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Martin Fuchs

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Martin Fuchs

Saarland University
Dep. of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax:	+ 49 681 302 4443
e-Mail:	preprint@math.uni-sb.de
WWW:	http://www.math.uni-sb.de/

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Abstract

We collect various Poincaré-type inequalities valid for fields of bounded deformation and give explicit upper bounds for the constants being involved.

1 Introduction

Variational problems arising in the theory of perfect plasticity are usually formulated in the space BD(Ω) consisting of all vectorfields ("deformations") $u : \Omega \to \mathbb{R}^n$, which belong to the class $L^1(\Omega; \mathbb{R}^n)$ and for which the symmetric gradient ("strain tensor")

$$\varepsilon(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^t\right) := \frac{1}{2}\left(\partial_i u^j + \partial_j u^i\right)_{1\leq i,j\leq n}$$

is a tensorvalued Radon measure of finite mass. Here Ω denotes a bounded domain in Euclidean space $\mathbb{R}^n$ with sufficiently regular (e.g. Lipschitz) boundary $\partial\Omega$, and the dimension n is equal to 2 or 3. We use the symbol ∇u to denote the Jacobian matrix of u, $(\nabla u)^t$ stands for its transpose. The Banach space BD(Ω) together with its natural norm

$$\|u\|_{BD(\Omega)} := \int\limits_\Omega |u|\, dx + \int\limits_\Omega |\varepsilon(u)|$$

has been introduced by Suquet [Su] and by Matthies, Strang and Christiansen [MSC], its role in perfect plasticity is outlined for example in the works of Temam and Strang [TS], Anzellotti and Giaquinta [AG] and of Seregin, we refer to the book [FuSe] for a historical overview and further references including Seregin's contributions.

A crucial tool for proving the coercivity on the space BD(Ω) of the energies occurring in plasticity theory consists of a collection of Poincaré-type inequalities, in which the L^1-norms of the deformations are estimated in terms of the total variations of the strains. In our paper we first want to give a short summary of the various estimates including some inequalities, which might be not so well-known. In a second major part consisting of three subsections we are going to obtain some explicit bounds for the constants being involved, and this aspect even seems to be of more importance for problems coming from applications. Let us start with a survey of the various (L^1-) Poincaré-type inequalities valid for fields from BD(Ω).

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THEOREM 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ denotes a bounded Lipschitz domain. Then there exist constants $C_i = C_i(\ldots)$ depending on the parameters specified below such that for $u \in BD\,(\Omega)$ we have the following estimates:*

a) $\|u\|_{L^1(\Omega)} \le C_1(n,\Omega)\int\limits_\Omega |\varepsilon(u)|$, *provided* $u|_{\partial\Omega} = 0$.

b) $\|u\|_{L^1(\Omega)} \le C_2(n,\Omega)\int\limits_\Omega |\varepsilon^D(u)|$, *where* $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}(\operatorname{div} u)\mathbf{1}, \mathbf{1} = (\delta_{ij})_{1\le i,j\le n}$, *and where again* $u|_{\partial\Omega} = 0$ *is required.*

c) $\|u - r_u\|_{L^1(\Omega)} \le C_3(n,\Omega)\int\limits_\Omega |\varepsilon(u)|$, *$r_u$ denoting a suitable rigid motion, i.e. an element of the kernel of ε, depending on u.*

d) $\|u - \kappa_u\|_{L^1(\Omega)} \le C_4(n,\Omega)\int\limits_\Omega |\varepsilon^D(u)|$ *for a Killing vector κ_u, i.e. an element of the kernel of the operator ε^D, which depends on the field u.*

e) $\|u\|_{L^1(\Omega)} \le C_5(n,\Omega,\Gamma)\int\limits_\Omega |\varepsilon(u)|$, *provided Γ is a part of $\partial\Omega$ having positive measure and u vanishes on Γ.*

Before commenting these results we wish to note that it is sufficient to study a) - e) for the smooth case, which is a consequence of the following approximation result due to Anzellotti and Giaquinta [AG] (see the comments stated after the proof of their Theorem 1.3).

Lemma 1.1. *For $u \in BD\,(\Omega)$ there exists a sequence $u_k \in C^\infty(\Omega;\mathbb{R}^n) \cap BD(\Omega)$ such that*

i) $u_k \to u \quad$ *in* $L^1(\Omega;\mathbb{R}^n)$,

ii) $u_k|_{\partial\Omega} = u|_{\partial\Omega} \quad$ *for all* k,

iii) $\int\limits_\Omega |\varepsilon_{ij}(u_k)|\,dx \to \int\limits_\Omega |\varepsilon_{ij}(u)|$ *as* $k \to \infty$ *for* $i,j = 1,\ldots,n$,

iv) $\int\limits_\Omega |\varepsilon^D(u)|\,dx \to \int\limits_\Omega |\varepsilon^D(u)|$ *as* $k \to \infty$.

REMARK 1.1. *a) By scaling it is easy to see that for $i = 1,\ldots,4$ the constants C_i can be chosen according to $C_i(n,\Omega) = \widetilde{C}_i(n)\mathrm{diam}(\Omega)$ with suitable constants $\widetilde{C}_i$ just depending on the dimension n. Here $\mathrm{diam}(\Omega)$ denotes the diameter of the domain Ω.*
b) For a more explicit description of the space of Killing vectors we refer to the papers of Reshetnyak [Re] and of Dain [Da]. For $n = 2$ this space consists of all holomorphic

mappings $\Omega \to \mathbb{R}^2$, whereas in higher dimensions it is finite dimensional.
c) Part a) Theorem 1.1 originates in the work of Strauss [Str] and in the present form it can be found in Proposition 1.2a) of Anzellotti and Giaquinta [AG]. Parts b) and d) recently have been established in [FuRe1] (see Theorem 2 and 3 of this paper). In case $n = 3$ the proof exploits representation formulas due to Reshetnyak [Re], whereas the 2D case was treated in [Fu].
For Theorem 1.1c) we refer to [AG] and [TS], where it is stated that the main idea of the proof actually goes back to the work of Kohn [Ko]. Finally, for Theorem 1.1e) the reader should consult Corollary 1.1 in [AG].
d) As discussed on p.231 of [FuRe1] it is unclear, if on the right-hand side of the inequality stated in Theorem 1.1e) the quantity $\varepsilon(u)$ can be replaced by the deviatoric part $\varepsilon^D(u)$.
e) We wish to note that L^p-variants of the above estimates for deformations from the Sobolev class $W^1_p(\Omega;\mathbb{R}^n)$ with $1 \leq p < \infty$ are valid and that the estimates of the constants C_i to be presented in Sections 2 - 4 can easily be adjusted to the p-case.

Before we turn to the derivation of upper bounds for some of the constants C_i let us recall the following result, which can be deduced from [Fu].

THEOREM 1.2. *Let $n = 2$ and consider a convex region Ω. Then we can choose*

$$C_i(2,\Omega) = \frac{1}{\sqrt{2}}\,\mathrm{diam}(\Omega) \tag{1.1}$$

for $i = 1, 2$ and 4.

Proof: Quoting Lemma 1.1 we may consider a smooth deformation. Then according to equation (3) in [Fu] it holds for $z \in \Omega$

$$u(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{u(w)}{w-z}\,dw - \frac{1}{\pi}\int_{\Omega} \frac{\partial_{\overline{z}} u(w)}{w-z}\,d\mathcal{L}^2(w)\,,$$

where we use standard complex notation. By $\int_{\partial\Omega}\ldots$ we denote the complex line integral and in the "volume integral" with respect to Lebesgue's measure $\mathcal{L}^2$ the symbol $\partial_{\overline{z}}$ stands for the Wirtinger operator.
Observing that $|\varepsilon^D(u)| = \sqrt{2}|\partial_{\overline{z}} u|$, where here and in what follows we always will make use of the Euclidean norms of the vectors and tensors under consideration, we obtain

$$|u(z) - \kappa_u(z)| \leq \frac{1}{\pi\sqrt{2}}\,V_{1/2}\left(|\varepsilon^D(u)|\right)(z)\,,$$

$\kappa_u(z)$ denoting the line integral $\frac{1}{2\pi i}\int_{\partial\Omega}\frac{u(w)}{w-z}\,dw$ and $V_{1/2}$ is the Riesz potential introduced in formula (7.31) of [GT] for the choices $\mu = 1/2$ and $n = 2$. Choosing $p = 1$ and $q = 1$ in (7.34) of [GT], inequality (1.1) is an immediate consequence of the foregoing considerations. $\square$

REMARK 1.2. *From the above proof we first obtain that $C_i(2,\Omega)$, $i = 1, 2$ and 4, can be chosen as*

$$\frac{\sqrt{2}}{\sqrt{\pi}}\,\sqrt{\mathcal{L}^2(\Omega)}\,.$$

If the isodiametric inequality (see, e.g.[Fe], 2.10.33 Corollary) is applied to estimate $\mathcal{L}^2(\Omega)$, then we arrive at (1.1).

As already remarked several times the purpose of the following sections is to derive some explicit choices for the constants C_1, C_3 and C_5 from Theorem 1.1, which means that for these values the inequalities from Theorem 1.1 are clearly satisfied. We guess that our results are far from being optimal, which can be seen by comparing the value of $C_1(2,\Omega)$ given in Theorem 1.2 with the one obtained in Section 3. Our discussion will not touch an estimation of the values of $C_2(3,\Omega)$ and $C_4(3,\Omega)$. In principle this can be done combining the ideas used in [FuRe1] with techniques as applied in the subsequent Sections 2 and 3.

2 A bound for the constant $C_3(n,\Omega)$

In this section we briefly review some results from [FuRe2]. This not only is done for completeness of the exposition, we also will make permanent use of the notation introduced below. From now on we assume that Ω is convex, a class of more general domains is treated in [FuRe2]. For smooth fields u we have according to equation (13) in the paper [MM] of Mosolov and Myasnikov the following representation: for x, $y \in \Omega$ and $i = 1, \dots, n$ it holds

$$\begin{aligned}(2.1)\qquad u^i(x) &= u^i(y) + \sum_{j=1}^n \omega_{ij}(y)(x_j - y_j)\\ &+ \sum_{k=1}^n \int_0^1 \left(\varepsilon_{ik}(z) - \sum_{j=1}^n \rho(y_j - x_j)\left\{\frac{\partial \varepsilon_{ik}}{\partial z_j}(z) - \frac{\partial \varepsilon_{kj}}{\partial z_i}(z)\right\}\right)(x_k - y_k)d\rho\,,\end{aligned}$$

where we have abbreviated

$$\varepsilon_{ij} := \varepsilon_{ij}(u), \omega_{ij} := \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} - \frac{\partial u^j}{\partial x_i}\right),\ z := x + \rho(y - x)\,.$$

Choose $q \in C^1_0(\Omega)$ such that

$$(2.2)\qquad 0 \le q \le 1,\ m(\Omega) := \int_\Omega q(y)\,dy > 0\,.$$

The reader should note that there is a large degree of freedom concerning the choice of admissible functions q: they just have to satisfy (2.2). We multiply equation (2.1) with

$q(y)$ and integrate the result with respect to $y \in \Omega$ ending up with (after passing to absolute values and performing a second integration with respect to the variable $x \in \Omega$)

$$\int_\Omega \left|u^i(x) - r_u^i(x)\right| \, dx \le \frac{1}{m(\Omega)} \, \left(T_1^i + T_2^i\right) , \ i = 1, \ldots, n \,, \tag{2.3}$$

where

$$r_u^i(x) := \frac{1}{m(\Omega)} \left\{ \int_\Omega u^i(y) q(y) \, dy + \sum_{j=1}^n \int_\Omega \omega_{ij}(y) q(y)(x_j - y_j) \, dy \right\} \tag{2.4}$$

defines the i^{th} component of a rigid motion r_u. We further have defined

$$T_1^i := \int_\Omega \int_\Omega \int_0^1 \sum_{k=1}^n |\varepsilon_{ik}(z)||x_k - y_k| q(y) d\rho \, dy \, dx \,,$$

$$T_2^i := \sum_{j,k=1}^n \int_\Omega \left| \int_\Omega \int_0^1 q(y)\rho(y_j - x_j)(y_k - x_k) \left\{ \frac{\partial \varepsilon_{ik}}{\partial z_j} \, (z) - \frac{\partial \varepsilon_{kj}}{\partial z_i} \, (z) \right\} \, d\rho \, dy \right| \, dx \,.$$

In order to continue we recall from [FuRe2] the following technical lemma, which in a different setting will also occur in Sections 3 and 4.

Lemma 2.1. *Let $f \in L^1(\Omega)$ such that $f \ge 0$ for almost all points in Ω. Then it holds*

$$\int_\Omega \int_\Omega \int_0^1 f(x + \rho(y - x)) d\rho \, dy \, dx \le \alpha_n |\Omega| \int_\Omega f(z) dz \,,$$

$$\alpha_n := 2 \frac{2^{n-1} - 1}{n - 1} \,, \tag{2.5}$$

$|\Omega|$ denoting the Lebesgue measure of Ω.

We apply Lemma 2.1 to the quantities T_1^i and T_2^i, which immediately yields

$$\begin{aligned} T_1^i &\le \operatorname{diam}(\Omega) \sum_{k=1}^n \int_\Omega \int_\Omega \int_0^1 \left|\varepsilon_{ik}\left(x + \rho(y - x)\right)\right| d\rho \, dy \, dx \\ &\le \ \alpha_n \operatorname{diam}(\Omega) |\Omega| \sum_{k=1}^n \int_\Omega |\varepsilon_{ik}(x)| \, dx \,. \end{aligned}$$

For handling T_2^i we perform an integration by parts in the expressions

$$\int_\Omega \int_0^1 q(y)\rho(y_j - x_j)(y_k - x_k)\{\ldots\} d\rho \, dy$$

observing the identities

$$\rho \frac{\partial \varepsilon_{ik}}{\partial z_j}(z) = \frac{\partial}{\partial y_j}\, \varepsilon_{ik}\left((1-\rho)x + \rho y\right) ,$$
$$\rho \frac{\partial \varepsilon_{kj}}{\partial z_i}(z) = \frac{\partial}{\partial y_i}\, \varepsilon_{kj}\left((1-\rho)x + \rho y\right) .$$

Then simple estimates in combination with Lemma 2.1 immediately show

$$T_2^i \leq \alpha_n \mathrm{diam}(\Omega)|\Omega|\Theta(q,\Omega) \sum_{j,k=1}^{n} \int_\Omega \left(|\varepsilon_{ik}| + |\varepsilon_{kj}|\right)\, dx ,$$

$$\Theta(q,\Omega) := 2 + \|\nabla q\|_{L^\infty(\Omega)}\ \mathrm{diam}(\Omega) . \tag{2.6}$$

With these estimates we return to (2.3) and recall that according to our previous convention $|\xi|$ denotes the Euclidean norm of a vector or tensor ξ. We therefore have

$$\int_\Omega |u - r_u|\, dx \leq n\, \alpha_n \frac{|\Omega|}{m(\Omega)} \mathrm{diam}(\Omega)\left(1 + 2n\Theta(q,\Omega)\right) \int_\Omega |\varepsilon(u)|\, dx . \tag{2.7}$$

Suppose now that u is a general function from the space $\mathrm{BD}(\Omega)$. Then (2.7) is true for a sequence of approximations u_k defined according to Lemma 1.1. In the second integral on the right-hand side of (2.4) (in the version for u_k) we can integrate by parts and pass to the limit $k \to \infty$, which finally will give

THEOREM 2.1. *Let Ω denote a bounded and convex region. Let*

$$C_3(n,\Omega) := n\, \alpha_n \frac{|\Omega|}{m(\Omega)} \mathrm{diam}(\Omega)\left(1 + 2n\Theta(q,\Omega)\right) \tag{2.8}$$

with $m(\Omega)$, α_n and $\Theta(q,\Omega)$ defined according to (2.2), (2.5) and (2.6) respectively. To a field $u \in BD(\Omega)$ we associate the rigid motion r_u with components

$$r_u^i(x) := \frac{1}{m(\Omega)} \Bigg\{ \int_\Omega u^i(y) q(y)\, dy$$
$$+ \sum_{j=1}^{n} \frac{1}{2} \int_\Omega \left(u^j(y) \frac{\partial}{\partial y_i}\{q(y)(x_j - y_j)\} - u^i(y) \frac{\partial}{\partial y_j}\{q(y)(x_j - y_j)\} \right) \Bigg\}\, dy ,$$

$i = 1, \ldots, n$. Then it holds

$$\int_\Omega |u - r_u|\, dx \leq C_3(n,\Omega) \int_\Omega |\varepsilon(u)| .$$

□

REMARK 2.1. *Of course the constant C_3 defined in (2.8) also depends on the choice of q, so a more adequate notion is $C_3(n,\Omega,q)$.*

3 The case of homogeneous boundary data: estimation of $C_1(n,\Omega)$

As in the previous section we consider a convex domain Ω. Suppose that $u \in C^\infty(\Omega;\mathbb{R}^n)\cap$ BD (Ω) has zero trace. We return to formula (2.1) and integrate this identity with respect to $y \in \Omega$ with the result

$$u^i(x) = r_u^i(x) + P^i(\varepsilon(u))(x),\ \ x\in\Omega, i = 1,\dots,n\,, \tag{3.1}$$

where we have set

$$r_u^i(x) = \fint_\Omega u^i(y)\ dy + \sum_{j=1}^n \fint_\Omega \omega_{ij}(y)(x_j - y_j)\ dy\,,$$
$$P^i(\varepsilon(u))(x) = \fint_\Omega \sum_{k=1}^n \int_0^1 \left(\varepsilon_{ik}(z) - \sum_{j=1}^n \rho(y_j - x_j)\left\{\frac{\partial\varepsilon_{ik}}{\partial z_j}\ (z) - \frac{\partial\varepsilon_{kj}}{\partial z_i}\ (z)\right\}\right)(x_k - y_k)\ d\rho\ dy\,,$$

ε_{ij}, ω_{ij} and z having the same meaning as in (2.1), $\fint_\Omega \dots$ denoting the mean value. Taking into account that $u|_{\partial\Omega} = 0$, we obtain for the rigid motion r_u

$$r_u = \frac{n+1}{2}\fint_\Omega u\ dy\,. \tag{3.2}$$

In fact, identity (3.2) immediately follows from

$$\begin{aligned}
&\int\limits_\Omega \omega_{ij}(y)(x_j - y_j)\ dy\\
&= \ -\frac{1}{2}\int\limits_\Omega \left(u^i(y)\frac{\partial}{\partial y_j}(x_j - y_j) - u^j(y)\frac{\partial}{\partial y_i}(x_j - y_j)\right)\ dy\\
&= \ -\frac{1}{2}\int\limits_\Omega \left(-u^i(y) + \delta_{ij}u^j(y)\right)\ dy\,,
\end{aligned}$$

hence

$$r_u^i(x) = \fint_\Omega u^i(y)\ dy + \frac{1}{2}\sum_{j=1}^n \fint_\Omega \left[u^i(y) - \delta_{ij}u^j(y)\right]\ dy = \fint_\Omega u^i(y)\ dy\left[1 + \frac{n}{2} - \frac{1}{2}\right]\,.$$

On the other hand, equation (3.1) implies

$$\int\limits_\Omega |u^i - r_u^i|\ dy = \int\limits_\Omega |P^i(\varepsilon(u))|\ dy \tag{3.3}$$

and in addition

$$\fint_\Omega u^i dy - \fint_\Omega r_u^i dy = \fint_\Omega P^i(\varepsilon(u))\ dy\,. \tag{3.4}$$

Combination of (3.2) and (3.4) then yields

$$\frac{-n+1}{2} \fint_{\Omega} u^i dy = \fint_{\Omega} P^i(\varepsilon(u))\, dy\,,$$

thus again by (3.2)

$$r_u^i = \frac{n+1}{1-n} \fint_{\Omega} P^i(\varepsilon(u))\, dy\,. \tag{3.5}$$

From (3.3) and (3.5) we get

$$\begin{aligned}
\|u^i\|_{L^1(\Omega)} &\leq \|u^i - r_u^i\|_{L^1(\Omega)} + \|r_u^i\|_{L^1(\Omega)} \\
&\leq \|P^i(\varepsilon(u))\|_{L^1(\Omega)} + \frac{n+1}{n-1}\left|\int_{\Omega} P^i(\varepsilon(u))\, dy\right| \\
&\leq \frac{2n}{n-1}\|P^i(\varepsilon(u))\|_{L^1(\Omega)}\,,
\end{aligned}$$

in other words

$$\int_{\Omega} |u^i|\, dy \leq \frac{2n}{n-1}\int_{\Omega} |P^i(\varepsilon(u))|\, dy,\ i = 1, \dots, n\,. \tag{3.6}$$

From the definition of $P(\varepsilon(u))$ stated after (3.1) we obtain

$$\begin{aligned}
&\int_{\Omega} \left|P^i(\varepsilon(u))\right|\, dx \leq \frac{1}{|\Omega|}\left(S_1^i + S_2^i\right)\,, \\
&S_1^i := \int_{\Omega}\int_{\Omega}\int_0^1 \sum_{k=1}^n |\varepsilon_{ik}(z)|\,|x_k - y_k|\, d\rho\, dy\, dx\,, \\
&S_2^i := \sum_{j,k=1}^n \int_{\Omega}\left|\int_{\Omega}\int_0^1 \rho(y_j - x_j)(y_k - x_k)\left\{\frac{\partial \varepsilon_{ik}}{\partial z_j}(z) - \frac{\partial \varepsilon_{kj}}{\partial z_i}(z)\right\}\, d\rho\, dy\right|\, dx\,,
\end{aligned}$$

and clearly we obtain the quantities S_j^i by formally letting $q \equiv 1$ in the definition of the items T_j^i stated after (2.4). Lemma 2.1 implies

$$S_1^i \leq \alpha_n |\Omega| \mathrm{diam}(\Omega) \sum_{k=1}^n \int_{\Omega} |\varepsilon_{ik}|\, dx$$

with α_n from (2.5), and since $u = 0$ on $\partial\Omega$ we can integrate by parts as done in Section 2 in order to handle S_2^i with the result ($S_{2,jk}^i$ denoting one term of the sum defining S_2^i)

$$\begin{aligned}
S_{2,jk}^i &\leq 2\ \mathrm{diam}(\Omega)\int_{\Omega}\int_{\Omega}\int_0^1 (|\varepsilon_{ik}(z)| + |\varepsilon_{kj}(z)|)\, d\rho\, dy\, dx \\
&\leq 2\alpha_n |\Omega|\ \mathrm{diam}(\Omega)\int_{\Omega} (|\varepsilon_{ik}| + |\varepsilon_{kj}|)\, dx\,,
\end{aligned}$$

where again Lemma 2.1 has been applied. Collecting terms we arrive at

$$\begin{aligned}\int_\Omega \left|P^i(\varepsilon(u))\right| dx \;\;&\le\;\; \alpha_n \operatorname{diam}(\Omega) \sum_{k=1}^n \int_\Omega |\varepsilon_{ik}|\, dx \\ &\qquad +2\alpha_n \operatorname{diam}(\Omega) \sum_{j,k=1}^n \int_\Omega (|\varepsilon_{ik}| + |\varepsilon_{kj}|)\, dx\,,\end{aligned}$$

hence by (3.6)

$$\int_\Omega |u^i|\, dx \le \frac{2n}{n-1}\alpha_n \operatorname{diam}(\Omega) \left\{ (1+2n) \sum_{k=1}^n \int_\Omega |\varepsilon_{ik}|\, dx + 2 \sum_{j,k=1}^n \int_\Omega |\varepsilon_{kj}|\, dx \right\}.$$

If we take the sum with respect to $i = 1, \ldots, n$ on both sides, we get $\Big($observing $\sum_{i,k=1}^n |\varepsilon_{ik}| \le n \left(\sum_{i,k=1}^n |\varepsilon_{ik}|^2\right)^{1/2}\Big)$

$$\sum_{i=1}^n \int_\Omega |u^i|\, dx \le n\frac{2n}{n-1}\alpha_n \operatorname{diam}(\Omega) \left\{ (1+2n) \int_\Omega |\varepsilon(u)|\, dx + 2n \int_\Omega |\varepsilon(u)|\, dx \right\},$$

which means by the definition of α_n (see (2.5))

$$\begin{aligned}\int_\Omega |u|\, dx \;\;&\le\;\; n\frac{2n}{n-1}\; \alpha_n (1+4n) \operatorname{diam}(\Omega) \int_\Omega |\varepsilon(u)|\, dx \\ &=\;\; \frac{4n^2}{(n-1)^2}\; (2^{n-1}-1)(1+4n) \operatorname{diam}(\Omega) \int_\Omega |\varepsilon(u)|\, dx\,.\end{aligned}$$

For $u \in \mathrm{BD}(\Omega)$ with $u|_{\partial\Omega} = 0$ we obtain the corresponding result by using Lemma 1.1, hence it is shown

THEOREM 3.1. *Let Ω denote a bounded convex region in $\mathbb{R}^n$. Then it holds*

$$\int_\Omega |u|\, dx \le \frac{4n^2}{(n-1)^2}\; (2^{n-1}-1)(1+4n)\; diam(\Omega) \int_\Omega |\varepsilon(u)|$$

for any $u \in BD\ (\Omega)$ such that $u|_{\partial\Omega} = 0$, and we can choose

(3.7) $$C_1(n,\Omega) = \frac{4n^2}{(n-1)^2}(2^{n-1}-1)(1+4n) \operatorname{diam}(\Omega)\,.$$

REMARK 3.1. *According to Theorem 1.2 we can select $C_1(2,\Omega)$ as $\frac{1}{\sqrt{2}} \operatorname{diam}(\Omega)$, whereas the choice of $C_1(2,\Omega)$ according to (3.7) leads to the value* $144 \operatorname{diam}(\Omega)$.

4 Mixed boundary conditions: estimation of the constant $C_5(n,\Omega,\Gamma)$ in terms of a finite dimensional variational problem

Let Ω denote a bounded convex domain in $\mathbb{R}^n$ and consider some (connected) part Γ of $\partial\Omega$ with positive measure. We will restrict ourselves to a special geometry assuming that

$$\Gamma \subset [x_n = 0],\ \Omega \subset [x_n > 0]\,. \tag{4.1}$$

If (4.1) is violated, then by a suitable transformation a sufficiently small part of Γ can be stratified and (4.1) holds at least locally. We leave it as an exercise to the reader to adjust the following considerations to this more general situation. Our first observation is

Lemma 4.1. *If $\mathcal{R}$ denotes the space of all rigid motions, then it holds*

$$K(n,\Omega,\Gamma) := \sup_{r\in\mathcal{R}-\{0\}} \frac{\|r\|_{L^1(\Omega)}}{\|r\|_{L^1(\Gamma)}} < \infty\,. \tag{4.2}$$

REMARK 4.1. *As we shall see below the calculation of $K(n,\Omega,\Gamma)$ is reduced to a more explicit finite dimensional extremal problem, from which upper bounds for K can be derived.*

Proof of Lemma 4.1: Let us consider the case $n = 2$. Then $r \in \mathcal{R}$ has the form

$$r(x) = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b =: A(x) + b$$

with $\alpha \in \mathbb{R}$ and $b \in \mathbb{R}^2$. In case $\alpha = 0$ and $b \neq 0$ we have

$$\frac{\|r\|_{L^1(\Omega)}}{\|r\|_{L^1(\Gamma)}} = \frac{\mathcal{L}^2(\Omega)}{\mathcal{H}^1(\Gamma)}\,,$$

where $\mathcal{H}^k$ denotes the Hausdorff measure of dimension k. If $\alpha \neq 0$, then it holds $A(x_0) = b$ for a suitable vector $x_0 \in \mathbb{R}^2$ and we obtain

$$\int_\Omega |r(x)|\,dx = \int_{x_0+\Omega} |A(y)|dy = |\alpha| \int_{x_0+\Omega} |y|dy\,,$$
$$\int_\Gamma |r(x)|d\mathcal{H}^1(x) = |\alpha| \int_{x_0+\Gamma} |y|d\mathcal{H}^1(y)\,.$$

So in order to verify (4.2), it remains to show that

$$\sup_{x\in\mathbb{R}^2} \left\{ \frac{\displaystyle\int_{x+\Omega} |y|dy}{\displaystyle\int_{x+\Gamma} |y|d\mathcal{H}^1(y)} \right\} < \infty\,. \tag{4.3}$$

Suppose on the contrary that (4.3) is false, which means

$$M(x_k)\Big/m(x_k) \overset{k\to\infty}{\longrightarrow} \infty\,,$$

$$M(x) := \int_{x+\Omega} |y|\,dy,\; m(x) := \int_{x+\Gamma} |y| d\mathcal{H}^1(y)\,,$$

for a suitable sequence of points x_k. In case $\sup_k |x_k| < \infty$ we may assume $x_k \to x_0$ for some $x_0 \in \mathbb{R}^2$, but then

$$M(x_k)\Big/m(x_k) \overset{k\to\infty}{\longrightarrow} M(x_0)\Big/m(x_0) < \infty\,.$$

Therefore we must have $|x_k| \to \infty$, and for $k \gg 1$ this implies

$$M(x_k) \leq 2|x_k|\mathcal{L}^2(\Omega),\; m(x_k) \geq \frac{1}{2}|x_k|\mathcal{H}^1(\Gamma)\,,$$

hence

$$M(x_k)\Big/m(x_k) \leq 4\mathcal{L}^2(\Omega)\Big/\mathcal{H}^1(\Gamma)\,.$$

So in both case we obtain a contradiction, thus (4.3) is established implying (4.2) in the 2D- case. Let us consider the situation for $n = 3$. In 3D a rigid motion r can be written as

$$r(x) = a \wedge x + b$$

with $a,\, b \in \mathbb{R}^3$, $\wedge$ denoting the cross product of vectors in $\mathbb{R}^3$. We introduce the quantities

$$\begin{aligned} M : \mathbb{R}^3 \times \mathbb{R}^3 &\to [0,\infty), \quad M(a,b) := \int_\Omega |a \wedge x + b|\,dx\,, \\ m : \mathbb{R}^3 \times \mathbb{R}^3 &\to [0,\infty), \quad m(a,b) := \int_\Gamma |a \wedge x + b| d\mathcal{H}^2(x)\,, \end{aligned}$$

and observe

$$m(a,b) = 0 \Longrightarrow a = b = 0\,. \tag{4.4}$$

In fact, $m(a,b) = 0$ implies

$$\begin{cases} a_2x_3 - x_2a_3 + b_1 = 0\,, \\ -a_1x_3 + a_3x_1 + b_2 = 0\,, \\ a_1x_2 - a_2x_1 + b_3 = 0 \end{cases}$$

on $\Gamma \subset \mathbb{R}^2 \times \{0\}$ (recall (4.1)), i.e.

$$\begin{cases} -x_2a_3 + b_1 = 0\,, \\ x_1a_3 + b_2 = 0\,, \\ a_1x_2 - a_2x_1 + b_3 = 0\,. \end{cases}$$

Obviously these equations can only hold in case $a = b = 0$. Now by (4.4) the ratio $M(a,b)\big/m(a,b)$ is well defined and positive for all $(a,b) \neq (0,0)$, and our claim (4.2) can be restated as

$$K(3,\Omega,\Gamma) = \sup_{(a,b)\neq(0,0)} M(a,b)\Big/m(a,b) < \infty\,. \tag{4.5}$$

Of course (4.5) is not an explicit formula for $K(3,\Omega,\Gamma)$, but it characterizes this quantity in terms of a finite dimensional extremal problem, from which upper bounds for $K(3,\Omega,\Gamma)$ can be deduced easily in concrete cases. As in the 2D-case we assume that (4.5) is wrong, hence

$$\Theta_k := M(a_k,b_k)\Big/m(a_k,b_k) \longrightarrow \infty,\ k \to \infty\,, \tag{4.6}$$

for a sequence $(a_k,b_k) \in \mathbb{R}^3 \times \mathbb{R}^3 - \{(0,0)\}$.

Case 1: $a_k = 0$ for infinitely many k
Then for such a subsequence we have

$$\Theta_k = \mathcal{L}^3(\Omega)\Big/\mathcal{H}^2(\Gamma) < \infty$$

contradicting (4.6).

Case 2: (w.l.o.g) $a_k \neq 0$ for all k
Observing that

$$\Theta_k = M(\widetilde{a}_k,\widetilde{b}_k)\Big/m(\widetilde{a}_k,\widetilde{b}_k),\ \widetilde{a}_k := \frac{a_k}{|a_k|},\ \widetilde{b}_k := \frac{b_k}{|a_k|}\,, \tag{4.7}$$

and that $\widetilde{a}_k \to: \widetilde{a}, |\widetilde{a}| = 1$, for a suitable subsequence, we distinguish two subcases:

Subcase a: $\widetilde{b}_k$ is bounded
Then - for a further subsequence - it holds $\widetilde{b}_k \longrightarrow: \widetilde{b} \in \mathbb{R}^3$ and

$$M(\widetilde{a}_k,\widetilde{b}_k)\Big/m(\widetilde{a}_k,\widetilde{b}_k) \longrightarrow M(\widetilde{a},\widetilde{b})\Big/m(\widetilde{a},\widetilde{b}) < \infty\,,$$

which together with (4.7) contradicts (4.6).

Subcase b: $|\widetilde{b}_k| \to \infty$ (for a subsequence)
In this case we estimate

$$\begin{aligned} M(\widetilde{a}_k,\widetilde{b}_k) &\leq |\widetilde{b}_k|\mathcal{L}^3(\Omega) + |\widetilde{a}_k| \int\limits_\Omega |y|\ dy\,, \\ m(\widetilde{a}_k,\widetilde{b}_k) &\geq |b_k|\mathcal{H}^2(\Gamma) - |\widetilde{a}_k| \int_\Gamma |y| d\mathcal{H}^2(y)\,, \end{aligned}$$

and from $|\widetilde{a}_k| = 1$ it follows (recall (4.7))

$$\limsup_{k\to\infty} \Theta_k \leq \mathcal{L}^3(\Omega) \Big/ \mathcal{H}^2(\Gamma)$$

again contradicting (4.6). This completes the proof of Lemma 4.1. □

Next we adjust Lemma 2.1 to the situation at hand.

Lemma 4.2. *Let $f \in C^0(\Omega) \cap L^1(\Omega)$ such that $f \geq 0$. Define the constant*

$$\widetilde{K}(n,\Omega,\Gamma) := \frac{2^{n-1}-1}{n-1}\,\mathcal{H}^{n-1}(\Gamma) + 2^{n-1}\int_{\Omega_0} \frac{1}{y_n}\,dy\,. \tag{4.8}$$

Then it holds

$$\int_\Gamma \int_0^1 \int_{\Omega_0} f\left(x+\rho(y-x)\right)\,dy\,d\rho\,d\mathcal{H}^{n-1}(x) \leq \widetilde{K}\int_\Omega f(z)\,dz\,. \tag{4.9}$$

Here Ω_0 denotes an arbitrary convex open subset of Ω such that $\overline{\Omega}_0 \subset \Omega$, in particular $\overline{\Omega}_0$ has positive distance to Γ.

REMARK 4.2. *More precisely we should write $\widetilde{K} = \widetilde{K}(n,\Omega,\Omega_0,\Gamma)$, since $\widetilde{K}$ also depends on the choice of Ω_0.*

Proof of Lemma 4.2: We split

$$\begin{aligned}
&\int_\Gamma \int_0^1 \int_{\Omega_0} f\left(x+\rho(y-x)\right)\,dy\,d\rho\,d\mathcal{H}^{n-1}(x)\\
&\quad= \int_{\Omega_0}\left\{\int_0^{1/2}\int_\Gamma f\left(x+\rho(y-x)\right)\,d\mathcal{H}^{n-1}(x)\,d\rho\right\}dy\\
&\qquad+\int_\Gamma\left\{\int_{1/2}^1\int_{\Omega_0} f\left(x+\rho(y-x)\right)\,dy\,d\rho\right\}d\mathcal{H}^{n-1}(x) =: T_1+T_2\,.
\end{aligned}$$

Let us abbreviate $S_{x,\rho}(y) := \rho y+(1-\rho)x, x\in\Gamma, 0\leq\rho\leq 1, y\in\Omega_0$. Observing $S_{x,\rho}(\Omega_0)\subset\Omega$ (at least for $\rho>0$) we find

$$\begin{aligned}
T_2 &= \int_\Gamma\int_{1/2}^1\left\{\int_{S_{x,\rho}(\Omega_0)} f(u)du\right\}\rho^{-n}d\rho\,d\mathcal{H}^{n-1}(x)\\
&\leq \int_\Omega f(u)du\int_\Gamma\int_{1/2}^1\rho^{-n}d\rho\,d\mathcal{H}^{n-1}(x)\,,
\end{aligned}$$

hence

$$T_2 \leq \frac{2^{n-1}-1}{n-1}\,\mathcal{H}^{n-1}(\Gamma)\int_\Omega f(x)\,dx\,. \tag{4.10}$$

Next let $R_{y,\rho}(x) := (1-\rho)x + \rho y$. We can write

$$\begin{aligned} T_1 &= \int_{\Omega_0} \int_0^{1/2} \int_\Gamma f\left(R_{y,\rho}(x)\right) d\mathcal{H}^{n-1}(x)\, d\rho\, dy \\ &= \int_{\Omega_0} \int_0^{1/2} (1-\rho)^{1-n} \int_{R_{y,\rho}(\Gamma)} f(u) d\mathcal{H}^{n-1}(u)\, d\rho\, dy\,. \end{aligned}$$

For $y \in \Omega_0$ we define the regions

$$\Omega_y := \{(1-\rho)x + \rho y : x \in \Gamma,\ 0 \le \rho \le 1/2\}\ .$$

For a point $u \in \Omega_y$ we denote by $\Phi(u)$ the unique point in Γ, where the ray starting in y and passing through u meets the boundary part Γ. Moreover we introduce on Ω_y the function $\rho(u)$ satisfying

$$\Phi(u)(1-\rho(u)) + \rho(u)y = u\,,$$

and from $\Phi^n(u) = 0$ it immediately follows that

$$\rho(u) = u_n \Big/ y_n\,. \tag{4.11}$$

We obtain (see, e.g. [Fe], 3.2.12 Theorem)

$$\begin{aligned} \int_0^{1/2} \int_{R_{y,t}(\Gamma)} f(u) d\mathcal{H}^{n-1}(u)\, dt &= \int_0^{1/2} \int_{\rho^{-1}(t)} f(u) d\mathcal{H}^{n-1}(u)\, dt \\ = \int_{\Omega_y} f(u) |\nabla \rho(u)|\, du &\overset{(4.11)}{\le} \frac{1}{y_n} \int_{\Omega_y} f(u) du \le \frac{1}{y_n} \int_\Omega f(u) du\,. \end{aligned}$$

This shows

$$\begin{aligned} T_1 &\le 2^{n-1} \int_{\Omega_0} \left\{ \int_0^{1/2} \int_{R_{y,\rho}(\Gamma)} f(u)\, d\mathcal{H}^{n-1}(u)\, d\rho \right\} dy \\ &\le 2^{n-1} \int_{\Omega_0} \frac{1}{y_n}\, dy \int_\Omega f(u)\, du\,, \end{aligned}$$

and together with (4.10) the inequality (4.9) is established. □

After these preparations we are now going to discuss the value of the constant C_5. As in Lemma 4.2 we fix Ω_0 and consider $u \in C^\infty(\Omega; \mathbb{R}^n) \cap \mathrm{BD}(\Omega)$ vanishing on Γ. Moreover, let us select $q \in C^1_0(\Omega_0)$ such that

$$0 \le q \le 1,\ m(\Omega_0) := \int_{\Omega_0} q(y)\, dy > 0\,. \tag{4.12}$$

Proceeding as done after (2.2) we obtain the representation

$$u = r_u + P(\varepsilon(u)) \text{ on } \Omega\,, \tag{4.13}$$

where for $i = 1, \dots, n$ and $x \in \Omega$

$$r_u^i(x) := \frac{1}{m(\Omega_0)} \left\{ \int_{\Omega_0} u^i(y) q(y)\, dy + \sum_{j=1}^n \int_{\Omega_0} \omega_{ij}(y) q(y)(x_j - y_j)\, dy \right\},$$

$$\begin{aligned} P^i(\varepsilon(u))(x) \quad := \quad & \frac{1}{m(\Omega_0)} \int_{\Omega_0} \int_0^1 \left(\sum_{k=1}^n \varepsilon_{ik}(x + \rho(y-x)) q(y)(x_k - y_k) \right. \\ & - \sum_{j,k=1}^n \rho(y_j - x_j)(x_k - y_k) q(y) \left\{ \frac{\partial \varepsilon_{ik}}{\partial z_j}(x + \rho(y-x)) \right. \\ & \left. \left. - \frac{\partial \varepsilon_{kj}}{\partial z_i}(x + \rho(y-x)) \right\} \right) d\rho\, dy, \end{aligned}$$

$$\varepsilon_{ij} := \varepsilon_{ij}(u), \omega_{ij} := \frac{1}{2} \left(\frac{\partial u^i}{\partial x_j} - \frac{\partial u^j}{\partial x_i} \right).$$

From (4.2), (4.13) and the fact that $r_u = P(\varepsilon(u))$ on Γ it follows

(4.14)
$$\begin{aligned} \|u\|_{L^1(\Omega)} \quad & \leq \quad \|r_u\|_{L^1(\Omega)} + \|P(\varepsilon(u))\|_{L^1(\Omega)} \\ & \leq \quad K\|r_u\|_{L^1(\Gamma)} + \|P(\varepsilon(u))\|_{L^1(\Omega)} \\ & \leq \quad K\|P(\varepsilon(u))\|_{L^1(\Gamma)} + \|P(\varepsilon(u))\|_{L^1(\Omega)}, \end{aligned}$$

and the items $\int\limits_\Omega |P^i(\varepsilon(u))(x)|\, dx$, $i = 1, \dots, n$, are bounded by the quantities $\frac{1}{m(\Omega_0)}(T_1^i + T_2^i)$ (compare the definition of these terms as stated after (2.4), where now the integration with respect to y only has to be performed over the set Ω_0). Proceeding as in Section 2 (see [FuRe2], Section 4 for details) we get

(4.15)
$$\begin{aligned} \sum_{i=1}^n \int\limits_\Omega |P^i(\varepsilon(u))|\, dx & \leq \frac{1}{m(\Omega_0)} \, (T_1^i + T_2^i) \\ & \leq \quad \frac{1}{2} \alpha_n \frac{|\Omega| + |\Omega_0|}{m(\Omega_0)} \, \mathrm{diam}(\Omega)(1 + 2n\Theta(q, \Omega)) \sum_{i,k=1}^n \int\limits_\Omega |\varepsilon_{ik}(x)|\, dx \end{aligned}$$

with α_n from (2.5) and $\Theta(q, \Omega)$ from (2.6). On account of (4.14) we next have to estimate the integrals $\int_\Gamma |P^i(\varepsilon(u))|\, d\mathcal{H}^{n-1}(x)$. Obviously we have

(4.16)
$$\int_\Gamma \left| P^i(\varepsilon(u)) \right| \, d\mathcal{H}^{n-1}(x) \leq \frac{1}{m(\Omega_0)} \left[\widetilde{T}_1^i + \widetilde{T}_2^i \right],$$

$$
\begin{aligned}
\widetilde{T}_1^i &:= \int_\Gamma \int_0^1 \int_{\Omega_0} \sum_{k=1}^n |\varepsilon_{ik}(x+\rho(y-x))|\, q(y) |x_k - y_k| \, dy \, d\rho \, d\mathcal{H}^{n-1}(x) \, , \\
\widetilde{T}_2^i &:= \sum_{j,k=1}^n \int_\Gamma \left| \int_0^1 \int_{\Omega_0} q(y) \rho (y_j - x_j)(y_k - x_k) \right. \\
& \left. \left\{ \frac{\partial \varepsilon_{ik}}{\partial z_j}(x+\rho(y-x)) - \frac{\partial \varepsilon_{kj}}{\partial z_i}(x+\rho(y-x)) \right\} dy \, d\rho \right| \, d\mathcal{H}^{n-1}(x) \, .
\end{aligned}
$$

With the help of Lemma 4.2 we obtain

$$
\begin{aligned}
\widetilde{T}_1^i &\leq \operatorname{diam}(\Omega) \int_\Gamma \int_0^1 \int_{\Omega_0} \sum_{k=1}^n |\varepsilon_{ik}(x+\rho(y-x))| \, dy \, d\rho \, d\mathcal{H}^{n-1}(x) \\
&\leq \widetilde{K} \operatorname{diam}(\Omega) \int_\Omega \sum_{k=1}^n |\varepsilon_{ik}| \, dx \, .
\end{aligned} \tag{4.17}
$$

$\widetilde{T}_2^i$ is handled via integration by parts with respect to the variable $y \in \Omega_0$ yielding

$$
\begin{aligned}
\widetilde{T}_2^i &\leq \Theta(q,\Omega) \operatorname{diam}(\Omega) \sum_{j,k=1}^n \int_\Gamma \int_0^1 \int_{\Omega_0} \Big\{ |\varepsilon_{ik}(x+\rho(y-x))| \\
& \quad + |\varepsilon_{kj}(x+\rho(y-x))| \Big\} \, dy \, d\rho \, d\mathcal{H}^{n-1}(x) \\
&\overset{(4.9)}{\leq} \widetilde{K} \Theta(q,\Omega) \operatorname{diam}(\Omega) \left[n \sum_{k=1}^n \int_\Omega |\varepsilon_{ik}| \, dx + \sum_{k,j=1}^n \int_\Omega |\varepsilon_{kj}| \, dx \right] .
\end{aligned}
$$

In combination with (4.17) this implies

$$
\begin{aligned}
\frac{1}{m(\Omega_0)} \left[\widetilde{T}_1^i + \widetilde{T}_2^i \right] &\leq \widetilde{K} \frac{\operatorname{diam}(\Omega)}{m(\Omega_0)} \Bigg[(1 + n\Theta(q,\Omega)) \\
& \quad \cdot \sum_{k=1}^n \int_\Omega |\varepsilon_{ik}| \, dx + \Theta(q,\Omega) \sum_{k,j=1}^n \int_\Omega |\varepsilon_{kj}| \, dx \Bigg] \, ,
\end{aligned}
$$

hence (recall (4.16))

$$
\int_\Gamma \sum_{i=1}^n \left| P^i(\varepsilon(u)) \right| \, d\mathcal{H}^{n-1}(x) \leq \widetilde{K} \frac{\operatorname{diam}(\Omega)}{m(\Omega_0)} \left(1 + 2n\Theta(q,\Omega)\right) \sum_{k,j=1}^n \int_\Omega |\varepsilon_{kj}| \, dx \, . \tag{4.18}
$$

Now we combine (4.15) and (4.18):

$$
\begin{aligned}
&K\|P(\varepsilon(u))\|_{L^1(\Gamma)} + \|P(\varepsilon(u))\|_{L^1(\Omega)} \\
&\quad\leq\; K\sum_{i=1}^{n}\int_\Gamma |P^i(\varepsilon(u))|\,d\mathcal{H}^{n-1} + \sum_{i=1}^{n}\int\limits_\Omega |P^i(\varepsilon(u))|\,dx \\
&\quad\leq\; \left\{K\widetilde{K} + \frac{1}{2}\alpha_n\left(|\Omega| + |\Omega_0|\right)\right\}\frac{\mathrm{diam}(\Omega)}{m(\Omega_0)} \\
&\qquad\quad \cdot\left(1 + 2n\Theta(q,\Omega)\right)\sum_{k,j=1}^{n}\int\limits_\Omega |\varepsilon_{kj}|\,dx \\
&\quad\leq\; n\left\{K\widetilde{K} + \frac{1}{2}\alpha_n\left(|\Omega| + |\Omega_0|\right)\right\}\frac{\mathrm{diam}(\Omega)}{m(\Omega_0)}\left(1 + 2n\Theta(q,\Omega)\right)\int\limits_\Omega |\varepsilon(u)|\,dx\,.
\end{aligned}
$$

If we insert this result into (4.14) and apply Lemma 1.1 to general u from $\mathrm{BD}(\Omega)$ such that $u|_\Gamma = 0$, it is finally shown:

THEOREM 4.1. *Consider a bounded convex domain Ω such that $\Omega \subset [x_n > 0]$ and with a flat boundary part Γ, i.e. $\Gamma \subset [x_n = 0]$, of positive measure. Moreover, fix a convex subregion Ω_0 and define the quantities α_n, q, $m(\Omega_0)$, $\Theta(q,\Omega)$, K and $\widetilde{K}$ according to (2.5), (4.12), (2.6), (4.2) and (4.8) respectively. Then we can choose*

$$
C_5(n,\Omega,\Gamma) = n\left\{K\widetilde{K} + \frac{1}{2}\alpha_n\left(|\Omega| + |\Omega_0|\right)\right\}\frac{\mathrm{diam}(\Omega)}{m(\Omega_0)}\left(1 + 2n\Theta(q,\Omega)\right)\,,
$$

which means that with this choice it holds

$$
\|u\|_{L^1(\Omega)} \leq C_5 \int\limits_\Omega |\varepsilon(u)|
$$

for all $u \in BD(\Omega)$ such that $u|_\Gamma = 0$.

$\square$

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