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Martin Fuchs and Guo Zhang

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On Entire Solutions Of The Equations For The Displacement Fields In The Deformation Theory Of Plasticity With Logarithmic Hardening

Martin Fuchs

Saarland University
Dep. of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Guo Zhang

University of Jyväskylä
Dept. of Mathematics and Statistics
P.O. Box 35 (MaD)
FI.-40014 University of Jyväskylä
Finland
guo.g.zhang@jyu.fi

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

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Abstract

Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ denote an entire solution of the homogeneous Euler-Lagrange equation associated to the energy used in the deformation theory of plasticity with logarithmic hardening. If $|u(x)|$ is of slower growth than $|x|$ as $|x| \to \infty$, then u must be constant. Moreover we show that u is affine if either $\sup_{\mathbb{R}^2} |\nabla u| < \infty$ or $\limsup\limits_{|x|\to\infty} |x|^{-1}|u(x)| < \infty$.

In their paper [FrSe] Frehse and Seregin propose to approximate the Hencky model used in perfect plasticity (cf. [DL], [He] or [Kl]) by a variational problem formulated in terms of the displacement fields, in which the energy density $G(\varepsilon(u))$ is of quadratic growth with respect to the trace of $\varepsilon(u)$ and of $L\log L$-growth with respect to the deviator $\varepsilon^D(u) = \varepsilon(u) - \frac{1}{n}(\operatorname{div} u)\mathbf{1}$ of $\varepsilon(u)$. Here u is a displacement field defined on some region in $\mathbb{R}^n$, $\varepsilon(u)$ denotes the symmetric part of the Jacobian matrix of u and $\mathbf{1}$ is the unit matrix. Modulo physical constants we have in the case of logarithmic hardening

$$G(\varepsilon) = h(|\varepsilon^D|) + \frac{1}{2}(\operatorname{trace} \varepsilon)^2 \tag{1}$$

for symmetric $(n \times n)$–matrices ε, where

$$h(t) = t\ln(1+t),\ \ t \geq 0\,. \tag{2}$$

Frehse and Seregin discuss solvability of the associated boundary value problems in suitable weak spaces and prove smoothness of local solutions at least in the case that $n = 2$. Later Seregin and the first author (see [FuSe1]) established partial regularity in the $3D$ case.

A related problem arises in the study of certain models describing the flow of generalized Newtonian fluids, for which the stress-strain relation takes the form

$$T^D = DH(\varepsilon)\,. \tag{3}$$

If we let

$$H(\varepsilon) = h(|\varepsilon|) \tag{4}$$

with h defined in equation (2), then (3) is the constitutive law for the so-called Prandtl–Eyring fluid, which has been the subject of the paper [FuSe1] and also of the monograph [FuSe2]. Very recently the authors discussed the behaviour of entire solutions of this fluid model at least in the stationary case for two spatial variables and proved Liouville-type results (see [FuZ]). The purpose of the present paper now is the investigation of planar entire solutions in the setting of plasticity with logarithmic hardening.

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DEFINITION 1. *A field $u : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^1 is an entire local minimizer of the energy*

$$I[v,\Omega] = \int_\Omega G(\varepsilon(v))\,dx \tag{5}$$

with density G defined according to equations (1) and (2), if for any bounded domain $\Omega \subset \mathbb{R}^2$ and all fields $v : \Omega \to \mathbb{R}^2$ such that $\operatorname{spt}(u-v)$ is compactly contained in Ω it holds

$$I[u,\Omega] \le I[v,\Omega]\,.$$

REMARK 1. *The smoothness assumption concerning u in Definition 1 is justified by the results in [FrSe].*

REMARK 2. *If u is an entire local I-minimizer, then it holds*

$$\int_\Omega DH(\varepsilon^D(u)) : \varepsilon^D(\varphi)\,dx + \int_\Omega \operatorname{div} u \operatorname{div} \varphi\,dx = 0 \tag{6}$$

for any domain $\Omega \subset \mathbb{R}^2$ and all fields $\varphi \in C_0^1(\Omega;\mathbb{R}^2)$. In equation (6) the symbol ":" is the scalar product of matrices and H is introduced in equation (4).

Now we can state our main results:

THEOREM 1. *Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ denote an entire local I-minimizer (cf. equation (5)) in the sense of Definition 1. If u satisfies the asymptotic condition*

$$\lim_{|x|\to\infty} \frac{|u(x)|}{|x|} = 0\,, \tag{7}$$

then the displacement field u is a constant vector. In particular, the boundedness of the field implies its constancy.

The next theorem concerns entire solutions satisfying a global Lipschitz condition:

THEOREM 2. *Consider an entire local I-minimizer $u : \mathbb{R}^2 \to \mathbb{R}^2$ in the sense of Definition1. If we know that $|\nabla u| \in L^\infty(\mathbb{R}^2)$, then u must be affine.*

Finally we relax the global boundedness of the gradient by imposing a growth condition on u:

THEOREM 3. *If the entire local I-minimizer $u : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $\limsup\limits_{|x|\to\infty} |x|^{-1}|u(x)| < \infty$, then u must be affine.*

REMARK 3. *It would be interesting to know what can be said about entire solutions in the $3D$-case. Due to the lack of regularity (cf. [FuSe1,2]) one either has to deal with weak local minimizers or the smoothness of u has to be imposed as a severe extra condition. In the latter case we think that for $n = 3$ condition (7) has to be replaced by $\lim\limits_{|x|\to\infty} \frac{|u(x)|}{\sqrt{|x|}} = 0$ in order to obtain the constancy of u, and this conclusion probably also holds in the case that $\limsup\limits_{|x|\to\infty} \frac{|u(x)|}{\sqrt{|x|}} < \infty$ (compare the proof of Theorem 3).*

For the proof of Theorem 1 we need two auxiliary results:

Lemma 1. *(Korn-type inequality)*
For fields $v : \mathbb{R}^2 \to \mathbb{R}^2$ with compact support it holds

$$\int_{\mathbb{R}^2} |\nabla v|^2 \, dx \le 2 \int_{\mathbb{R}^2} |\varepsilon^D(v)|^2 \, dx \, . \tag{8}$$

Korn-type inequalities involving ε^D have been established by Reshetnyak [Re] in a much more general setting. Recently Dain rediscovered these estimates in the L^2-setting (see [Da]), and the first author together with Bildhauer proved variants in the context of Orlicz-Sobolev spaces (cf. [FuB]).

The next lemma is essentially due to Giaquinta and Modica (compare Lemma 0.5 in [GM]), in the formulation given below it corresponds to Lemma 3.1 in [FuZ].

Lemma 2. *Let f, f_1, ..., f_ℓ denote non-negative functions from the space $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and suppose that we are given exponents $\alpha_1, \dots, \alpha_\ell > 0$. Then we can find a number $\delta_0 > 0$ depending on $\alpha_1, \dots, \alpha_\ell$ as follows: if for $\delta \in (0, \delta_0)$ it is possible to calculate a constant $c(\delta) > 0$ such that the inequality*

$$\int_{Q_R(z)} f \, dx \le \delta \int_{Q_{2R}(z)} f \, dx + c(\delta) \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j \, dx$$

holds for any choice of $Q_R(z) := \{x \in \mathbb{R}^2 : |x_i - z_i| < R, \, i = 1, 2\}$, then there is a constant $c > 0$ with the property

$$\int_{Q_R(z)} f \, dx \le c \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j \, dx$$

again for all squares $Q_R(z)$.

REMARK 4. *Of course Lemma 2 extends to $\mathbb{R}^n$, $n \ge 3$, replacing squares by cubes, and it is easy to see that estimate (8) remains valid in higher dimensions.*

Now we pass to the **proof of Theorem 1** proceeding in several steps.

Step 1. a growth estimate for the energy

We fix a square $Q_{2R}(x_0)$ and choose $\eta \in C^1_0(Q_{2R}(x_0))$ such that $\eta = 1$ on $Q_R(x_0), 0 \le \eta \le 1, |\nabla \eta| \le c/R$. Then we apply equation (6) by selecting $\varphi = \eta^2 u$. We get with H defined in (4)

$$(9) \qquad \int_{Q_{2R}(x_0)} \eta^2 DH\left(\varepsilon^D(u)\right) : \varepsilon^D(u)\,dx + \int_{Q_{2R}(x_0)} \eta^2 (\operatorname{div} u)^2\,dx$$
$$= -2\int_{Q_{2R}(x_0)} \eta DH\left(\varepsilon^D(u)\right) : (\nabla\eta\otimes u)^D\,dx - 2\int_{Q_{2R}(x_0)} \eta \operatorname{div} u \nabla\eta\cdot u\,dx$$
$$\leq c\left[\int_{Q_{2R}(x_0)} \eta h'\left(|\varepsilon^D(u)|\right)|\nabla\eta||u|\,dx + \int_{Q_{2R}(x_0)} \eta|\operatorname{div} u||\nabla\eta||u|\,dx\right].$$

Using Young's inequality we obtain for any $\delta > 0$

$$\begin{aligned} \eta h'\left(|\varepsilon^D(u)|\right)|\nabla\eta||u| &\leq \delta\eta^2 h'\left(|\varepsilon^D(u)|\right)|\varepsilon^D(u)| + \delta^{-1}|\nabla\eta|^2 \tfrac{h'(|\varepsilon^D(u)|)}{|\varepsilon^D(u)|}|u|^2\,, \\ \eta|\operatorname{div} u||\nabla\eta||u| &\leq \delta\eta^2(\operatorname{div} u)^2 + \delta^{-1}|\nabla\eta|^2|u|^2\,. \end{aligned}$$

Inserting these estimates in inequality (9) observing that $\frac{h'(t)}{t} \leq 2$, we deduce after appropriate choice of δ and recalling the properties of η

$$(10) \qquad \begin{aligned} \int_{Q_R(x_0)} G\left(\varepsilon(u)\right)\,dx &= \int_{Q_R(x_0)} \left[H\left(\varepsilon^D(u)\right) + \frac{1}{2}\left(\operatorname{div} u\right)^2\right]\,dx \\ &\leq cR^{-2}\int_{Q_{2R}(x_0)-\overline{Q}_R(x_0)} |u|^2\,dx\,. \end{aligned}$$

In particular, if we choose $x_0 = 0$ and abbreviate

$$\Theta(R) := \sup\left\{|x|^{-1}|u(x)| : x\in\mathbb{R}^2 - \overline{Q}_R\right\}\,,$$

then (10) implies

$$(11) \qquad \int_{Q_R} G\left(\varepsilon(u)\right)\,dx \leq cR^2\Theta(R)^2$$

with $\lim\limits_{R\to\infty}\Theta(R) = 0$ according to our hypothesis (7).

Step 2. discussion of the second derivatives

Returning to equation (6) and performing an integration by parts we get for $\alpha = 1$, 2 and $\varphi\in C_0^1(Q_{\frac{3}{2}R}(x_0))$

$$(12) \qquad 0 = \int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon^D(u)\right)\left(\varepsilon^D(\partial_\alpha u),\,\varepsilon^D(\varphi)\right)\,dx + \int_{Q_{\frac{3}{2}R}(x_0)} \operatorname{div}(\partial_\alpha u)\operatorname{div}\varphi\,dx\,.$$

In equation (12) we choose $\varphi = \eta^2\partial_\alpha u$ (from now on summation with respect to $\alpha = 1, 2$), where η is as in Step 1 with $2R$ replaced by $\frac{3}{2}R$. From (12) we easily obtain by applying the Cauchy-Schwarz inequality to the quantity

$$D^2H\left(\varepsilon^D(u)\right)\left(\eta\varepsilon^D(\partial_\alpha u),\,(\nabla\eta\otimes\partial_\alpha u)^D\right)$$

and appropriate use of Young's inequality (observing the boundedness of $|D^2H(\varepsilon^D(u))|$)

$$\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\partial_\alpha \varepsilon^D(u), \partial_\alpha \varepsilon^D(u)\right)\eta^2\,dx$$

$$+ \int_{Q_{\frac{3}{2}R}(x_0)} \eta^2 |\nabla(\operatorname{div} u)|^2\,dx \le c\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla \eta|^2 |\nabla u|^2\,dx\,,$$

hence by the properties of η

$$\int_{Q_R(x_0)} D^2H\left(\varepsilon^D(u)\right)\left(\varepsilon^D(\partial_\alpha u), \varepsilon^D(\partial_\alpha u)\right)\,dx + \int_{Q_R(x_0)} |\nabla(\operatorname{div} u)|^2\,dx \le cR^{-2}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx\,, \tag{13}$$

and inequality (13) holds for all squares $Q_R(x_0)$. Note that (13) implies that entire local minimizers having finite Dirichlet integral must be affine. This follows by letting $R \to \infty$ and observing that on the right-hand side of (13) the domain of integration can be replaced by $Q_{\frac{3}{2}R}(x_0) - \overline{Q_R(x_0)}$. In order to control $\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx$ we choose $\Psi \in C_0^1(Q_{2R}(x_0))$ such that $0 \le \Psi \le 1$, $\Psi = 1$ on $Q_{\frac{3}{2}R}(x_0)$ and $|\nabla \Psi| \le c/R$. From estimate (8) in Lemma 1 we obtain

$$\begin{aligned}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx &\le c\left[\int_{Q_{2R}(x_0)} |\nabla(\Psi u)|^2\,dx + \int_{Q_{2R}(x_0)} |\nabla\Psi|^2|u|^2\,dx\right]\\ &\le c\left[\int_{Q_{2R}(x_0)} |\varepsilon^D(\Psi u)|^2\,dx + \int_{Q_{2R}(x_0)} |\nabla\Psi|^2|u|^2\,dx\right]\\ &\le c\left[\int_{Q_{2R}(x_0)} \Psi^2|\varepsilon^D(u)|^2\,dx + R^{-2}\int_{Q_{2R}(x_0)} |u|^2\,dx\right]\end{aligned}$$

or by the support properties of Ψ

$$\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx \le c\left[\int_{Q_{2R}(x_0)} \Psi^2|\varepsilon^D(u)|^2\,dx + R^{-2}\int_{Q_{2R}(x_0)-\overline{Q_{\frac{3}{2}R}(x_0)}} |u|^2\,dx\right]. \tag{14}$$

In order to proceed we observe

$$\varepsilon^D_{ij}(u) = \frac{1}{2}\left(\frac{\partial u^j}{\partial x_i} + \frac{\partial u^i}{\partial x_j}\right) - \frac{1}{2}(\operatorname{div} u)\delta_{ij}\,,$$

hence by the symmetry of $\varepsilon^D(u)$ and the fact that $\varepsilon^D_{ij}(u)\delta_{ij} = 0$

$$\begin{aligned}&\int_{Q_{2R}(x_0)} \Psi^2|\varepsilon^D(u)|^2\,dx\\ &= \frac{1}{2}\int_{Q_{2R}(x_0)} \Psi^2\left\{\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) - (\operatorname{div} u)\delta_{ij}\right\}\varepsilon^D_{ij}(u)\,dx\\ &= -\int_{Q_{2R}(x_0)} \partial_i\left(\Psi^2\varepsilon^D_{ij}(u)\right)u^j\,dx\,.\end{aligned}$$

This yields

$$\begin{aligned}
\int_{Q_{2R}(x_0)} \Psi^2 |\varepsilon^D(u)|^2\,dx \le c &\left[\int_{Q_{2R}(x_0)} |\nabla \Psi^2||u||\varepsilon^D(u)|\,dx \right.\\
&\left. + \int_{Q_{2R}(x_0)} \Psi^2 |\nabla \varepsilon^D(u)||u|\,dx\right]\\
\le\ c &\left[R^{-1}\int_{Q_{2R}(x_0)} |u||\varepsilon^D(u)|\,dx + \delta \int_{Q_{2R}(x_0)} \frac{|\nabla\varepsilon^D(u)|^2}{1+|\varepsilon^D(u)|}\,dx\right.\\
&\left. +\delta^{-1}\int_{Q_{2R}(x_0)} |u|^2\left(1+|\varepsilon^D(u)|\right)\,dx\right].
\end{aligned}$$

Let $\omega := D^2 H\left(\varepsilon^D(u)\right)\left(\partial_\alpha \varepsilon^D(u), \partial_\alpha \varepsilon^D(u)\right)$. If we combine (13), (14) and the inequalities from above, we obtain for any $\delta > 0$ and all squares $Q_R(x_0)$

$$\begin{aligned}
(15)\qquad & \int_{Q_R(x_0)} \omega\,dx + \int_{Q_R(x_0)} |\nabla(\operatorname{div} u)|^2\,dx\\
\le\ & c\left[R^{-4}\int_{Q_{2R}(x_0)} |u|^2\,dx + R^{-2}\delta\int_{Q_{2R}(x_0)} \omega\,dx\right.\\
& \left. + R^{-3}\int_{Q_{2R}(x_0)} |u||\varepsilon^D(u)|\,dx + R^{-2}\delta^{-1}\int_{Q_{2R}(x_0)} |u|^2\left(1+|\varepsilon^D(u)|\right)\,dx\right].
\end{aligned}$$

Replacing δ by $\delta' R^2$ an application of Lemma 2 yields

$$\begin{aligned}
(15')\qquad & \int_{Q_R(x_0)} \omega\,dx + \int_{Q_R(x_0)} |\nabla(\operatorname{div} u)|^2\,dx\\
\le\ & c\left[R^{-4}\int_{Q_{2R}(x_0)} |u|^2\,dx + R^{-3}\int_{Q_{2R}(x_0)} |u||\varepsilon^D(u)|\,dx\right.\\
& \left. + R^{-4}\int_{Q_{2R}(x_0)} |u|^2\left(1+|\varepsilon^D(u)|\right)\,dx\right].
\end{aligned}$$

Let $x_0 = 0$ and $R \ge 1$. From our hypothesis (7) we obtain $|u(x)| \le cR$ on Q_{2R}. Therefore (15') implies

$$(16)\qquad \int_{Q_R} \omega\,dx + \int_{Q_R} |\nabla(\operatorname{div} u)|^2\,dx \le c\left[R^{-4}R^4 + R^{-2}\int_{Q_{2R}} |\varepsilon^D(u)|\,dx\right].$$

Clearly we have $\left(Q^+ := Q_{2R} \cap \left[|\varepsilon^D(u)| \ge 1\right], Q^- := \ldots\right)$

$$\begin{aligned}
\int_{Q_{2R}} |\varepsilon^D(u)|\,dx &= \int_{Q^+} |\varepsilon^D(u)|\,dx + \int_{Q^-} |\varepsilon^D(u)|\,dx\\
&\le \left(\int_{Q^-} 1\,dx\right)^{1/2}\left(\int_{Q^-} |\varepsilon^D(u)|^2\,dx\right)^{1/2} + \frac{1}{\ln 2}\int_{Q^+} H\left(\varepsilon^D(u)\right)\,dx\\
&\le cR\left(\int_{Q_{2R}} H\left(\varepsilon^D(u)\right)\,dx\right)^{1/2} + \frac{1}{\ln 2}\int_{Q_{2R}} H\left(\varepsilon^D(u)\right)\,dx,
\end{aligned}$$

and if we use (10), we find

$$\int_{Q_{2R}} |\varepsilon^D(u)|\, dx \le cR^2 \,. \tag{17}$$

This shows that the right-hand side of (16) stays bounded as $R \to \infty$, which means

$$\int_{\mathbb{R}^2} \omega\, dx + \int_{\mathbb{R}^2} |\nabla(\operatorname{div} u)|^2\, dx < \infty \,. \tag{18}$$

Step 3. conclusion

We claim that the integral in (18) vanishes. In order to prove this we choose $x_0 = 0$ and return to inequality (13) recalling that in place of (13) we actually have

$$\int_{Q_R} |\nabla\,(\operatorname{div} u)\,|^2\, dx + \int_{Q_R} \omega\, dx \le cR^{-2} \int_{Q_{\frac{3}{2}R}-\overline{Q}_R} |\nabla u|^2\, dx \,.$$

Let $\Psi \in C^1_0(Q_{2R} - \overline{Q}_{R/2})$ such that $0 \le \Psi \le 1$ and $\Psi = 1$ on $Q_{\frac{3}{2}R} - \overline{Q}_R$ together with $|\nabla\Psi| \le c/R$. Observing

$$\int_{Q_{\frac{3}{2}R}-\overline{Q}_R} |\nabla u|^2\, dx \le c\left[\int_{Q_{2R}-\overline{Q}_{R/2}} |\nabla(\Psi u)|^2\, dx + R^{-2}\int_{Q_{2R}-\overline{Q}_{R/2}} |u|^2\, dx\right]$$

we obtain a variant of (14), in which now the term $\int_{Q_{2R}-\overline{Q}_{R/2}} \Psi^2 |\varepsilon^D(u)|^2\, dx$ occurs on the right-hand side. Proceeding as before we get in place of (15)

$$\begin{aligned} \int_{Q_R} \omega\, dx &+ \int_{Q_R} |\nabla(\operatorname{div} u)|^2\, dx \\ \le\;\; & c\left[R^{-4}\int_{Q_{2R}-\overline{Q}_{R/2}} |u|^2\, dx + R^{-2}\delta \int_{Q_{2R}-\overline{Q}_{R/2}} \omega\, dx \right.\\ & \left. +R^{-3}\int_{Q_{2R}-\overline{Q}_{R/2}} |u||\varepsilon^D(u)|\, dx + R^{-2}\delta^{-1}\int_{Q_{2R}-\overline{Q}_{R/2}} |u|^2\left(1+|\varepsilon^D(u)|\right)\, dx\right] . \end{aligned} \tag{19}$$

Let $\delta := \frac{1}{2c}R^2$. Inequality (19) then yields

$$\begin{aligned} \int_{Q_R} \omega\, dx + \int_{Q_R} |\nabla(\operatorname{div} u)|^2\, dx &\le \frac{1}{2}\int_{Q_{2R}-\overline{Q}_{R/2}} \omega\, dx \\ &+c\left[\Theta^2(R/2) + \Theta(R/2)R^{-2}\int_{Q_{2R}} |\varepsilon^D(u)|\, dx \right.\\ &\left. +\Theta^2(R/2)R^{-2}\int_{Q_{2R}} \left(1+|\varepsilon^D(u)|\right)\, dx\right] , \end{aligned} \tag{20}$$

and if we use (17) and (18) together with the hypothesis that $\lim_{R\to\infty} \Theta(R) = 0$, estimate (20) implies after passing to the limit $R \to \infty$ that ω as well as $\nabla(\operatorname{div} u)$ must vanish, thus $\nabla\varepsilon(u) \equiv 0$. But then it holds $\nabla^2 u \equiv 0$, which means that u is affine and thereby constant on account of our assumption (7). This completes the proof of Theorem 1.

For **proving Theorem 2** we observe that boundedness of $|\nabla u|$ implies the estimate

$$|u(x)| \le cR,\ x \in Q_{2R}\,,$$

provided $R \ge 1$. Using this information we again arrive at inequality (18), and this estimate can be restated in the form (recall (1) and (2))

$$\int_{\mathbb{R}^2} D^2G(\varepsilon(u))(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u))\, dx < \infty\,.$$

Note that this is also a direct consequence of estimate (13). Using $|\nabla^2 u| \le c|\nabla\varepsilon(u)|$ together with the boundedness of $\varepsilon(u)$ we get

$$\int_{\mathbb{R}^2} \left|\nabla^2 u\right|^2\, dx < \infty\,. \tag{21}$$

Similar to equation (12) it holds ($\alpha = 1, 2$)

$$0 = \int_{Q_{2R}} D^2G(\varepsilon(u))\,(\partial_\alpha\varepsilon(u), \varepsilon(\varphi))\, dx\,,$$

and we may choose $\varphi = \eta^2\partial_\alpha u$ with $\eta \in C_0^1(Q_{2R})$ such that $\eta = 1$ on Q_R, $0 \le \eta \le 1$ and $|\nabla\eta| \le c/R$. We get

$$\begin{aligned}
&\int_{Q_{2R}} D^2G(\varepsilon(u))\,(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u))\,\eta^2\, dx \\
&\quad = \; -2\int_{Q_{2R}-\overline{Q}_R} D^2G\,(\varepsilon(u))\,(\partial_\alpha\varepsilon(u), \nabla\eta\otimes\partial_\alpha u)\,\eta\, dx\,,
\end{aligned}$$

and the boundedness of $\varepsilon(u)$ yields (recall $|\nabla^2 u| \le c|\nabla\varepsilon(u)|$)

$$\begin{aligned}
\int_{Q_R} |\nabla^2 u|^2\, dx &\le cR^{-1}\int_{Q_{2R}-\overline{Q}_R} |\nabla^2 u||\nabla u|\, dx \\
&\le \; cR^{-1}\left(\int_{Q_{2R}-\overline{Q}_R} \left|\nabla^2 u\right|^2\, dx\right)^{1/2}\left(\int_{Q_{2R}-\overline{Q}_R} |\nabla u|^2\, dx\right)^{1/2}\,,
\end{aligned}$$

hence by the boundedness of the gradient

$$\int_{Q_R} \left|\nabla^2 u\right|^2\, dx \le c\left(\int_{Q_{2R}-\overline{Q}_R} \left|\nabla^2 u\right|^2\, dx\right)^{1/2}\,. \tag{22}$$

On account of (21) the right-hand side of (22) vanishes as $R \to \infty$, thus $\nabla^2 u \equiv 0$, which proves Theorem 2.

$\square$

Let us finally discuss the **proof of Theorem 3**. As remarked in the beginning of the proof of Theorem 2 the growth condition imposed now on u is still sufficient to get

$$\int_{\mathbb{R}^2} D^2G\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u)\right)\, dx < \infty\,. \tag{23}$$

We return to equation (12) choosing $x_0 = 0$, $R \geq 1$, and select η as done after (12). The Cauchy-Schwarz inequality together with Hölder's estimate implies ($\xi_\alpha := \partial_\alpha u \otimes \nabla\eta^2$)

$$\begin{aligned}
&\int_{Q_R} D^2G\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u)\right)\, dx\\
&= \int_{Q_R} \omega\, dx + \int_{Q_R} \left|\nabla(\operatorname{div} u)\right|^2\, dx\\
&\leq c\left[\left(\int_{Q_{\frac{3}{2}R}-\overline{Q}_R} \omega\, dx\right)^{1/2}\left(\int_{Q_{\frac{3}{2}R}-\overline{Q}_R} D^2H\left(\varepsilon^D(u)\right)\left(\xi^D_\alpha, \xi^D_\alpha\right)\, dx\right)^{1/2}\right.\\
&\qquad\left.+\left(\int_{Q_{\frac{3}{2}R}-\overline{Q}_R} \left|\nabla(\operatorname{div} u)\right|^2\, dx\right)^{1/2}\left(\int_{Q_{\frac{3}{2}R}-\overline{Q}_R} |\nabla u|^2|\nabla\eta|^2\, dx\right)^{1/2}\right],
\end{aligned}$$

hence we find

$$\begin{aligned}
&\int_{Q_R} \omega\, dx + \int_{Q_R} \left|\nabla(\operatorname{div} u)\right|^2\, dx\\
&\leq cR^{-1}\left(\int_{Q_{\frac{3}{2}R}-\overline{Q}_R} |\nabla u|^2\, dx\right)^{1/2}\left\{\int_{Q_{\frac{3}{2}R}-\overline{Q}_R} \omega\, dx + \int_{Q_{\frac{3}{2}R}-\overline{Q}_R} \left|\nabla(\operatorname{div} u)\right|^2\, dx\right\}^{1/2},
\end{aligned} \tag{24}$$

and with (23) and (24) our claim will follow by letting $R \to \infty$ as soon as we can show

$$\int_{Q_R} |\nabla u|^2\, dx \leq cR^2 \tag{25}$$

for all $R \geq 1$. For proving (25) we recall that by (14) it holds

$$\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\, dx \leq c\left\{\int_{Q_{2R}} \Psi^2\left|\varepsilon^D(u)\right|^2\, dx + R^{-2}\int_{Q_{2R}-\overline{Q}_{\frac{3}{2}R}} |u|^2\, dx\right\} \tag{26}$$

with Ψ defined after (13), and from (26) in combination with our growth assumption imposed on u we infer that it remains to bound the quantity $\int_{Q_{2R}} \Psi^2\left|\varepsilon^D(u)\right|^2\, dx$ in terms of R^2. Proceeding as done after inequality (14) we find

$$\begin{aligned}\int_{Q_{2R}} \Psi^2 \left|\varepsilon^D(u)\right|^2 \, dx \le c\Bigg\{ & \int_{Q_{2R}} |\nabla\Psi^2||u||\varepsilon^D(u)| \, dx \\ & + \int_{Q_{2R}} \Psi^2 |u||\nabla\varepsilon^D(u)| \, dx \Bigg\} \\ \le \;\; c\Bigg\{ & \frac{1}{R}\int_{Q_{2R}} |u||\varepsilon^D(u)| \, dx + \left(\int_{Q_{2R}} \frac{|\nabla\varepsilon^D(u)|^2}{1+|\varepsilon^D(u)|} \, dx\right)^{1/2} \\ & \cdot \left(\int_{Q_{2R}} |u|^2 \left(1+|\varepsilon^D(u)|\right) \, dx\right)^{1/2} \Bigg\} .\end{aligned}$$

On Q_{2R} it holds $|u(x)| \le cR$, hence

$$\int_{Q_{2R}} \Psi^2 \left|\varepsilon^D(u)\right|^2 \, dx \le c\left\{ \int_{Q_{2R}} \left|\varepsilon^D(u)\right| \, dx + R\left(\int_{Q_{2R}} \left(1+|\varepsilon^D(u)|\right) \, dx\right)^{1/2} \right\} , \tag{27}$$

where we also made use of (23) to bound the term involving $\nabla\varepsilon^D(u)$. The discussion following inequality (16) shows

$$\int_{Q_{2R}} \left|\varepsilon^D(u)\right| \, dx \le cR\left(\int_{Q_{2R}} H\left(\varepsilon^D(u)\right) \, dx\right)^{1/2} + \frac{1}{\ln 2}\int_{Q_{2R}} H\left(\varepsilon^D(u)\right) \, dx \, , \tag{28}$$

and if we go back to estimate (10) (being valid without any growth hypothesis imposed on u) we find that now this inequality yields

$$\int_{Q_{2R}} H\left(\varepsilon^D(u)\right) \, dx \le cR^2 \, . \tag{29}$$

Finally, we combine the inequalities (28) and (29) with the result that

$$\int_{Q_{2R}} \left|\varepsilon^D(u)\right| \, dx \le cR^2 \, , \tag{30}$$

and we see that (27) and (30) imply the correct bound for $\int_{Q_{2R}} \Psi^2 \left|\varepsilon^D(u)\right|^2 dx$. Thus we have established (25) and the proof of Theorem 3 is complete.

□

REMARK 5. *We leave it to the reader to discuss Theorem 1, 2 and 3 for nonlinear Hencky materials, which means that the energy density from equation (1) is replaced by*

$$W(\varepsilon) := \Phi\left(|\varepsilon^D|\right) + \frac{1}{2}(tr\ \varepsilon)^2$$

for a "general" N-function Φ (compare, e.g. [Ad] for a definition). We refer to the article [BF], where one will find natural hypotheses to be imposed on Φ under which one can expect a Liouville-type result.

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