circ}{W}{}^{1,h}(\Omega; \mathbb{R}^n)$, the Poincaré-type inequality from [24] (Lemma 2.4) in combination with the Korn-type inequality (2.7) yields

$$\|u_m - u_0\|_h \leqslant k\|\nabla u_m - \nabla u_0\|_h \leqslant k\|\mathcal{E}^D u_m - \mathcal{E}^D u_0\|_h.$$

On the other hand, $J[u_m] \leqslant c$, which implies

$$\|\mathcal{E}^D u_m\|_h \leqslant \int_\Omega h(|\mathcal{E}^D u_m|)\,\mathrm{d}x + 1 \leq J[u_m] + 2 \leqslant c$$

for all $m \gg 1$. By using (2.7) once more, we deduce that u_m is bounded in $W^{1,h}(\Omega; \mathbb{R}^n)$ so that we have $u_m \xrightarrow{m}: u$ with a function $u \in W^{1,h}(\Omega; \mathbb{R}^n)$ (at least for a subsequence). Note that $W^{1,h}(\Omega; \mathbb{R}^n)$ is reflexive since h satisfies global Δ_2- and ∇_2-conditions (as a consequence of (H3) and (H4); compare Remark 1.1 iii)); see [1]. Moreover, from $u_m - u_0 \in \overset{\circ}{W}{}^{1,h}(\Omega; \mathbb{R}^n)$ we infer $u - u_0 \in \overset{\circ}{W}{}^{1,h}(\Omega; \mathbb{R}^n)$ (compare Theorem 2.1 in [24]) so that the lower semicontinuity of J shows that u is a J-minimizer in the class $\mathbb{K}$. Finally, the uniqueness of u is a consequence of (H1).

4 Regularization and higher integrability

Let u be a (local) J-minimizer under the assumptions (H1)–(H4) with $\omega < 4/n$ and let $(u)_\rho$ denote a mollification of u. As usual we consider for a fixed ball $B_R = B_R(x_0) \Subset \Omega$ the more regular functional (compare [8])

$$J_\delta[w] := \int_{B_R} H_\delta(\mathcal{E}^D w)\,\mathrm{d}x$$

among vector fields $w \in (u)_\rho + \overset{\circ}{W}{}^{1,h}(B_R; \mathbb{R}^n)$, where

$$H_\delta(\sigma) := H(\sigma) + \delta(1 + |\sigma|^2)^{q/2} \quad (\sigma \in \mathbb{M}^n)$$

with exponent $q \geqslant 2$ as in (1.9) and

$$\delta = \delta(\rho) := \left(1 + \rho^{-1} + \|\mathcal{E}^D (u)_\rho\|_{q;B_R}^{2q}\right)^{-1}.$$

Then H_δ is strictly convex and of isotropic q-growth and J_δ admits a unique minimizer u_δ in the class $(u)_\rho + \overset{\circ}{W}{}^{1,h}(B_R; \mathbb{R}^n)$. As in [44] we see that u_δ enjoys the regularity properties collected in the following lemma.

Lemma 4.1 *Let (H1)–(H4) hold, and let* $\Gamma_\delta := 1 + |\mathcal{E}^D u_\delta|^2$.

a) $u_\delta \in W^{2,2}_{\mathrm{loc}}(B_R;\mathbb{R}^n)$ *and* $\tau_\delta := DH_\delta(\mathcal{E}^D u_\delta) \in W^{1,q/(q-1)}_{\mathrm{loc}}(B_R;\mathbb{M}^n)$.

b) $\Gamma_\delta^{q/4} \in W^{1,2}_{\mathrm{loc}}(B_R)$.

c) $u_\delta \rightharpoonup u$ *in* $W^{1,2}(B_R;\mathbb{R}^n)$ *as* $\delta \searrow 0$.

d) $\int_{B_R} h(|\mathcal{E}^D u_\delta|)\,\mathrm{d}x$ *is uniformly bounded and* $\delta \int_{B_R} \Gamma_\delta^{q/2}\,\mathrm{d}x \to 0$ *as* $\delta \searrow 0$.

The main aim in this section is the following theorem.

Theorem 4.2 *Let (H1)–(H4) be fulfilled with* $\omega < 4/n$.

a) $h(|\mathcal{E}^D u|) \in L^{n/(n-2)}_{\mathrm{loc}}(\Omega)$.

b) $\psi := \int_0^{|\mathcal{E}^D u|} \sqrt{\frac{h'(t)}{t}}\,\mathrm{d}t \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

In the proof we need the Caccioppoli-type inequality contained in the following lemma.

Lemma 4.3 *Let (H1)–(H4) hold. Then for all* $\eta \in C_0^\infty(B_R)$ *and* $\chi \in \mathcal{K}_{B_R}$ *we have the estimate*

$$\begin{aligned}
&\int_{B_R} \eta^2 D^2 H_\delta(\mathcal{E}^D u_\delta)(\partial_k \mathcal{E}^D u_\delta, \partial_k \mathcal{E}^D u_\delta)\,\mathrm{d}x \\
&\quad \leqslant c \int_{B_R} |\nabla \eta|^2 \Gamma_\delta^{\omega/2} \frac{h'(|\mathcal{E}^D u_\delta|)}{|\mathcal{E}^D u_\delta|} |\nabla u_\delta - \nabla \chi|^2\,\mathrm{d}x \\
&\qquad + c\delta \int_{B_R} |\nabla \eta|^2 \Gamma_\delta^{q/2-1} |\nabla u_\delta - \nabla \chi|^2\,\mathrm{d}x,
\end{aligned}$$

where c is independent of δ *and* R.

Proof From (5.7) in [44] we deduce the starting inequality

$$\begin{aligned}
&\int_{B_R} \eta^2 D^2 H_\delta(\mathcal{E}^D u_\delta)(\partial_k \mathcal{E}^D u_\delta, \partial_k \mathcal{E}^D u_\delta)\,\mathrm{d}x \\
&\quad \leqslant -2 \int_{B_R} \eta D^2 H_\delta(\mathcal{E}^D u_\delta)(\partial_k \mathcal{E}^D u_\delta, (\nabla \eta \odot \partial_k (u_\delta - \chi))^D)\,\mathrm{d}x.
\end{aligned} \tag{4.1}$$

On the right-hand side of (4.1) we apply the Cauchy-Schwarz and Young's inequality. After absorbing terms on the left-hand side (4.1) turns into

$$\begin{aligned}
&\int_{B_R} \eta^2 D^2 H_\delta(\mathcal{E}^D u_\delta)(\partial_k \mathcal{E}^D u_\delta, \partial_k \mathcal{E}^D u_\delta)\,\mathrm{d}x \\
&\quad \leqslant c \int_{B_R} D^2 H_\delta(\mathcal{E}^D u_\delta)((\nabla \eta \odot \partial_k (u_\delta - \chi))^D, (\nabla \eta \odot \partial_k (u_\delta - \chi))^D)\,\mathrm{d}x
\end{aligned}$$

so that, using (1.8) on the right-hand side, the desired inequality follows at once.

Proof (of Theorem 4.2) We fix a ball $B_\rho(\overline{x}) \Subset B_R$ and a function $\eta \in C_0^\infty(B_R)$, $\eta \geqslant 0$, with $\eta \equiv 1$ in $B_\rho(\overline{x})$ for some radius $\rho < r$, $\eta \equiv 0$ outside $B_r(\overline{x})$, and $|\nabla\eta| \leqslant c/(r-\rho)$. Since $h(|\mathcal{E}^D u_\delta|)^{1/2} \in W^{1,2}(B_R)$, we get with $\kappa := n/(n-2)$

$$
\begin{aligned}
&\int_{B_\rho(\overline{x})} h(|\mathcal{E}^D u_\delta|)^\kappa \, \mathrm{d}x \leqslant c\left(\int_{B_r(\overline{x})} |\nabla[\eta h(|\mathcal{E}^D u_\delta|)^{1/2}]|^2 \, \mathrm{d}x \right)^\kappa \\
&\leqslant c\left(\int_{B_r(\overline{x})} |\nabla\eta|^2 h(|\mathcal{E}^D u_\delta|) \, \mathrm{d}x + \int_{B_r(\overline{x})} \eta^2 \frac{h'(|\mathcal{E}^D u_\delta|)}{|\mathcal{E}^D u_\delta|} |\nabla \mathcal{E}^D u_\delta|^2 \, \mathrm{d}x \right)^\kappa \\
&\leqslant c\left(\int_{B_r(\overline{x})} |\nabla\eta|^2 h(|\mathcal{E}^D u_\delta|) \, \mathrm{d}x + \int_{B_r(\overline{x})} |\nabla\eta|^2 \Gamma_\delta^{\omega/2} \frac{h'(|\mathcal{E}^D u_\delta|)}{|\mathcal{E}^D u_\delta|} |\nabla u_\delta - \nabla\chi|^2 \, \mathrm{d}x \right. \\
&\quad \left. + \,\delta \int_{B_r(\overline{x})} |\nabla\eta|^2 \Gamma_\delta^{q/2-1} |\nabla u_\delta - \nabla\chi|^2 \, \mathrm{d}x \right)^\kappa \\
&=: c(I_1 + I_2 + \delta I_3)^\kappa,
\end{aligned}
$$

where we used (1.6) as well as (1.8) combined with the Caccioppoli-type inequality from Lemma 4.3. According to part d) of Lemma 4.1 we clearly have $I_1 \leqslant c/(r-\rho)^2$. To estimate I_2 we distinguish the cases $x \in B_r(\overline{x}) \cap [|\mathcal{E}^D u_\delta| \leqslant |\nabla u_\delta - \nabla\chi|]$ and $x \in B_r(\overline{x}) \cap [|\mathcal{E}^D u_\delta| > |\nabla u_\delta - \nabla\chi|]$. In the first case the monotonicity of $t \mapsto h'(t)/t$ together with the lower bound in (1.6) gives us

$$
\Gamma_\delta^{\omega/2} \frac{h'(|\mathcal{E}^D u_\delta|)}{|\mathcal{E}^D u_\delta|} |\nabla u_\delta - \nabla\chi|^2 \leqslant K\overline{h}(|\nabla u_\delta - \nabla\chi|),
$$

where $\overline{h}(t) := (1+t^2)^{\omega/2} h(t)$. In the latter case we obtain

$$
\Gamma_\delta^{\omega/2} \frac{h'(|\mathcal{E}^D u_\delta|)}{|\mathcal{E}^D u_\delta|} |\nabla u_\delta - \nabla\chi|^2 \leqslant K\overline{h}(|\mathcal{E}^D u_\delta|)
$$

so that

$$
I_2 \leqslant \frac{c}{(r-\rho)^2} \left(\int_{B_r(\overline{x})} \overline{h}(|\nabla u_\delta - \nabla\chi|) \, \mathrm{d}x + \int_{B_r(\overline{x})} \overline{h}(|\mathcal{E}^D u_\delta|) \, \mathrm{d}x \right).
$$

Now, we observe that $\overline{h}$ is an N-function that satisfies the conditions (H1)–(H4) so that we may apply the Korn-type inequality (2.2), that is, we may choose $\chi \in \mathcal{K}_{B_r(\overline{x})}$ such that

$$
\int_{B_r(\overline{x})} \overline{h}(|u_\delta - \chi|) \, \mathrm{d}x \leqslant c \int_{B_r(\overline{x})} \overline{h}(|\mathcal{E}^D u_\delta|) \, \mathrm{d}x, \tag{4.2}
$$

where c does not depend on δ. By combining (4.2) with the Korn-type inequality (2.3), we get

$$
I_2 \leqslant \frac{c}{(r-\rho)^2} \int_{B_r(\overline{x})} \overline{h}(|\mathcal{E}^D u_\delta|) \, \mathrm{d}x.
$$

By applying Young's inequality we can estimate

$$\delta I_3 \leqslant \delta \frac{c}{(r-\rho)^2}\left(\int_{B_r(\overline{x})} |\nabla u_\delta - \nabla\chi|^q \,\mathrm{d}x + \int_{B_r(\overline{x})} \Gamma_\delta^{q/2} \,\mathrm{d}x\right).$$

For the first integral on the right-hand side we obtain

$$\begin{aligned}\int_{B_r(\overline{x})} |\nabla u_\delta - \nabla\chi|^q \,\mathrm{d}x &\leqslant c\left(\int_{B_r(\overline{x})} |u_\delta - \chi|^2 \,\mathrm{d}x + \int_{B_r(\overline{x})} |\mathcal{E}^D u_\delta|^q \,\mathrm{d}x\right)\\ &\leqslant c\left(\int_{B_r(\overline{x})} \overline{h}(|u_\delta - \chi|) \,\mathrm{d}x + \int_{B_r(\overline{x})} \Gamma_\delta^{q/2} \,\mathrm{d}x\right)\\ &\leqslant c\left(\int_{B_r(\overline{x})} \overline{h}(|\mathcal{E}^D u_\delta|) \,\mathrm{d}x + \int_{B_r(\overline{x})} \Gamma_\delta^{q/2} \,\mathrm{d}x\right),\end{aligned}$$

where we combined the interpolation inequality (2.2) from [44] (recall $q \geqslant 2$) with (1.5) and (4.2). Summarizing the various estimates we have established:

$$\int_{B_\rho(\overline{x})} h(|\mathcal{E}^D u_\delta|)^\kappa \,\mathrm{d}x \leqslant \frac{c}{(r-\rho)^{2\kappa}}\left(1 + \int_{B_r(\overline{x})} \overline{h}(|\mathcal{E}^D u_\delta|) \,\mathrm{d}x + \delta \int_{B_r(\overline{x})} \Gamma_\delta^{q/2} \,\mathrm{d}x\right)^\kappa$$

and since the last integral on the right-hand side is uniformly bounded according to Lemma 4.1 d), we end up with

$$\int_{B_\rho(\overline{x})} h(|\mathcal{E}^D u_\delta|)^\kappa \,\mathrm{d}x \leqslant \frac{c}{(r-\rho)^{2\kappa}}\left[1 + \left(\int_{B_r(\overline{x})} h(|\mathcal{E}^D u_\delta|)|\mathcal{E}^D u_\delta|^\omega \,\mathrm{d}x\right)^\kappa\right].$$

But now, we are exactly in the same situation as in (2.7) of [8], and we can proceed as in [8] (recall $\omega < 4/n$) with the result:

$$h(|\mathcal{E}^D u_\delta|) \in L^{n/(n-2)}_{\mathrm{loc}}(B_R) \text{ uniformly with respect to } \delta. \tag{4.3}$$

Going through the above calculations, we get from the Caccioppoli-type inequality from Lemma 4.3:

$$\begin{aligned}&\int_{B_\rho(\overline{x})} D^2 H(\mathcal{E}^D u_\delta)(\partial_k \mathcal{E}^D u_\delta, \partial_k \mathcal{E}^D u_\delta) \,\mathrm{d}x\\ &\quad\leqslant \frac{c}{(r-\rho)^2}\left[1 + \int_{B_r(\overline{x})} h(|\mathcal{E}^D u_\delta|)|\mathcal{E}^D u_\delta|^\omega \,\mathrm{d}x\right],\end{aligned} \tag{4.4}$$

where the integral on the right-hand side is uniformly bounded on account (4.3) and our assumption $\omega < 4/n$. Hence, from (1.8) we infer $\partial_k \mathcal{E}^D u_\delta \in L^2_{\mathrm{loc}}(B_R; \mathbb{R}^n)$ uniformly with respect to δ for each $k \in \{1, \ldots, n\}$ so that according to Theorem 2.1 in [44] (recall also Lemma 4.1, c)) $u_\delta \in W^{2,2}_{\mathrm{loc}}(B_R; \mathbb{R}^n)$ uniformly with respect to δ. Therefore,

$$\begin{aligned}&u \in W^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n), \quad u_\delta \rightharpoonup u \text{ in } W^{2,2}_{\mathrm{loc}}(B_R; \mathbb{R}^n),\\ &\text{and} \quad \nabla u_\delta \to \nabla u \text{ a.e. in } B_R \text{ as } \delta \searrow 0,\end{aligned} \tag{4.5}$$

where the convergences hold after passing to a subsequence of u_δ not being relabeled. Hence, part a) of Theorem 4.2 follows by combining (4.5) with (4.3). To prove b), we observe that on account of (4.4) and (1.8) the functions

$$\psi_\delta := \int_0^{|\mathcal{E}^D u|} \sqrt{\frac{h'(t)}{t}}\,\mathrm{d}t$$

are bounded in $W^{1,2}_{\mathrm{loc}}(B_R)$ uniformly with respect to δ so that $\psi_\delta \rightharpoonup \psi$ in $W^{1,2}_{\mathrm{loc}}(B_R)$ as $\delta \searrow 0$ (recall (4.5)).

Remark 4.4 From the Caccioppoli-type inequality stated in Lemma 4.3 we deduce the following limit version by passing to the limit $\delta \searrow 0$ (compare Remark 2.1 in [8], or the proof of Lemma 6.3 in [44]): For each $\eta \in C_0^\infty(B_R)$ and $\chi \in \mathcal{K}_{B_R}$ it holds

$$\begin{aligned}&\int_{B_R} \eta^2 D^2 H(\mathcal{E}^D u)(\partial_k \mathcal{E}^D u, \partial_k \mathcal{E}^D u)\,\mathrm{d}x \\ &\qquad \leqslant \int_{B_R} |\nabla \eta|^2 |D^2 H(\mathcal{E}^D u)| |\nabla u - \nabla \chi|^2\,\mathrm{d}x.\end{aligned} \tag{4.6}$$

Alternatively, we may replace $D^2H(\mathcal{E}^D u)(\partial_k \mathcal{E}^D u, \partial_k \mathcal{E}^D u)$ on the left-hand side of (4.6) by $|\nabla \psi|^2$ or $|\nabla \mathcal{E}^D u|^2$ as a consequence of (1.8) and (1.9), respectively. Moreover, on the right-hand side we may replace $\nabla \chi$ by an arbitrary matrix $Q \in \mathbb{R}^{n\times n}$.

5 Partial Regularity: proof of Theorem 1.3, b)

Following the lines of [8] and consider the excess

$$(Eu)_{x_0,r} := \fint_{B_r(x_0)} |\mathcal{E}^D u - (\mathcal{E}^D u)_{x_0,r}|^2\,\mathrm{d}x + \fint_{B_r(x_0)} \widetilde{h}(|\mathcal{E}^D u - (\mathcal{E}^D u)_{x_0,r}|)\,\mathrm{d}x,$$

for balls $B_r(x_0) \Subset \Omega$, where $\widetilde{h}(t) := h(t)t^\omega$ is an N-function. Note that $(Eu)_{x_0,r}$ is well-defined since $u \in W^{1,\widetilde{h}}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ as a consequence of Theorem 4.2 a) combined with (2.1) (with h replaced by $\overline{h}$ defined in the proof of Theorem 4.2; note $\widetilde{h} \leqslant \overline{h}$) and Poincaré's inequality in Orlicz spaces [24] (Lemma 2.4).

Lemma 5.1 *Let $L > 0$ be given. Then there is a positive constant $c_* = c_*(L)$ with the property: To each $\tau \in (0,1)$ there exists a positive number $\varepsilon = \varepsilon(L,\tau)$ such that for every ball $B_r(x_0) \Subset \Omega$ for which*

$$|(\mathcal{E}^D u)_{x_0,r}| < L \quad \textit{and} \quad (Eu)_{x_0,r} < \varepsilon$$

hold, we have

$$(Eu)_{x_0,\tau r} \leqslant c_* \tau^2 (Eu)_{x_0,r}.$$

From the above lemma we deduce by a standard iteration procedure (compare [29]) that $\mathcal{E}^D u$ is of class $C^{0,\alpha}$ on the set

$$\Omega_0 := \Big\{ x \in \Omega \,:\, \sup_{r>0} |(\mathcal{E}^D u)_{x,r}| < \infty \text{ and } \liminf_{r \searrow 0} (Eu)_{x,r} = 0 \Big\}.$$

Moreover, Ω_0 is an open set of full Lebesgue measure. On account of $u \in W^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ (recall (4.5)) we then can argue as in [44] (Section 6) to obtain the statement in part b) of Theorem 1.3.

Proof (of Lemma 5.1) As usual (compare [8] or [44]) we argue by contradiction: Assume that for some $\tau \in (0,1)$ there exists a sequence of balls $B_{r_m}(x_m) \Subset \Omega$ such that

$$\begin{gathered} |(\mathcal{E}^D u)_{x_m,r_m}| < L, \quad (Eu)_{x_m,r_m} =: \lambda_m^2 \xrightarrow{m} 0, \\ (Eu)_{x_m,\tau r_m} > c_* \tau^2 \lambda_m^2. \end{gathered} \tag{5.1}$$

We define

$$u_m(z) := \frac{u(x_m + r_m z) - r_m A_m z - \chi_m(z)}{\lambda_m r_m} \quad (z \in B_1),$$

where $A_m := (\mathcal{E}^D u)_{x_m,r_m}$ and $\chi_m \in \mathcal{K}_{B_1}$ is chosen according to Theorem 2.1 such that (compare Remark 2.1 vii))

$$\int_{B_1} h(|\lambda_m u_m|)\,\mathrm{d}z + \int_{B_1} h(|\lambda_m \nabla u_m|)\,\mathrm{d}z \leqslant c(h) \int_{B_1} h(|\lambda_m \mathcal{E}^D u_m|)\,\mathrm{d}z \tag{5.2}$$

for all N-functions h satisfying the conditions (H1)-(H4).
Observing $\mathcal{E}^D u_m = \lambda_m^{-1}[\mathcal{E}^D u(x_m + r_m z) - A_m]$, the definition of λ_m implies

$$\fint_{B_1} |\mathcal{E}^D u_m|^2\,\mathrm{d}z + \lambda_m^{-2} \fint_{B_1} \widetilde{h}(|\mathcal{E}^D u_m|)\,\mathrm{d}z = 1, \tag{5.3}$$

which together with (5.2) with $h(t) = t^2$ and (5.3) leads to

$$\begin{aligned} \int_{B_1} |\nabla u_m|^2\,\mathrm{d}z &= \lambda_m^{-2} \int_{B_1} |\lambda_m \nabla u_m|^2\,\mathrm{d}z \\ &\leqslant c\lambda_m^{-2} \int_{B_1} |\lambda_m \mathcal{E}^D u_m|^2\,\mathrm{d}z = c \int_{B_1} |\mathcal{E}^D u_m|^2\,\mathrm{d}z \leqslant c. \end{aligned}$$

Hence, we have boundedness of (u_m) in $W^{1,2}(B_1;\mathbb{R}^n)$ and therefore (at least for a subsequence being not relabeled)

$$\begin{aligned} u_m &\xrightarrow{m}: \overline{u} \quad \text{in } W^{1,2}(B_1;\mathbb{R}^n) \\ \lambda_m \mathcal{E}^D u_m &\xrightarrow{m} 0 \quad \text{in } L^2(B_1;\mathbb{M}^n) \text{ and a.e. in } B_1. \end{aligned} \tag{5.4}$$

Moreover, $A_m \xrightarrow{m}: A$ (for a subsequence) with a matrix $A \in \mathbb{M}^n$, $|A| \leqslant L$, and $\overline{u}$ fulfills

$$\int_{B_1} D^2 H(A)(\mathcal{E}^D \overline{u}, \mathcal{E}^D \varphi)\,\mathrm{d}z = 0 \tag{5.5}$$

for all $\varphi \in C^1_0(B_1;\mathbb{R}^n)$, which can be shown as in [4] (Proposition 5.1).
By virtue of (5.5) and Lemma 4.1 in [44] $\overline{u}$ belongs to $C^\infty(B_1;\mathbb{R}^n)$ and satisfies

$$\fint_{B_\tau} |\mathcal{E}^D\overline{u} - (\mathcal{E}^D\overline{u})_{0,\tau}|^2 \,\mathrm{d}z \leqslant c^*\tau^2 \fint_{B_1} |\mathcal{E}^D\overline{u} - (\mathcal{E}^D\overline{u})_{0,1}|^2 \,\mathrm{d}z \leqslant c^*\tau^2 \tag{5.6}$$

with a constant $c^* = c^*(L)$, where in the last step we used $(\mathcal{E}^D\overline{u})_{0,1} = 0$ (which follows from $(\mathcal{E}^D u_m)_{0,1} = 0$ and (5.4)) as well as (5.3). Suppose that we can show

$$\mathcal{E}^D u_m \xrightarrow{m} \mathcal{E}^D\overline{u} \text{ in } L^2_{\mathrm{loc}}(B_1;\mathbb{M}^n), \tag{5.7}$$

$$\lambda_m^{-2} \fint_{B_r} \widetilde{h}(\lambda_m|\mathcal{E}^D u_m|)\,\mathrm{d}z \xrightarrow{m} 0 \text{ for } r<1. \tag{5.8}$$

Then (5.6) turns into

$$\lim_m \fint_{B_\tau} |\mathcal{E}^D u_m - (\mathcal{E}^D u_m)_{0,\tau}|^2 \,\mathrm{d}z + \lambda_m^{-2} \fint_{B_\tau} \widetilde{h}(|\mathcal{E}^D u_m - (\mathcal{E}^D u_m)_{0,\tau}|)\,\mathrm{d}z \leqslant c^*\tau^2.$$

But then, choosing $c_* = 2c^*$, we get a contradiction to our assumption (5.1) since the third condition in (5.1) is equivalent to

$$\fint_{B_\tau} |\mathcal{E}^D u_m - (\mathcal{E}^D u_m)_{0,\tau}|^2 \,\mathrm{d}z + \lambda_m^{-2} \fint_{B_\tau} \widetilde{h}(|\mathcal{E}^D u_m - (\mathcal{E}^D u_m)_{0,\tau}|)\,\mathrm{d}z > c_*\tau^2.$$

Therefore, it remains to prove (5.7) and (5.8). For this purpose we use (4.6) in the version with $|\nabla\mathcal{E}^D u|^2$ on the left- and a matrix $Q \in \mathbb{R}^{n\times n}$ on the right-hand side. If we consider a radius $0<t<1$ and choose η with $\eta \equiv 1$ in $B_{tr_m}(x_m)$ and $|\nabla\eta| \leqslant c/(r_m(1-t))$ in this version of (4.6), we obtain after scaling the inequality (compare the proof of Lemma 6.3 in [44] for similar calculations)

$$\begin{aligned}
&\int_{B_t} |\nabla\mathcal{E}^D u_m(z)|^2 \,\mathrm{d}z \\
&\qquad \leqslant c(1-t)^{-2}\lambda_m^{-2} \int_{B_1} |D^2H(\mathcal{E}^D u(x_m + r_m z))||\nabla u(x_m + r_m z) - Q|^2 \,\mathrm{d}z,
\end{aligned}$$

from which we deduce

$$\int_{B_t} |\nabla\mathcal{E}^D u_m|^2 \,\mathrm{d}z \leqslant c(1-t)^{-2} \int_{B_1} |D^2H(A_m + \lambda_m\,\mathcal{E}^D u_m)||\nabla u_m|^2 \,\mathrm{d}z \tag{5.9}$$

by choosing $Q \coloneqq A_m + r_m^{-1}\nabla\chi_m$. To estimate the right-hand side, we observe that according to (1.8) and the monotonicity of $t \mapsto h'(t)/t$ (recall $|A_m| < L$) we have

$$|D^2H(A_m + \lambda_m\,\mathcal{E}^D u_m)||\nabla u_m|^2 \leqslant c(K)\,|\nabla u_m|^2$$

on the set $[\lambda_m|\mathcal{E}^D u_m| \leqslant K]$, whereas on the set $[\lambda_m|\mathcal{E}^D u_m| \geqslant K]$ it holds (for sufficiently large K)

$$\begin{aligned}
&|D^2 H(A_m + \lambda_m \mathcal{E}^D u_m)||\nabla u_m|^2 \\
&\quad \leqslant c(K)\left[1 + (\lambda_m|\mathcal{E}^D u_m|)^\omega \frac{h'(\lambda_m|\mathcal{E}^D u_m|)}{\lambda_m|\mathcal{E}^D u_m|}\right]|\nabla u_m|^2 \\
&\quad \leqslant c(K)\,\lambda_m^{-2}\,\overline{h}(\lambda_m|\nabla u_m|).
\end{aligned}$$

Here, we abbreviated $\overline{h}(t) \coloneqq (1+t^2)^{\omega/2} h(t)$, which is an N-function satisfying the conditions (H1)–(H4). From (5.9) we infer (compare (3.20) in [18])

$$\int_{B_t} |\nabla \mathcal{E}^D u_m|^2 \,\mathrm{d}z \leqslant c(1-t)^{-2} \lambda_m^{-2} \int_{B_1} \overline{h}(\lambda_m|\nabla u_m|)\,\mathrm{d}z$$

so that, using (5.2) together with (5.3), we obtain

$$\begin{aligned}
\int_{B_t} |\nabla \mathcal{E}^D u_m|^2 \,\mathrm{d}z &\leqslant c(1-t)^{-2} \lambda_m^{-2} \int_{B_1} \overline{h}(\lambda_m|\mathcal{E}^D u_m|)\,\mathrm{d}z \\
&\leqslant c(1-t)^{-2}\left(\int_{B_1} |\mathcal{E}^D u_m|^2 \,\mathrm{d}z + \lambda_m^{-2}\int_{B_1} \widetilde{h}(\lambda_m|\mathcal{E}^D u_m|)\,\mathrm{d}z\right) \\
&\leqslant c(1-t)^{-2},
\end{aligned} \tag{5.10}$$

Note that in the second step we distinguished the cases $\lambda_m|\mathcal{E}^D u_m| \leqslant 1$ and $\lambda_m|\mathcal{E}^D u_m| \geqslant 1$, where in the first case we used $\overline{h}(t) \leqslant ct^2$ (valid for all $t \leqslant 1$ according to (1.6) and (1.7)) and $\overline{h}(t) \leqslant c\widetilde{h}(t)$ (valid for all $t \geqslant 1$) in the latter case.
Owing to (5.10) $|\nabla \mathcal{E}^D u_m|$ is bounded in $L^2_{\mathrm{loc}}(B_1)$ uniformly with respect to m. Consequently, by combining Theorem 2.1 a) (with $h(t) = t^2$) with (5.4), we get with respect to m uniform $W^{2,2}_{\mathrm{loc}}$-bounds of u_m. Upon passing to a subsequence (being not relabeled) we have

$$u_m \overset{m}{\rightharpoonup} \overline{u} \text{ in } W^{2,2}_{\mathrm{loc}}(B_1;\mathbb{R}^n),$$

which leads to the desired local strong convergence (5.7) according to Kondrachov's Theorem.
Now, we introduce the auxiliary functions

$$\Psi_m \coloneqq \lambda_m^{-1}\left(\int_0^{|A_m+\lambda_m \mathcal{E}^D u_m|} \sqrt{\frac{h'(t)}{t}}\,\mathrm{d}t - \int_0^{|A_m|} \sqrt{\frac{h'(t)}{t}}\,\mathrm{d}t\right)$$

and observe that according to Remark 4.4 the estimate (5.9) remains valid if we replace $\nabla \mathcal{E}^D u_m$ by $\nabla \Psi_m$ on the left-hand side. Hence, we also get a corresponding variant of (5.10), from which we deduce

$$\int_{B_t} |\nabla \Psi_m|^2 \,\mathrm{d}z \leqslant c(t). \tag{5.11}$$

On the other hand, following the lines of [18] (after (3.22)) with ∇ replaced by $\mathcal{E}^D$, we obtain $\int_{B_1} |\Psi_m|^2 \,\mathrm{d}z \leqslant c$ so that with (5.11) we end up with

$$\|\Psi_m\|_{1,2;B_t} \leqslant c(t) < \infty \quad \text{for all } t \in (0,1).$$

But now we can argue exactly as in [18] (after (3.23)) to get the desired local strong convergence (5.8). Note that the condition

$$t^\omega \leqslant c\big(1 + h(t)^2\big) \quad (t \geqslant 0)$$

required in [18] clearly is satisfied in our context as a consequence of the superquadratic growth of h (recall (1.7)) and our hypothesis $\omega < 4/n$. This completes the proof of the blow-up lemma Lemma 5.1, and thus of our partial regularity result Theorem 1.3 b).

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