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Abstract

We prove the existence of weak solutions with homogeneous boundary data for the following equations of Navier-Stokes type equations

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = \nabla \pi + (\nabla \mathbf{u})\mathbf{u},$$

where $\mathbf{u} : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ is the velocity field satisfying $\operatorname{div} \mathbf{u} = 0$ and $\mathbf{f} : \Omega \to \mathbb{R}^2$ and $\pi : \Omega \to \mathbb{R}$ are the external volume forces resp. the pressure. In order to model the behaviour of Prandtl-Eyring fluids we assume the validity of the stress-strain relation

$$\boldsymbol{\sigma} = DW(\boldsymbol{\varepsilon}(\mathbf{u})), \quad W(\boldsymbol{\varepsilon}) = |\boldsymbol{\varepsilon}| \ln(1 + |\boldsymbol{\varepsilon}|).$$

The crucial tool in our approach is a solenoidal Lipschitz truncation.

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Keywords: equations of Navier-Stokes type, generalized Newtonian fluids, steady flows, existence of weak solutions, solenoidal Lipschitz truncation.

1 Introduction

The stationary flow of a fluid of Prandtl-Eyring type in a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is - according to Eyring [Eyr36] - described by the following set of equations: let $\mathbf{u} : \Omega \to \mathbb{R}^d$ denote the velocity field. Since the fluid is incompressible, we have

$$\operatorname{div} \mathbf{u} = 0. \tag{1.1}$$

For a given system of volume forces $\mathbf{f} : \Omega \to \mathbb{R}^d$ the fluid has to obey the equation of motion

$$\operatorname{div} \boldsymbol{\tau} + \mathbf{f} = (\nabla \mathbf{u})\mathbf{u}, \quad \boldsymbol{\tau} = \boldsymbol{\sigma} - \pi I \tag{1.2}$$

where $(\nabla \mathbf{u})\mathbf{u} := (\partial_i \mathbf{u}^j \mathbf{u}^i)_{1 \le j \le n}$ denotes the convective term, $\boldsymbol{\sigma}$ is the stress deviator and π the pressure. In order to characterize the specific fluid under consideration we need a constitutive law, which relates $\boldsymbol{\sigma}$ and the symmetric gradient

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\partial_i \mathbf{u}^j + \partial_j \mathbf{u}^i)_{1 \le i,j \le n}.$$

Here this relation reads as

$$\boldsymbol{\sigma} = \eta_0 \frac{\operatorname{arsinh}(\lambda |\boldsymbol{\varepsilon}(\mathbf{u})|)}{\lambda |\boldsymbol{\varepsilon}(\mathbf{u})|} \boldsymbol{\varepsilon}(\mathbf{u}) \tag{1.3}$$

with physical constants $\eta_0, \lambda > 0$. Eyring [Eyr36] obtained this law by a molecular theory, similar relations are due to Prandtl (compare [BB35] for an overview). Equation (1.3) means that the viscosity $\nu : \Omega \to \mathbb{R}$ of the fluid can be described by the function (depending on the shear rate $|\boldsymbol{\varepsilon}(\mathbf{u})|$)

$$\nu = \eta_0 \frac{\operatorname{arsinh}(\lambda|\boldsymbol{\varepsilon}(\mathbf{u})|)}{\lambda|\boldsymbol{\varepsilon}(\mathbf{u})|}. \tag{1.4}$$

(1.4) shows that the fluid is very shear thinning and such a behaviour can be observed, for example, in the motion of lubricants. Furthermore one can use the model as an approximation for perfectly plastic fluids introduced in [vM13]. Similar approximations are used in the study of plastic material behaviour, compare [FS98] and [FS99] for a mathematical approach.
Letting

$$W(\boldsymbol{\varepsilon}) := \eta_0 \int\limits_0^{|\boldsymbol{\varepsilon}|} \frac{1}{\lambda} \operatorname{arsinh}(\lambda t)\, dt \tag{1.5}$$

for $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ (:= space of symmetric $d \times d$-matrices) we can replace (1.3) by the equation

$$\boldsymbol{\sigma} = DW(\boldsymbol{\varepsilon}(\mathbf{u})). \tag{1.6}$$

Finally we consider the case of non-slip boundary conditions, i.e., we require

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{1.7}$$

If in addition the flow is also slow, which means that we can neglect the convective term $(\nabla u)u$, then inspired by ideas of Frehse and Seregin [FS98] it is shown in [FS99] how to reduce (1.1)-(1.7) to a variational problem and thereby obtaining a weak solution $\mathbf{u}$ in the natural function space (see Appendix for a precise definition)

$$V^{1,h}_{0,\mathrm{div}} := \left\{ \mathbf{w} \in L^1(\Omega) : \int_\Omega h(|\boldsymbol{\varepsilon}(\mathbf{w})|)\, dx < \infty,\ \operatorname{div} \mathbf{w} = 0,\ \mathbf{w}|_{\partial\Omega} = 0 \right\},$$
$$h(t) := t \ln(1+t),\ t \geq 0,$$

which is a smooth function, if $d = 2$, and partially of class C^1, if the 3D-case is considered. Note that we can replace the energy W from (1.5) by the more convenient expression

$$W(\boldsymbol{\varepsilon}) = h(|\boldsymbol{\varepsilon}|) \tag{1.8}$$

since all our arguments actually work for potentials of the form $g(|\boldsymbol{\varepsilon}|)$ with g being C^2-close to the function h.

For the natural case $(\nabla u)u \neq 0$ it is not immediate how to find a solution of (1.1)-(1.3) and (1.7) with W defined in (1.5) or (1.8). In order to get an idea of how to proceed, let us assume that $\mathbf{u}$ is a sufficiently smooth solution. Then we obtain

$$\int_\Omega DW(\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx = \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi}\, dx + \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\, dx + \int_\Omega \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx \tag{1.9}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$. If we restrict ourselves to test functions $\boldsymbol{\varphi}$ with $\operatorname{div} \boldsymbol{\varphi} = 0$, then the term involving the pressure vanishes and we get

$$\int_\Omega DW(\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx = \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\, dx + \int_\Omega \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx \tag{1.10}$$

valid for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$, and according to the results of Section 4 we see that in case $d = 2$ all terms in (1.9) and (1.10) are well-defined, provided we choose $\mathbf{u}$ from the space $V^{1,h}_{0,\mathrm{div}}(\Omega)$, π from $L^h_0(\Omega)$ (see Section 2 for its definition) and require

$$\mathbf{f} \in L^{p_0}(\Omega) \tag{1.11}$$

for some $p_0 > 1$. Our main result now states that actually such a weak solution exists.

THEOREM 1.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain and consider volume forces $\mathbf{f}$ satisfying (1.11). Moreover, let W be defined according to (1.5) or (1.8). Then there exists a velocity field $\mathbf{u} \in V^{1,h}_{0,\mathrm{div}}(\Omega)$ and a pressure $\pi \in L^h_0(\Omega)$ satisfying* (1.9) *for all fields $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$ and* (1.10) *for all fields*

$$\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega) := \{\boldsymbol{\psi} \in C_0^\infty(\Omega) : \ \operatorname{div} \boldsymbol{\psi} = 0\}\,.$$

Remark 1.2. *The reason for condition (1.11) is the choice of our approximation. We stabilize the equation by adding a quadratic term in the main part. Hence we obtain a sequence $(\mathbf{v}^n) \subset W^{1,2}_{0,\mathrm{div}}(\Omega)$ and the term $\int_\Omega \mathbf{f} \cdot \mathbf{v}^n\, dx$ is well-defined by (1.11). We expect that it is possible to weaken this assumption. Precisely, it suffices to suppose $\mathbf{f} = \operatorname{div} \mathbf{F}$ with $\mathbf{F} \in L^1(\Omega)$.*

We will prove Theorem 1.1 by approximation, i.e., by replacing (1.10) through a sequence of more regular problems with corresponding solutions $\mathbf{u}^n$. It turns out that the sequence $(\mathbf{u}^n)$ is bounded in the space $V^{1,h}_{0,\mathrm{div}}(\Omega)$, and in Theorem 4.7 we will investigate spaces like $V^{1,h}_{0,\mathrm{div}}(\Omega)$ with the result that $V^{1,h}_{0,\mathrm{div}}(\Omega)$ is compactly embedded in the space $L^2(\Omega)$. Hence it holds (for a subsequence)

$$\int_\Omega \mathbf{u}^n \otimes \mathbf{u}^n : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx \ \longrightarrow \ \int_\Omega \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx, \quad n \to \infty,$$

with a suitable limit function $\mathbf{u}$, which turns out to be in the class $V^{1,h}_{0,\mathrm{div}}(\Omega)$. The main task is the proof of

$$\int_\Omega DW(\boldsymbol{\varepsilon}(\mathbf{u}^n)) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\,dx \;\longrightarrow\; \int_\Omega DW(\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\,dx,\; n \to \infty.$$

We profit from the Lipschitz truncation method of Acerbi and Fusco [AF84] which was used in the context of fluid mechanics in [FMS03] and advanced in [DMS08]. The two latter papers are dealing with the situation

$$W(\boldsymbol{\varepsilon}) \approx |\boldsymbol{\varepsilon}|^p$$

for a power $p > 1$ (but in 2D arbitrary close to 1). This situation is much better than our case, since the spaces L^p (for $p > 1$) feature a more nice behaviour than the space $L \ln L$, which is the natural space (for the symmetric gradient) in our setting. Due to the lack of a Korn-type inequality in $L \ln L$ (see [BD11]), we are not able to bound $M(\nabla \mathbf{u}^n)$ in L^1 (M is the Hardy-Littlewood maximal function, which we will define in section 2). This means that an ordinary Lipschitz-truncation is not possible. The main idea to overcome this difficulty is instead of approximating $\mathbf{u}^n$ by a sequence of Lipschitz continuous functions, to use functions only having a bounded symmetric gradient (instead of a bounded gradient).
In equation (1.10) (and also in our approximated version, see section 3) only solenoidal test functions are admissible. Since the Lipschitz truncation is based on a nonlinear extension operator it does not preserve the incompressibility condition of the solution. In the p-fluid situation there are two ways to overcome this difficulty:

- introducing the pressure function π which belongs to L^s for some $s > 1$;
- correcting by the Bogovskiĭ operator [Bog80].

In case of p-fluids both methods are applicable but they fail for Prandtl-Eyring fluids. Neither the pressure belongs to the correct space (see end of section 3) nor the Bogovskiĭ operator is continuous. This strongly motivates us to construct a solenoidal Lipschitz-truncation which is also a very useful advancement of the Lipschitz truncation method by itself. Here the main idea is a local projection procedure by means of finite dimensional function spaces (with globally bounded dimensions). In these spaces the Bogovskiĭ operator is continuous independent of the applied norm.
In connection with Theorem 1.1 we mention four open problems:

i) What are the smoothness properties of the specific weak solution $\mathbf{u}$ constructed in the proof of Theorem 1.1? We conjecture that $\mathbf{u}$ is locally of class C^1.

ii) Can we prove the existence of solutions for stationary 3D flows?

iii) The logarithmic potential $|\boldsymbol{\varepsilon}(\mathbf{u})| \ln(1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)$ serves as an approximation for perfectly plastic fluids with potential $|\boldsymbol{\varepsilon}(\mathbf{u})|$. Is it possible to handle the linear case with similar arguments?

iv) Can we obtain similar results for nonstationary Prandtl-Eyring fluids? In the paper [DRW10] a parabolic version of the Lipschitz-truncation was developed in order to consider unsteady flows of power-law fluids. The result is an improvement of [Wol07] which was based on a L^∞-truncation. Unfortunately our technique to produce a solenoidal Lipschitz truncation cannot be extended to unsteady problems.

2 The solenoidal Lipschitz truncation

Note that the results of this section are not restricted to $\mathbb{R}^2$ but hold on $\mathbb{R}^d$. Lipschitz truncations of Sobolev functions are used in various areas of analysis in different aspects and go back to Acerbi and Fusco [AF84]. In the context of fluid mechanics the method was firstly used in [FMS03] in order to conclude the almost everywhere convergence $\boldsymbol{\varepsilon}(\mathbf{u}^n) \to \boldsymbol{\varepsilon}(\mathbf{u})$ of the approximating sequence leading to the identification of the weak limit. This technique was later simplified and improved in [DMS08].
The main idea in the method of Lipschitz truncation is to approximate a Sobolev function $\mathbf{w}$ by Lipschitz-continuous functions which differ from $\mathbf{w}$ only on a set of small Lebesgue measure. This is achieved by redefining the function on the set $\{M(\nabla \mathbf{w}) > \lambda\}$ by a suitable Lipschitz extension. Here M denotes the Hardy-Littlewood maximal operator defined by

$$(Mv)(x) := \sup_{r>0} \fint_{B_r(x)} |v| \, dy, \quad v \in L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

The result is an approximation $\mathbf{w}_\lambda$ of $\mathbf{w}$ whose gradients are bounded by a constant times λ. For the application of the Lipschitz truncation to the equation it is important that the function $\lambda \chi_{\{M(\nabla v) > \lambda\}}$ is small for certain large $\lambda > 0$ in the corresponding function space. In the setting of [FMS03] and [DMS08] the corresponding space was the Lebesgue space L^p with $1 < p < \infty$. On such spaces M is a bounded operator and Korn's inequality holds. However, in our situation the corresponding function space is the Orlicz class L^h. Unfortunately, M is not unbounded on L^h and Korn's inequality is not valid on L^h (see [BD11]). Therefore, we need to refine the method of [DMS08]

even further. We compensate the unboundedness of M on L^h using refined weak-type $L^h \to L^1$ estimates for the maximal operator.
To overcome the lack of Korn's inequality, we have to work directly with function spaces defined in terms of $\boldsymbol{\varepsilon}(\mathbf{u})$ rather than $\nabla \mathbf{u}$. In particular, we redefine $\mathbf{w}$ on $\{M(\boldsymbol{\varepsilon}(\mathbf{w})) > \lambda\}$. As a consequence our truncation is not Lipschitz continuous but has bounded symmetric gradient, i.e. $\boldsymbol{\varepsilon}(\mathbf{w}_\lambda) \in L^\infty$.
Another difficulty of the Lipschitz truncation method in the context of fluid mechanics is that the truncation of a solenoidal field $\mathbf{w}$ is no longer solenoidal. This problem can usually be overcome by the use of the Bogovskiĭ solution operator "Bog_Ω" of the divergence equation $\operatorname{div} \mathbf{z} = f$ on Ω with zero boundary values. In particular, the Lipschitz truncation $\mathbf{w}_\lambda$ will be corrected by the solution $\mathbf{z}$ of $\operatorname{div} \mathbf{z} = \chi_{\mathbf{w} \neq \mathbf{w}_\lambda} \operatorname{div} \mathbf{w}_\lambda$. However, the operator Bog_Ω is only bounded in the L^p setting but not in the L^h setting. Therefore, it is not possible to simply truncate $\mathbf{w}$ and afterwards correct the divergence of the truncation to zero. To solve this problem, we develop a modified version of the Lipschitz truncation method, which is able to approximate solenoidal functions by solenoidal truncations. We refer to this modified Lipschitz truncation also as *solenoidal Lipschitz truncation.*
Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with Lipschitz boundary. We denote by $L^h(\Omega)$ the Orlicz space generated by $h(t) := t \ln(1+t),\ t \geq 0$ equipped with the *Luxemburg norm* (cf. [Ada75])

$$\|\mathbf{w}\|_{L^h(\Omega)} := \inf \{k > 0 : \rho_h(\mathbf{w}/k) \leq 1\},$$

where

$$\rho_h(\mathbf{w}) := \int_\Omega h(|\mathbf{w}|)\, dx.$$

The functional ρ_h is called the modular of L^h. Note that $\|\cdot\|_h$ is just the Minkowski functional of the set $\{\mathbf{w} :\ \rho_h(\mathbf{w}) \leq 1\}$. We write $L^h_0(\Omega)$ and $L^p_0(\Omega)$ for the subspace of $L^h(\Omega)$ and $L^p(\Omega)$, respectively, consisting of those functions whose integral over Ω vanishes.
Following ideas developed by Frehse and Seregin [FS98] we define the space

$$V^{1,h}(\Omega) := \{\mathbf{w} \in L^1(\Omega)\ :\ |\boldsymbol{\varepsilon}(\mathbf{w})| \in L^h(\Omega)\}.$$

By letting

$$\|\mathbf{u}\|_{V(\Omega)} := \|\mathbf{u}\|_{L^1(\Omega)} + \|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^h(\Omega)} \tag{2.1}$$

$V^{1,h}(\Omega)$ turns into a Banach space. We define

$$V^{1,h}_0(\Omega) := \overline{\{\mathbf{w} \in C^\infty_0(\Omega)\}}^{V^{1,h}(\Omega)},$$

$$V^{1,h}_{0,\mathrm{div}}(\Omega) := \{\mathbf{w} \in V^{1,h}_0(\Omega) \,:\, \operatorname{div} \mathbf{w} = 0\}.$$

In the appendix we summarize further properties of these spaces. Among other things we show that $V^{1,h}_0(\Omega)$ is the subspace of $V^{1,h}(\Omega)$ consisting of all fields with vanishing trace and $V^{1,h}_{0,\mathrm{div}}(\Omega)$ is the closure of $C^\infty_{0,\mathrm{div}}(\Omega)$-functions. Let $\mathbf{w} \in V^{1,h}_{0,\mathrm{div}}(B)$ for some ball $B \subset \mathbb{R}^d$. We define our *bad set* by $\mathcal{O}_\lambda := \{M(\boldsymbol{\varepsilon}(\mathbf{w})) > \lambda\}$. We do not have to truncate our function $\mathbf{w}$ if $\mathcal{O}_\lambda$ is empty. So we can assume in the following that $\mathcal{O}_\lambda \neq \emptyset$. We decompose the open set $\mathcal{O}_\lambda$ into a family of dyadic closed cubes $\{Q_j\}_j$ with side length $\ell(Q_j)$ such that

(W1) $\bigcup_j \overline{Q_j} = \mathcal{O}_\lambda$ and the Q_j's have disjoint interiors.

(W2) $8\sqrt{d}\ell(Q_j) \leq \operatorname{dist}(Q_j, \partial\mathcal{O}_\lambda) \leq 32\sqrt{d}\ell(Q_j)$. In particular, if $c_d := 2 + 32\sqrt{d}$, then $(c_d Q_j) \cap (\mathbb{R}^d \setminus \mathcal{O}_\lambda) \neq \emptyset$.

(W3) If the boundaries of two cubes Q_j and Q_k touch, then

$$\frac{1}{2} \leq \frac{\ell(Q_j)}{\ell(Q_k)} \leq 2.$$

(W4) For a given Q_j there exists at most $(3^d - 1)2^d$ Q_k's that touch Q_j.

We can get this family as follows: take the family of closed dyadic cubes as in [Gra04] and subdivide each of these cubes into 8^d dyadic sub-cubes. (The constants in [Gra04] are $\frac{1}{4}$ and 4 instead of $\frac{1}{2}$ and 2, respectively, but this is due to the use of $\leq$ in the first step of line 10 of page A-35 in [Gra04] instead of a sharper $<$.)
Define $Q^*_k := \frac{9}{8}Q_j$, then we have the following properties:

(W5) $\bigcup_j \overline{Q^*_j} = \mathcal{O}_\lambda$

(W6) If Q^*_j and Q^*_k intersect, then the boundaries of Q_j and Q_k touch and $Q^*_j \subset 5Q^*_k$, moreover $\ell(Q^*_j) \sim \ell(Q^*_k)$ and $|Q^*_j \cap Q^*_k| \sim |Q^*_j| \sim |Q^*_k|$ (here $\sim$ means that two quantities can be bounded vice versa).

(W7) The family Q^*_j is locally 6^d finite.

(W8) $\sum_j \mathcal{L}^d(Q^*_j) \leq c(d)\mathcal{L}^d(\mathcal{O}_\lambda)$.

Let $\widetilde{\varphi}_j \in C^\infty_0(\mathbb{R}^d)$ such that

- $\operatorname{supp} \widetilde{\varphi}_j = Q^*_j$.

- $\chi_{\frac{7}{9}Q_j^*} = \chi_{\frac{7}{8}Q_j} \leq \varphi_j \leq \chi_{\frac{9}{8}Q_j} = \chi_{Q_j^*}$.
- All $\widetilde{\varphi}_j$ are up to translation and dyadic scaling the same function.

Define $\gamma := \sum_j \widetilde{\varphi}_j$ and $\varphi_j := \frac{\widetilde{\varphi}_j}{\gamma}$. Then

- $1 \leq \gamma \leq 6^d$,
- $|\nabla\gamma|\,\chi_{Q_j^*} \leq c\frac{1}{\ell(Q_j)}$ for all $j \in \mathbb{N}$,

and φ_j defines a partition of unity with the following properties:

(U1) $\varphi_j \in C_0^\infty(\mathbb{R}^d)$.

(U2) $\operatorname{supp}\varphi_j = Q_j^*$.

(U3) $\chi_{\frac{7}{9}Q_j^*} = \chi_{\frac{7}{8}Q_j} \leq \varphi_j \leq \chi_{\frac{9}{8}Q_j} = \chi_{Q_j^*}$.

(U4) $|\nabla\varphi_j| \leq \dfrac{c\,\chi_{Q_j^*}}{\ell(Q_j^*)}$.

(U5) $|\nabla^2\varphi_j| \leq \dfrac{c\,\chi_{Q_j^*}}{\ell(Q_j^*)^2}$.

We abbreviate $r_j := \ell(Q_j^*)$.
We define $\mathcal{R}_{Q_j^*}\mathbf{w}$ as the $L^2(Q_j^*)$-orthonormal projection of $\mathbf{w}$ onto the space of rigid motions $\mathcal{R}$, i.e.,

$$\big(\mathcal{R}_{Q_j^*}\mathbf{w}\big)(x) := \sum_l \left(\int_{Q_j^*} \mathbf{R}_l^j \cdot \mathbf{w}\,dy\right)\mathbf{R}_l^j(x),$$

where $(\mathbf{R}_l^j)$ is an $L^2(Q_j^*)$-orthonormal base of $\mathcal{R}$. This operator is also well defined for $\mathbf{w} \in L^1(Q_j^*)$. Moreover, it is continuous from L^1 to $W^{1,\infty}$ and

$$\|\mathcal{R}_{Q_j^*}\mathbf{w}\|_{L^\infty(Q_j^*)} + r_j\|\nabla\mathcal{R}_{Q_j^*}\mathbf{w}\|_{L^\infty(Q_j^*)} \leq c \fint_{Q_j^*} |\mathbf{w}|\,dx \qquad \text{for all } \mathbf{w} \in L^1(Q_j^*). \tag{2.2}$$

Since $\mathcal{R}_{Q_j^*}$ is the identity on constants it follows easily from (2.2) that $\mathcal{R}_{Q_j^*}$ is also $W^{1,1}$-stable in the sense that

$$\fint_{Q_j^*} |\nabla\mathcal{R}_{Q_j^*}w|\,dx \leq \fint_{Q_j^*} |\nabla w|\,dx. \tag{2.3}$$

Moreover, it follows from (2.2) and the fact that $\mathcal{R}_{Q_j^*}$ is the identity on $\mathcal{R}$ that

$$\fint_{Q_j^*} |\mathbf{w} - \mathcal{R}_{Q_j^*}\mathbf{w}|\,dx \le c \inf_{\mathbf{R}\in\mathcal{R}} \fint_{Q_j^*} |\mathbf{w} - \mathbf{R}|\,dx. \tag{2.4}$$

Now we can define for $\mathbf{w} \in W_0^{1,1}(B)$ our preliminary Lipschitz truncation operator T^λ by

$$T^\lambda \mathbf{w} = \begin{cases} \mathbf{w} & \text{on } \mathbb{R}^d \setminus \mathcal{O}_\lambda \\ \sum\limits_j \varphi_j \mathbf{w}_j & \text{on } \mathcal{O}_\lambda, \end{cases}$$

where $\mathbf{w}_j := \mathcal{R}_{Q_j^*}\mathbf{w}$ for $j \in \mathbb{N}$.
We will see later that $T^\lambda \mathbf{w} \in W_0^{1,1}(\mathbb{R}^d)$. Let us remark that $\operatorname{div} T^\lambda \neq 0$ for our preliminary Lipschitz truncation. The following Lemma provides some important estimates for T^λ.

Lemma 2.1. *Let* $\mathbf{w} \in W_0^{1,1}(B)$.

(a) *For all j it holds*

$$\fint_{Q_j^*} \left| \frac{\mathbf{w} - \mathbf{w}_j}{r_j} \right| dx \le c \fint_{Q_j^*} |\boldsymbol{\varepsilon}(\mathbf{w})|\,dx \le c\, M(\boldsymbol{\varepsilon}(\mathbf{w}))(y) \qquad \textit{for all } y \in Q_j^*.$$

(b) *For all j it holds*

$$\fint_{Q_j^*} |\boldsymbol{\varepsilon}(\mathbf{w})|\,dx \le \fint_{c_d Q_j} |\boldsymbol{\varepsilon}(\mathbf{w})|\,dx \le c\,\lambda.$$

(c) *For all j and k with $Q_j^* \cap Q_k^* \neq \emptyset$ it holds*

$$\|\mathbf{w}_j - \mathbf{w}_k\|_{L^\infty(Q_j^*)} \sim \fint_{Q_j^*} |\mathbf{w}_j - \mathbf{w}_k|\,dx \le c \fint_{Q_j^*} \left| \frac{\mathbf{w} - \mathbf{w}_j}{r_j} \right| dx + c \fint_{Q_k^*} \left| \frac{\mathbf{w} - \mathbf{w}_j}{r_j} \right| dx.$$

Proof. (a): The inequality follows by the Poincaré-Korn inequality, see [ST81].
(b): The first estimate is obvious. The second estimate follows from the fact that $c_d Q_j$ intersects $\mathbb{R}^d \setminus \mathcal{O}_\lambda \subset \{M(\boldsymbol{\varepsilon}(\mathbf{w})) \le \lambda\}$.

(c): The equivalence follows from the fact that $\mathcal{R}$ is finite dimensional (so all norms are equivalent) and a suitable scaling argument. It remains to show the second estimate.
Since $|Q_j^* \cap Q_k^*| \sim |Q_j^*| \sim |Q_k^*|$ by (W6), it follows by the fact that all norms on the finite dimensional $\mathcal{R}$ are equivalent and a simple scaling argument that for every $\mathbf{R} \in \mathcal{R}$ it holds that

$$\int_{Q_j^*} |\mathbf{R}|\,dx \sim \int_{Q_j^* \cap Q_k^*} |\mathbf{R}|\,dx.$$

As a consequence

$$\begin{aligned}
\fint_{Q_j^*} \frac{|\mathbf{w}_j - \mathbf{w}_k|}{r_j}\,dx &\leq c\,|Q_j^*| \int_{Q_j^* \cap Q_k^*} \frac{|\mathbf{w}_j - \mathbf{w}_k|}{r_j}\,dx \\
&\leq c\,|Q_j^*| \int_{Q_j^* \cap Q_k^*} \frac{|\mathbf{w} - \mathbf{w}_j|}{r_j}\,dx + c\,|Q_j^*| \int_{Q_j^* \cap Q_k^*} \frac{|\mathbf{w} - \mathbf{w}_k|}{r_j}\,dx \\
&\leq c \fint_{Q_j^*} \frac{|\mathbf{w} - \mathbf{w}_j|}{r_j}\,dx + c \fint_{Q_k^*} \frac{|\mathbf{w} - \mathbf{w}_k|}{r_j}\,dx,
\end{aligned}$$

where we used (a) and (b) in the last step. □

The next Lemma shows that although $T^\lambda \mathbf{w}$ is defined on two different sets it is a global Sobolev function.

Lemma 2.2. *Let* $\mathbf{w} \in W_0^{1,1}(B)$*, then* $T^\lambda \mathbf{w} - \mathbf{w} \in W_0^{1,1}(\mathcal{O}_\lambda)$ *and* $T^\lambda \mathbf{w} \in W_0^{1,1}(\mathbb{R}^d)$*.*

Proof. It suffices to show that $T^\lambda \mathbf{w} - \mathbf{w} \in W_0^{1,1}(\mathcal{O}_\lambda)$. Let J be a finite subset of $\mathbb{N}$. We have pointwise

$$\nabla(\mathbf{w}_\lambda - \mathbf{w}) = \nabla\Big(\sum_{j\in\mathbb{N}} \varphi_j(\mathbf{w}_j - \mathbf{w})\Big) = \sum_{j\in\mathbb{N}} \big((\nabla\varphi_j)(\mathbf{w}_j - \mathbf{w}) + \varphi_j(\nabla\mathbf{w}_j - \nabla\mathbf{w})\big).$$

Since every summand in the last sum belongs to $W_0^{1,1}(\mathcal{O}_\lambda)$, it suffices to show that the last sum converges absolutely in L^1. Let $J \subset \mathbb{N}$ be finite. Then we obtain

$$\begin{aligned}
(I) &:= \int \sum_{j\in\mathbb{N}\setminus J} \big|(\nabla\varphi_j)(\mathbf{w}_j - \mathbf{w}) + \varphi_j(\nabla\mathbf{w}_j - \nabla\mathbf{w})\big|\,dx \\
&\leq \sum_{j\in\mathbb{N}\setminus J} \int_{Q_j^*} \big|(\nabla\varphi_j)(\mathbf{w}_j - \mathbf{w})\big|\,dx + \int_{Q_j^*} \sum_{j\in\mathbb{N}\setminus J} |\varphi_j(\nabla\mathbf{w}_j - \nabla\mathbf{w})|\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j\in\mathbb{N}\setminus J}\int_{Q_j^*}\frac{|\mathbf{w}_j-\mathbf{w}|}{r_j}\,dx+\sum_{j\in\mathbb{N}\setminus J}\int_{Q_j^*}|\nabla\mathbf{w}_j-\nabla\mathbf{w}|\,dx\\
&=: (II)+(III).
\end{aligned}
$$

Now, by (a) it follows that

$$
\begin{aligned}
(II) &\leq \sum_{j\in\mathbb{N}\setminus J}\int_{Q_j^*}|\boldsymbol{\varepsilon}(\mathbf{w})|\,dx=\int_{\mathcal{O}_\lambda}\sum_{j\in\mathbb{N}\setminus J}\chi_{Q_j^*}|\varepsilon(\mathbf{w})|\,dx\\
&\leq c\int_{\mathcal{O}_\lambda}\chi_{\cup_{j\in\mathbb{N}\setminus J}Q_j^*}|\nabla\mathbf{w}|\,dx.
\end{aligned}
$$

On the other hand with (2.3) we estimate

$$
(III)\leq \sum_{j\in\mathbb{N}\setminus J}\int_{Q_j^*}|\nabla\mathbf{w}|\,dx\leq c\int_{\mathcal{O}_\lambda}\chi_{\cup_{j\in\mathbb{N}\setminus J}Q_j^*}|\nabla\mathbf{w}|\,dx.
$$

Overall, we have shown that

$$
(I)\leq c\int_{\mathcal{O}_\lambda}\chi_{\cup_{j\in\mathbb{N}\setminus J}Q_j^*}|\nabla\mathbf{w}|\,dx.
$$

Since $\chi_{\cup_{j\in\mathbb{N}\setminus J}}\to 0$ for $J\to\mathbb{N}$ and $\nabla\mathbf{w}\in L^1$, it follows by dominated convergence that $(I)\to 0$ for $J\to\mathbb{N}$. In particular, we have shown that $\sum_{j\in\mathbb{N}}\varphi_j(\mathbf{w}_j-\mathbf{w})$ converges unconditionally in the gradient norm $\|\nabla\cdot\|_1$ and therefore in $W_0^{1,1}(\mathcal{O}_\lambda)$. □

If follows from the previous lemma that

$$
\nabla T^\lambda\mathbf{w}=\chi_{\mathbb{R}^d\setminus\mathcal{O}_\lambda}\nabla\mathbf{w}+\chi_{\mathcal{O}_\lambda}\sum_j\nabla(\varphi_j\mathbf{w}_j). \tag{2.5}
$$

We define the set of neighbors of Q_j^* (including Q_j^* itself) by

$$
A_j:=\{k\in\mathbb{N}\,:\,Q_j^*\cap Q_k^*\neq\emptyset\}.
$$

Lemma 2.3. *If $\mathbf{w}\in V_0^{1,h}(B)$, then it holds*

(a) $\|T^\lambda\mathbf{w}\|_1\leq c\,\|\mathbf{w}\|_1$.

(b) $|\boldsymbol{\varepsilon}(T^\lambda(\mathbf{w}))|\leq\sum_{k\in A_j}⨍_{Q_k^*}\frac{|\mathbf{w}-\mathbf{w}_k|}{r_k}\,dx$ *on Q_j^* for every $j\in\mathbb{N}$.*

(c) $|\boldsymbol{\varepsilon}(T^\lambda\mathbf{w})|\leq c\,\lambda\chi_{\mathcal{O}_\lambda}+|\boldsymbol{\varepsilon}(\mathbf{w})|\chi_{\mathbb{R}^d\setminus\mathcal{O}_\lambda}$ *and $|\boldsymbol{\varepsilon}(T^\lambda\mathbf{w})|\leq c\,\lambda$ almost everywhere.*

(d) $\|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})\|_h \le c\,\|\varepsilon(\mathbf{w})\|_h$ *and* $\rho_h(\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})) \le c\,\rho_h(\boldsymbol{\varepsilon}(\mathbf{w}))$.

Proof. (a): By definition of T^λ we have

$$\int |T^\lambda \mathbf{w}|\,dx \le \int_{\mathbb{R}^d \setminus \mathcal{O}_\lambda} |\mathbf{w}|\,dx + \sum_j \int_{Q_j^*} |\varphi_j \mathcal{R}_{Q_j^*}\mathbf{w}|\,dx.$$

Now it follows with (2.2) and the local finiteness of the Q_j^* that

$$\int |T^\lambda \mathbf{w}|\,dx \le \int_{\mathbb{R}^d \setminus \mathcal{O}_\lambda} |\mathbf{w}|\,dx + \sum_j \int_{Q_j^*} |\mathbf{w}|\,dx \le c \int_{\mathbb{R}^d} |\mathbf{w}|\,dx.$$

(b): Fix $j \in \mathbb{N}$. Then on Q_j^* it holds

$$\boldsymbol{\varepsilon}(T^\lambda \mathbf{w}) = \boldsymbol{\varepsilon}\Big(\sum_k \varphi_k \mathbf{w}_k\Big) = \boldsymbol{\varepsilon}\Big(\sum_k \varphi_k(\mathbf{w}_k - \mathbf{w}_j)\Big) = \sum_k \nabla\varphi_k \otimes^{\mathrm{sym}} (\mathbf{w}_k - \mathbf{w}_j),$$

where we used $\boldsymbol{\varepsilon}(\mathbf{w}_j) = 0$ and $\sum_k \varphi_k = 1$ on $\mathcal{O}_\lambda$. Therefore, with the local finiteness of the Q_k^* and with Lemma 2.1 (a) and (c) it follows

$$|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})| \le c \sum_{j\,:\,Q_j^* \cap Q_k^* \neq \emptyset} \frac{\|\mathbf{w}_j - \mathbf{w}_k\|_{L^\infty(Q_j^*)}}{r_j} \le c \sum_{k \in A_j} \fint_{Q_k^*} \frac{|\mathbf{w} - \mathbf{w}_k|}{r_k}\,dx.$$

(c): It follows by (b) and Lemma 2.1 (b) that $|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})| \le c\,\lambda$ on Q_j^*. Since $\bigcup_k Q_k^* = \mathcal{O}_\lambda$ we get $|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})| \le c\,\lambda$ on $\mathcal{O}_\lambda$. As a consequence $|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})| \le c\,\lambda \chi_{\mathcal{O}_\lambda} + |\boldsymbol{\varepsilon}(\mathbf{w})| \chi_{\mathbb{R}^d \setminus \mathcal{O}_\lambda}$. On $\mathbb{R}^d \setminus \mathcal{O}_\lambda$ we have $|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})| = |\boldsymbol{\varepsilon}(\mathbf{w})| \le M(\boldsymbol{\varepsilon}(\mathbf{w})) \le \lambda$. So we get $|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})| \le c\,\lambda$ on all of $\mathbb{R}^d$.
(d): We estimate with (c)

$$\begin{aligned}
\|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})\|_h &\le \|\chi_{\mathbb{R}^d \setminus \mathcal{O}_\lambda} \boldsymbol{\varepsilon}(\mathbf{w})\|_h + \|\chi_{\mathcal{O}_\lambda} \boldsymbol{\varepsilon}(T^\lambda \mathbf{w})\|_h \\
&\le \|\boldsymbol{\varepsilon}(\mathbf{w})\|_h + c\,\|\chi_{\mathcal{O}_\lambda} \lambda\|_h. \\
&\le \|\boldsymbol{\varepsilon}(\mathbf{w})\|_h + c\,\|\chi_{\{M(\boldsymbol{\varepsilon}(\mathbf{w})) > \lambda\}} \lambda\|_h.
\end{aligned}$$

Now the weak type estimate for the norm of the maximal function proves

$$\|\boldsymbol{\varepsilon}(T^\lambda \mathbf{w})\|_h \le c\,\|\boldsymbol{\varepsilon}(\mathbf{w})\|_h.$$

The estimate for ρ_h follows analogously using the weak type estimate for the modular of the maximal function. $\square$

It follows immediately from Lemma 2.3 that $T^\lambda \mathbf{w} \in W_0^{1,1}(\mathbb{R}^d)$ with $\boldsymbol{\varepsilon}(T^\lambda \mathbf{w}) \in L^\infty(\mathbb{R}^d)$. So from the point of regularity $T^\lambda \mathbf{w}$ qualifies as a test function of system (1.10). However, the operator T^λ destroys the divergence of $\mathbf{w}$. In particular, $T^\lambda \mathbf{w}$ is not necessarily solenoidal if $\mathbf{w}$ is solenoidal.
We want to construct a *solenoidal Lipschitz truncation* T^λ_{div} such that $T^\lambda_{\mathrm{div}} \mathbf{w}$ of a solenoidal $\mathbf{w}$ is again solenoidal while preserving all nice properties of the Lipschitz truncation. In particular, we want that T^λ_{div} maps $V^{1,h}_{0,\mathrm{div}}$ into $V^{1,\infty}_{0,\mathrm{div}}$, where[1]

$$V^{1,\infty}_{0,\mathrm{div}}(\Omega) := \{\mathbf{w} \in W^{1,1}_{0,\mathrm{div}}(\Omega) \,:\, \boldsymbol{\varepsilon}(\mathbf{w}) \in L^\infty(\Omega)\}.$$

The idea is to correct locally the destroyed divergence of $T^\lambda \mathbf{w}$. The simplest way would be to correct the divergence of $\varphi_j \mathbf{w}_j$ such that it becomes solenoidal. However, the sum of such corrections will not converge in $W^{1,1}$ due an extra r_j^{-1} at every summand.
A better approach is to use the fact that for any solenoidal $\mathbf{w}$ we have the identity

$$\operatorname{div}(T^\lambda \mathbf{w}) = \chi_{\mathcal{O}_\lambda} \operatorname{div}(T^\lambda \mathbf{w}) = \sum_j \varphi_j \operatorname{div}(T^\lambda \mathbf{w})$$

and to correct the contributions $\varphi_j \operatorname{div}(T^\lambda \mathbf{w})$ by suitable solutions $\mathbf{z} \in W_0^{1,1}(Q_j^*)$ of the divergence equation $\operatorname{div} \mathbf{z} = g$. However, the solvability of the divergence equation requires $\int_{Q_j^*} g\, dx = 0$ and $\varphi_j \operatorname{div}(T^\lambda \mathbf{w})$ does not satisfy this constraint. To overcome this problem we use ideas from the construction of divergence preserving interpolation operators in the context of finite elements [BF91]. In the first step we define $T_0^\lambda \mathbf{w}$ by

$$T_0^\lambda \mathbf{w} := T^\lambda \mathbf{w} + \Pi(\mathbf{w} - T^\lambda \mathbf{w}),$$

where Π is a local projection, which ensures that $\varphi_j \operatorname{div} T_0^\lambda \mathbf{w}$ satisfies the constraint $\int_{Q_j^*} g\, dx = 0$. In the second step we correct the divergence of T_0^λ by

$$T^\lambda_{\mathrm{div}} \mathbf{w} := T_0^\lambda \mathbf{w} - \sum_j \operatorname{Bog}_j\big(\varphi_j \operatorname{div} T_0^\lambda \mathbf{w}\big),$$

where Bog_j is the local solution operator of the divergence equation.

[1] Actually, functions from $V^{1,\infty}_{0,\mathrm{div}}$ need not be Lipschitz. Nevertheless, we use the term *Lipschitz truncation* for historical reasons.

We begin with the construction of the local projection Π. For $j \in \mathbb{N}$ we define

$$\begin{aligned} A_j' &:= A_j \setminus \{j\}, \\ X_j &:= \operatorname{span}\{(\nabla \varphi_k)|_{Q_j^*} \,:\, k \in A_j'\} \subset L^1(Q_j^*). \end{aligned}$$

Note that $k \in A_j'$ is equivalent to $j \in A_k'$ and $k \in A_j$ is equivalent to $j \in A_k$. The set $\{\nabla \varphi_k\}_{k \in A_j'}$ is a basis of X_j. For $\mathbf{f}, \mathbf{g} \in L^2(Q_j^*)$ let

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\varphi_j} := \int (\mathbf{f} \cdot \mathbf{g}) \varphi_j \, dx.$$

Then $\langle \cdot, \cdot \rangle_{\varphi_j}$ is a scalar product on X_j, where we use that $\operatorname{supp} \varphi_j = Q_j^*$. By $\|\cdot\|_{\varphi_j}$ we denote the induced norm and by Π_{φ_j} we denote the $\langle \cdot, \cdot \rangle_{\varphi_j}$-orthogonal projection of $L^2(Q_j^*)$ onto X_j. In particular,

$$\langle \mathbf{g}, \Pi_{\varphi_j} \mathbf{w} \rangle_{\varphi_j} = \langle \mathbf{g}, \mathbf{w} \rangle_{\varphi_j} \qquad \text{for all } \mathbf{w} \in X_j.$$

Lemma 2.4. *For all* $\mathbf{g} \in X_j$ *it holds*

$$\|\Pi_{\varphi_j} \mathbf{g}\|_\infty^2 |Q_j^*| \sim \|\Pi_{\varphi_j} \mathbf{g}\|_{\varphi_j}^2 = \int |\mathbf{g}|^2 \varphi_j \, dx.$$

Proof. By a simple scaling and translation argument it suffices to consider the case, where Q_j^* is the unit cube. Since X_j is finite dimensional (and the dimension is bounded independent of j), the norms $\|\Pi_{\varphi_j} \mathbf{g}\|_\infty$ and $\|\mathbf{g}\|_{\varphi_j} |Q_j^*|^{-1/2}$ must be equivalent. Note that only finitely many situations w.r.t. the geometric configuration of a cube Q_j^* and its neighbours can appear due to our assumptions concerning the Whitney covering. $\square$

Lemma 2.5. *For all* $\mathbf{g} \in L^2(Q_j^*)$ *it holds*

$$\|\Pi_{\varphi_j} \mathbf{g}\|_\infty \leq c \fint_{Q_j^*} |\mathbf{g}| \, dx,$$

thus Π_{φ_j} *is well defined from* $L^1(Q_j^*)$ *to* X_j *with the same estimates.*

Proof. From Lemma 2.4 and the definition of Π_j we obtain

$$\|\Pi_{\varphi_j} \mathbf{g}\|_\infty^2 |Q_j^*| \leq c \, \|\Pi_{\varphi_j} \mathbf{g}\|_{\varphi_j}^2 = \langle \Pi_{\varphi_j} \mathbf{g}, \mathbf{g} \rangle \leq \|\Pi_{\varphi_j} \mathbf{g}\|_\infty \|\mathbf{g}\|_{L^1(Q_j^*)} \|\varphi_j\|_\infty.$$

This proves the claim. $\square$

Lemma 2.6. *For all $\mathbf{g} \in X_j$ we have*

$$r_j \|\nabla \mathbf{g}\|_\infty \leq c\, \|\mathbf{g}\|_\infty.$$

Proof. The estimate is a consequence of the definition of X_j and the properties (U4) and (U5). $\square$

Lemma 2.7. *For all $j, k \in \mathbb{N}$ and any $\mathbf{w} \in W^{1,1}(Q_j^*)$ it holds*

$$\langle \varphi_k, \operatorname{div}(\varphi_j \Pi_{\varphi_j} \mathbf{w})\rangle = \langle \varphi_k, \operatorname{div}(\varphi_j \mathbf{w})\rangle.$$

Proof. Fix $j \in \mathbb{N}$. For all $k \in A_j'$ it follows by partial integration and the definition of Π_{φ_j} that

$$\langle \varphi_k, \operatorname{div}(\varphi_j \Pi_{\varphi_j} \mathbf{w})\rangle = -\big\langle \nabla\varphi_k, \Pi_{\varphi_j} \mathbf{w}\big\rangle_{\varphi_j} = -\big\langle \nabla\varphi_k, \mathbf{w}\big\rangle_{\varphi_j} = \langle \varphi_k, \operatorname{div}(\varphi_j \mathbf{w})\rangle.$$

This proves the claim for all $k \in A_j'$. Since

$$\langle 1, \operatorname{div}(\varphi_j \Pi_{\varphi_j} \mathbf{w})\rangle = 0 = \langle 1, \operatorname{div}(\varphi_j \mathbf{w})\rangle$$

and $\varphi_j = 1 - \sum_{k \in A_j'} \varphi_k$ on Q_j^*, we get

$$\begin{aligned}
\langle \varphi_j, \operatorname{div}(\varphi_j \Pi_{\varphi_j} \mathbf{w})\rangle &= \langle 1 - \sum_{k \in A_j'} \varphi_k, \operatorname{div}(\varphi_j \Pi_{\varphi_j} \mathbf{w})\rangle \\
&= \langle 1 - \sum_{k \in A_j'} \varphi_k, \operatorname{div}(\varphi_j \mathbf{w})\rangle \\
&= \langle \varphi_j, \operatorname{div}(\varphi_j \mathbf{w})\rangle.
\end{aligned}$$

This proves the case $k = j$. The case $k \in \mathbb{N} \setminus A_j$ is obvious. $\square$

For $\mathbf{w} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ we define

$$\Pi \mathbf{w} := \sum_j \varphi_j \Pi_{\varphi_j} \mathbf{w}. \tag{2.6}$$

Corollary 2.8. *For all $\mathbf{w} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and any $k \in \mathbb{N}$ it holds*

$$\int \varphi_k \operatorname{div}(\Pi \mathbf{w})\, dx = \int \varphi_k \operatorname{div} \mathbf{w}\, dx.$$

Proof. Note that for every $k \in \mathbb{N}$ we have $\sum_j \varphi_j = 1$ on $Q_k^* = \operatorname{supp}(\varphi_k)$. So the claim follows by summing the equality of Lemma 2.7 over all $j \in \mathbb{N}$. $\square$

We define for all $\mathbf{w} \in W_0^{1,1}(\mathbb{R}^d)$

$$T_0^\lambda \mathbf{w} := T^\lambda \mathbf{w} + \Pi(\mathbf{w} - T^\lambda \mathbf{w}).$$

The next lemma shows that $\varphi_k \operatorname{div}(T_0^\lambda \mathbf{w})$ satisfies for every $k \in \mathbb{N}$ the desired constraint $\int_{Q_k^*} g\,dx = 0$.

Lemma 2.9. *For all $\mathbf{w} \in W_{0,\operatorname{div}}^{1,1}(\mathbb{R}^d)$ and any $k \in \mathbb{N}$ it holds*

$$\int \varphi_k \operatorname{div}(T_0^\lambda \mathbf{w})\,dx = 0.$$

Proof. For all $k \in \mathbb{N}$ it follows from Corollary 2.8 that

$$\begin{aligned}
\int \varphi_k \operatorname{div}(T_0^\lambda \mathbf{w})\,dx &= \int \varphi_k \operatorname{div}(T^\lambda \mathbf{w})\,dx + \int \varphi_k \operatorname{div}\big(\Pi(\mathbf{w} - T^\lambda \mathbf{w})\big)\,dx \\
&= \int \varphi_k \operatorname{div}(T^\lambda \mathbf{w})\,dx + \int \varphi_k \operatorname{div}(\mathbf{w} - T^\lambda \mathbf{w})\,dx \\
&= \int \varphi_k \operatorname{div} \mathbf{w}\,dx \\
&= 0.
\end{aligned}$$

□

We want to show that T_0^λ has basically the same properties as T^λ.

Lemma 2.10. *If $\mathbf{w} \in V_0^{1,h}(B)$, then*

(a) $\|T_0^\lambda \mathbf{w}\|_1 \leq c\,\|\mathbf{w}\|_1$.

(b) $|\boldsymbol{\varepsilon}(T_0^\lambda(\mathbf{w}))| \leq \sum_{k \in A_j} \fint_{Q_k^*} \frac{|\mathbf{w}-\mathbf{w}_k|}{r_k}\,dx$ *on Q_j^* for every $j \in \mathbb{N}$.*

(c) $|\boldsymbol{\varepsilon}(T_0^\lambda \mathbf{w})| \leq c\,\lambda \chi_{\mathcal{O}_\lambda} + |\boldsymbol{\varepsilon}(\mathbf{w})|\chi_{\mathbb{R}^d \setminus \mathcal{O}_\lambda}$ *and* $|\boldsymbol{\varepsilon}(T_0^\lambda \mathbf{w})| \leq c\,\lambda$ *almost everywhere.*

(d) $\|\boldsymbol{\varepsilon}(T_0^\lambda \mathbf{w})\|_h \leq c\,\|\varepsilon(\mathbf{w})\|_h$ *and* $\rho_h(\boldsymbol{\varepsilon}(T_0^\lambda \mathbf{w})) \leq c\,\rho_h(\boldsymbol{\varepsilon}(\mathbf{w}))$.

Proof. (a): We estimate

$$|T_0^\lambda \mathbf{w}| \leq |T^\lambda \mathbf{w}| + \sum_j |\Pi_j(\mathbf{w} - T^\lambda \mathbf{w})|.$$

Now, the L^1-estimate of $T_0^\lambda \mathbf{w}$ follows by the $L^1(Q_j^*)$-stability of Π_j (which is a consequence of Lemma 2.5), the local finiteness of the Q_j^* and the L^1-estimate for $T^\lambda \mathbf{w}$ in Lemma 2.1.

(b): For $j \in \mathbb{N}$ it follows by Lemma 2.6, Lemma 2.5 and Lemma 2.1 that

$$\begin{aligned}
|\boldsymbol{\varepsilon}(\varphi_j \Pi_{\varphi_j}(\mathbf{w} - T^\lambda \mathbf{w}))| &\leq c \fint_{Q_j^*} \frac{|\Pi_{\varphi_j}(\mathbf{w} - T^\lambda \mathbf{w})|}{r_j}\, dx + \fint_{Q_j^*} |\nabla \Pi_j(\mathbf{w} - T^\lambda \mathbf{w})|\, dx \\
&\leq c \fint_{Q_j^*} \frac{|\Pi_{\varphi_j}(\mathbf{w} - T^\lambda \mathbf{w})|}{r_j}\, dx \\
&\leq c \fint_{Q_j^*} \frac{|\mathbf{w} - T^\lambda \mathbf{w}|}{r_j}\, dx \\
&\leq c \sum_{k \in A_j} \fint_{Q_k^*} \left| \frac{\mathbf{w} - \mathbf{w}_k}{r_k} \right| dx.
\end{aligned}$$

(c): Summing the estimate of (b) over all j and using Lemma 2.1 we deduce that $|\boldsymbol{\varepsilon}(\Pi(\mathbf{w} - T^\lambda \mathbf{w}))| \leq \lambda \chi_{\mathcal{O}_\lambda}$. This and the estimate for $T^\lambda \mathbf{w}$ stated in Lemma 2.3 (c) prove (c).
(d): This follows from (c) exactly as in Lemma 2.3. $\square$

Let Bog_j denote the Bogovskiĭ operator [Bog80] on Q_j^* generated from one fixed Bogovskiĭ operator on $[0,1]^n$ by means of translation and dyadic scaling. In particular, Bog_j is the solution operator to the divergence equation $\operatorname{div} \mathrm{Bog}_i g = g$ in the Sobolev space with zero boundary values. Note that Bog_i is continuous from $L_0^p(Q_i^*)$ to $W_0^{1,p}(Q_i^*)$ for $p > 1$ but not from $L_0^h(Q_i^*)$ to $W_0^{1,h}(Q_i^*)$.
We define for $\mathbf{w} \in W^{1,1}_{0,\operatorname{div}}(B)$

$$T^\lambda_{\operatorname{div}} \mathbf{w} := T_0^\lambda \mathbf{w} - \sum_j \mathrm{Bog}_j\big(\varphi_j \operatorname{div} T_0^\lambda \mathbf{w}\big).$$

This expression is well defined, since $\varphi_j \operatorname{div}(T_0^\lambda \mathbf{w}) \in L_0^\infty(Q_j^*)$ by Lemma 2.9 and Lemma 2.10. Then we obtain

$$\begin{aligned}
\operatorname{div} T^\lambda_{\operatorname{div}} \mathbf{w} &= \operatorname{div} T_0^\lambda \mathbf{w} - \sum_k \operatorname{div} \mathrm{Bog}_k\big(\varphi_k \operatorname{div} T_0^\lambda \mathbf{w}\big) \\
&= \operatorname{div} T_0^\lambda \mathbf{w} - \sum_k \varphi_k \operatorname{div} T_0^\lambda \mathbf{w} \\
&= \operatorname{div} T_0^\lambda \mathbf{w} - \operatorname{div} T_0^\lambda \mathbf{w} \\
&= 0,
\end{aligned} \tag{2.7}$$

in particular $T^\lambda_{\operatorname{div}}$ is solenoidal.
We show now that additionally $T^\lambda_{\operatorname{div}}$ has basically the same properties as T^λ.

Lemma 2.11. *If* $\mathbf{w} \in V^{1,h}_{0,\mathrm{div}}(B)$, *then* $T^\lambda_{\mathrm{div}}\mathbf{v} \in V^{1,\infty}_{0,\mathrm{div}}(\mathbb{R}^d)$ *and*

(a) $\|T^\lambda_{\mathrm{div}}\mathbf{w}\|_1 \le c\,\|\mathbf{w}\|_1$.

(b) $|\boldsymbol{\varepsilon}(T^\lambda_{\mathrm{div}}(\mathbf{w}))| \le \sum_{k\in A_j} \fint_{Q_k^*} \frac{|\mathbf{w}-\mathbf{w}_k|}{r_k}\,dx$ *on* Q_j^* *for every* $j \in \mathbb{N}$.

(c) $|\boldsymbol{\varepsilon}(T^\lambda_{\mathrm{div}}\mathbf{w})| \le c\,\lambda\chi_{\mathcal{O}_\lambda} + |\boldsymbol{\varepsilon}(\mathbf{w})|\chi_{\mathbb{R}^d\setminus\mathcal{O}_\lambda}$ *and* $|\boldsymbol{\varepsilon}(T^\lambda_{\mathrm{div}}\mathbf{w})| \le c\,\lambda$ *almost everywhere.*

(d) $\|\boldsymbol{\varepsilon}(T^\lambda_{\mathrm{div}}\mathbf{w})\|_h \le c\,\|\varepsilon(\mathbf{w})\|_h$ *and* $\rho_h(\boldsymbol{\varepsilon}(T^\lambda_{\mathrm{div}}\mathbf{w})) \le c\,\rho_h(\boldsymbol{\varepsilon}(\mathbf{w}))$.

Proof. (b) and (c): Note that Bog_j is continuous from $L^q_0(Q_i^*)$ to $W^{1,q}_0(Q_i^*)$ for any $q \in (1,\infty)$. If $q = \infty$, then we only have BMO for the gradients of Bog_j. However, in the definition of T^λ_{div}, we only apply Bog_j to the special functions $\varphi_j \operatorname{div}(T^\lambda_0\mathbf{w})$, which are from a finite dimensional subspace of smooth functions. As in the derivation of the estimates for φ_j the special geometric properties of the cubes Q_j^* together with the properties of φ_j imply, that (up to translation and dyadic scaling) only finitely many different finite dimensional subspaces of smooth functions occur here. Now, we can use the property that Bog_j also maps $L^q_0(Q_j^*)\cap W^{1,q}(Q_j^*)$ to $W^{1,q}_0(Q_j^*)\cap W^{2,q}(Q_j^*)$ for some $q > n$ to see that Bog_j acts as a mapping from $L^\infty_0(Q_j^*)$ to $W^{1,\infty}_0(Q_j^*)$ on these finite dimensional subspaces. In particular, we have

$$\|\nabla \mathrm{Bog}_j(\varphi_j \operatorname{div}(T^\lambda_0\mathbf{w}))\|_{\infty,Q_j^*} \le c\,\|\varphi_j \operatorname{div}(T^\lambda_0\mathbf{w})\|_{\infty,Q_j^*}. \tag{2.8}$$

Hence, with Lemma 2.10 (c)

$$\|\nabla \mathrm{Bog}_j(\varphi_j \operatorname{div}(T^\lambda_0\mathbf{w}))\|_{\infty,Q_j^*} \le c\,\|\boldsymbol{\varepsilon}(T^\lambda_0\mathbf{w})\|_{L^\infty(Q_j^*)}.$$

Now, this estimate and Lemma 2.10 (b) and (c) prove (b) and (c), respectively.
(a): Using Poincaré's inequality on Q_j^* and the estimate (2.8) for Bog_j, we get

$$\|\mathrm{Bog}_j(\varphi_j \operatorname{div}(T^\lambda_0\mathbf{w}))\|_{\infty,Q_j^*} \le c\,r_j\|\nabla \mathrm{Bog}_j(\varphi_j \operatorname{div}(T^\lambda_0\mathbf{w}))\|_{\infty,Q_j^*} \le c\,r_j\|\boldsymbol{\varepsilon}(T^\lambda\mathbf{w})\|_{\infty,Q_j^*}.$$

Now, from Lemma 2.10 (b) and the L^1-stability of $\mathcal{R}_{Q_j^*}$, see (2.2), we get

$$\|\mathrm{Bog}_j(\varphi_j \operatorname{div}(T^\lambda_0\mathbf{w}))\|_{\infty,Q_j^*} \le c\sum_{k\in A_j}\fint_{Q_k^*}|\mathbf{w}-\mathbf{w}_k|\,dx \le c\fint_{5Q_k^*}|\mathbf{w}|\,dx.$$

This, the locally finiteness of the Q_j^* and the L^1-stability of T^λ_0 prove the L^1-stability of T^λ_{div}.
(d): This follows from (c) exactly as in Lemma 2.3.
It follows from (2.7) that $\operatorname{div} T^\lambda_{\mathrm{div}}\mathbf{w} = 0$, hence $T^\lambda_{\mathrm{div}} \in V^{1,\infty}_{0,\mathrm{div}}(\mathbb{R}^d)$. □

Remark 2.12. *Let* $\mathbf{w} \in V^{1,h}_{0,\mathrm{div}}(B)$*. If we want to use* $T^{\lambda}_{\mathrm{div}}\mathbf{w}$ *as a test function to a PDE, then it is useful that the support of* $T^{\lambda}_{\mathrm{div}}\mathbf{w}$ *does not become too big. This can be ensured by choosing* λ *large enough. Indeed, if* $\lambda > \fint_B |\boldsymbol{\varepsilon}(\mathbf{w})|\,dx$*, then* $\mathcal{O}_\lambda = \{M(\boldsymbol{\varepsilon}(\mathbf{w})) > \lambda\} \subset 2B$*. Therefore,* $\mathrm{supp}(T^{\lambda}_{\mathrm{div}}\mathbf{w}) \subset \overline{B} \cup \mathcal{O}_\lambda \subset \overline{2B}$*, which implies* $T^{\lambda}_{\mathrm{div}}\mathbf{w} \in V^{1,\infty}_{0,\mathrm{div}}(2B)$*.*

We will now apply the solenoidal Lipschitz truncation $T^{\lambda}_{\mathrm{div}}$ to a weak null sequence in $V^{1,h}_{0,\mathrm{div}}(\Omega)$.

THEOREM 2.13. *Let* $(\mathbf{w}^n) \subset V^{1,h}_{0,\mathrm{div}}(B)$ *be a bounded sequence which converges strongly to zero in* $L^1(B)$ *. Then there is a double sequence* $(\lambda_{n,j}) \subset \mathbb{R}$ *and* $j_0 \in \mathbb{N}$ *and null sequences* $\kappa_j, \widetilde{\kappa}_j \to 0$ *such that the sequence* $\mathbf{w}^{n,j} := T^{\lambda_{n,j}}_{\mathrm{div}}\mathbf{w} \in V^{1,\infty}_{0,\mathrm{div}}(2B)$ *satisfies the following properties. We have for* $j \geq j_0$

(a) $\mathbf{w}^{n,j} \in V^{1,\infty}_{0,\mathrm{div}}(2B)$ *and* $\mathbf{w}^{n,j} = \mathbf{w}^n$ *on* $\mathbb{R}^d \setminus \{M(\boldsymbol{\varepsilon}\mathbf{w}^n) > \lambda_{n,j}\}$*,*

(b) $\|\boldsymbol{\varepsilon}(\mathbf{w}^{n,j})\|_\infty \leq c\lambda_{n,j}$ *where* $2^{2^j} \leq \lambda_{n,j} \leq 2^{2^{j+1}}$*,*

(c) $\boldsymbol{\varepsilon}(\mathbf{w}^{n,j}) \overset{*}{\rightharpoonup} 0$ *for* $n \to \infty$ *in* $L^\infty(2B)$*,*

(d) *There exists a (non-relabeled) subsequence of* $\mathbf{w}^n$ *which satisfies* $\limsup_{n\to\infty} \int h\big(|\lambda_{n,j}\chi_{\{\mathbf{w}^{n,j}\neq\mathbf{w}^n\}}|\big)\,dx \leq \kappa_j$ *and* $\limsup_{n\to\infty} \|\lambda_{n,j}\chi_{\{\mathbf{w}^{n,j}\neq\mathbf{w}^n\}}\|_h \leq \widetilde{\kappa}_j$*.*

Proof. We will construct below a double sequence $\lambda_{n,j}$ with $2^{2^j} \leq \lambda_{n,j} \leq 2^{2^{j+1}}$ and define $\mathbf{w}^{n,j} := T^{\lambda_{n,j}}_{\mathrm{div}}\mathbf{w}^n$. Choose j_0 such that $\sup_n \fint_B |\boldsymbol{\varepsilon}(\mathbf{w}^n)|\,dx \leq 2^{2^{j_0}}$. Properties (a) and (b) follow immediately from Lemma 2.11 and Remark 2.12 for $j \geq j_0$.
Since $\mathbf{w}^n$ is bounded in $V^{1,h}_{0,\mathrm{div}}(B)$, it follows that there exists a subsequence of $\mathbf{w}^n$ such that the corresponding subsequence of $\boldsymbol{\varepsilon}(\mathbf{w}^n)$ converges weakly in $L^1(B)$. Due to the L^1-convergence of $\mathbf{w}^n$, this limit must be zero. Since, we can apply this argument to any subsequence of $\mathbf{w}^n$, it follows that the whole sequence $\boldsymbol{\varepsilon}(\mathbf{w}^n)$ converges weakly in L^1 to zero.
It follows from $\|\mathbf{w}^{n,j}\|_1 \leq c\,\|\mathbf{w}\|_1$ (by Lemma 2.11), that $\mathbf{w}^{n,j} \overset{n}{\to} 0$ in L^1. Moreover, $(\mathbf{w}^{n,j})_n$ is (by Lemma 2.11) for every $j \geq j_0$ bounded in $V^{1,\infty}_0(2B)$. Therefore, there exists a subsequence such that $\boldsymbol{\varepsilon}(\mathbf{v}^n)$ converges $*$-weakly. As in the argument used above, this implies that the whole sequence $\boldsymbol{\varepsilon}(\mathbf{v}^n)$ converges $*$-weakly to zero, which proves (c).
Since M is bounded from $L^h(2B)$ to $L^1(2B)$ (see [Ste93] I 8.14(a)), we have

$$K := \sup_n \|M(\boldsymbol{\varepsilon}(\mathbf{w}^n))\|_1 < \infty.$$

Next, we observe that for any $g \in L^1(2B)$ we have

$$
\begin{aligned}
\|g\|_1 &= \int_{2B} \int_0^\infty \chi_{\{|g|>t\}}\, dt\, dx \\
&\geq \int_{2B} \sum_{m\in\mathbb{Z}} 2^m\, \chi_{\{|g|>2^{m+1}\}}\, dx \\
&\geq \sum_{j\in\mathbb{N}} \sum_{k=2^j}^{2^{j+1}-1} \int_{2B} 2^k\, \chi_{\{|g|>2\cdot 2^k\}}\, dx.
\end{aligned}
\tag{2.9}
$$

The choice $g = \chi_{2B} M(\boldsymbol{\varepsilon}(\mathbf{w}^n))$ implies

$$
\sum_{j\in\mathbb{N}} \sum_{k=2^j}^{2^{j+1}-1} \int_{2B} 2^k\, \chi_{\{|M(\boldsymbol{\varepsilon}(\mathbf{w}^n))|>2\cdot 2^k\}}\, dx \leq K.
$$

We can rewrite the last inequality as

$$
\sum_{j\in\mathbb{N}} b_j^n \leq K
$$

with an obvious definition for b_j^n.
Since the sum in the definition of b_j contains 2^j summands, there is at least one index $k_{n,j}$ such that

$$
\int_{2B} 2^{k_{n,j}}\, \chi_{\{|M(\boldsymbol{\varepsilon}(\mathbf{w}^n))|>2\cdot 2^{k_{n,j}}\}}\, dx \leq 2^{-j}\, b_j^n. \tag{2.10}
$$

which is equivalent to

$$
\int_{2B} h\left(2^{k_{n,j}}\right)\, \chi_{\{|M(\boldsymbol{\varepsilon}(\mathbf{w}^n))|>2\cdot 2^{k_{n,j}}\}}\, dx \leq \ln(1+2^{k_{n,j}})\, 2^{-j}\, b_j^n. \tag{2.11}
$$

Note that $\ln(1+2^{k_{n,j}})\, 2^{-j} \leq 3$ on account of $k_{n,j} \leq 2^{j+1}$; thus we get

$$
\int_{2B} h\left(2^{k_{n,j}}\right)\, \chi_{\{|M(\boldsymbol{\varepsilon}(\mathbf{w}^n))|>2\cdot 2^{k_{n,j}}\}}\, dx \leq 3\, b_j^n. \tag{2.12}
$$

Define $\delta_1 := \liminf_n b_1^n$. Then there exists a subsequence (not relabeled) with

$$
\limsup_n b_1^n = \liminf_n b_1^n = \delta_1.
$$

This proves

$$
\limsup_n \int_{2B} h\left(2^{k_{n,1}}\right)\, \chi_{\{|M(\boldsymbol{\varepsilon}(\mathbf{w}^n))|>2\cdot 2^{k_{n,1}}\}}\, dx \leq 3 \limsup_n b_1^n = 3\, \delta_1.
$$

Next, define $\delta_2 := \liminf_n b_2^n$ and by passing to a further subsequence we get

$$\limsup_n \int_{2B} h\left(2^{k_{n,2}}\right) \chi_{\{|M(\boldsymbol{\varepsilon}(\mathbf{w}^n))|>2\cdot 2^{k_{n,2}}\}}\, dx \le 3 \limsup_n b_2^n = 3\,\delta_2.$$

Using this iterative argument we can construct a diagonal sequence (not relabeled) such that for every j

$$\limsup_n \int_{2B} h\left(2^{k_{n,j}}\right) \chi_{\{|M(\boldsymbol{\varepsilon}(\mathbf{w}^n))|>2\cdot 2^{k_{n,j}}\}}\, dx \le 3 \limsup_n b_j^n = 3\,\delta_j. \tag{2.13}$$

From now on we will use the diagonal sequence. The Lemma of Fatou gives

$$K \ge \liminf_n \sum_j b_j^n \ge \sum_j \liminf_n b_j^n = \sum_j \delta_j,$$

hence, δ_j is a null sequence. Define $\kappa_j := 3\,\delta_j$ and $\lambda_{n,j} := 2^{k_{n,j}}$. Then (2.13) proves the integral estimate of (d). The norm estimate is a direct consequence. □

Remark 2.14. *Note that it is not possible to show* (d) *of Theorem 2.13 by the technique of [DMS08], since there the boundedness of the maximal function it used, which does not hold in L^h. Therefore, we must apply a more subtle weak type argument.*

Remark 2.15. *Replacing $\mathbf{w}_j$ in the definition of T^λ by mean values (instead of rigid motions) we get from Theorem 2.13 the following result for $1 < p < \infty$.*
Let $(\mathbf{w}^n) \subset W^{1,p}_{0,\mathrm{div}}(B)$ be a bounded sequence which converges strongly to zero in $L^1(B)$. Then there is a double sequence $(\lambda_{n,j}) \subset \mathbb{R}$ and $j_0 \in \mathbb{N}$, a null sequence $\kappa_j \to 0$ and a sequence $\mathbf{w}^{n,j} \in W^{1,\infty}_{0,\mathrm{div}}(2B)$ satisfying the following properties. We have for $j \ge j_0$

(a) $\mathbf{w}^{n,j} \in W^{1,\infty}_{0,\mathrm{div}}(2B)$ *and* $\mathbf{w}^{n,j} = \mathbf{w}^n$ *on* $\mathbb{R}^d \setminus \{M(\nabla \mathbf{w}^n) > \lambda_{n,j}\}$,

(b) $\|\nabla \mathbf{w}^{n,j}\|_\infty \le c\lambda_{n,j}$ *where* $2^{2^j} \le \lambda_{n,j} \le 2^{2^{j+1}}$,

(c) $\nabla \mathbf{w}^{n,j} \overset{*}{\rightharpoonup} 0$ *for* $n \to \infty$ *in* $L^\infty(2B)$,

(d) $\limsup_{n\to\infty} \|\lambda_{n,j}\chi_{\{\mathbf{w}^{n,j}\neq \mathbf{w}^n\}}\|_p \le \kappa_j 2^{-j}$.

The name Lipschitz truncation *originates from this situation, where the $\mathbf{w}^{n,j}$ are Lipschitz.*

3 Existence of weak solutions

In this section we prove Theorem 1.1. In particular, we show the existence of a weak solution $\mathbf{v} \in V^{1,h}_{0,\mathrm{div}}(\Omega)$ to the equation

$$\int_\Omega DW(\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx = \int_\Omega \mathbf{f} \cdot \varphi\, dx + \int_\Omega \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx \tag{3.1}$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain. We start by approximating this equation. We consider solutions $\mathbf{v}^n \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ of the system

$$\int_\Omega \big(DW(\boldsymbol{\varepsilon}(\mathbf{u})) + n^{-1}\boldsymbol{\varepsilon}(\mathbf{u})\big) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx = \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\, dx + \int_\Omega \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx. \tag{3.2}$$

The existence of solutions to this system can easily be verified due to the quadratic growth of the main part by means of monotone operators. An important advantage of this approximation consists in the fact that the space of test functions coincides with the space where the solution is constructed. Moreover, all $\mathbf{v}^n$ satisfy the uniform estimate

$$\int_\Omega h(|\boldsymbol{\varepsilon}(\mathbf{v}^n)|)\, dx + n^{-1} \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{v}^n)|^2\, dx \le c,$$

which follows from testing (3.2) by $\mathbf{v}^n$. Consequently, we get

$$\begin{aligned} \|\boldsymbol{\varepsilon}(\mathbf{v}^n)\|_h &\le c, \\ \|n^{-1/2}\boldsymbol{\varepsilon}(\mathbf{v}^n)\|_2 &\le c. \end{aligned}$$

This estimate and Theorem 4.7 imply the existence of $\mathbf{v} \in V^{1,h}_{0,\mathrm{div}}(\Omega)$, and a (not relabeled) subsequence $\{\mathbf{v}^n\}$ such that

$$\begin{aligned} \mathbf{v}^n &\to \mathbf{v} && \text{in } L^2(\Omega), \\ \boldsymbol{\varepsilon}(\mathbf{v}^n) &\rightharpoonup \boldsymbol{\varepsilon}(\mathbf{v}) && \text{in } L^1(\Omega), \\ n^{-1}\boldsymbol{\varepsilon}(\mathbf{v}^n) &\to 0 && \text{in } L^2(\Omega). \end{aligned}$$

It follows from these convergences that

$$\begin{aligned} &\frac{1}{n}(\boldsymbol{\varepsilon}(\mathbf{v}^n), \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) \to 0 && \text{and} \\ &(\mathbf{v}^n \otimes \mathbf{v}^n, \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) \to (\mathbf{v} \otimes \mathbf{v}, \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) && \text{for all } \boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega). \end{aligned}$$

Clearly these statements extend to $\boldsymbol{\varphi} \in V^{1,\infty}_{0,\mathrm{div}}(\Omega)$.

Next, to prove that also

$$(DW(\boldsymbol{\varepsilon}(\mathbf{v}^n)),\boldsymbol{\varepsilon}(\boldsymbol{\varphi})) \to (DW(\boldsymbol{\varepsilon}(\mathbf{v})),\boldsymbol{\varepsilon}(\boldsymbol{\varphi})) \quad \text{for all } \boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega) \tag{3.3}$$

it suffices, by virtue of $\|\boldsymbol{\varepsilon}(\mathbf{v}^n)\|_h \le c$ and Vitali's theorem, to show at least for a subsequence that $\boldsymbol{\varepsilon}(\mathbf{v}^n) \to \boldsymbol{\varepsilon}(\mathbf{v})$ almost everywhere. This follows, see for example [DMM98] for details, from the strict monotonicity of the operator DW provided that for a certain $\theta \in (0,1]$ and every ball $B \subset \Omega$ with $4B \subset \Omega$

$$\limsup_n \int_B \Big((DW(\boldsymbol{\varepsilon}(\mathbf{v}^n)) - DW(\boldsymbol{\varepsilon}(\mathbf{v}))) : (\boldsymbol{\varepsilon}(\mathbf{v}^n) - \boldsymbol{\varepsilon}(\mathbf{v}))\Big)^\theta \, dx = 0\,. \tag{3.4}$$

To verify equation (3.4), let $\eta \in C^\infty_0(2B)$ with $\chi_B \le \eta \le \chi_{2B}$ and $|\nabla \eta| \le c\,R^{-1}$, where R is the radius of B. We define

$$\mathbf{w}^n := \eta(\mathbf{v}^n - \mathbf{v}) - \mathrm{Bog}_{2B}(\nabla\eta \cdot (\mathbf{v}^n - \mathbf{v})),$$

where Bog_{2B} is the Bogovskiĭ operator on $2B$ from $L^2_0(2B)$ to $W^{1,2}_0(2B)$. Since $\nabla\eta\cdot(\mathbf{v}^n - \mathbf{v})$ is bounded in $L^2_0(2B)$, we have that $\mathbf{w}^n$ is bounded in $V^{1,h}_{0,\mathrm{div}}(2B)$. Moreover, $\mathbf{v}^n \to \mathbf{v}$ in L^2 and the continuity of Bog implies $\mathbf{w}^n \to 0$ in L^1. In particular, we can apply our solenoidal Lipschitz truncation of Theorem 2.13 to get a suitable double sequence $\mathbf{w}^{n,j} \in V^{1,\infty}_{0,\mathrm{div}}(4B)$.
The weak formulation of the approximative problem (3.2) with $\mathbf{w}^{n,j}$ as a test function can be rewritten as

$$\begin{aligned}(DW(\boldsymbol{\varepsilon}(\mathbf{v}^n)) - DW(\boldsymbol{\varepsilon}(\mathbf{v})),\boldsymbol{\varepsilon}(\mathbf{w}^{n,j})) = &-(DW(\boldsymbol{\varepsilon}(\mathbf{v})),\boldsymbol{\varepsilon}(\mathbf{w}^{n,j})) \\ &- \frac{1}{n}(\boldsymbol{\varepsilon}(\mathbf{v}^n),\boldsymbol{\varepsilon}(\mathbf{w}^{n,j})) + (\mathbf{f},\mathbf{w}^{n,j}) \\ &+ (\mathbf{v}^n \otimes \mathbf{v}^n, \boldsymbol{\varepsilon}(\mathbf{w}^{n,j})).\end{aligned}$$

It follows from the properties of $\mathbf{w}^{n,j}$ and $\mathbf{v}^n$ that the right-hand side converges for fixed j to zero as $n \to \infty$. So we get

$$\lim_{n\to\infty} (DW(\boldsymbol{\varepsilon}(\mathbf{v}^n)) - DW(\boldsymbol{\varepsilon}(\mathbf{v})),\boldsymbol{\varepsilon}(\mathbf{w}^{n,j})) = 0. \tag{3.5}$$

We decompose the set $4B$ into $\{\mathbf{w} \ne \mathbf{w}^{n,j}\}$ and $4B \cap \{\mathbf{w} = \mathbf{w}^{n,j}\}$ to get

$$\begin{aligned}(I) &:= \limsup_n \left| \int\limits_{4B\cap\{\mathbf{w}=\mathbf{w}^{n,j}\}} \eta \big(DW(\boldsymbol{\varepsilon}(\mathbf{v}^n))\big) - DW(\boldsymbol{\varepsilon}(\mathbf{v})\big) : (\boldsymbol{\varepsilon}(\mathbf{v}^n) - \boldsymbol{\varepsilon}(\mathbf{v}))\, dx \right| \\ &= \limsup_n \left| \int\limits_{\{\mathbf{w}\ne\mathbf{w}^{n,j}\}} \big(DW(\boldsymbol{\varepsilon}(\mathbf{v}^n))\big) - DW(\boldsymbol{\varepsilon}(\mathbf{v})\big) : \boldsymbol{\varepsilon}(\mathbf{w}^{n,j})\, dx \right|\end{aligned}$$

$$+\limsup_n \left| \int\limits_{4B\cap\{\mathbf{w}=\mathbf{w}^{n,j}\}} \big(DW(\boldsymbol{\varepsilon}(\mathbf{v}^n))\big) - DW(\boldsymbol{\varepsilon}(\mathbf{v})\big) : \big(\nabla\eta \otimes^{\mathrm{sym}} (\mathbf{v}^n - \mathbf{v})\big)\, dx \right|$$

$$+\limsup_n \left| \int\limits_{4B\cap\{\mathbf{w}=\mathbf{w}^{n,j}\}} \big(DW(\boldsymbol{\varepsilon}(\mathbf{v}^n))\big) - DW(\boldsymbol{\varepsilon}(\mathbf{v})\big) : \boldsymbol{\varepsilon}\big(\mathrm{Bog}_{2B}(\nabla\eta \cdot (\mathbf{v}^n - \mathbf{v}))\big)\, dx \right|$$

$$=: (II) + (III) + (IV).$$

Since $\nabla\eta \otimes (\mathbf{v}^n - \mathbf{v}) \overset{n}{\to} 0$ in L^2, we have $(III) + (IV) \overset{n}{\to} 0$, where we also used the continuity of Bog_{2B} from $L^2_0(2B)$ to $W^{1,2}_0(2B)$.
By Young's inequality

$$(II) \le \limsup_n \big(\|DW(\boldsymbol{\varepsilon}(\mathbf{v}^n))\|_{h^*} + \|DW(\boldsymbol{\varepsilon}(\mathbf{v}))\|_{h^*}\big)\, \|\chi_{\{\mathbf{w}^n \ne \mathbf{w}^{n,j}\}}\boldsymbol{\varepsilon}(\mathbf{w}^{n,j})\|_h,$$

where h^* is the conjugate N-function of h. Since

$$h^*(|DW(\boldsymbol{\varepsilon})|) \le h^*(h'(|\boldsymbol{\varepsilon}|)) \le h(2|\boldsymbol{\varepsilon}|) \le c\, h(|\boldsymbol{\varepsilon}|),$$

we deduce from the uniform boundedness of $\mathbf{w}^n$ and $\mathbf{w}$ in $V^{1,h}_0(\Omega)$ that $DW(\boldsymbol{\varepsilon}(\mathbf{w}^n))$ and $DW(\boldsymbol{\varepsilon}(\mathbf{w}))$ are uniformly bounded in L^{h^*}. On the other hand by Theorem 2.13

$$\|\chi_{\{\mathbf{w}^n \ne \mathbf{w}^{n,j}\}}\boldsymbol{\varepsilon}(\mathbf{w}^{n,j})\|_h \le c\, \|\chi_{\{\mathbf{w}^n \ne \mathbf{w}^{n,j}\}}\lambda\|_h \le c\,\widetilde{\kappa}_j$$

for a null sequence $\widetilde{\kappa}_j$. This proves $(II) \le c\,\widetilde{\kappa}_j$. Overall we get

$$\limsup_n \left| \int\limits_{4B\cap\{\mathbf{w}=\mathbf{w}^{n,j}\}} \eta\, \big(DW(\boldsymbol{\varepsilon}(\mathbf{v}^n))\big) - DW(\boldsymbol{\varepsilon}(\mathbf{v})\big) : (\boldsymbol{\varepsilon}(\mathbf{v}^n) - \boldsymbol{\varepsilon}(\mathbf{v}))\, dx \right| \le c\,\widetilde{\kappa}_j. \tag{3.6}$$

Let $\theta \in (0,1)$. We claim that the previous estimate implies

$$\limsup_n \int\limits_{4B} \big(\eta\, \big(DW(\boldsymbol{\varepsilon}(\mathbf{v}^n))\big) - DW(\boldsymbol{\varepsilon}(\mathbf{v})\big) : (\boldsymbol{\varepsilon}(\mathbf{v}^n) - \boldsymbol{\varepsilon}(\mathbf{v}))\big)^\theta\, dx = 0. \tag{3.7}$$

Let z^n denote the integrand of the integral in (3.6). Then

$$\limsup_n \left| \int\limits_{4B\cap\{\mathbf{w}=\mathbf{w}^{n,j}\}} z^n\, dx \right| \le c\,\widetilde{\kappa}_j. \tag{3.8}$$

Hölder's inequality implies

$$\int_{4B} (z^n)^\theta\, dx \le \Bigg(\int\limits_{4B\cap\{\mathbf{w}=\mathbf{w}^{n,j}\}} z^n\, dx \Bigg)^\theta |4B|^{1-\theta} + \Bigg(\int\limits_{\{\mathbf{w}\ne\mathbf{w}^{n,j}\}} z^n\, dx \Bigg)^\theta |\{\mathbf{w} \ne \mathbf{w}^{n,j}\}|^{1-\theta}.$$

From $\limsup_{n\to\infty} \rho_h(\lambda_{n,j}\chi_{\{\mathbf{w}^{n,j}\neq\mathbf{w}^n\}}) \leq \kappa_j$ we deduce $|\{\mathbf{w} \neq \mathbf{w}^{n,j}\}| \leq \kappa_j 2^{-2^j} \leq \kappa_j$. Overall, we get together with (3.8) after passing ot the limit $j \to \infty$

$$\limsup_n \int_{4B} (z^n)^\theta \, dx = 0.$$

This proves (3.7). Now, (3.4) is a consequence of $\eta \geq \chi_B$, which in turn implies the almost every convergence of $\boldsymbol{\varepsilon}(\mathbf{v}^n) \to \boldsymbol{\varepsilon}(\mathbf{v})$. So we can pass to the limit in (3.3) as desired, which shows that $\mathbf{v}$ is a weak solution of (3.1). The proof for the existence of $\mathbf{v}$ is complete.
It remains to reconstruct the pressure. Standard arguments applied to (3.2) show the existence of a sequence $(\pi^n) \subset L_0^2(\Omega)$ with

$$\begin{aligned} \int_\Omega \mathbf{H}^n : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx &= \int_\Omega \pi^n \operatorname{div} \boldsymbol{\varphi}\, dx, \\ \mathbf{H}^n &:= DW(\boldsymbol{\varepsilon}(\mathbf{v}^n)) + n^{-1}\boldsymbol{\varepsilon}(\mathbf{v}^n) - \mathbf{F} - \mathbf{v}^n \otimes \mathbf{v}^n, \end{aligned} \tag{3.9}$$

for all $\boldsymbol{\varphi} \in W_0^{1,2}(\Omega)$ (where $\mathbf{F} = \nabla(\Delta^{-1}\mathbf{f}) \in L^{p_0}(\Omega)$). In order to show bounds for the pressure we need the continuity of

$$\mathrm{Bog}_\Omega : L_0^{\mathrm{Exp}}(\Omega) \to W_0^{1,\mathrm{Exp}^{1/2}}(\Omega), \tag{3.10}$$

where

$$L^{\mathrm{Exp}^\alpha}(\Omega) := \left\{ \mathbf{u} \in L^1(\Omega);\ \int_\Omega \exp\left(\left[\frac{|\mathbf{u}|}{\lambda} \right]^\alpha \right) < \infty \text{ for some } \lambda < \infty \right\}.$$

The corresponding Sobolev space has an obvious meaning, where the zero stands for zero boundary values in the sense of $W^{1,1}$-traces. If Ω is star-shaped[2] with respect to a ball B_0, we have

$$\big(\nabla \mathrm{Bog}_\Omega(\psi)\big)_{ij}(x) = \int k_{ij}(x, x-y)\psi(y)\, dy + \psi(x)\Big(\frac{z_i z_j}{|z|^2} * \omega\Big)(x)$$

for a suitable $\omega \in C_0^\infty(B_0)$, where the integral with k_{ij} is a singular integral operator, see [Sch07, equation (3.5)], [Gal94, Lemma III 3.1]. These singular integral operators are continuous from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ where the operator norm is $O(p)$ (see [Ste93] I 8.13 or [Duo01], section 6.5). From [CUK02] (introduction) we quote that $\mathbf{u} \in L^{\mathrm{Exp}^\alpha}$ iff

$$\sup_p p^{-\frac{1}{\alpha}} \|\mathbf{u}\|_p < \infty. \tag{3.11}$$

[2]Note that domains with Lipschitz boundary are the finite union of such domains.

Now let $\mathbf{u} \in L^{\mathrm{Exp}}(\Omega)$ with $\|\mathbf{u}\|_{L^{\mathrm{Exp}}} \leq 1$, then (3.11) implies

$$\sup_p p^{-2}\|\nabla \mathrm{Bog}_\Omega \mathbf{u}\|_p \leq c \sup_p p^{-1}\|\mathbf{u}\|_p < \infty.$$

As a consequence we get (3.10) using (3.11) again. In the general case, where Ω has Lipschitz boundary, we use the fact that Ω is the finite union of domains of the above type.
This enables us to bound the $L^h(\Omega)$-norm of the pressure using the dualities $(L^h)^* = L^{\mathrm{Exp}}$ and $(L^{t\ln^2(t)})^* = L^{\mathrm{Exp}^{1/2}}$ as follows:

$$\begin{aligned}
&\|\pi^n\|_{L^h} = \sup_{\boldsymbol{\varphi} \in L_0^{\mathrm{Exp}}, \|\boldsymbol{\varphi}\| \leq 1} \int_\Omega \pi^n \boldsymbol{\varphi}\, dx = \sup_{\boldsymbol{\varphi} \in L_0^{\mathrm{Exp}}, \|\boldsymbol{\varphi}\| \leq 1} \int_\Omega \pi^n \operatorname{div} \mathrm{Bog}_\Omega \boldsymbol{\varphi}\, dx \\
&= \sup_{\boldsymbol{\varphi} \in L_0^{\mathrm{Exp}}, \|\boldsymbol{\varphi}\| \leq 1} \int_\Omega \mathbf{H}^n : \boldsymbol{\varepsilon}(\mathrm{Bog}_\Omega \boldsymbol{\varphi})\, dx \leq \sup_{\boldsymbol{\varphi} \in L_0^{\mathrm{Exp}}, \|\boldsymbol{\varphi}\| \leq 1} \|\mathbf{H}^n\|_{L^{t\ln^2(t)}} \|\nabla \mathrm{Bog}_\Omega \boldsymbol{\varphi}\|_{L^{\mathrm{Exp}^{1/2}}} \\
&\leq c \sup_{\boldsymbol{\varphi} \in L_0^{\mathrm{Exp}}, \|\boldsymbol{\varphi}\| \leq 1} \|\mathbf{H}^n\|_{L^{t\ln^2(t)}} \|\boldsymbol{\varphi}\|_{L^{\mathrm{Exp}}} \leq c\|\mathbf{H}^n\|_{L^{t\ln^2(t)}}.
\end{aligned}$$

Using the definition of $\mathbf{H}^n$, we see that the critical part is $\mathbf{v}^n \otimes \mathbf{v}^n$ which is bounded in $L^{t\ln^2(t)}(\Omega)$ by Lemma 4.6 and Lemma 4.5. This finally gives boundedness of π^n in $L^h(\Omega)$. On account of the De La Vallée Poussin Lemma we can choose a subsequence and a function $\pi \in L^h(\Omega)$ with

$$\pi^n \rightharpoonup: \pi \quad \text{in} \quad L^1(\Omega). \tag{3.12}$$

Combining (3.12) with (3.3) we get

$$\langle DW(\boldsymbol{\varepsilon}(\mathbf{u})) - (\mathbf{u} \otimes \mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\rangle - \langle \mathbf{f}, \boldsymbol{\varphi}\rangle = \langle \pi, \operatorname{div} \boldsymbol{\varphi}\rangle \qquad \text{for all } \boldsymbol{\varphi} \in C_0^\infty(\Omega), \tag{3.13}$$

which proves (1.9).

4 Appendix

4.1 Korn's inequality

Lemma 4.1. *Let $Q \subset \mathbb{R}^d$ be a an cube (or a ball). Then there is $c > 0$ such that for all $\mathbf{w} \in V^{1,h}(Q)$ it holds*

$$\fint_Q |\nabla(\mathbf{w} - \mathcal{R}_Q \mathbf{w})|\, dx \leq c \fint_Q |M(\boldsymbol{\varepsilon}(\mathbf{w}))|\, dx,$$

where c does not depend on Q.

Proof. By scaling it suffers to consider the unit cube (or ball). In [FB11] (Lemma 3.1) it is shown that

$$\int_Q |\nabla \mathbf{w}|\, dx \le c \left(\int_Q h(|\boldsymbol{\varepsilon}(\mathbf{w})|)\, dx + \int_Q |\mathbf{w}|\, dx + 1 \right) \tag{4.1}$$

holds for all $\mathbf{w} \in C^\infty(\overline{Q})$. This is a consequence of a representation formula from [Res70] and the continuity of singular integral operators from L^h to L^1. From (4.1) and closeness of smooth functions we deduce the boundedness of the mapping

$$V^{1,h}(Q) \ni \mathbf{w} \mapsto \nabla \mathbf{w} \in L^1(Q).$$

This implies

$$\|\nabla \mathbf{w}\|_1 \le c \|\mathbf{w}\|_{V^{1,h}}, \quad \mathbf{w} \in V^{1,h}(Q).$$

As a consequence we get for all $\mathbf{w} \in V^{1,h}(Q)$

$$\begin{aligned} \|\nabla(\mathbf{w} - \mathcal{R}_Q \mathbf{w})\|_1 &\le c\|\boldsymbol{\varepsilon}(\mathbf{w})\|_h + c\|\mathbf{w} - \mathcal{R}_Q \mathbf{w}\|_1 \\ &\le c\|\boldsymbol{\varepsilon}(\mathbf{w})\|_h + c\|\boldsymbol{\varepsilon}(\mathbf{w})\|_1 \\ &\le c\|\boldsymbol{\varepsilon}(\mathbf{w})\|_h. \end{aligned}$$

Here we used a variant of Korn's inequality for BD (see, i.e., [FS99], Theorem A.3.1). The claim follows since $\|\cdot\|_h$ and $\|M(\cdot)\|_1$ are equivalent on bounded domains (see [Ste93] I 8.14(a)). □

Remark 4.2. *Since the results from [FB11] quoted in the proof given above actually hold for all star-shaped domains we are able to consider a much more general case as stated in Lemma 4.1. In particular the inequality holds for all bounded Lipschitz domains Ω, which are finite unions of star-shaped domains.*

Corollary 4.3. *In case of zero boundary data we get immediately*

$$\begin{aligned} \fint_Q |\nabla \mathbf{w}|\, dx = \fint_Q |\nabla \mathbf{w} - \langle \nabla \mathbf{w} \rangle_Q|\, dx &\le 2 \inf_{A \in \mathbb{R}^{d\times d}} \fint_Q |\nabla \mathbf{w} - A|\, dx \\ &\le 2 \fint_Q |\nabla(\mathbf{w} - \mathcal{R}_Q \mathbf{w})|\, dx \le c \fint_Q |M(\boldsymbol{\varepsilon}(\mathbf{w}))|\, dx, \end{aligned}$$

where $\langle \nabla \mathbf{w} \rangle_Q$ is the mean of $\nabla \mathbf{w}$ over Q.

Corollary 4.4. *Since $\|M(\cdot)\|_1$ is equivalent to $\|\cdot\|_h$ on bounded domains (see [Ste93] I 8.14(a)) we further obtain*

$$\int_\Omega |\nabla(\mathbf{w} - \mathcal{R}_\Omega \mathbf{w})|\, dx \le c\, \|\boldsymbol{\varepsilon}(\mathbf{w})\|_{L^h(\Omega)}, \quad \mathbf{w} \in V^{1,h}(\Omega),$$

$$\int_\Omega |\mathbf{w}|\, dx \le c\, \|\boldsymbol{\varepsilon}(\mathbf{w})\|_{L^h(\Omega)}, \quad \mathbf{w} \in V_0^{1,h}(\Omega).$$

4.2 Function spaces involving symmetric gradients

In this subsection we summarize further properties of $V_0^{1,h}(\Omega)$ and $V_{0,\mathrm{div}}^{1,h}(\Omega)$. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with Lipschitz boundary.
Clearly $V^{1,h}(\Omega)$ is a Banach space being a proper subspace of the space $\mathrm{BD}(\Omega)$ containing all functions of bounded deformation introduced by Suquet [Suq78] and by Matthies, Strang, Christiansen [MSC78]. The class $\mathrm{BD}(\Omega)$ has been widely considered in the literature in connection with problems from plasticity, we refer to the works of Anzellotti and Giaquinta [AG80], Teman and Strang [ST81] and Teman [Tem85]. The space $\mathrm{BD}(\Omega)$ is equipped with the norm

$$\|\mathbf{u}\|_{\mathrm{BD}(\Omega)} := \|\mathbf{u}\|_{L^1(\Omega)} + \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{u})| \,, \tag{4.2}$$

where $\int_\Omega |\boldsymbol{\varepsilon}(\mathbf{u})|$ is the total variation of the matrix valued measure $\boldsymbol{\varepsilon}(\mathbf{u})$. From the above references we deduce

Lemma 4.5. *The space* $\mathrm{BD}(\Omega$ *is continuously embedded into the Lebesgue space* $L^{d/(d-1)}(\Omega)$*. For* $1 \leq p < d/(d-1)$ *the embedding* $\mathrm{BD}(\Omega) \hookrightarrow L^p(\Omega)$ *is compact.*

To functions $\mathbf{u}$ from $\mathrm{BD}(\Omega)$ we can associate a trace $\mathbf{u}|_{\partial\Omega}$ in $L^1(\partial\Omega)$, and in case $\mathbf{u}|_{\partial\Omega} = 0$ it holds (see, e.g. [AG80])

$$\|\mathbf{u}\|_{L^{d/(d-1)}(\Omega)} \leq c(d,\Omega) \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{u})| \,. \tag{4.3}$$

From (4.3) it follows that on the subspace $\mathrm{BD}(\Omega) \cap \{\mathbf{u} : \mathbf{u}|_{\partial\Omega} = 0\}$ the BD–norm defined in (4.2) can be replaced by the equivalent norm $\int_\Omega |\boldsymbol{\varepsilon}(\cdot)|$. We observe that (cf. [FS00], Lemma 4.1.6)

$$V_0^{1,h}(\Omega) = \{\mathbf{u} \in V^{1,h}(\Omega) : \mathbf{u}|_{\partial\Omega} = 0\} \,, \tag{4.4}$$

where $\mathbf{u}|_{\partial\Omega}$ has to be understood in the BD–trace sense. We therefore have inequality (4.3) for functions $\mathbf{u} \in V_0^{1,h}(\Omega)$, which means (recall (2.1)) that

$$\|\mathbf{u}\|_{V_0^{1,h}(\Omega)} := \|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^h(\Omega)} \tag{4.5}$$

is a norm equivalent to $\|\cdot\|_{V_0^{1,h}(\Omega)}$ on the class $V_0^{1,h}(\Omega)$.
From Korn's inequality (see Corollary 4.3) it follows that $V_0^{1,h}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$. Another consequence of Korn's inequality is:

Lemma 4.6. *Let* $\mathbf{u} \in V_0^{1,h}(\Omega)$. *Then the field* $\mathbf{w} := \ln(1+|\mathbf{u}|)\mathbf{u}$ *belongs to the space* $\mathrm{BD}(\Omega)$, *and the total variation* $\int_\Omega |\boldsymbol{\varepsilon}(\mathbf{w})|$ *of* $\mathbf{w}$ *is bounded in terms of* $\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^h(\Omega)}$, *i.e. we have*

$$\int_\Omega |\boldsymbol{\varepsilon}(\mathbf{w})| \le C\left(\|\mathbf{u}\|_{V_0^{1,h}(\Omega)}\right) . \tag{4.6}$$

Proof. Consider first the case $\mathbf{u} \in C_0^\infty(\Omega)$. Then it holds

$$\boldsymbol{\varepsilon}(\mathbf{w}) = \ln(1+|\mathbf{u}|)\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{1}{2}\left(\mathbf{u}^i\partial_j \ln(1+|\mathbf{u}|) + \mathbf{u}^j\partial_i \ln(1+|\mathbf{u}|)\right)_{1\le i,j\le n} ,$$

hence

$$|\boldsymbol{\varepsilon}(\mathbf{w})| \le \ln(1+|\mathbf{u}|)|\boldsymbol{\varepsilon}(\mathbf{u})| + c(n)\frac{|\mathbf{u}|}{1+|\mathbf{u}|}|\nabla\mathbf{u}| .$$

From Young's inequality for N–functions we get for s, $t \ge 0$

$$h'(t)s \le h^*(h'(t)) + h(s) ,$$

h^* denoting the conjugate function of h. Moreover we have

$$h^*(h'(t)) = th'(t) - h(t) \le h(t) .$$

These inequalities imply

$$\ln(1+|\mathbf{u}|)|\boldsymbol{\varepsilon}(\mathbf{u})| \le h'(|\mathbf{u}|)|\boldsymbol{\varepsilon}(\mathbf{u})| \le h(|\mathbf{u}|) + h(|\boldsymbol{\varepsilon}(\mathbf{u})| ,$$

hence

$$\int_\Omega |\boldsymbol{\varepsilon}(\mathbf{w})|\, dx \le \int_\Omega h(|\mathbf{u}|)\, dx + \int_\Omega h(|\boldsymbol{\varepsilon}(\mathbf{u})|)\, dx + c(n)\int_\Omega |\nabla\mathbf{u}|\, dx .$$

The quantity $\int_\Omega h(|\boldsymbol{\varepsilon}(\mathbf{u})|)\, dx$ can be estimated in terms of $\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^h(\Omega)}$ (and vice versa), to $\int_\Omega |\nabla\mathbf{u}|\, dx$ we apply Lemma 4.1, and finally observe that $\int_\Omega h(|\mathbf{u}|)\, dx$ ist bounded e.g. by $\int_\Omega |\mathbf{u}|^{d/d-1}\, dx$ and this integral can be handled via (4.3). Altogether we have (4.6) for the smooth case.
If $\mathbf{u} \in V_0^{1,h}(\Omega)$ is arbitrary, then we choose $\mathbf{u}_\nu \in C_0^\infty(\Omega)$ such that $\|\mathbf{u} - \mathbf{u}_\nu\|_{V_0^{1,h}(\Omega)} \to 0$ as $\nu \to \infty$. This in particular gives $\|\mathbf{u}_\nu\|_{V_0^{1,h}} \to \|\mathbf{u}\|_{V_0^{1,h}(\Omega)}$, and (4.6) shows that

$$\sup_\nu \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{w}_\nu)|\, dx < \infty . \tag{4.7}$$

If we apply (4.3) to $\mathbf{u}_\nu - \mathbf{u}$, we get $\mathbf{u}_\nu \to \mathbf{u}$ in $L^{d/(d-1)}(\Omega)$, and for a suitable subsequence it holds $\mathbf{u}_\nu \to \mathbf{u}$ a.e., and therefore $\mathbf{w}_\nu \to \mathbf{w}$ a.e. By (4.7) and (4.3) we see that $\{\mathbf{w}_\nu\}$ is bounded sequence in $\mathrm{BD}(\Omega)$, thus there is a strongly convergent subsequence in $L^1(\Omega)$, (see Lemma 4.5), which means that there exists $\widetilde{\mathbf{w}} \in \mathrm{BD}(\Omega)$ such that $\mathbf{w}_\nu \to \widetilde{\mathbf{w}}$ in $L^1(\Omega)$. The finiteness of $\int_\Omega |\boldsymbol{\varepsilon}(\widetilde{\mathbf{w}})|$ follows by lower semi-continuity, i.e.

$$\int_\Omega |\boldsymbol{\varepsilon}(\widetilde{\mathbf{w}})| \leq \liminf_{\nu\to\infty} \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{w}_\nu)|\, dx\,. \tag{4.8}$$

Clearly we have $\widetilde{\mathbf{w}} = \mathbf{w}$, and (4.6) for $\mathbf{w}$ follows from (4.8) and the version of (4.6) for $\mathbf{w}_\nu$. □

Now we can prove the main result of this section:

THEOREM 4.7. *The embedding $V_0^{1,h}(\Omega) \hookrightarrow L^{d/(d-1)}(\Omega)$ is compact. More precisely, if $\mathbf{u}_\nu$ denotes a bounded sequence in $V_0^{1,h}(\Omega)$, then there exists a subsequence $\mathbf{u}_\nu$ (not relabeled) and a function $\mathbf{u} \in V_0^{1,h}(\Omega)$ such that $\mathbf{u}_\nu \to \mathbf{u}$ in $L^{d/(d-1)}(\Omega)$ and $\boldsymbol{\varepsilon}(\mathbf{u}_\nu) \rightharpoonup \boldsymbol{\varepsilon}(\mathbf{u})$ in $L^1(\Omega)$ for $\nu \to \infty$.*

Proof. Suppose that $\sup_{\nu\in\mathbb{N}} \|\mathbf{u}_\nu\|_{V_0^{1,h}(\Omega)} < \infty$. From Lemma 4.6 we deduce the existence of a field $\mathbf{u} \in L^1(\Omega)$ such that

$$\mathbf{u}_\nu \to \mathbf{u} \quad \text{in } L^1(\Omega) \quad \text{and a.e.}\,, \tag{4.9}$$

where here and in what follows we will pass to subsequences whenever this is necessary. According to the De La Vallée Poussin criterion for weak compactness in L^1 or by a theorem of Dunford and Pettis (cf. [AFP00], Theorem 1.38) we get from

$$\sup_{\nu\in\mathbb{N}} \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{u}_\nu)| \ln(1 + |\boldsymbol{\varepsilon}(\mathbf{u}_\nu)|)\, dx < \infty$$

that $\boldsymbol{\varepsilon}(\mathbf{u}_\nu) \rightharpoonup: \boldsymbol{\sigma}$ in $L^1(\Omega)$, and clearly $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\mathbf{u})$. Moreover, by lower semi-continuity it holds

$$\int_\Omega h(|\boldsymbol{\varepsilon}(\mathbf{u})|)\, dx \leq \liminf_{\nu\to\infty} \int_\Omega h(|\boldsymbol{\varepsilon}(\mathbf{u}_\nu)|)\,,$$

so that $\mathbf{u}$ is an element of the space $V^{1,h}(\Omega)$. In order to show $\mathbf{u} \in V_0^{1,h}(\Omega)$, we follow the arguments of Frehse and Seregin [FS98]: since $\boldsymbol{\varepsilon}(\mathbf{u}_\nu) \rightharpoonup \boldsymbol{\varepsilon}(\mathbf{u})$ in $L^1(\Omega)$ we can find a sequence $\{\boldsymbol{\sigma}_\mu\}$, $\boldsymbol{\sigma}_\mu$ being an element of the convex hull of $\{\boldsymbol{\varepsilon}(\mathbf{u}_\nu) : \nu \geq \mu\}$, such that $\boldsymbol{\sigma}_\mu \to \boldsymbol{\varepsilon}(\mathbf{u})$ in $L^1(\Omega)$. This follows from the well–known Banach–Saks lemma. We have

$$\boldsymbol{\sigma}_\mu = \sum_{\nu=\mu}^{N(\mu)} \lambda_\nu^\mu \boldsymbol{\varepsilon}(\mathbf{u}_\nu),\ \sum_{\nu=\mu}^{N(\mu)} \lambda_\nu^\mu = 1,\ 0 \leq \lambda_\nu^\mu \leq 1$$

with suitable coefficients λ_ν^μ and integers $N(\mu) \geq \mu$. Let

$$\overline{\mathbf{u}}_\mu := \sum_{\nu=\mu}^{N(\mu)} \lambda_\nu^\mu \, \mathbf{u}_\nu.$$

These functions belong to $V_0^{1,h}(\Omega)$ and satisfy

$$\|\overline{\mathbf{u}}_\mu - \mathbf{u}\|_{L^1(\Omega)} \leq \sum_{\nu=\mu}^{N(\mu)} \lambda_\nu^\mu \|\mathbf{u}_\nu - \mathbf{u}\|_{L^1(\Omega)} \to 0, \quad \mu \to \infty,$$

which is a consequence of (4.9). Moreover it holds

$$\int_\Omega |\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_\mu)| \, dx = \int_\Omega |\boldsymbol{\sigma}_\mu| \, dx \to \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{u})| \, dx, \; \mu \to \infty \,,$$

and according to [AG80] these two convergences imply the L^1–convergence of the traces of $\mathbf{u}_\mu$ towards the trace of $\mathbf{u}$. In conclusion $\mathbf{u}|_{\partial\Omega} = 0$, hence $\mathbf{u} \in V_0^{1,h}(\Omega)$, and it remains to show that

$$\mathbf{u}_\nu \to \mathbf{u} \quad \text{in } L^{d/(d-1)}(\Omega) \tag{4.10}$$

holds. From our assumption combined with (4.6) we get

$$\sup_{\nu \in \mathbb{N}} \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{w}_\nu)| < \infty \,, \tag{4.11}$$

$\mathbf{w}_\nu := \ln(1 + |\mathbf{u}_\nu|)\mathbf{u}_\nu$, and (4.11) together with the first part of Lemma 4.6 gives

$$\sup_{\nu \in \mathbb{N}} \|\mathbf{w}_\nu\|_{L^{d/(d-1)}(\Omega)} < \infty \,. \tag{4.12}$$

Let $\Gamma(t) := h\left(t^{\frac{d-1}{d}}\right)^{d/(d-1)}, \; t \geq 0$. Then

$$\frac{\Gamma(t)}{t} = \left[\frac{h\left(t^{\frac{d-1}{d}}\right)}{t^{\frac{d-1}{d}}} \right]^{\frac{d}{d-1}} \longrightarrow \infty, \; t \to \infty \,, \tag{4.13}$$

and (compare (4.12))

$$\int_\Omega \Gamma\left(|\mathbf{u}_\nu|^{\frac{d}{d-1}}\right) \, dx = \int_\Omega h(|\mathbf{u}_\nu|)^{\frac{d}{d-1}} \, dx = \int_\Omega |\mathbf{w}_\nu|^{\frac{d}{d-1}} \, dx \leq \text{ const } < \infty \,, \tag{4.14}$$

therefore $|\mathbf{u}_\nu|^{d/(d-1)} \rightharpoonup: g$ weakly in $L^1(\Omega)$ by quoting the De La Vallée Poussin criterion one more time. By (4.9) we must have $g = |\mathbf{u}|^{d/(d-1)}$, since $|\mathbf{u}_\nu|^{d/(d-1)} \to |\mathbf{u}|^{d/(d-1)}$ a.e. on Ω. This in particular implies

$$\|\mathbf{u}_\nu\|_{L^{d/(d-1)}(\Omega)} \to \|\mathbf{u}\|_{L^{d/(d-1)}(\Omega)}, \ \nu \to \infty ,$$

where we combined (4.13) and (4.14) with Vitali's Theorem. At the same time it follows from

$$\sup_{\nu \in \mathbb{N}} \|\mathbf{u}_\nu\|_{L^{d/(d-1)}(\Omega)} < \infty$$

and (4.9), that $\mathbf{u}_\nu \rightharpoonup \mathbf{u}$ in $L^{d/(d-1)}(\Omega)$. Putting both convergences together, the Radon–Riesz lemma (cf. [GMS98], p.47, Proposition 3) gives our claim (4.10), and Theorem 4.7 is proved. □

In the setting of Prandtl–Eyring fluids we have to work in the space $V^{1,h}_{0,\mathrm{div}}(\Omega)$ which according to Lemma 4.1.6 in [FS00] is the closure of $C^\infty_{0,\mathrm{div}}(\Omega)$ in the class $V^{1,h}(\Omega)$ w.r.t. the norm $\|\cdot\|_{V^{1,h}(\Omega)}$ defined in (2.1). From Theorem 4.7 it follows

Corollary 4.8. *The statement of Theorem 4.7 remains valid, if the space $V^{1,h}_0(\Omega)$ is replaced by the subclass $V^{1,h}_{0,\mathrm{div}}(\Omega)$.*

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