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A Note On A Liouville-Type Result Of Gilbarg And Weinberger For The Stationary Navier-Stokes Equations In $2D$

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Abstract

We show that a velocity field u satisfying the stationary Navier–Stokes equations on the entire plane must be constant under the growth condition $\limsup\limits_{|x|\to\infty} |x|^{-\alpha}\,|u(x)| < \infty$ for some $\alpha \in [0, 1/7)$.

1 Introduction

In our note we investigate entire solutions $u : \mathbb{R}^2 \to \mathbb{R}^2$, $\pi : \mathbb{R}^2 \to \mathbb{R}$ of the stationary Navier–Stokes equations in the plane and discuss conditions under which the Liouville property holds. To be precise we assume that

$$
\left\{\begin{array}{l} \nu\Delta u = u \cdot \nabla u + \nabla\pi\,, \\ \operatorname{div} u = 0 \end{array}\right. \tag{1.1}
$$

holds in $\mathbb{R}^2$, u denoting the velocity field of an incompressible Newtonian fluid with constant viscosity $\nu > 0$, and π stands for the pressure function. An explanation of the mathematical and physical background of equations (1.1) is given for example in the textbooks of Ladyzhenskaya [La] and of Galdi[Ga1,2]. In their famous paper [GW] Gilbarg and Weinberger showed:

THEOREM 1.1. *Consider solutions u and π of equations (1.1) defined over the entire plane and assume that*

$$
\int_{\mathbb{R}^2} |\nabla u|^2\, dx < \infty\,. \tag{1.2}
$$

Then u and π are constant.

Let us note that their proof makes extensive use of the fact that the vorticity function satisfies a nice elliptic equation to which a maximum principle applies.

A Liouville theorem being more in the spirit of the classical one for entire analytic functions was obtained by Koch [Ko] and by Koch, Nadirashvili, Seregin, Sverăk [KNSS] as a byproduct of their work on the instationary case:

THEOREM 1.2. *The conclusion of Theorem 1.1 holds, if condition (1.2) is replaced by the requirement*

$$
\sup_{\mathbb{R}^2} |u| < \infty\,. \tag{1.3}
$$

In the papers [Fu] and [FuZ] we recently gave extensions of Theorems 1.1 and 1.2 to the case of generalized Newtonian fluids, where the viscosity is a function depending on the shear rate. Now we return to the standard Navier–Stokes equations (1.1) for which we like to show with the help of techniques due to Gilbarg and Weinberger that the condition of boundedness (1.3) actually can be weakened. More precisely we have:

THEOREM 1.3. *Let u and π denote entire solutions of (1.1) in 2D. If we know that*

$$\limsup_{|x|\to\infty} |x|^{-\alpha}\, |u(x)| < \infty \tag{1.4}$$

for some $\alpha \in [0, 1/7)$, then u and π must be constant.

The proof of Theorem 1.3 is based on three ingredients:

(i) energy estimates giving precise bounds for the Dirichlet integral of u calculated over squares of side length R;

(ii) energy estimates and local L^∞-bounds for the vorticity ω;

(iii) scaling properties of equations (1.1).

For generalized Newtonian fluids we have no substitute for (ii) and (iii) and therefore it is a challenging task to discuss Theorem 1.3 in this setting. Another open question is, if condition (1.4) can be improved to

$$\limsup_{|x|\to\infty} |x|^{-\alpha}\, |u(x)| < \infty \tag{1.5}$$

for some $\alpha \geq 1/7$. By letting $u(x_1, x_2) := (-x_1, x_2)$, $\pi(x_1, x_2) := -\frac{1}{2}\,(x_1^2 + x_2^2)$ it becomes clear that we can not include $\alpha = 1$. A list of further open problems is presented in Section 5.

2 Estimates for the Dirichlet integral of the velocity field

We start with a collection of auxiliary results. The first lemma is a slight extension of a contribution due to Giaquinta and Modica (compare Lemma 0.5 in [GM]). In this lemma and also during this section we abbreviate

$$Q_R(z) := \left\{ x \in \mathbb{R}^2 : |x_i - z_i| < R,\ i = 1, 2 \right\},\ z \in \mathbb{R}^2, R > 0\,.$$

Lemma 2.1. *Let $f,\ f_1, \ldots, f_\ell$ denote non-negative functions from the space $L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Suppose further that we are given exponents $\alpha_1, \ldots, \alpha_\ell > 0$. Then we can find a number*

$\delta_0 > 0$ *depending on* $\alpha_1, \dots, \alpha_\ell$ *as follows: if for* $\delta \in (0, \delta_0)$ *it is possible to calculate a constant* $c(\delta) > 0$ *such that the inequality*

$$\int_{Q_R(z)} f\,dx \le \delta \int_{Q_{2R}(z)} f\,dx + c(\delta) \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j\,dx$$

holds for any choice of $Q_R(z) \subset \mathbb{R}^2$*, then there is a constant* c *with the property*

$$\int_{Q_R(z)} f\,dx \le c \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j\,dx$$

for all squares $Q_R(z)$*.*

REMARK 2.1. *Of course Lemma 2.1 extends to* $\mathbb{R}^n$*,* $n \ge 3$*, replacing squares by cubes.*

Proof of Lemma 2.1: see [FuZ], Appendix. $\square$

Next we recall a standard result concerning the “divergence equation”, see e.g. [Ga1] or [La].

Lemma 2.2. *Consider a function* $f \in L^2(Q_R(z))$ *such that* $\int_{Q_R(z)} f\,dx = 0$*. Then there exists a field* $v \in \overset{\circ}{W}{}^1_2(Q_R(z);\mathbb{R}^2)$ *and a constant* C *independent of* $Q_R(z)$ *such that we have* $\operatorname{div} v = f$ *on* $Q_R(z)$ *together with the estimate*

$$\int_{Q_R(z)} |\nabla v|^2\,dx \le C \int_{Q_R(z)} f^2\,dx\,.$$

Lemma 2.3. *Under the assumptions and with the notation from Theorem 1.3 it holds*

$$\int_{Q_R(0)} |\nabla u|^2\,dx \le cR^{1+3\alpha} \tag{2.1}$$

for any $R \ge 1$ *with a positive constant* c *being independent of* R*.*

Proof: Letting $\nu = 1$ in equations (1.1) we obtain

$$0 = \int_{Q_{2R}(x_0)} \nabla u : \nabla\varphi\,dx + \int_{Q_{2R}(x_0)} u^k \partial_k u^i \varphi^i\,dx \tag{2.2}$$

(from now on summation with respect to indices repeated twice) for all φ with compact support in $Q_{2R}(x_0)$ satisfying $\operatorname{div}\varphi = 0$, where $Q_R(x_0)$ denotes an arbitrary square. We let $\eta \in C^1_0(Q_{2R}(x_0))$ with $\eta = 1$ on $Q_R(x_0)$, $0 \le \eta \le 1$ and $|\nabla\eta| \le c/R$. In equation (2.2) we choose

$$\varphi := \eta^2 u - w$$

with w defined according to Lemma 2.2, where $f := \operatorname{div}(\eta^2 u)$ and with $Q_R(z)$ being replaced by $Q_{2R}(x_0)$. This gives

$$\begin{aligned}
&\int_{Q_{2R}(x_0)} \eta^2 \left|\nabla u\right|^2 \, dx + 2 \int_{Q_{2R}(x_0)} \eta \nabla u : (\nabla \eta \otimes u) \, dx \\
&\quad - \int_{Q_{2R}(x_0)} \nabla u : \nabla w \, dx + \int_{Q_{2R}(x_0)} u^k \partial_k u^i u^i \eta^2 \, dx \\
&\quad - \int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i \, dx = 0 \, ,
\end{aligned}$$

and an application of Young's inequality implies

$$\begin{aligned}
\int_{Q_{2R}(x_0)} \eta^2 \left|\nabla u\right|^2 \, dx &\le c \left[R^{-2} \int_{Q_{2R}(x_0)} |u|^2 \, dx \right. \\
&\quad + \int_{Q_{2R}(x_0)} |\nabla u| |\nabla w| \, dx + \left| \int_{Q_{2R}(x_0)} u^k \partial_k u^i u^i \eta^2 \, dx \right| \\
&\quad \left. + \left| \int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i \, dx \right| \right] =: c \sum_{i=1}^{4} T_i \, .
\end{aligned} \tag{2.3}$$

From the definition of w it follows

$$\begin{aligned}
T_2 &\le \delta \int_{Q_{2R}(x_0)} |\nabla u|^2 \, dx + \delta^{-1} \int_{Q_{2R}(x_0)} |\nabla w|^2 \, dx \\
&\le \delta \int_{Q_{2R}(x_0)} |\nabla u|^2 \, dx + c \, \delta^{-1} R^{-2} \int_{Q_{2R}(x_0)} |u|^2 \, dx \, ,
\end{aligned} \tag{2.4}$$

moreover we have

$$\begin{aligned}
T_3 &= \left| \frac{1}{2} \int_{Q_{2R}(x_0)} u^k \partial_k |u|^2 \eta^2 \, dx \right| \\
&= \left| \frac{1}{2} \int_{Q_{2R}(x_0)} u^k |u|^2 \partial_k \eta^2 \, dx \right| \le c \, R^{-1} \int_{Q_{2R}(x_0)} |u|^3 \, dx \, .
\end{aligned} \tag{2.5}$$

For T_4 we observe the identity

$$\int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i \, dx = - \int_{Q_{2R}(x_0)} u^k u^i \partial_k w^i \, dx \, ,$$

hence by the properties of w and by Hölder's inequality

$$\begin{aligned}
T_4 &\le \left(\int_{Q_{2R}(x_0)} |u|^4 \, dx \right)^{1/2} \left(\int_{Q_{2R}(x_0)} |\nabla w|^2 \, dx \right)^{1/2} \\
&\le c \, R^{-1} \left[\int_{Q_{2R}(x_0)} |u|^4 \, dx \int_{Q_{2R}(x_0)} |u|^2 \, dx \right]^{1/2} \\
&\le c \, R^{-1} \left[\int_{Q_{2R}(x_0)} |u|^4 \, dx + \int_{Q_{2R}(x_0)} |u|^2 \, dx \right] .
\end{aligned} \tag{2.6}$$

With (2.4), (2.5) and (2.6) we return to (2.3) and get for any $\delta > 0$

$$
\begin{aligned}
\int_{Q_R(x_0)} |\nabla u|^2\, dx &\le \delta \int_{Q_{2R}(x_0)} |\nabla u|^2\, dx \\
&+ \ c\left[\delta^{-1} R^{-2} \int_{Q_{2R}(x_0)} |u|^2\, dx + R^{-1} \int_{Q_{2R}(x_0)} |u|^3\, dx \right. \\
&+ \ \left. R^{-1} \int_{Q_{2R}(x_0)} |u|^4\, dx + R^{-1} \int_{Q_{2R}(x_0)} |u|^2\, dx \right]
\end{aligned} \tag{2.7}
$$

valid for all squares $Q_{2R}(x_0) \subset \mathbb{R}^2$. Lemma 2.1 combined with inequality (2.7) implies

$$
\begin{aligned}
\int_{Q_R(x_0)} |\nabla u|^2\, dx \ \le \ c &\left[R^{-1} \int_{Q_{2R}(x_0)} \left(|u|^2 + |u|^3 + |u|^4\right)\, dx \right. \\
&\left. + R^{-2} \int_{Q_{2R}(x_0)} |u|^2\, dx \right] .
\end{aligned} \tag{2.8}
$$

In (2.8) we fix $x_0 = 0$ and consider $R \ge 1$. Let us further assume the validity of (1.5) for some $\alpha \in (0,1)$ which means

$$
|u(x)| \le c\, R^\alpha \text{ on } Q_{2R}(0)\,. \tag{2.9}
$$

Using the bound (2.9) in estimate (2.8) we find

$$
\int_{Q_R(0)} |\nabla u|^2\, dx \le c\, R^{1+4\alpha}\,. \tag{2.10}
$$

Obviously (2.10) does not yield our claim (2.1). In order to improve (2.10) we return to (2.7) and observe that according to (2.6) we get the following variant of (2.7):

$$
\begin{aligned}
\int_{Q_R(0)} |\nabla u|^2\, dx \quad &\le \quad \delta \int_{Q_{2R}(0)} |\nabla u|^2\, dx \\
&\qquad + c\left[\delta^{-1} R^{-2} \int_{Q_{2R}(0)} |u|^2\, dx + R^{-1} \int_{Q_{2R}(0)} |u|^3\, dx \right. \\
&\qquad \left. + R^{-1} \left(\int_{Q_{2R}(0)} |u|^4\, dx \int_{Q_{2R}(0)} |u|^2\, dx \right)^{1/2} \right] \\
&\overset{(2.9),(2.10)}{\le} \quad c\,\delta R^{1+4\alpha} + c\Big[\delta^{-1} R^{2\alpha} + R^{1+3\alpha} \\
&\qquad + R^{-1} \left(R^{2+4\alpha} R^{2+2\alpha}\right)^{1/2}\Big] \\
&\le \quad c\,\delta R^{1+4\alpha} + c\Big[\delta^{-1} R^{2\alpha} + R^{1+3\alpha}\Big]\,,
\end{aligned}
$$

and with $\delta := R^{-\alpha}$ we find

$$
\int_{Q_R(0)} |\nabla u|^2\, dx \le c R^{1+3\alpha}\,,
$$

which gives our claim (2.1) even without the restriction $\alpha < 1/7$. $\square$

3 Energy estimates for the vorticity function

Let $\omega := \partial_2 u^1 - \partial_1 u^2$ be the vorticity function of a velocity field $u : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying equations (1.1). Then we have:

Lemma 3.1. *Suppose that we have the validity of inequality (1.4) for some exponent $\alpha \geq 0$. Then it holds*

$$\int_{Q_R(0)} |\nabla \omega|^2 \, dx \leq c R^{4\alpha} \tag{3.1}$$

for any $R \geq 1$ with a positive constant c not depending on R.

Proof: The vorticity ω satisfies the equation

$$\Delta \omega - u \cdot \nabla \omega = 0 \, ,$$

hence

$$\int_{Q_{2R}(0)} \nabla \omega \cdot \nabla \left(\eta^2 \omega \right) \, dx + \int_{Q_{2R}(0)} u \cdot \nabla \omega \, \eta^2 \omega \, dx = 0 \, ,$$

where $\eta \in C_0^1 \left(Q_{2R}(0) \right)$, $\eta = 1$ on $Q_R(0)$, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq c/R$. Young's inequality gives

$$\int_{Q_{2R}(0)} \eta^2 |\nabla \omega|^2 \, dx \leq c \left[R^{-2} \int_{Q_{2R}(0)} \omega^2 \, dx + \left| \int_{Q_{2R}(0)} u \cdot \nabla \omega \, \eta^2 \omega \, dx \right| \right] . \tag{3.2}$$

For the second integral on the right-hand side of (3.2) we observe

$$\int_{Q_{2R}(0)} u \cdot \nabla \omega \, \eta^2 \omega \, dx = \frac{1}{2} \int_{Q_{2R}(0)} u \cdot \nabla |\omega|^2 \, \eta^2 \, dx = -\frac{1}{2} \int_{Q_{2R}(0)} u \cdot \nabla \, \eta^2 |\omega|^2 \, dx \, ,$$

and (1.4) together with (2.1) implies

$$\left| \int_{Q_{2R}(0)} u \cdot \nabla \omega \, \eta^2 \omega \, dx \right| \leq c R^{3\alpha+1} R^{-1} R^{\alpha} = c R^{4\alpha} \, .$$

Returning to (3.2) and using (2.1) one more time, we find

$$\int_{Q_R(0)} |\nabla \omega|^2 \, dx \leq c \left[R^{3\alpha - 1} + R^{4\alpha} \right] \leq c R^{4\alpha} \, ,$$

and inequality (3.1) follows. $\square$

4 Proof of Theorem 1.3

Let $v : \mathbb{R}^2 \to \mathbb{R}^2$ denote an entire solution of (1.1) with vorticity ω. From the maximum-principle we get (cf. [GT], Theorem 8.16)

$$\sup_{B_R(0)} |\omega| = \sup_{\partial B_R(0)} |\omega| \tag{4.1}$$

for any radius $R > 0$. If $\omega(r, \Theta)$ denotes the representation of ω in polar coordinates, we have similar to the first inequality from the proof of Lemma 2.4 in [GW]:

$$\begin{aligned} &\int_1^2 \frac{dr}{r} \int_0^{2\pi} \big(r^2\omega^2 + 2r|\omega\partial_\Theta\omega|\big)\, d\Theta \\ &\quad \leq \int_{1<|x|<2} \big(\omega^2 + 2|\omega\nabla\omega|\big)\, dx\,, \end{aligned} \tag{4.2}$$

where on the right-hand side of (4.2) the integration is performed over the annulus $B_2(0) - \overline{B_1(0)}$. For a suitable radius $r \in (1, 2)$ inequality (4.2) implies

$$\begin{aligned} &\int_0^{2\pi} \big[r^2\omega^2(r,\Theta) + 2r|\omega(r,\Theta)\partial_\Theta\omega(r,\Theta)|\big]\, d\Theta \\ &\quad \leq \frac{1}{\ln 2} \int_{1<|x|<2} \big(\omega^2 + 2|\omega\nabla\omega|\big)\, dx\,. \end{aligned} \tag{4.3}$$

At the same time the estimate stated after (2.17) in [GW] yields for all Θ

$$\omega^2(r,\Theta) \leq \frac{1}{2\pi} \int_0^{2\pi} \omega^2(r,\varphi)d\varphi + 2 \int_0^{2\pi} |\omega(r,\varphi)\partial_\varphi\omega(r,\varphi)|\, d\varphi\,. \tag{4.4}$$

If we combine (4.1) with (4.3) and (4.4) we arrive at

$$\sup_{B_r(0)} \omega^2 \leq c \int_{B_2(0)} \big(\omega^2 + |\omega\nabla\omega|\big)\, dx$$

with c independent of ω, in particular it holds

$$\sup_{B_1(0)} \omega^2 \leq c \int_{B_2(0)} \big(\omega^2 + |\omega\nabla\omega|\big)\, dx\,. \tag{4.5}$$

Now consider u as in Theorem 1.3 with vorticity ω. Then for $R \geq 1$ we let $u_R(x) := Ru(Rx)$ and observe that u_R (together with $\pi_R(x) := R^2\pi(Rx)$) is an entire solution of

(1.1) with vorticity ω_R given by $\omega_R(x) = R^2\omega(Rx)$. We apply (4.5) to u_R and ω_R with the result

$$\begin{aligned} R^4 \sup_{B_R(o)} \omega^2 &= \sup_{B_1(0)} \omega_R^2 \\ &\le c \left\{ \int\limits_{B_2(0)} R^4\omega^2(Rx)\,dx + \|\omega_R\|_{L^2(B_2(0))}\|\nabla\omega_R\|_{L^2(B_2(0))} \right\} . \end{aligned}$$

This implies

$$\sup_{B_R(0)} \omega^2 \le c \left\{ R^{-2} \int\limits_{B_{2R}(0)} \omega^2\,dx + R^{-1}\|\omega\|_{L^2(B_{2R}(0))}\|\nabla\omega\|_{L^2(B_{2R}(0))} \right\} ,$$

and from the estimates (2.1) and (3.1) we deduce

$$\sup_{B_R(0)} \omega^2 \le c \left\{ R^{3\alpha-1} + R^{-1}R^{\frac{1}{2}+\frac{3}{2}\alpha+2\alpha} \right\} ,$$

hence $\omega \equiv 0$ in case $\alpha < 1/7$. This together with $\operatorname{div} u = 0$ shows that $u^1 - iu^2$ is an entire analytic function, which must be constant on account of (1.4). □

5 Some open problems

The statement of Theorem 1.3 seems to be far away from being optimal, and we therefore suggest to investigate the following problems for entire solutions $u : \mathbb{R}^2 \to \mathbb{R}^2$ of equations (1.1):

I. Suppose that $\lim\limits_{|x|\to\infty} \frac{|u(x)|}{|x|} = 0$. Does the constancy of u follow?

II. Is u an affine function in case $\limsup\limits_{|x|\to\infty} \frac{|u(x)|}{|x|} < \infty$ or under the stronger hypothesis $\sup\limits_{\mathbb{R}^2} |\nabla u| < \infty$?

III. What can be said about u if we require $\lim\limits_{|x|\to\infty} \frac{|u(x)|}{|x|^\alpha} = 0$ for some $\alpha > 1$ or $\sup\limits_{\mathbb{R}^2} |\nabla^k u| < \infty$ for some $k \ge 2$?

Next we look at generalized Newtonian fluids replacing (1.1) by

$$\text{(5.1)} \qquad \begin{cases} -\operatorname{div}\left[T^D\left(\varepsilon(u)\right)\right] + u^k\partial_k u + \nabla\pi = 0\,, \\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^2\,. \end{cases}$$

Here T denotes the Cauchy stress tensor, T^D its deviatoric part, and $\varepsilon(u)$ is the symmetric gradient of the velocity field u. We impose the constitutive relation

$$T^D(\varepsilon) = DH(\varepsilon)$$

for a potential H of the form $H(\varepsilon) = h(|\varepsilon|)$, hence the viscosity function is given by

$$\mu(t) = \frac{h'(t)}{t}, \; t \geq 0 \, .$$

As in the papers [Fu] and [FuZ] we assume that the fluid is of shear thickening or of shear thinning type, and that we have an entire solution u of (5.1). A first goal might be an extension of the Liouville theorem 1.1 of Gilbarg and Weinberger to the situation at hand:

IV. Assume that $\int_{\mathbb{R}^2} h\,(|\varepsilon(u)|)\; dx < \infty$. Can we prove that u is an rigid motion? If so, then $\int_{\mathbb{R}^2} h\,(|\nabla u|)\; dx < \infty$ implies the constancy of the field u.

An even more challenging task is

V. the discussion of I - III in the setting of generalized Newtonian fluids.

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