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Abstract

We discuss several variants of the TV-regularization model used in image recovery. The proposed alternatives are either of nearly linear growth or even of linear growth, but with some weak ellipticity properties. The main feature of the paper is the investigation of the analytic properties of the corresponding solutions.

1 Introduction

In our note we investigate the existence and the regularity of solutions for some variational problems related to variational and PDE methods used in image recovery as outlined for example in the papers of Rudin, Osher and Fatemi [ROF], Aubert and Vese [AV], Vese [Ve], Chan, Shen and Vese [CSV] and of Kawohl [Ka].

Suppose that we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ together with a function $f\colon \Omega \to \mathbb{R}$, for which we assume

$$f \in L^2(\Omega)\,. \tag{1.1}$$

Here f acts as an "observed image", and our goal is to reconstruct the "original image" $u\colon \Omega \to \mathbb{R}$ from f, where the quality of "data fitting" is measured through the quantity

$$E[u,\Omega] := \int_\Omega (u-f)^2 \,\mathrm{d}x\,.$$

In the variational approach towards the deconvolution of images in its most elementary form one tries to find the original image u by minimizing a functional of the type

$$E[w,\Omega] + \alpha \int_\Omega \Psi(|\nabla w|)\,\mathrm{d}x$$

among functions $w\colon \Omega \to \mathbb{R}$ from a suitable space. Here α is a (small) positive parameter, and Ψ is supposed to be an increasing and convex function being under our disposal.

i) A common choice is to perturb E with Dirichlet's energy or – as a first generalization – to work with the power function

$$\Psi_p(t) := t^p\,, \quad t \geq 0\,, \tag{1.2}$$

for a fixed exponent $1 < p < \infty$. This is the classical setting and quite well understood.

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ii) In the limit case, i.e. if $p = 1$ is included in (1.2), we obtain the so-called TV-regularization, which means that we are looking for minimizers u of (for notational simplicity we let $\alpha = 1$)

$$\int_\Omega (w - f)^2 \, \mathrm{d}x + \int_\Omega |\nabla w|$$

in the space $\mathrm{BV}(\Omega)$ ($= \mathrm{TV}(\Omega)$) of functions of bounded (= finite total) variation. For a definition and the investigation of this function space we refer to the textbook of Giusti ([Gi]). Here we may also pass to more regular (in the sense of ellipticity) energy densities still having linear growth, where the minimal surface integrand serves as the most prominent example.

iii) Of course the regularizing function has not to be of power growth as stated in (1.2). In particular, we can study examples satisfying a "nearly linear growth" condition like ("$L \log L$ case")

$$h(t) := t \ln(1 + t), \quad t \geq 0, \tag{1.3}$$

or any finite iteration of the logarithm.

In all these cases, the minimization is done w.r.t. comparison functions of the natural energy class and, as a model problem, we may fix Dirichlet boundary data u_0 in a suitable sense. However, unconstrained problems without boundary data are included in our considerations as well.

In the following we both discuss the $L \log L$ and some linear growth alternatives for the TV-regularization, which have to be handled in quite different analytical frameworks.

Nevertheless, there is a one-parameter family of regularizations connecting these cases and a series of numerical experiments may be inspired by this family exploiting the interesting features depending on the range of the parameter being involved.

Before going into details we want to describe and interprete our analytical results.

i) Superlinear problems in the sense of $L \log L$ or with densities of p-growth for some exponent $p > 1$ on the one hand admit regular solutions. This "nice" analytical behavior on the other hand may lead to the problem of "over smoothing" which means that bad data always produce smooth solutions. Here $L \log L$-growth should be seen as a first compromise between the cases $p > 1$ and $p = 1$ in (1.2): going through the regularity proof in the $L \log L$ case it becomes evident that, although we have a priori estimates, the constants are quite bad in comparison to those obtained for superlinear power growth.

ii) In the linear growth situation we are faced with several difficulties: for instance, working with a Dirichlet-boundary condition, the boundary data in general are not attained as the trace of a minimizer. Instead of this a penalty term measures the deviation from the boundary data.

Another feature is more serious: even though we will approximate the TV-density $|\nabla w|$ by a strictly convex integrand $F(\nabla w)$, the required linear growth of F allows only very weak and anisotropic ellipticity conditions, thus the study of smoothness properties such as continuity and differentiability of generalized minimizers is a quite delicate problem, and in Theorem 1.5 we will clearly see, how regularity depends on the modulus of ellipticity.

Roughly speaking, we have to distinguish two subcases being related to the exponent $\mu > 1$ introduced in (1.11) giving a lower bound for $D^2F(\nabla u)$ in terms of $(1 + |\nabla u|)^{-\mu}$:

(a) If F is "close" to $|\nabla u|\ln(1+|\nabla u|)$, i.e. if we require $\mu \in (1, 2)$, then – despite of the other particular problems occuring in connection with integrands of linear growth – the regularity theory can be carried out along similar lines as in the $L \log L$-case leading to comparable results, but with a reduction of over smoothing effects.

(b) In case $\mu \geq 2$ we still have the following results for the exact minimizer u:

- The solution u locally is of class L^t for any finite exponent t.
- If the observed image f has first weak derivatives in the space L^2_{loc}, then so does u.
- It is well known (see, e.g., [AFP], Lemma 3.76, p. 170) that subsets of Ω, which in a measure theoretic sense are smaller than an edge, are not recognized by the vector measure ∇u, which means that ∇u cannot concentrate on sets of vanishing $\mathcal{H}^1$-measure.
- Suppose that the solution u coincides with the observed image f on some disk B in Ω. Then on B the variation of the vector measure ∇f can be bounded in terms of the one-dimensional Hausdorff-measure.
- If we know the local boundedness of the solution, then we have local bounds for the measure $|\nabla u|$ through the measure $\mathcal{H}^1$.

We think that for the choice $\mu = 2$ we have reached a kind of limit case in the sense that for values of μ in the interval $[2, \infty)$ no further smoothness properties of a BV-minimizer can be expected, and this even concerns the existence of the first weak partial derivatives in the space L^1_{loc}: as it is outlined in [Bi1], examples of BV-solutions exist at least for $\mu > 3$, whose distributional gradient is not generated by a function in L^1_{loc}.

However, there is another "regularity feature" of our problem, which is valid for all exponents $\mu > 1$. Suppose we have a minimizing sequence usually consisting of smooth functions as constructed for example in Lemma 3.2 below. Then this minimizing sequence converges towards the BV-minimizer strongly in L^2, i.e. we have convergence in the norm induced by the quantity E, which in particular is the best degree of convergence we can expect for numerical approximations.

Now let us be more precise: logarithmic growth is treated in Part I of our note, in Part II we concentrate on linear growth problems.

Part I. The logarithmic regularization (1.3) on the one hand is very close to the linear growth of Ψ_1 (recall (1.2)), on the other hand it still leads to minimizers with rather good smoothness properties. We consider (1.3) as a model case for "nearly linear growth", however, we could also include the second iteration

$$\widetilde{h}(t) := t \ln\left(1 + \ln(1+t)\right)$$

or any other finite iteration of the logarithm into our discussion without essential changes in the arguments.

We wish to note that the logarithmic regularization of variational problems having linear growth in the first weak partial derivatives is well established in the mechanics of solids and fluids. As it is outlined in the paper [FrSe] of Frehse and Seregin, "plasticity with logarithmic hardening" serves as a model for perfect plasticity, and later Seregin and the second author (see [FuS1], [FuS2]) studied the Prandtl-Eyring fluid model as an logarithmic approximation of perfectly plastic fluids. From this maybe more philosophical point of view it seems reasonable to discuss the logarithmic version of the TV-regularization used in the variational approach of the deconvolution of images.

Referring to standard textbooks (see, e.g., [Ad]) we introduce the Orlicz space

$$L_h(\Omega) := \left\{ u :\, \Omega \to \mathbb{R} \text{ measurable} :\, \int_\Omega h\left(|u|\right)\, \mathrm{d}x < \infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{L_h(\Omega)} := \inf \left\{ l > 0 :\, \int_\Omega h\left(\frac{|u|}{l}\right)\, \mathrm{d}x \le 1 \right\}$$

and the Orlicz-Sobolev space

$$W^1_h(\Omega) := \left\{ v \in W^1_1(\Omega) :\, |v|,\, |\nabla v| \in L_h(\Omega) \right\}$$

with corresponding norm $\|v\|_{W^1_h(\Omega)} := \|v\|_{L_h(\Omega)} + \|\nabla v\|_{L_h(\Omega)}$. By definition a function $v \in W^1_h(\Omega)$ belongs to the standard Sobolev space $W^1_1(\Omega)$ and therefore it has a trace $u|_{\partial\Omega}$ in $L^1(\partial\Omega)$. Let us finally define the subspace

$$\overset{\circ}{W}{}^1_h(\Omega) := \text{closure of } C^\infty_0(\Omega) \text{ in } W^1_h(\Omega)$$

w.r.t. $\|\cdot\|_{W^1_h(\Omega)}$. In order to formulate our variational problem we may assume in addition to (1.1) that we are given a boundary datum u_0 with the property

$$u_0 \in W^1_h(\Omega)\,. \tag{1.4}$$

On the class $u_0 + \overset{\circ}{W}{}^1_h(\Omega)$ we then consider the energy

$$J[w,\Omega] := \int_\Omega (w-f)^2 \, \mathrm{d}x + \int_\Omega h\left(|\nabla w|\right) \mathrm{d}x \tag{1.5}$$

and observe that $J[w,\Omega]$ is well-defined by the continuity of the embedding

$$W^1_1(\Omega) \hookrightarrow L^2(\Omega)\,.$$

Let us note that actually the space $W^1_h(\Omega)$ is compactly embedded in the class $L^2(\Omega)$ on account of [Ad], 8.32 Theorem. Now we can state our existence result:

Theorem 1.1 *Let (1.1) and (1.4) hold and consider the energy J from (1.5) with h defined according to (1.3). Then the problem*

$$J[\cdot,\Omega] \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}^1_h(\Omega)$$

admits a unique solution u, and the same is true, if we minimize the functional $J[\cdot,\Omega]$ on the entire space $W^1_h(\Omega)$.

Proof of Theorem 1.1. We can refer to Theorem 3.1 from [FO] by remarking that the quantity $\int_\Omega (w-f)^2 \, \mathrm{d}x$ causes no difficulty. $\square$

Next we discuss the smoothness of our solution:

Theorem 1.2 *Let the assumptions of Theorem 1.1 hold. Suppose further that $f \in L^\infty_{\mathrm{loc}}(\Omega)$ holds. Then the minimizer u belongs to the space $W^1_{2,\mathrm{loc}} \cap W^2_{1,\mathrm{loc}}(\Omega)$ satisfying*

$$\frac{|\nabla^2 u|^2}{1+|\nabla u|} \in L^1_{\mathrm{loc}}(\Omega)\,. \tag{1.6}$$

In particular we have

$$\sqrt{1+|\nabla u|} \in W^1_{2,\mathrm{loc}}(\Omega)\,. \tag{1.7}$$

From (1.7) we immediately deduce

Corollary 1.1 *The function $|\nabla u|$ is in any class $L^p_{\mathrm{loc}}(\Omega)$, $p < \infty$, in particular it holds $u \in C^{0,\lambda}(\Omega)$ for all $\lambda \in (0,1)$.*

Remark 1.1 *The statements of Theorem 1.2 and of Corollary 1.1 extend to local minimizers $v \in W^1_{h,\mathrm{loc}}(\Omega)$ of our functional $J[\cdot,\Omega]$ from (1.5).*

The results of Theorem 1.2 can be improved considerably:

Theorem 1.3 *Under the assumptions and with the notation from Theorem 1.2, the minimizer u from Theorem 1.1 is of class $C^{1,\lambda}(\Omega)$ for any exponent $\lambda \in (0,1)$.*

Remark 1.2 *We wish to note that the proof of Theorem 1.3 heavily relies on the preliminary results stated in (1.6) and (1.7).*

Remark 1.3 *Without changes Theorem 1.1 – 1.3 extend to the case of vectorvalued images, i.e. f, u: $\Omega \to \mathbb{R}^M$ for some $M \geq 2$.*

Part II. Let us now discuss alternative approximations (compare Remark 1.6 for a list of examples) of the TV-density $\Psi_1(|\nabla u|) = |\nabla u|$ being of linear growth in ∇u but with a better degree of ellipticity compared to Ψ_1. In what follows we directly concentrate on vectorvalued images u, f: $\Omega \to \mathbb{R}^M$ with $M \geq 1$ and assume (compare (1.1) and (1.4)) that our data satisfy

$$u_0 \in W^1_2(\Omega; \mathbb{R}^M)\,, \quad f \in L^2 \cap L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^M)\,. \tag{1.8}$$

Here we assume w.l.o.g. that the boundary values u_0 are of class W^1_2, a suitable approximation procedure for W^1_1 data is outlined, for instance, in [Bi2]. Note that the presence of the function u_0 in (1.8) gives us the flexibility to include boundary conditions, but as before we can also work in the appropriate unrestricted classes. We further introduce the energy

$$I[w, \Omega] := \int_\Omega |w - f|^2 \,\mathrm{d}x + \int_\Omega F(\nabla w)\,\mathrm{d}x \tag{1.9}$$

for functions w from the space $W^1_1(\Omega; \mathbb{R}^M)$, where the density F: $\mathbb{R}^{2M} \to [0, \infty)$ is of class C^2 with $DF(0) = 0$ and satisfies the following set of assumptions: there exist positive constants ν_1, ν_2, ν_3 and a real number $\mu > 1$ such that for any Y, $Z \in \mathbb{R}^{2M}$ we have

$$|DF(Z)| \leq \nu_1 \tag{1.10}$$

and

$$\nu_2 \frac{1}{(1 + |Z|)^\mu} |Y|^2 \leq D^2F(Z)(Y, Y) \leq \nu_3 \frac{1}{1 + |Z|} |Y|^2\,. \tag{1.11}$$

Moreover, in the vector case $M \geq 2$ we assume that

$$F(Z) = \Phi(|Z|) \tag{1.12}$$

for a convex and increasing C^2-function Φ: $[0, \infty) \to [0, \infty)$. Of course these hypotheses look rather technical and therefore need some comments:

Remark 1.4 *It is easy to show that F is of linear growth in the sense that*

$$a|Z| - b \leq F(Z) \leq A|Z| + B$$

holds for all $Z \in \mathbb{R}^{2M}$ with suitable constants a, $A > 0$, b, $B \in \mathbb{R}$.

Remark 1.5 *Using (1.11) we obtain the inequalities*

$$\begin{aligned} DF(Z) : Z &\geq \nu_4 |Z| - \nu_5 \,, \\ \left|D^2F(Z)\right| |Z|^2 &\leq \nu_6 \left(1 + F(Z)\right) \,, \end{aligned}$$

again for arbitrary matrices Z and with constants ν_4, ν_5 and ν_6 being positive. Since we suppose $DF(0) = 0$, we may even choose $\nu_5 = 0$ (see, e.g., [Bi1], p. 98). The second estimate shows that F automatically satisfies a "balancing condition".

Remark 1.6 *The most prominent example for which we have (1.10) – (1.12) is the minimal surface integrand given by $F(Z) := \sqrt{1 + |Z|^2}$. In this case (1.11) holds for the optimal choice $\mu = 3$.*

Suppose next that we are given a number $\mu > 1$ and let

$$\Phi_\mu(t) := \int_0^t \int_0^s (1 + r)^{-\mu} \,\mathrm{d}r \,\mathrm{d}s \,, \quad t \geq 0 \,, \tag{1.13}$$

together with

$$F_\mu(Z) := \Phi_\mu(|Z|) \,, \quad Z \in \mathbb{R}^{2M} \,. \tag{1.14}$$

Then it holds:

the density F_μ from (1.14) with Φ_μ defined in (1.13) satisfies (1.10) and (1.11) exactly with the prescribed parameter μ.

For $\mu \neq 2$ we have the formula

$$\Phi_\mu(t) = \frac{t}{\mu - 1} + \frac{1}{\mu - 1}\frac{1}{\mu - 2}(t + 1)^{-\mu+2} - \frac{1}{\mu - 1}\frac{1}{\mu - 2} \,, \tag{1.15}$$

whereas

$$\Phi_2(t) = t - \ln(1 + t) \,,$$

and we see that

$$(\mu - 1)F_\mu(Z) \to |Z| \,, \quad \mu \to \infty \,, \tag{1.16}$$

for matrices $Z \in \mathbb{R}^{2M}$. For this reason and also with respect to the explicit formula (1.15) the density $F_\mu(\nabla u)$ serves as a very good candidate for an approximation of $|\nabla u|$ by more regular integrands of linear growth.

There is another interesting feature of the functions Φ_μ: if we formally let $\mu = 1$ in (1.13), then we obtain

$$\Phi_1(t) = t \ln(1 + t) + \ln(1 + t) - t \,,$$

which means that up to lower order terms the function Φ_1 coincides with the logarithmic density h defined in formula (1.3). Therefore and with respect to (1.16) we can *interprete the family of densities Φ_μ as a smooth curve in the space of integrands deforming the logarithmic density into the density $|\nabla u|$ occurring in the TV-regularization model.*

A slight modification of the functions Φ_μ from (1.13) is given by

$$\widetilde{\Phi}_\mu(t) := \int_0^t \int_0^s (1+r^2)^{-\mu/2}\,\mathrm{d}r\,\mathrm{d}s\,, \quad t \geq 1\,,$$

where as before $\mu > 1$. It is easy to check that (1.10) and (1.11) hold for the corresponding integrands $\widetilde{F}_\mu(Z) := \widetilde{\Phi}_\mu(|Z|)$, $Z \in \mathbb{R}^{2M}$. For $\mu = 3$ we obtain the minimal surface density, whereas

$$\widetilde{F}_2(Z) = |Z| \arctan |Z| - \frac{1}{2} \ln\left(1+|Z|^2\right)\,.$$

After these preparations we look at the variational problem

$$I[\cdot,\Omega] \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^M) \tag{1.17}$$

or its unconstrained version

$$I[\cdot,\Omega] \to \min \quad \text{in } W^1_1(\Omega;\mathbb{R}^M)\,. \tag{1.17*}$$

We cannot expect solvability of these problems in the non-reflexive space $W^1_1(\Omega;\mathbb{R}^M)$ unless we impose more restrictive assumptions (see Corollary 1.2). In general we only have existence results for the BV-variants of (1.17) and (1.17*).

Theorem 1.4 *Suppose that the density F satisfies (1.10) and (1.11) (together with (1.12), if the case $M > 1$ is considered) for some number $\mu > 1$. Let (1.8) hold for the data u_0 and f. Then the variational problems*

$$\int_\Omega F(\nabla^a w)\,\mathrm{d}x + \int_\Omega F_\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)\,\mathrm{d}|\nabla^s w| + \int_\Omega |f-w|^2\,\mathrm{d}x \\ + \int_{\partial\Omega} F_\infty\left((u_0 - w)\otimes \nu\right)\,\mathrm{d}\mathcal{H}^1 \to \min \tag{1.18}$$

and

$$\int_\Omega F(\nabla^a w)\,\mathrm{d}x + \int_\Omega F_\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)\,\mathrm{d}|\nabla^s w| + \int_\Omega |f-w|^2\,\mathrm{d}x \to \min \tag{1.18*}$$

are uniquely solvable in $\mathrm{BV}(\Omega;\mathbb{R}^M)$. Here F_∞ denotes the recession function of F, i.e.

$$F_\infty(Z) := \lim_{t\to\infty} \frac{1}{t} F(tZ)\,,\ Z \in \mathbb{R}^{2M}\,,$$

$\otimes$ is the tensor product of vectors and ν stands for the exterior normal of $\partial\Omega$. Moreover, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff-measure and $\nabla^a w$ ($\nabla^s w$) is the regular (singular) part of ∇w w.r.t. Lebesgue's measure $\mathcal{L}^2$.

Remark 1.7 *Clearly* (1.18^*) *is the appropriate extension of* (1.17^*) *to the space* $\mathrm{BV}(\Omega;\mathbb{R}^M)$. *The formal difference between the variational problems (1.17) and (1.18) obviously is the fact that in (1.18) the boundary condition enters implicitly through the penalty term*

$$\int_{\partial\Omega} F_\infty\left((u-u_0)\otimes\nu\right) dH^1 \,.$$

To give a clearer interpretation of this quantity, let us look at the scalar case $M=1$ *and let us also assume that* F *satisfies (1.12). Then it holds*

$$F_\infty(Z) = \Phi_\infty|Z| \,,\ Z\in\mathbb{R}^2 \,,\ \Phi_\infty := \lim_{t\to\infty}\frac{1}{t}\Phi(t) \,,$$

and $(u_0-w)\otimes\nu$ *just reduces to* $(u_0-w)\nu$, *which implies the validity of*

$$\int_{\partial\Omega} F_\infty\left((u_0-w)\otimes\nu\right) \mathrm{d}\mathcal{H}^1 = \Phi_\infty \int_{\partial\Omega} |u_0-w| \,\mathrm{d}\mathcal{H}^1$$

in this special case, for which the variational problems (1.18) and (1.18^*) *read as*

$$\int_\Omega \Phi(|\nabla^a w|)\,\mathrm{d}x + \Phi_\infty\int_\Omega |\nabla^s w| + \int_\Omega |f-w|^2\,\mathrm{d}x$$
$$\left(+ \Phi_\infty \int_{\partial\Omega} |u_0-w| \,\mathrm{d}\mathcal{H}^1 \right) \to \min \quad \text{in } \mathrm{BV}(\Omega)\,.$$

Moreover, we may choose $\Phi = (\mu-1)\Phi_\mu$ *with* Φ_μ *from (1.13) so that* $\Phi_\infty = 1$.

The next result summarizes the local regularity properties of our solutions:

Theorem 1.5 *Let the assumptions of Theorem 1.4 hold and let* $u\in\mathrm{BV}(\Omega;\mathbb{R}^M)$ *denote either the solution of problem (1.18) or of* (1.18^*).

i) Let μ *denote any number greater than* 1.

  (a) The function u *is in the space* $L^t_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$ *for any finite exponent* t.

  (b) Suppose that $f\in W^1_{2,\mathrm{loc}}(\Omega^*;\mathbb{R}^M)$ *for some open subset* Ω^* *of* Ω. *Then the same is true for the minimizer* u.

ii) If the case $\mu<2$ *is considered, then* u *belongs to the class* $W^1_{2,\mathrm{loc}}(\Omega;\mathbb{R}^M)$. *For the limit case* $\mu=2$ *the same is true under the smallness condition* $\|f\|_{L^\infty(\Omega)} < \sqrt{2\nu_2}$ *with* ν_2 *from (1.11).*

Remark 1.8 *The reader will find some global higher integrability results for problem* (1.18^*) *in Remark 3.5.*

Observing that $\mathrm{BV}\cap W^1_{2,\mathrm{loc}}(\Omega;\mathbb{R}^M)$ is a subspace of $W^1_1(\Omega;\mathbb{R}^M)$, we get

Corollary 1.2 *If we are in the situation ii) of Theorem 1.5 or if we consider the case i), (b), together with* $\Omega^* = \Omega$*, then* (1.17^*) *in particular admits a unique solution in the space* $W^1_1(\Omega;\mathbb{R}^M)$*.*

In general the vector measure ∇u behaves as follows:

Theorem 1.6 *Under the assumptions and with the notation of Theorem 1.4 consider the solution* u *of problem (1.18) or of* (1.18^*)*.*

i) For any subregion $\Omega^* \Subset \Omega$ *and all* $\alpha \in (0,1)$ *there exists a finite constant* $c(\alpha,\Omega^*)$ *such that*

$$|\nabla u|(B_r(x)) \le c(\alpha,\Omega^*)r^\alpha \tag{1.19}$$

is true for all disks $B_r(x) \subset \Omega^*$*.*

ii) Assume that $u = f$ *on some open subset* Ω^* *of* Ω*. Fix a domain* $\tilde{\Omega} \Subset \Omega^*$*. Then there is a constant* $c(\tilde{\Omega})$ *such that*

$$|\nabla f|(B_r(x)) \le c(\tilde{\Omega})r \tag{1.20}$$

is valid for all disks $B_r(x)$ *in* $\tilde{\Omega}$*. In particular we obtain*

$$|\nabla f|(E) \le c(\tilde{\Omega})\mathcal{H}^1(E) \tag{1.21}$$

for all Borel sets $E \subset \tilde{\Omega}$*.*

iii) If u *is locally bounded (compare Theorem 1.8 for sufficient conditions), then for any subdomain* $\Omega^* \Subset \Omega$ *there is a constant* $c(\Omega^*)$ *with the properties*

$$\begin{aligned} |\nabla u|(B_r(x)) &\le c(\Omega^*)r\,, \quad B_r(x) \subset \Omega^*\,, & (1.22)\\ |\nabla u|(E) &\le c(\Omega^*)\mathcal{H}^1(E)\,, \quad E \textit{ Borel subset of } \Omega^*\,. & (1.23) \end{aligned}$$

Remark 1.9 *The reader should note that ii) of Theorem 1.6 does not follow from i). In order to derive ii) we essentially use the local boundedness of* f*, and the local boundedness of* u *is needed to justify iii). We like to remark that estimate (1.22) reflects in somes sense a "*W^1_2*-behaviour" of the solution* u*: if we know that* ∇u *belongs to the space* $L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^{2M})$ *(compare Theorem 1.5 i), b), ii)), then (1.22) is a consequence of Hölder's inequality.*

Let us now state a surprising compactness property of minimizing sequences.

Theorem 1.7 *Under the hypotheses of Theorem 1.4 consider any minimizing sequence* $\{u_m\}$ *for (1.17) or for* (1.17^*)*. Then* $\{u_m\}$ *converges strongly in* $L^2(\Omega;\mathbb{R}^M)$ *to the unique* BV*-solution* u *of (1.18) or of* (1.18^*)*, respectively.*

We finish the introduction with some "maximum-principles" applicable to nearly linear and linear growth problems.

Theorem 1.8 *i) Let $M = 1$ and consider f such that $0 \le f \le 1$ almost everywhere in Ω. Then the same inequality is true for the unconstrained J-minimizer, i.e. the solution of*

$$J[\cdot, \Omega] \to \min \quad \text{in}\, W^1_h(\Omega)$$

with J from (1.5), and also for the BV*-solution u of* (1.18*)*, provided F satisfies (1.10) – (1.12).*

ii) Let $M \ge 1$, consider F with (1.10) – (1.12) and assume that

$$L := \|f\|_{L^\infty(\Omega;\mathbb{R}^M)} < \infty \,.$$

Then the BV-solution u of (1.18*) *satisfies*

$$|u| \le L$$

almost everywhere in Ω.

If in addition to (1.8) we know that $u_0 \in L^\infty(\Omega;\mathbb{R}^M)$, then we get for the BV*-solution w of (1.18) the bound*

$$\|w\|_{L^\infty(\Omega;\mathbb{R}^M)} \le \max\Big\{L, \|u_0\|_{L^\infty(\Omega;\mathbb{R}^M)}\Big\} \,.$$

In the subsequent sections we are going to present the proofs of these results following the natural subdivision into nearly linear growth and linear growth energy densities.

2 Part I. Nearly linear growth

2.1 Proof of Theorem 1.2

Let the assumptions of Theorem 1.2 hold with a given boundary function u_0, the unconstrained case follows by simplification of the arguments. Unfortunately our minimizer u does not have a sufficient degree of initial regularity in order to carry out the subsequent calculations. So we have to introduce a sequence of regularized variational problems, whose solutions are smooth and in addition converge towards u. Such a regularization can be done locally, i.e. on subdomains of Ω, providing also a proof of Remark 1.1. We refer to [FuS2], Section 3, for an outline of the details of this procedure. For technical simplicity we prefer the global technique applied in [FO], which requires the slightly stronger hypothesis (recall (1.4)) that we know $u_0 \in W^1_2(\Omega)$. Then we define for $\delta > 0$

$$J_\delta[w,\Omega] := \frac{\delta}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + J[w,\Omega]$$

on the class $u_0 + \overset{\circ}{W}{}^1_2(\Omega)$ and denote by u_δ the unique J_δ-minimizer in this space. (In the absence of the boundary condition we just minimize $J_\delta[\cdot,\Omega]$ on the entire space $W^1_2(\Omega)$

again with unique solution u_δ.) For later purposes we already remark that we have $u_\delta \in W^2_{2,\mathrm{loc}}(\Omega)$ which follows from standard results on elliptic equations (see, e.g., [GT] or [LU]).

The next lemma summarizes the convergence properties of the approximation:

Lemma 2.1 *With the notation introduced above we have*

$$u_\delta \rightharpoondown u \quad \text{in } W^1_1(\Omega) \quad \textit{and} \quad \sup_{0<\delta<1} \|u_\delta\|_{W^1_h(\Omega)} < \infty\,, \tag{2.1}$$

$$\delta \int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x \quad \to \quad 0\,, \tag{2.2}$$

$$J_\delta[u_\delta,\Omega] \quad \to \quad J[u,\delta] \tag{2.3}$$

as $\delta \downarrow 0$, where u denotes the $J[\cdot,\Omega]$-minimizer from Theorem 1.1.

Proof of Lemma 2.1. The proof is completely analogous to the one of Lemma 3.1 in [FO]. □

In order to give a clearer exposition of the following calculations, we will drop the index δ in all places, and we will also neglect the quantity $\frac{\delta}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x$ occurring in the perturbed functionals J_δ. However, the reader should always keep in mind that we actually work with the sequence of regularizations. In particular we must justify that all constants c_1, c_2, ... occurring during our estimates are independent of the parameter δ. We also agree to take the sum w.r.t. indices being repeated twice.

For notational simplicity we assume that f is a globally bounded function, otherwise we have to work on subdomains compactly contained in Ω.

Letting $H(p) := h(|p|)$, $p \in \mathbb{R}^2$, the J-minimality of u implies

$$0 = \int_\Omega DH(\nabla u)\cdot \nabla\varphi \,\mathrm{d}x + 2\int_\Omega (u-f)\varphi \,\mathrm{d}x$$

for any $\varphi \in C^\infty_0(\Omega)$, and if we replace φ by $\partial_\alpha\varphi$, $\alpha = 1$, 2, we obtain after an integration by parts

$$\int_\Omega D^2H(\nabla u)(\partial_\alpha\nabla u, \nabla\varphi)\,\mathrm{d}x = 2\int_\Omega (u-f)\partial_\alpha\varphi \,\mathrm{d}x\,. \tag{2.4}$$

Note that (2.4) actually is true for $\varphi \in W^1_2(\Omega)$ with compact support, which follows via approximation recalling the smoothness of the regularizations u $(= u_\delta)$. Fix a disk $B := B_R(x_0)$ such that $2B := B_{2R}(x_0)$ is compactly contained in Ω. We let $\eta \in C^\infty_0(2B)$ such that $\eta = 1$ on B, $0 \le \eta \le 1$ on $2B$ and $|\nabla\eta| \le c_1/R$, and define $\varphi := \eta^2\partial_\alpha u$. From (2.4) we obtain

$$\begin{aligned}\int_{2B} D^2H(\nabla u)(\partial_\alpha\nabla u, \partial_\alpha\nabla u)\eta^2 \,\mathrm{d}x &= -2\int_{2B} D^2H(\nabla u)(\partial_\alpha\nabla u, \nabla\eta)\eta\partial_\alpha u \,\mathrm{d}x \\ &\quad +2\int_{2B}(u-f)\partial_\alpha\left(\eta^2\partial_\alpha u\right)\,\mathrm{d}x \\ &=: T_1 + T_2\,. \end{aligned}\tag{2.5}$$

In order to estimate T_1 we apply the Cauchy-Schwarz inequality to the bilinear form $D^2H(\nabla u)$ and use Young's inequality with parameter $\varepsilon > 0$, hence

$$|T_1| \leq \varepsilon \int_{2B} D^2H(\nabla u)\,(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\,\eta^2\,\mathrm{d}x + c_2\varepsilon^{-1} \int_{2B} D^2H(\nabla u)\,(\nabla\eta, \nabla\eta)\,|\nabla u|^2\,\mathrm{d}x\,,$$

and with the choice $\varepsilon = 1/2$ we deduce from (2.5)

$$\begin{aligned}
&\int_{2B} \eta^2 D^2H(\nabla u)\,(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\,\mathrm{d}x \\
&\quad \leq c_3 \left[\int_{2B} D^2H(\nabla u)\,(\nabla\eta, \nabla\eta)\,|\nabla u|^2\,\mathrm{d}x \right] + 2T_2\,. \qquad (2.6)
\end{aligned}$$

For p, $q \in \mathbb{R}^2$ it holds

$$\frac{1}{1+|p|}|q|^2 \leq D^2H(p)(q,q) \leq 2\frac{\ln(1+|p|)}{|p|}|q|^2\,. \qquad (2.7)$$

Applying the ellipticity estimate (2.7) to the integrals from (2.6) involving $D^2H(\nabla u)$, we get from inequality (2.6)

$$\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{1+|\nabla u|}\,\mathrm{d}x \leq c_4 \left[R^{-2} \int_{2B} h\left(|\nabla u|\right)\,\mathrm{d}x \right] + 2T_2 =: c_5(R) + 2T_2\,. \qquad (2.8)$$

Now let us look at T_2. It holds

$$\begin{aligned}
T_2 &= -2\int_{2B} \partial_\alpha u \eta^2 \partial_\alpha u\,\mathrm{d}x - 2\int_{2B} f\partial_\alpha\left(\eta^2\partial_\alpha u\right)\,\mathrm{d}x \\
&= -2\int_{2B} |\nabla u|^2\,\eta^2\,\mathrm{d}x - 4\int_{2B} f\eta\partial_\alpha\eta\partial_\alpha u\,\mathrm{d}x - 2\int_{2B} f\eta^2\Delta u\,\mathrm{d}x\,. \qquad (2.9)
\end{aligned}$$

Observing the sign of the first term of the r.h.s. of (2.9) and applying Young's inequality one more time, we deduce from (2.8) and (2.9)

$$\begin{aligned}
&\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{1+|\nabla u|}\,\mathrm{d}x + 2\int_{2B} \eta^2|\nabla u|^2\,\mathrm{d}x \\
&\quad \leq c_5(R) + c_6\left[\varepsilon\int_{2B} \eta^2|\nabla u|^2\,\mathrm{d}x + \varepsilon^{-1}R^{-2}\int_{2B} f^2\,\mathrm{d}x + \int_{2B} \eta^2|f||\nabla^2 u|\,\mathrm{d}x \right]. \qquad (2.10)
\end{aligned}$$

Choosing ε small enough we get from (2.10)

$$\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{1+|\nabla u|}\,\mathrm{d}x + \int_{2B} \eta^2|\nabla u|^2\,\mathrm{d}x \leq c_5(R) + c_7\left[1 + \int_{2B} \eta^2|\nabla^2 u|\,\mathrm{d}x \right], \qquad (2.11)$$

where the boundedness of f has been used. A final application of Young's inequality yields for any $\varepsilon > 0$

$$\int_{2B} \eta^2 |\nabla^2 u| \, \mathrm{d}x \le \varepsilon \int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{1 + |\nabla u|} \, \mathrm{d}x + c_8 \varepsilon^{-1} \int_{2B} \eta^2 (1 + |\nabla u|) \, \mathrm{d}x \,, \tag{2.12}$$

and with (2.11) and (2.12) we arrive at

$$\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{1 + |\nabla u|} \, \mathrm{d}x + \int_{2B} \eta^2 |\nabla u|^2 \, \mathrm{d}x \le c_9(R) \,,$$

thus (introducing the index δ again)

$$\sup_{0<\delta<1} \left[\int_B \frac{|\nabla^2 u_\delta|^2}{1 + |\nabla u_\delta|} \, \mathrm{d}x + \int_B |\nabla u_\delta|^2 \, \mathrm{d}x \right] < \infty \,. \tag{2.13}$$

Note that inequality (2.13) is in correspondence to estimate (3.13) from the paper [FuS2], and as demonstrated there we get our claims (1.6) and (1.7) from (2.13). □

2.2 Proof of Theorem 1.3

Let the assumptions of Theorem 1.3 hold with fixed Dirichlet-boundary data u_0. We recall Lemma 2.1 and continue to work with the approximations u_δ for which we have shown the validity of (2.13), which means (compare (1.7)) that

$$\sqrt{1 + |\nabla u_\delta|} \in W^1_{2,\mathrm{loc}}(\Omega)$$

holds uniformly with respect to the parameter δ. This implies by Sobolev's embedding theorem

$$u_\delta \in L^\infty_{\mathrm{loc}}(\Omega)$$

uniformly, and for technical simplicity we may just assume that

$$\sup_{0<\delta<1} \|u_\delta\|_{L^\infty(\Omega)} < \infty \,, \quad \|f\|_{L^\infty(\Omega)} < \infty \,,$$

since otherwise we may pass to a subdomain Ω^* with compact closure in Ω. We finally let

$$\widetilde{f_\delta} := u_\delta - f \,.$$

As usual we drop the index δ keeping in mind that all our estimates are uniform w.r.t. δ. Consider a disk $B := B_R(x_0)$ such that $2B := B_{2R}(x_0) \Subset \Omega$, let $T := 2B - \overline{B}$ and choose $\eta \in C^\infty_0(2B)$ such $0 \le \eta \le 1$, $\eta \equiv 1$ on B and $|\nabla \eta| \le c/R$. As testfunction in (2.4) we select

$$\varphi := \eta^2 (\partial_\alpha u - \xi_\alpha) \,, \quad \xi := \fint_T \nabla u \, \mathrm{d}x \,, \quad \alpha = 1, 2 \,,$$

and obtain (after returning to summation with respect to α)

$$\begin{aligned}\int_{2B} \eta^2\omega^2 \,\mathrm{d}x \;&\leq\; c\left[\frac{1}{R}\int_T \left|D^2H(\nabla u)\right| \left|\nabla^2 u\right| |\nabla u - \xi| \,\mathrm{d}x + \int_{2B} \left|\partial_\alpha \left(\eta^2 \left[\partial_\alpha u - \xi_\alpha\right]\right)\right| \,\mathrm{d}x\right] \\ &=: \; c\left[S_1 + S_2\right] \end{aligned} \tag{2.14}$$

with c depending on $\|\widetilde{f}\|_{L^\infty(\Omega)}$. Here we have abbreviated

$$\omega := \left(D^2H(\nabla u)\left(\partial_\alpha \nabla u, \partial_\alpha \nabla u\right)\right)^{\frac{1}{2}}\,, \quad H(p) := h\left(|p|\right),\; p \in \mathbb{R}^2\,.$$

From (2.7) we deduce

$$\begin{aligned} S_1 \;&\leq\; c\frac{1}{R}\int_T \frac{\ln\left(1 + |\nabla u|\right)}{|\nabla u|} \left|\nabla^2 u\right| |\nabla u - \xi| \,\mathrm{d}x \\ &\leq\; c\frac{1}{R}\int_T \omega \frac{\sqrt{1 + |\nabla u|}}{|\nabla u|} \ln\left(1 + |\nabla u|\right) |\nabla u - \xi| \,\mathrm{d}x \\ &\leq\; c\frac{1}{R}\int_T \omega |\nabla u - \xi| \,\mathrm{d}x\,, \end{aligned}$$

where we have used the boundedness of the quantity

$$\frac{\sqrt{1 + |\nabla u|}}{|\nabla u|} \ln\left(1 + |\nabla u|\right)\,.$$

With the help of Hölder's inequality and a proper application of the Sobolev-Poincaré estimate we get

$$S_1 \leq c\frac{1}{R}\left(\int_T \omega^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \int_T \left|\nabla^2 u\right| \,\mathrm{d}x\,.$$

Let $\Theta := \sqrt{1 + |\nabla u|}$ and observe the validity of

$$\left|\nabla^2 u\right| \leq \Theta\omega\,.$$

Then we arrive at

$$S_1 \leq c\frac{1}{R}\left(\int_T \omega^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \int_T \omega\Theta \,\mathrm{d}x\,. \tag{2.15}$$

Next we have by Poincaré's inequality

$$\begin{aligned} S_2 \;&\leq\; c\left[\frac{1}{R}\int_T |\nabla u - \xi| \,\mathrm{d}x + \int_{2B} \eta^2 \left|\nabla^2 u\right| \,\mathrm{d}x\right] \\ &\leq\; c\left[\int_T \left|\nabla^2 u\right| \,\mathrm{d}x + \int_{2B} \eta^2 \left|\nabla^2 u\right| \,\mathrm{d}x\right] \\ &\leq\; c\left[\int_T \omega\Theta \,\mathrm{d}x + \int_{2B} \eta^2 \omega\Theta \,\mathrm{d}x\right]. \end{aligned}$$

Inserting this bound and also (2.15) into (2.14) we find

$$\int_{2B} \eta^2\omega^2 \,\mathrm{d}x \le \frac{c}{R}\left[\int_T \omega^2 \,\mathrm{d}x + R^2\right]^{\frac{1}{2}} \int_T \omega\Theta \,\mathrm{d}x + c\int_{2B} \eta^2\omega\Theta \,\mathrm{d}x\,. \tag{2.16}$$

To the last integral on the r.h.s. of (2.16) we can apply Young's inequality and get after putting the $\eta^2\omega^2$-term into the l.h.s. of (2.16):

$$\int_{B} \omega^2 \,\mathrm{d}x \le \frac{c}{R}\left[\int_T \omega^2 \,\mathrm{d}x + R^2\right]^{1/2} \int_T \omega\Theta \,\mathrm{d}x + c\int_{2B} \Theta^2 \,\mathrm{d}x\,. \tag{2.17}$$

Since $|\nabla u|$ is (locally) in L^p for any finite p (uniformly w.r.t. the approximation parameter) it follows that

$$\int_{2B} \Theta^2 \,\mathrm{d}x \le c_\varepsilon R^{2-\varepsilon}\,,$$

where $\varepsilon > 0$ can be chosen arbitrarily close to zero. With the exception of the additional contribution $R^{2-\varepsilon}$ generated by $\int_{2B}\Theta^2\,\mathrm{d}x$, inequality (2.17) corresponds to estimate (4.22) in Lemma 4.1 of [FrSe]. But as outlined in [ABF], p.295, or in [BF5], p.1615, the conclusion of the Frehse-Seregin lemma is still valid, i.e. we obtain

$$\int_{B_R(x_0)} \omega^2 \,\mathrm{d}x \le c\frac{1}{\ln(1/R)} \tag{2.18}$$

at least locally and uniformly w.r.t. to δ. Let $\sigma := DH(\nabla u)$. From

$$|\nabla\sigma| \le \sqrt{2}\,\omega$$

combined with (2.18) it follows that

$$\int_{B_R(x_0)} |\nabla\sigma|^2 \,\mathrm{d}x \le c\frac{1}{\ln(1/R)}\,,$$

and the continuity of σ follows from a result due to Frehse [Fr]. We emphasize again that actually the local modulus of continuity of σ_δ can be bounded independent of the parameter δ. But then we can complete the proof of Theorem 1.3 along the lines of e.g. [BF4], step 3 in the proof of Theorem 1, where the obvious modifications are left to the reader. □

3 Part II. The case of linear growth

3.1 Preliminaries, existence and convergences

We start with a collection of auxiliary results which we need for establishing Theorem 1.4 and Theorem 1.5. For simplicity we assume

$$F \text{ satisfies (1.10), (1.11) (and (1.12) in case } M > 1)\,, \tag{3.1}$$

$$u_0 \in W_2^1(\Omega;\mathbb{R}^M) \tag{3.2}$$

and

$$f \in L^2 \cap L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^M)\,, \tag{3.3}$$

where in (1.11) μ denotes any number in $(1,\infty)$. We further let (recall (1.9))

$$I[u,\Omega] \;:=\; \int_\Omega |u-f|^2\,\mathrm{d}x + \int_\Omega F(\nabla u)\,\mathrm{d}x\,,\;\; u \in W_1^1(\Omega;\mathbb{R}^M)\,, \tag{3.4}$$

$$\begin{aligned} K[w,\Omega] \;:=\;& \int_\Omega |w-f|^2\,\mathrm{d}x + \int_\Omega F(\nabla^a w)\,\mathrm{d}x + \int_\Omega F_\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right)\mathrm{d}|\nabla^s w| \\ &+ \int_{\partial\Omega} F_\infty\left((u_0-w)\otimes\nu\right)\,\mathrm{d}\mathcal{H}^1\,,\;\; w \in \;\mathrm{BV}\;(\Omega;\mathbb{R}^M)\,. \end{aligned} \tag{3.5}$$

For the notation used in (3.5) we refer to Theorem 1.4. Let us emphasize one more time that our arguments will not rely on the presence of the boundary function u_0:

in the absence of the boundary condition we just drop the boundary integral in (3.5) and the regularized variational problems introduced below are studied on the entire space $W_2^1(\Omega;\mathbb{R}^M)$.

The next lemma essentially has been shown in [BF2], Theorem 1.2:

Lemma 3.1 *Let (3.1) – (3.3) hold and define I and K according to (3.4) and (3.5).*

i) The variational problem

$$K[\cdot,\Omega] \to \min \quad \text{in } \mathrm{BV}(\Omega;\mathbb{R}^M)$$

admits a unique solution.

ii) It holds

$$\inf_{u_0+\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^M)} I[\cdot,\Omega] = \inf_{\mathrm{BV}(\Omega;\mathbb{R}^M)} K[\cdot,\Omega]\,.$$

iii) Consider the set

$$\begin{aligned} \mathcal{M} \;:=\;& \{u \in \mathrm{BV}(\Omega;\mathbb{R}^M) : \textit{ u is the } L^1\textit{-limit of an } I[\cdot,\Omega]\textit{- minimizing sequence} \\ & \textit{from the space } u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^M)\}\,. \end{aligned}$$

Then it holds: $u \in \mathcal{M}$ if and only if u is $K[\cdot,\Omega]$-minimizing in $\mathrm{BV}(\Omega;\mathbb{R}^M)$.

Proof. The existence result from i) as well the statements ii) and iii) of Lemma 3.1 have been established in [BF2] in the absence of the f-term given by $\int_\Omega |f-u|^2\,\mathrm{d}x$. However, this quantity causes no difficulties during the calculations, and since it is strictly convex w.r.t. the function u, we also obtain the uniqueness of the $K[\cdot,\Omega]$-minimizer. $\square$

Remark 3.1 *From i) and iii) it follows that $\mathcal{M}$ exactly consists of the $K[\cdot,\Omega]$-minimizing function u and that actually each $I[\cdot,\Omega]$-minimizing sequence from $u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^M)$ converges in the space $L^1(\Omega;\mathbb{R}^M)$ towards the function u.*

Remark 3.2 *The reader should note that Lemma 3.1 actually holds under less restrictive assumptions on the data as stated in (3.1) – (3.3), we again refer to [BF2].*

Remark 3.3 *Clearly Lemma 3.1 implies the statement of Theorem 1.4.*

As done in Section 2.1 we next introduce the global quadratic regularizations of the problem

$$I[\cdot,\Omega]\to\min \quad \text{in } u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^M)\,,$$

i.e. for $\delta\in(0,1]$ we let

$$I_\delta[w,\Omega]:=\frac{\delta}{2}\int_\Omega|\nabla w|^2\,\mathrm{d}x+I[w,\Omega]$$

and denote by u_δ the unique minimizer of $I_\delta[\cdot,\Omega]$ in class $u_0+\overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^M)$.

Lemma 3.2 *Suppose that we have (3.1) – (3.3). Then it holds (passing to appropriate subsequences):*

i) $\sup\limits_{0<\delta<1}\|u_\delta\|_{W^1_1(\Omega;\mathbb{R}^M)}<\infty$;

ii) $\delta\int_\Omega|\nabla u_\delta|^2\,\mathrm{d}x\to 0$ *as* $\delta\to 0$;

iii) $\{u_\delta\}$ *is an I-minimizing sequence in the class* $u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^M)$;

iv) $u_\delta\in W^2_{2,\mathrm{loc}}(\Omega;\mathbb{R}^M)$.

Proof. Let us discuss the statements of Lemma 3.2 in the case that the boundary condition does not occur, which means that u_δ is the unique $I_\delta[\cdot,\Omega]$-minimizer in the space $W^1_2(\Omega;\mathbb{R}^M)$.

Claim *i)* is immediate and *iv)* follows with the help of the difference quotient technique applied to equation (3.9) below.

In order to show *ii)* and *iii)* we have to introduce some notation following [ET] or [FuS2]. Let

$$\tau_\delta:=DF(\nabla u_\delta)\quad\text{and}\quad\sigma_\delta:=\delta\nabla u_\delta+\tau_\delta\,.$$

By the uniform estimate

$$I_\delta[u_\delta,\Omega]\le I_\delta[0,\Omega]=I[0,\Omega]$$

we have as $\delta \to 0$ (compare [Bi1])

$$\|\delta \nabla u_\delta\|^2_{L^2(\Omega;\mathbb{R}^{2M})} = \delta \left[\delta \int_\Omega |\nabla u_\delta|^2 \, \mathrm{d}x \right] \to 0\,, \tag{3.6}$$

$$\tau_\delta \overset{*}{\rightharpoondown}: \tau \quad \text{in } L^\infty(\Omega;\mathbb{R}^{2M})\,, \tag{3.7}$$

$$\sigma_\delta \rightharpoondown: \sigma = \tau \quad \text{in } L^2(\Omega;\mathbb{R}^{2M}) \tag{3.8}$$

as well as

$$\|u_\delta\|_{W^1_1(\Omega;\mathbb{R}^M)} \le c$$

for a suitable constant c independent of δ. Here and in what follows we always pass to convergent subsequences whenever this is necessary without relabeling the quantities under consideration.

Denote by u a BV-limit of the sequence $\{u_\delta\}$ and note that by the definition of u_δ and σ_δ the equation

$$\int_\Omega \sigma_\delta : \nabla\varphi \, \mathrm{d}x + 2\int_\Omega \varphi \cdot (u_\delta - f)\, \mathrm{d}x = 0 \tag{3.9}$$

holds for all $\varphi \in W^1_2(\Omega;\mathbb{R}^M)$. Passing to the limit $\delta \to 0$ and using the convergences (3.6) – (3.8) we see the validity of

$$\int_\Omega \tau : \nabla\varphi \, \mathrm{d}x + 2\int_\Omega \varphi \cdot (u - f)\, \mathrm{d}x = 0 \tag{3.10}$$

first for smooth φ and then by approximation for all $\varphi \in W^1_1(\Omega;\mathbb{R}^M)$.

Following the arguments given, for instance, in [Bi1], p. 18, we now need a careful analysis of the additional term $\int_\Omega |u-f|^2 \, \mathrm{d}x$ which causes some difficulties.

For $(v,\varkappa) \in W^1_1(\Omega;\mathbb{R}^M) \times L^\infty(\Omega;\mathbb{R}^{2M})$ the Lagrangian is defined by

$$l(v,\varkappa) = \int_\Omega \big[\varkappa : \nabla v - F^*(\varkappa)\big]\, \mathrm{d}x + \int_\Omega |v-f|^2 \, \mathrm{d}x\,,$$

where F^* denotes the conjugate function of F. A discussion of the properties of F^* can be found in [ET].

Letting

$$R[\varkappa] = \inf\big\{ l(v,\varkappa) :\, v \in W^1_1(\Omega;\mathbb{R}^M) \big\}$$

it is shown in [FuS2], Theorem 1.2.1, that

$$\sup_{\varkappa \in L^\infty(\Omega;\mathbb{R}^{2M})} R[\varkappa] = \inf_{w \in W^1_1(\Omega;\mathbb{R}^M)} I[w,\Omega]\,, \tag{3.11}$$

and, using the duality relation

$$\tau_\delta : \nabla u_\delta - F^*(\tau_\delta) = F(\nabla u_\delta)\,,$$

we obtain as in [Bi1] the representation

$$I_\delta[u_\delta,\Omega] = \underbrace{\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2\,\mathrm{d}x}_{=:\Theta_\delta} + \int_\Omega \big[\tau_\delta : \nabla u_\delta - F^*(\tau_\delta)\big]\,\mathrm{d}x + \int_\Omega |u_\delta - f|^2\,\mathrm{d}x\,.$$

The definition of σ_δ and τ_δ, respectively, and (3.9) applied to $\varphi = u_\delta$ then give

$$\begin{aligned} I_\delta[u_\delta,\Omega] &= -\Theta_\delta + \int_\Omega \big[\sigma_\delta : \nabla u_\delta - F^*(\tau_\delta)\big]\,\mathrm{d}x + \int_\Omega |u_\delta - f|^2\,\mathrm{d}x \\ &= -\Theta_\delta - \int_\Omega F^*(\tau_\delta)\,\mathrm{d}x - \int_\Omega |u_\delta|^2\,\mathrm{d}x + \int_\Omega |f|^2\,\mathrm{d}x\,. \end{aligned} \tag{3.12}$$

If we pass to the limit $\delta \to 0$ and use upper semicontinuity arguments (observe that we have the appropriate signs in (3.12)), equation (3.12) implies

$$\lim_{\delta\to 0} I_\delta[u_\delta,\Omega] \le -\lim_{\delta\to 0}\Theta_\delta - \int_\Omega F^*(\tau)\,\mathrm{d}x - \int_\Omega |u|^2\,\mathrm{d}x + \int_\Omega |f|^2\,\mathrm{d}x\,. \tag{3.13}$$

Now, given any $v \in W^1_1(\Omega;\mathbb{R}^M)$, we apply (3.10) to v and (3.13) yields

$$\begin{aligned} \lim_{\delta\to 0} I_\delta[u_\delta,\Omega] &\le -\lim_{\delta\to 0}\Theta_\delta + \int_\Omega \big[\tau : \nabla v - F^*(\tau)\big]\,\mathrm{d}x + \int_\Omega |v - f|^2\,\mathrm{d}x \\ &\quad - \int_\Omega |u - v|^2\,\mathrm{d}x \\ &= -\lim_{\delta\to 0}\Theta_\delta + l(v,\tau) - \int_\Omega |u - v|^2\,\mathrm{d}x \\ &\le -\lim_{\delta\to 0}\Theta_\delta + l(v,\tau)\,. \end{aligned} \tag{3.14}$$

Taking the infimum in (3.14) w.r.t. all admissible $v \in W^1_1(\Omega;\mathbb{R}^M)$, the inf-sup relation (3.11) finally shows

$$\sup_{\varkappa\in L^\infty(\Omega;\mathbb{R}^{2M})} R[\varkappa] \le \lim_{\delta\to 0} I_\delta[u_\delta,\Omega] \le -\lim_{\delta\to 0}\theta_\delta + R[\tau] \tag{3.15}$$

which proves the R-maximality of τ and our claims *ii*) and *iii*) as well. $\square$

The following Corollary is a first application of Lemma 3.2.

Corollary 3.1 *Under the assumptions of Lemma 3.1 let u^* denote the unique $K[\cdot,\Omega]$-minimizer from the space* $\mathrm{BV}(\Omega;\mathbb{R}^M)$*. Then it holds*

$$\lim_{\delta\to 0} \|u_\delta - u^*\|_{L^1(\Omega)} = 0\,. \tag{3.16}$$

Proof. From Lemma 3.2 i) we deduce the existence of $\widetilde{u} \in \mathrm{BV}\,(\Omega;\mathbb{R}^M)$ such that for a suitable sequence $\delta = \delta_m$ we have $u_\delta \to \widetilde{u}$ in $L^1(\Omega;\mathbb{R}^M)$. By iii) of Lemma 3.2 we have $\widetilde{u} \in \mathcal{M}$, thus $\widetilde{u} = u^*$, and therefore the whole family $\{u_\delta\}$ must converge towards u^* in the space $L^1(\Omega;\mathbb{R}^M)$, thus (3.16) follows. $\square$

Proof of Theorem 1.7. We now prove the much stronger result

$$\lim_{\delta\to 0} \|u_\delta - u^*\|_{L^2(\Omega)} = 0\,, \tag{3.17}$$

and (3.17) extends to any minimizing sequence.

Let us first recall the well known representation formula (see [ET])

$$\int_\Omega F(p)\,\mathrm{d}x = \sup_{\varkappa\in L^\infty(\Omega;\mathbb{R}^{2M})} \left[\int_\Omega \varkappa : p\,\mathrm{d}x - \int_\Omega F^*(\varkappa)\,\mathrm{d}x\right] \quad \text{for all } p \in L^1(\Omega;\mathbb{R}^{2M})\,. \tag{3.18}$$

In (3.14) it is actually shown, that for any $v \in W^1_1(\Omega;\mathbb{R}^M)$ we have

$$\lim_{\delta\to 0} I[u_\delta,\Omega] \le \int_\Omega \big[\tau : \nabla v - F^*(\tau)\big]\,\mathrm{d}x + \int_\Omega |v-f|^2\,\mathrm{d}x - \int_\Omega |u^*-v|^2\,\mathrm{d}x\,. \tag{3.19}$$

In particular u_δ is admissible for v in (3.19),

$$\lim_{\delta\to 0} I[u_\delta,\Omega] \le \int_\Omega \big[\tau : \nabla u_\delta - F^*(\tau)\big]\,\mathrm{d}x + \int_\Omega |u_\delta-f|^2\,\mathrm{d}x - \int_\Omega |u^*-u_\delta|^2\,\mathrm{d}x\,, \tag{3.20}$$

and (3.18), (3.20) give

$$\begin{aligned}
\lim_{\delta\to 0} I[u_\delta,\Omega] &\le \sup_{\varkappa\in L^\infty(\Omega;\mathbb{R}^{2M})} \left[\int_\Omega \varkappa : \nabla u_\delta\,\mathrm{d}x - \int_\Omega F^*(\varkappa)\,\mathrm{d}x\right] \\
&\quad + \int_\Omega |u_\delta - f|^2\,\mathrm{d}x - \int_\Omega |u^*-u_\delta|^2\,\mathrm{d}x \\
&= \int_\Omega F(\nabla u_\delta)\,\mathrm{d}x + \int_\Omega |u_\delta-f|^2\,\mathrm{d}x - \int_\Omega |u^*-u_\delta|^2\,\mathrm{d}x \\
&= I[u_\delta,\Omega] - \int_\Omega |u^*-u_\delta|^2\,\mathrm{d}x\,.
\end{aligned}$$

Passing also to the limit on the r.h.s. we have established (3.17), and the second claim follows in the same way, if in (3.19) v is replaced by the elements of any minimizing sequence. $\square$

3.2 Uniform local higher integrability of u_δ

Here we are going to prove uniform higher integrability of u_δ implying claim i), (a), of Theorem 1.5 (compare Section 3.4). We emphasize again that the next result is valid without any restriction on the modulus of ellipticity $\mu > 1$.

Theorem 3.1 *With the assumptions of Theorem 1.5, i), and for the minimizing sequence* $\{u_\delta\}$ *discussed in Lemma 3.1 we have for any* $t < \infty$ *and for any domain* $\Omega^* \Subset \Omega$

$$\|u_\delta\|_{L^t(\Omega^*)} \le c(\Omega^*, t)\,,$$

where the constant is not depending on δ.

The main tool is Lemma 3.3, which is very much in the spirit of [Bi2], Lemma 2.

Lemma 3.3 *With the notation and under the assumptions of Theorem 3.1 suppose that* $u_\delta \in L^\gamma_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$ *uniformly for some* $\gamma \ge 2$. *Then for any ball* $B := B_r(x_0)$ *such that* $2B := B_{2r}(x_0) \Subset \Omega$ *the estimate*

$$\int_{B_r(x_0)} (1+|\nabla u_\delta|^2)^{\frac{1}{2}} |u_\delta|^\gamma \, \mathrm{d}x + \delta \int_{B_r(x_0)} |\nabla u_\delta|^2 |u_\delta|^\gamma \, \mathrm{d}x + \int_{B_r(x_0)} |u_\delta|^{\gamma+2} \, \mathrm{d}x \le c \qquad (3.21)$$

holds with a constant c *which is not depending on* δ.

Proof of Lemma 3.3. As usual we may assume w.l.o.g. that f is in the space $L^\infty(\Omega;\mathbb{R}^M)$. Let us start with the scalar case $M = 1$ and choose $\eta \in C^\infty_0(2B)$ such that $\eta \equiv 1$ on B, $0 \le \eta \le 1$ on $2B$, $|\nabla \eta| \le c/r$. Moreover, fix $k \in \mathbb{N}$. Then the function $\varphi = |u_\delta|^\gamma u_\delta \eta^{2k}$ is an admissible choice in the Euler equation

$$\delta \int_\Omega \nabla u_\delta \cdot \nabla \varphi \, \mathrm{d}x + \int_\Omega DF(\nabla u_\delta) \cdot \nabla \varphi \, \mathrm{d}x + 2 \int_\Omega (u_\delta - f)\varphi \, \mathrm{d}x = 0 \qquad (3.22)$$

for any $\varphi \in C^\infty_0(\Omega)$. In the scalar case we trivially have

$$\nabla(|u_\delta|^\gamma u_\delta) = (\gamma+1)|u_\delta|^\gamma \nabla u_\delta \qquad (3.23)$$

and with the above choice of φ we obtain on account of the boundedness of $|DF|$

$$\begin{aligned} &(\gamma+1) \int_{2B} DF(\nabla u_\delta) \nabla u_\delta |u_\delta|^\gamma \eta^{2k} \, \mathrm{d}x \\ &+ \quad \delta(\gamma+1) \int_{2B} |\nabla u_\delta|^2 |u_\delta|^\gamma \eta^{2k} \, \mathrm{d}x + 2 \int_{2B} |u_\delta|^{\gamma+2} \eta^{2k} \, \mathrm{d}x \\ &\quad \le c(I_1 + I_2 + I_3)\,, \end{aligned} \qquad (3.24)$$

where

$$I_1 := \int_{2B} |u_\delta|^{\gamma+1} \eta^{2k-1} |\nabla \eta| \, \mathrm{d}x\,, \qquad (3.25)$$

$$I_2 := \delta \int_{2B} |\nabla u_\delta| |u_\delta|^{\gamma+1} \eta^{2k-1} |\nabla \eta| \, \mathrm{d}x\,, \qquad (3.26)$$

$$I_3 := \int_{2B} |f| |u_\delta|^{\gamma+1} \eta^{2k} \, \mathrm{d}x\,. \qquad (3.27)$$

By (1.11) (compare Remark 1.5) we have a lower bound for the l.h.s. of (3.24), thus

$$\int_{2B}(1+|\nabla u_\delta|^2)^{\frac{1}{2}}|u_\delta|^\gamma\eta^{2k}\,\mathrm{d}x+\delta\int_{2B}|\nabla u_\delta|^2|u_\delta|^\gamma\eta^{2k}\,\mathrm{d}x$$
$$+\int_{2B}|u_\delta|^{\gamma+2}\eta^{2k}\,\mathrm{d}x\le c(1+I_1+I_2+I_3) \tag{3.28}$$

and we have to find suitable estimates for I_1, I_2 and I_3.

We start we I_1 from (3.25) using Hölder's and Sobolev's inequality:

$$\begin{aligned}
I_1 &\le \left[\int_{2B}|u_\delta|^{\gamma+2}\eta^{4k-2}\,\mathrm{d}x\right]^{\frac{1}{2}}\underbrace{\left[\int_{2B}|u_\delta|^\gamma|\nabla\eta|^2\,\mathrm{d}x\right]^{\frac{1}{2}}}_{\le c(r)=c}\\
&\le c\int_{2B}\left|\nabla\left(|u_\delta|^{\frac{\gamma}{2}+1}\eta^{2k-1}\right)\right|\mathrm{d}x\\
&\le c\left[\int_{2B}|u_\delta|^{\frac{\gamma}{2}}|\nabla u_\delta|\eta^{2k-1}\,\mathrm{d}x+\int_{2B}|u_\delta|^{\frac{\gamma}{2}+1}\eta^{2k-2}|\nabla\eta|\,\mathrm{d}x\right].
\end{aligned}\tag{3.29}$$

Since we have $\gamma\ge 2$, the uniform integrability of u_δ gives a uniform bound for the second term in (3.29) and Young's inequality shows for any $\varepsilon>0$

$$I_1\le c\left[1+\varepsilon\int_{2B}|u_\delta|^\gamma(1+|\nabla u_\delta|^2)^{\frac{1}{2}}\eta^{2k}\,\mathrm{d}x+\varepsilon^{-1}\int_{2B}|\nabla u_\delta|\eta^{2(k-1)}\,\mathrm{d}x\right]. \tag{3.30}$$

If ε is small enough, then the first integral on the r.h.s. of (3.30) can be absorbed in the l.h.s. of (3.28), whereas the second integral remains uniformly bounded.

Now let us have a look at I_2 from (3.26): again by Hölder's and Sobolev's inequality we obtain ($c=c(r)$)

$$\begin{aligned}
I_2 &\le c\delta\left[\int_{2B}|\nabla u_\delta|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{2B}|u_\delta|^{2\gamma+2}\eta^{4k-2}\,\mathrm{d}x\right]^{\frac{1}{2}}\\
&\le c\delta^{\frac{1}{2}}\int_{2B}\left|\nabla\left(|u_\delta|^{\gamma+1}\eta^{2k-1}\right)\right|\mathrm{d}x\\
&\le c\delta^{\frac{1}{2}}\left[\int_{2B}|u_\delta|^\gamma|\nabla u_\delta|\eta^{2k-1}\,\mathrm{d}x+\int_{2B}|u_\delta|^{\gamma+1}\eta^{2k-2}|\nabla\eta|\,\mathrm{d}x\right]\\
&\le c\delta^{\frac{1}{2}}\left[\int_{2B}\left[\varepsilon\delta^{\frac{1}{2}}|u_\delta|^\gamma|\nabla u_\delta|^2\eta^{2k}+\varepsilon^{-1}\delta^{-\frac{1}{2}}|u_\delta|^\gamma\eta^{2k-2}\right]\mathrm{d}x\right.\\
&\qquad\left.+\int_{2B}|u_\delta|^{\gamma+1}\eta^{2k-2}|\nabla\eta|\,\mathrm{d}x\right]=:c\sum_{i=1}^3 I_2^i
\end{aligned}\tag{3.31}$$

where we also made use of Young's inequality. Here $\varepsilon > 0$ is chosen sufficiently small such that I_2^1 can be absorbed on the l.h.s. of (3.28), I_2^2 is uniformly bounded by assumption and we discuss I_2^3 similar as above supposing $k > 1$:

$$\begin{aligned}
I_2^3 &\leq c\delta^{\frac{1}{2}}\left[\int_{2B}|u_\delta|^{\gamma+2}\eta^{4k-4}\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{2B}|u_\delta|^{\gamma}\,\mathrm{d}x\right]^{\frac{1}{2}} \\
&\leq c\delta^{\frac{1}{2}}\int_{2B}\Big|\nabla\big(|u_\delta|^{\frac{\gamma}{2}+1}\eta^{2k-2}\big)\Big|\,\mathrm{d}x \\
&\leq c\delta^{\frac{1}{2}}\int_{2B}\Big[|\nabla u_\delta||u_\delta|^{\frac{\gamma}{2}}\eta^{2k-2}+|u_\delta|^{\frac{\gamma}{2}+1}\eta^{2k-3}|\nabla\eta|\Big]\,\mathrm{d}x \\
&\leq c\delta^{\frac{1}{2}}\left[\left(\int_{2B}|\nabla u_\delta|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{2B}|u_\delta|^{\gamma}\,\mathrm{d}x\right)^{\frac{1}{2}}+\int_{2B}|u_\delta|^{\gamma}\,\mathrm{d}x\right] \\
&\leq c\,.
\end{aligned} \tag{3.32}$$

Let us finally consider I_3 from (3.27): it holds

$$I_3 \leq \varepsilon\int_{2B}|u_\delta|^{\gamma+2}\eta^{2k}\,\mathrm{d}x + c(\varepsilon)\int_{2B}|u_\delta|^{\gamma}|f|^2\eta^{2k}\,\mathrm{d}x\,,$$

and using the boundedness of f as well as the uniform local higher integrability of $|u_\delta|^\gamma$, our claim (3.21) follows by choosing ε sufficiently small.

In the vector case $M > 1$ we just need a counterpart for (3.23), the rest of our arguments stays the same. Letting $F_\delta(Z) := \frac{\delta}{2}|Z|^2 + F(Z) =: \Phi_\delta\big(|Z|\big)$ we have for $\psi = |u_\delta|^\gamma u_\delta$ a.e.

$$\begin{aligned}
DF_\delta(\nabla u_\delta):\nabla\psi &= \frac{\Phi_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|}\nabla u_\delta:\nabla\psi \\
&= \frac{\Phi_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|}\big[\partial_\alpha u_\delta^i\partial_\alpha u_\delta^i|u_\delta|^\gamma+\gamma|u_\delta|^{\gamma-2}[\partial_\alpha u_\delta^i u_\delta^i][\partial_\alpha u_\delta^j u_\delta^j]\big]\,.
\end{aligned} \tag{3.33}$$

On account of

$$\Phi_\delta'(|Z|) \geq 0 \quad \text{for all } Z \in \mathbb{R}^{2M}$$

(compare the properties of Φ stated after (1.12)) we obtain

$$\begin{aligned}
DF_\delta(\nabla u_\delta):\nabla\psi &\geq \frac{\Phi_\delta'(|\nabla u_\delta|)}{|\nabla u_\delta|}\partial_\alpha u_\delta^i\partial_\alpha u_\delta^i|u_\delta|^\gamma \\
&= DF_\delta(\nabla u_\delta):\nabla u_\delta|u_\delta|^\gamma
\end{aligned} \tag{3.34}$$

from (3.33). Replacing (3.23) by (3.34) the proof is completed in the vector case as well. □

Proof of Theorem 3.1. Starting with $\gamma = 2$, Lemma 3.3 in particular implies that $|u_\delta|$ is in the space $L^4_{\mathrm{loc}}(\Omega)$ uniformly w.r.t. the parameter δ. Proceeding by induction, the claim of Theorem 3.1 follows. □

3.3 Conclusions from Caccioppoli's inequality

The main ingredient for proving the higher weak differentiability results from Theorem 1.5 is Caccioppoli's inequalitiy which as in Section 2 is based on the appropriate variant of the differentiated form (2.4) of the Euler equation. Here we collect the main conclusions which can be drawn from this inequality for the different linear growth cases under consideration.

In what follows we assume that the hypotheses of Therorem 1.4 are valid, the additional assumptions concerning f and the values of μ will be specified from line to line. As usual we work with the regularization introduced im Lemma 3.2, but for notational simplicity we drop the index δ.

Instead of (2.7) we now have (1.11) and the counterpart of (2.8) reads as

$$\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x \leq c(R) + 2\int_{2B} (u-f)\partial_\alpha(\eta^2\partial_\alpha u)\,\mathrm{d}x\,, \tag{3.35}$$

where – without a further explicit numeration of constants – we used the same notation as in Section 2.1. Note that the constant $c(R)$ on the r.h.s. can be calculated with the help of the second inequality of Remark 1.5. We emphasize that all constants are uniform in δ.

i) Assume that f locally is of class W^1_2. In this case we may write for any $\varepsilon > 0$

$$\begin{aligned}\int_{2B} (u-f)\cdot\partial_\alpha(\eta^2\partial_\alpha u)\,\mathrm{d}x &= -\int_{2B} |\nabla u|^2\eta^2\,\mathrm{d}x + \int_{2B} \partial_\alpha f\cdot\partial_\alpha u\eta^2\,\mathrm{d}x\\ &\leq \;\; -\int_{2B} |\nabla u|^2\eta^2\,\mathrm{d}x + \varepsilon\int_{2B}\eta^2|\nabla u|^2\,\mathrm{d}x + c\varepsilon^{-1}\int_{2B}\eta^2|\nabla f|^2\,\mathrm{d}x\,,\end{aligned}$$

i.e. choosing ε sufficiently small, (3.35) gives

$$\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x + \int_{2B}\eta^2|\nabla u|^2\,\mathrm{d}x \leq c(R) \tag{3.36}$$

for any $\mu > 1$.

ii) For $1 < \mu < 2$ and just assuming the local boundedness of f we proceed as in Section 2.1 and obtain instead of (2.11)

$$\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x + \int_{2B}\eta^2|\nabla u|^2\,\mathrm{d}x \leq c(R) + c\left[1+\int_{2B}\eta^2|\nabla^2 u|\,\mathrm{d}x\right].$$

Now observe for any $\varepsilon > 0$

$$\int_{2B}\eta^2|\nabla^2 u|\,\mathrm{d}x \leq \varepsilon\int_{2B}\eta^2\frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x + c\varepsilon^{-1}\int_{2B}\eta^2(1+|\nabla u|)^\mu\,\mathrm{d}x\,,$$

thus for ε sufficiently small

$$\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x + \int_{2B} \eta^2 |\nabla u|^2\,\mathrm{d}x \le c(R) + c\left[1 + \int_{2B} \eta^2 |\nabla u|^\mu\,\mathrm{d}x\right]. \qquad (3.37)$$

Now, on account of $\mu < 2$, we may apply Young's inequality one more time and get (3.36) for $\mu < 2$ and locally bounded f as well.

iii) In the case $\mu \ge 2$ the best we can show is

$$\begin{aligned} &\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x + \int_{2B} \eta^2 |\nabla u|^2\,\mathrm{d}x \\ &\quad \le c(R) + c\left[1 + \|f\|_\infty \int_{2B} \eta^2 |\nabla u|^\mu\,\mathrm{d}x\right]. \end{aligned} \qquad (3.38)$$

Here the Dirichlet part on the l.h.s. does not provide any additional information at all (see (31), p. 116, of [Bi1]).

If $f \equiv 0$, then (3.38) implies

$$\int_{2B} \eta^2 \frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x \le c(R). \qquad (3.39)$$

Note that (3.39) exactly corresponds to Caccioppoli's inequality valid for functionals just depending on the gradient, i.e. for minimizers of the energy

$$\int_\Omega F(\nabla w)\,\mathrm{d}x$$

(compare [Bi1], Lemma 4.19, i) in the case $s = 0$). Once we have the version (3.39) of Caccioppoli's inequality, then we are led to ($\Omega^* \Subset \Omega$)

$$\int_{\Omega^*} |\nabla u^*| \ln^2(1 + |\nabla u^*|^2) \le c(\Omega^*) < \infty$$

up to the limit case $\mu = 3$ (see [Bi2]), where u^* is the solution of problem (1.18) or of (1.18*), respectively.

3.4 Proof of Theorems 1.5 and 1.6

Suppose that we have (3.1) – (3.3) for some parameter $\mu > 1$, let $u^* \in \mathrm{BV}(\Omega; \mathbb{R}^M)$ denote the solution of (1.18) or of (1.18*) with associated regularizing sequence $\{u_\delta\}$ as in Lemma 3.2.

Proof of Theorem 1.5. The claim of Theorem 1.5, i), (a) is an immediate consequence of Theorem 3.1 combined with Corollary 3.1. If the function f is of class $W^1_{2,\mathrm{loc}}(\Omega^*;\mathbb{R}^M)$ for some subdomain Ω^* of Ω, then inequality (3.36) implies

$$\int_{2B} \eta^2 |\nabla u_\delta|^2 \,\mathrm{d}x \le c(R)$$

for disks $2B \Subset \Omega^*$ and cut-off functions η, thus u^* belongs to $W^1_{2,\mathrm{loc}}(\Omega^*;\mathbb{R}^M)$, and Theorem 1.5, i), (b) follows.

Since estimate (3.36) extends to the case of bounded functions f, provided we choose $\mu \in (1,2)$, we get the statement of part ii) in Theorem 1.5. The discussion of the limit case $\mu = 2$ is left to the reader. $\square$

Remark 3.4 *We conjecture that in the situation of Theorem 1.5, ii), minima even have a higher degree of regularity, which may be verified along the lines of [BF1].*

Remark 3.5 *An inspection of the arguments used for the proof of Theorem 1.5, i), (a), gives the following "global higher integrability" result, which is already in the spirit of Theorem 1.8:*

suppose that $M = 1$, let F satisfy (3.1) and assume that $f \in L^\infty(\Omega)$. Then the solution of (1.18^*) is in the same space.

To see this, we choose $\eta = 1$ in the proof of Lemma 3.3 and get from (3.24) the inequality $(\gamma \ge 2)$

$$\int_\Omega |u_\delta|^{\gamma+2} \,\mathrm{d}x \le \|f\|_{L^\infty(\Omega)} \int_\Omega |u_\delta|^{\gamma+1} \,\mathrm{d}x \,,$$

hence

$$\int_\Omega |u_\delta|^{\gamma+2} \,\mathrm{d}x \le \|f\|^2_{L^\infty(\Omega)} \int_\Omega |u_\delta|^{\gamma} \,\mathrm{d}x \,.$$

In order to justify this calculation we use the testfunction $\varphi := (v_k)^\gamma u_\delta$ in (3.22) with

$$v_k := \begin{cases} |u_\delta| & \text{on} \quad [|u_\delta| \le k] \\ \;\;k & \text{on} \quad [|u_\delta| \ge k] \,, \end{cases}$$

k denoting a non-negative number.

Then φ is in the space $W^1_2(\Omega)$ and thereby admissible in (3.22). Passing to the limit $k \to \infty$, the above estimates for u_δ easily follow. Starting with $\gamma = 2$ iteration of the inequalities leads to

$$\fint_\Omega |u_\delta|^{2+2m} \,\mathrm{d}x \le \|f\|^{2m}_{L^\infty(\Omega)} \fint_\Omega |u_\delta|^2 \,\mathrm{d}x$$

for any $m \in \mathbb{N}$, thus by passing to the limit $m \to \infty$

$$\|u_\delta\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)} \,,$$

and another application of Corollary 3.1 yields this bound also for the minimizer of (1.18^*)*.*

It should be clear that, if we replace our assumption on f *by the requirement that* $f \in L^t(\Omega)$ *for some "large"* t*, then a "finite iteration" leads to an improvement of the global integrability of the solution of* (1.18^*) *up to the exponent* t*. This can be justified formally by using the starting inequality (compare (3.24))*

$$\int_\Omega |u_\delta|^{\gamma+2}\,\mathrm{d}x \le \int_\Omega |f||u_\delta|^{\gamma+1}\,\mathrm{d}x\,,$$

from which we deduce

$$\int_\Omega |u_\delta|^{\gamma+2}\,\mathrm{d}x \le \varepsilon \int_\Omega |u_\delta|^{(\gamma+1)\frac{t}{t-1}}\,\mathrm{d}x + c(\varepsilon)\int_\Omega |f|^t\,\mathrm{d}x$$

valid for any $\varepsilon > 0$*. If we choose* $\gamma = t-2$*, then* $\gamma + 2 = (\gamma+1)t/(t-1)$*, and we can absorb the* ε*-term in the l.h.s. of the above inequality. This immediately implies* $u_\delta \in L^t(\Omega)$ *uniformly in* δ*.*

Proof of Theorem 1.6. Ad i). Consider a disk $B_r(x_0)$ and a subregion Ω^* such that $B_{2r}(x_0) \Subset \Omega^* \Subset \Omega$. We then easily obtain from the proof of Lemma 3.3 using the test function $\varphi = \eta u_\delta$ and estimating terms involving u_δ with the help of Theorem 3.1

$$\begin{aligned} |\nabla u_\delta|\big(B_r(x_0)\big) = \int_{B_r(x_0)} |\nabla u_\delta|\,\mathrm{d}x \;&\le\; c\left[\frac{1}{r}\int_{B_{2r}(x_0)-B_r(x_0)} |u_\delta|\,\mathrm{d}x + \int_{B_{2r}(x_0)} |u_\delta|\,\mathrm{d}x\right] \\ &\le\; c(\alpha,\Omega^*)r^\alpha \end{aligned} \tag{3.40}$$

for any exponent $\alpha \in (0,1)$. After passing to the limit $\delta \to 0$ we get from (3.40) the inequality

$$|\nabla u^*|\big(B_r(x_0)\big) \le c(\alpha,\Omega^*)r^\alpha\,, \tag{3.41}$$

and (3.41) exactly is the desired estimate (1.19).

Ad ii). Suppose that $u^* = f$ on the open set $\Omega^* \subset \Omega$. The arguments from i) now applied on a subregion $\tilde\Omega \Subset \Omega^*$ lead to the estimate

$$\int_{B_r(x_0)} |\nabla u_\delta|\,\mathrm{d}x \le c\left[\frac{1}{r}\int_{B_{2r}(x_0)-B_r(x_0)} |u_\delta|\,\mathrm{d}x + \int_{B_{2r}(x_0)} |u_\delta|\,\mathrm{d}x\right]$$

for disks $B_{2r}(x_0) \Subset \tilde\Omega$. Using Corollary 3.1 as well as $u^* = f$ on Ω^*, we find

$$|\nabla u^*|\big(B_r(x_0)\big) \le c\left[\frac{1}{r}\int_{B_{2r}(x_0)-B_r(x_0)} |f|\,\mathrm{d}x + \int_{B_{2r}(x_0)} |f|\,\mathrm{d}x\right],$$

and the boundedness of f yields

$$|\nabla u^*|\big(B_r(x_0)\big) \le c(\tilde\Omega)r\,. \tag{3.42}$$

With (3.42) we have established (1.20), and (1.21) follows from (1.20) along the same lines as (1.23) from (1.22) in part iii).

For iii) we just observe that the local boundedness of u implies the validity of (3.42) for disks in an arbitrary subdomain Ω^* with compact closure in Ω.

Consider a Borel set $E \subset \Omega^*$. In case $\mathcal{H}^1(E) = \infty$ the desired estimate obviously holds. Assume next that $\mathcal{H}^1(E) < \infty$. Then we select another domain Ω^{**} such that $\Omega^* \Subset \Omega^{**} \Subset \Omega$ and choose a finite collection of disks $B_{r_i}(x_i) \subset \Omega^{**}$ covering E and satisfying

$$c \sum_{i=1}^{N} r_i \leq \mathcal{H}^1(E) + \varepsilon \, ,$$

where c denotes a positive constant and where $\varepsilon > 0$ is given. An application of (3.42) (to disks contained in Ω^{**}) gives the claim of iii) after passing to the limit $\varepsilon \to 0$. □

4 Proof of Theorem 1.8

We start with claim a) and consider first the unconstrained logarithmic case with solution $u \in W^1_h(\Omega)$. We have

$$J[u, \Omega] \leq J[v, \Omega] \tag{4.1}$$

with $v := \max\{0, u\}$. Clearly it holds

$$\int_\Omega h(|\nabla v|) \, \mathrm{d}x \leq \int_\Omega h(|\nabla u|) \, \mathrm{d}x \, .$$

On the set $[u \geq 0]$ we have $u - f = v - f$, whereas on $[u \leq 0]$ we deduce from $f \geq 0$

$$|v - f| = f \leq |u - f| \, ,$$

hence

$$\int_\Omega (v - f)^2 \, \mathrm{d}x \leq \int_\Omega (u - f)^2 \, \mathrm{d}x \, .$$

This shows

$$J[v, \Omega] \leq J[u, \Omega]$$

and thereby $v = u$, which means $u \geq 0$. Applying (4.1) to the choice $v := \min\{1, u\}$, we get that $u \leq 1$.

In the case of linear growth we find along the same lines

$$0 \leq u_\delta \leq 1 \, , \tag{4.2}$$

where u_δ is the I_δ-minimizer on the whole space $W^1_2(\Omega)$ (compare Lemma 3.2). Quoting Theorem 1.7 we have proved (4.2) for the minimizer u as well.

The proof of b) is in the spirit of the paper [BF3] (see also [BF6]). Let u denote the solution of (1.18*) with corresponding approximations u_δ. Recalling our assumption that $L := \|f\|_{L^\infty(\Omega;\mathbb{R}^M)} < \infty$, we define the projection

$$\pi : \mathbb{R}^M \to \mathbb{R}^M\,, \quad \pi(y) := \begin{cases} y & \text{if} \quad |y| \leq L \\ L\dfrac{y}{|y|} & \text{if} \quad |y| \geq L\,, \end{cases}$$

and observe $\mathrm{Lip}(\pi) = 1$, hence by the chain rule

$$\big|\nabla(\pi \circ u_\delta)\big| \leq \mathrm{Lip}(\pi)|\nabla u_\delta| = |\nabla u_\delta|\,,$$

which means ($F_\delta(Z) := \frac{1}{2}\delta|Z|^2 + \Phi(|Z|)$, $Z \in \mathbb{R}^{2M}$)

$$F_\delta\big(\nabla(\pi \circ u_\delta)\big) \leq F_\delta(\nabla u_\delta)\,.$$

An elementary calculations shows $|\pi(u_\delta) - f| \leq |u_\delta - f|$, hence $\pi \circ u_\delta = u_\delta$ by the unique solvability of the approximate problems. This gives $|u_\delta| \leq L$, hence $|u| \leq L$.

If we consider problem (1.18) with bounded trace u_0, then the claim follows along the same lines. □

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