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Abstract

This paper is devoted to a proof of optimal regularity, near the initial state, for weak solutions to the two-phase parabolic obstacle problem. The approach used here is general enough to allow us to consider the initial data belonging to the class $C^{1,1}$.

1 Introduction.

We consider a weak solution of the two-phase parabolic obstacle problem

$$H[u] = f(u) = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} \quad \text{in} \quad \mathcal{C}_{10} = B_{10} \times]0,1], \tag{1}$$

$$u = \varphi \quad \text{on} \quad B_{10}, \tag{2}$$

and u satisfies some boundary conditions on the lateral surface of the cylinder $\mathcal{C}_{10}$. Here $H[u] = \Delta u - \partial_t u$ is the heat operator, $\lambda^\pm$ are non-negative constants such that $\lambda^+ + \lambda^- > 0$, χ_E is the characteristic function of the set E, and $B_{10} = \{x : |x| < 10\}$. Observe that equation (1) is understood in distributional sence. We suppose that a given function φ satisfies

$$\varphi \in C^{1,1}(B_{10}). \tag{3}$$

We suppose also that $\sup\limits_{\mathcal{C}_{10}} |u| \leqslant M$ with $M \geqslant 1$.

For a $C^1_x \cap C^0_t$-function u defined in $\mathcal{C}_{10}$ we introduce the following sets:

$$\begin{aligned} \Omega^\pm(u) &= \{(x,t) \in \mathcal{C}_{10} : \pm u(x,t) > 0\} \\ \Lambda(u) &= \{(x,t) \in \mathcal{C}_{10} : u(x,t) = |Du(x,t)| = 0\} \\ \Gamma(u) &= \partial\{(x,t) \in \mathcal{C}_{10} : u(x,t) \neq 0\} \cap \mathcal{C}_{10} \quad \text{is the free boundary.} \end{aligned}$$

We emphasize that in the two-phase case we do not have the property that the gradient vanishes on the free boundary, as it was in the classical one-phase case; this causes difficulties. Therefore, we will distinguish the following parts of Γ:

$$\Gamma^0(u) = \Gamma(u) \cap \Lambda(u), \qquad \Gamma^*(u) = \Gamma(u) \setminus \Gamma^0(u).$$

From [SUW09] it follows that in some suitable rotated coordinate system in $\mathbb{R}^n$ the set $\Gamma^*(u)$ can be locally described as $x_1 = f(x_2, \dots, x_n, t)$ with $f \in C^{1,\alpha}$ for any $\alpha < 1$.

1.1 Background and main result.

In this paper we are interested in uniform L^∞-estimates near the initial state for the derivatives D^2u and $\partial_t u$ of the function u satisfying (1)-(2).
Relative interior estimates were obtained in [SUW09]. The corresponding estimates up to the lateral surface were proved in [Ura07] for zero Dirichlet data, and in [AU09] for general Dirichlet data satisfying certain structure conditions, respectively. Unfortunately the proofs presented in [SUW09], [Ura07] and [AU09] do not work near the initial state. To this end, the additional investigation of the behaviour of the solution u close to the initial state is required.
Speaking about regularity up to the initial state, we are only aware of the results of [Ura07], [NPP10], [Sha08] and [Nys08]. In the papers [Nys08] and [Sha08] the authors studied the parabolic obstacle problem near the initial state for quasilinear and fully nonlinear equations, respectively. In both cases, the estimates of the second derivatives D^2u were not considered, and hence, only the gradient Du and the time derivative $\partial_t u$ were estimated. The results in [NPP10] are most close to those obtained here. Indeed, the authors of [NPP10] considered the parabolic obstacle problem with more general differential operator of Kolmogorov type and established the L^∞-estimates of D^2u and $\partial_t u$ under the assumption that the initial data φ belongs to the space $C^{2,\alpha}$. It remains only to note that the two-phase parabolic problem with the initial data $\varphi \in C^{2,\alpha}$ was studied in [Ura07] under the additional structure assumption that φ vanishes together with its gradient.

Now we formulate the main result of the paper.

Theorem 1. *Let u be a weak solution of (1)-(2) with a function φ satisfying the assumption (3). Suppose also that $\sup\limits_{\mathcal{C}_{10}} |u| \leqslant M$.*
Then there exists a positive constant c completely defined by n, M, $\lambda^\pm$, and φ such that

$$ess \sup_{\mathcal{C}_1} \left\{ |D^2u| + |\partial_t u| \right\} \leqslant c.$$

Remark 1.1. *The result of Theorem 1 is optimal in the sense that we require from the initial function φ as much regularity as we want to prove for the solution.*

Remark 1.2. *The cylinder $\mathcal{C}_{10}$ is chosen only for simplicity. In fact, the problem (1)-(2) can be treated in $\mathcal{C}_{1+\delta}$ for arbitrary $\delta > 0$. In this case, the constant c in Theorem 1 will also depend on δ.*

Remark 1.3. *From Theorem 1 and the well-known interpolation theory it follows that $Du \in C^{1,1/2}_{x,t}(\mathcal{C}_1)$.*

The main strategy used in the present paper follows. At first we prove the estimate of the time derivative. We do this in §2 with the help of regularizations. The next step is to obtain the estimates of the second derivatives. The analysis of the second derivatives in §3 is based essentially on the famous local monotonicity formula due to L. Caffarelli.

1.2 Notation.

Throughout this paper we use the following notation:
$z = (x, t)$ are points in $\mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^1$;
$|x|$ is the Euclidean norm of x in $\mathbb{R}^n$;
χ_E denotes the characteristic function of the set $E \subset \mathbb{R}^{n+1}$;
$v_+ = \max\{v, 0\}$; $\qquad v_- = \max\{-v, 0\}$;
$B_r(x^0)$ denotes the open ball in $\mathbb{R}^n$ with center x^0 and radius r;
$B_r = B_r(0)$;
$\mathcal{C}_r = B_r \times]0, 1]$;
$\partial' \mathcal{C}_r$ is the parabolic boundary, i.e., the topological boundary minus the top of the cylinder;
$Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0]$. Since our main interest are the estimates near the initial state, the radius r in $Q_r(z^0)$ will be always chosen such that $t^0 - r^2 = 0$.
D_i denotes the differential operator with respect to x_i; $\partial_t = \frac{\partial}{\partial t}$;
$D = (D_1, D_2, \ldots, D_n)$ denotes the spatial gradient;
$D^2 u = D(Du)$ denotes the Hessian of u;
D_ν stands for the operator of differentiation along the direction $\nu \in \mathbb{R}^n$, i.e., $|\nu| = 1$ and

$$D_\nu u = \sum_{i=1}^{n} \nu_i D_i u.$$

$\| \cdot \|_{p, E}$ denotes the norm in $L^p(E)$, $1 < p \leqslant \infty$;

$\fint_E \ldots$ stands for the average integral over the set E, i.e.,

$$\fint_E \cdots = \frac{1}{\text{meas}\,\{E\}} \int_E \ldots;$$

$\xi = \xi(|x|)$ stands for a time-independent cut-off function belonging $C^2(B_2)$, having support in B_2, and satisfying $\xi \equiv 1$ in B_1.

$\xi_{r,x^0}(x) = \xi\left(\dfrac{|x - x^0|}{r}\right)$. It is clear that in the annular $B_{2r}(x^0) \setminus B_r(x^0)$ the function ξ_{r,x^0} satisfies the inequalities

$$|D\xi_{r,x^0}(x)| \leqslant c(n)r^{-1}, \qquad |\Delta\xi_{r,x^0}(x)| \leqslant c(n)r^{-2}. \tag{4}$$

For future reference, we introduce the fundamental solution

$$G(x,t) = \begin{cases} \dfrac{\exp\left(-|x|^2/4t\right)}{(4\pi t)^{n/2}}, & \text{for} \quad t > 0, \\ 0, & \text{for} \quad t \leqslant 0 \end{cases} \tag{5}$$

to the heat equation.
We use letters M, N, and c (with or without indices) to denote various constants. To indicate that, say, c depends on some parameters, we list them in the parentheses: $c(\dots)$. We will write $c(\varphi)$ to indicate that c is defined by the sum $\|D^2\varphi\|_{\infty,B_{10}} + \|\varphi\|_{\infty,B_{10}}$.

1.3 Useful facts

For the readers convenience and for future references we recall and explain some facts.

Fact 1.4. *Let u be a solution of Equation (1), and let e be a direction in $\mathbb{R}^n$. Then*

$$H\left[\,(D_e u)_\pm\,\right] \geqslant 0 \quad in \quad \mathcal{C}_{10}, \tag{6}$$

where the inequalities (6) are understood in the sense of distributions.

Proof. The proof of this assertion can be found in [SUW09] (see also [AU09]). □

We denote

$$I(r,v,z^0) = \int\limits_{t^0-r^2}^{t^0} \int\limits_{\mathbb{R}^n} |Dv(x,t)|^2 G(x-x^0, t^0-t)\,dxdt,$$

where $r \in]0,1]$, $z^0 = (x^0,t^0)$ is a point in $\mathbb{R}^{n+1}$, a function v is defined in the strip $\mathbb{R}^n \times [t^0-1,t^0]$, and the heat kernel $G(x,t)$ is defined by (5).
To prove the main theorem, we need the following monotonicity formula for pairs of disjointly supported subsolutions of the heat equation.

Fact 1.5. *Let $\xi := \xi(|x|)$ be a standard time-independent cut-off function (see Notation), and let h_1, h_2 be nonnegative, sub-caloric and continuous functions in $\mathcal{C}_2$, satisfying*

$$h_1(0,1) = h_2(0,1) = 0, \qquad h_1(x,t) \cdot h_2(x,t) = 0 \quad \textit{in} \quad \mathcal{C}_2.$$

Then, for $0 < r \leqslant 1$ the functional

$$\Psi(r) = \Psi(r, h_1, h_2, \xi, 0, 1) = \frac{1}{r^4} I(r, \xi h_1, 0, 1) I(r, \xi h_2, 0, 1)$$

satisfies the inequality

$$\Psi(r) \leqslant \Psi(1) + N(n) \|h_1\|^2_{2,\mathcal{C}_2} \|h_2\|^2_{2,\mathcal{C}_2}. \tag{7}$$

Proof. For the proof of this statement we refer the reader to (the proof of) Theorem 1.1.4 [CK98] (see also Theorem 12.12 in [CS05]). $\square$

Remark 1.6. *By rescaling one can easily derive the following modification of the local monotonicity formula (7):*

$$\Phi(r, \zeta_R) \leqslant \Phi(R, \zeta_R) + \frac{N(n)}{R^{2n+8}} \|h_1\|^2_{2,Q_{2R}(z^0)} \|h_2\|^2_{2,Q_{2R}(z^0)} \qquad \forall r \leqslant R = \sqrt{t^0}.$$

Here $\zeta_R(x) = \xi_{R,x^0}(|x|)$ is a standard cut-off function (see Notation) and

$$\Phi(r, \zeta_R) = \Phi(r, h_1, h_2, \zeta_R, z^0) = \frac{1}{r^4} I(r, \zeta_R h_1, z^0) I(r, \zeta_R h_2, z^0). \tag{8}$$

2 Estimate of the time derivative

For $\varepsilon > 0$ we consider the regularized problem

$$H[u^\varepsilon] = f^\varepsilon(u^\varepsilon) \quad \text{in} \quad \mathcal{C}_9, \tag{9}$$

$$u^\varepsilon = u \quad \text{on} \quad \partial B_9 \times]0,1], \tag{10}$$

$$u^\varepsilon = \varphi_\varepsilon \quad \text{on} \quad B_9 \times \{0\}, \tag{11}$$

where f^ε is a smooth non-decreasing function such that $f^\varepsilon(s) = \lambda^+$ as $s \geqslant \varepsilon$ and $f^\varepsilon(s) = -\lambda^-$ as $s \leqslant \varepsilon$; while φ_ε is a mollifier of φ with the radius depending on the distance to ∂B_9 such that

$$\sup_{B_9} |\varphi - \varphi_\varepsilon| \leqslant \varepsilon,$$

and u satisfies (1)-(2).
By the parabolic theory, for each $\varepsilon > 0$, the regularized problem (9)-(11) has a solution u^ε with Du^ε and $\partial_t u^\varepsilon$ belonging to $L^2(\mathcal{C}_9)$.

Lemma 2.1. *Let $\varepsilon > 0$, let u satisfy (1)-(2), and let u^ε be a solution of (9)-(11). Then*

$$\sup_{\mathcal{C}_9} |u^\varepsilon - u| \leqslant \varepsilon. \tag{12}$$

Proof. Setting $w = u^\varepsilon - u$ we observe that $w\big|_{\partial'\mathcal{C}_9} = u_{\{\varepsilon\}} - u$, and, consequently, $(w-\varepsilon)_+\big|_{\partial'\mathcal{C}_9} = 0$. Then Eqs. (9) and (1) together with integration by parts provide for arbitrary $t \in]0,1]$ the following identity

$$\begin{aligned} \iint\limits_{B_9\times]0,t]} [f(u) - f^\varepsilon(u^\varepsilon)]\,(w-\varepsilon)_+\,dz &= \iint\limits_{B_9\times]0,t]} -H[w]\,(w-\varepsilon)_+\,dz \\ &= \frac{1}{2}\int\limits_{B_9} [(w-\varepsilon)_+]^2\Big|_0^t dx + \iint\limits_{\{w>\varepsilon\}} |Dw|^2 dz. \end{aligned} \tag{13}$$

Taking into account the relations $f(u) - f^\varepsilon(u^\varepsilon) \leqslant 0$ on the set $\{u^\varepsilon > \varepsilon\} \cup \{u < 0\}$ and $(w-\varepsilon)_+ = 0$ on the set $\{u^\varepsilon \leqslant \varepsilon\} \cap \{u \geqslant 0\}$, we conclude that the left-hand side of identity (13) is nonpositive. The latter means that

$$\sup_{\mathcal{C}_9} w \leqslant \varepsilon. \tag{14}$$

Replacing in identity (13) the term $(w-\varepsilon)_+$ by $(w+\varepsilon)_-$ and repeating the above arguments we end up with

$$\inf_{\mathcal{C}_9} w \geqslant -\varepsilon. \tag{15}$$

Combination inequalities (14) and (15) finishes the proof. $\square$

We observe also that

$$\partial_t u^\varepsilon\big|_{t=0} = -f^\varepsilon(u^\varepsilon)\big|_{t=0} + \Delta\varphi_\varepsilon, \qquad x \in B_9.$$

Thanks to condition (3) we may conclude that $\partial_t u^\varepsilon\big|_{t=0}$ are bounded uniformly with respect to ε. Moreover, for each small $\delta > 0$ the functions u^ε are smooth in the closure of the cylinder $\mathcal{C}_{9-\delta}$.
With the estimate (12) at hands it is easy to check that

$$\|\partial_t u^\varepsilon\|_{2,\mathcal{C}_8} \leqslant c,$$

and the latter inequality is also uniform with respect to ε. Hence we can easily deduce the following result.

Lemma 2.2. *For each small $\delta > 0$ the uniform estimate*

$$\sup_{\mathcal{C}_{8-\delta}} |\partial_t u^\varepsilon| \leqslant N_1(n, M, \lambda^\pm, \varphi, \delta) \tag{16}$$

holds true for solutions u^ε of the regularized problem (9)-(11).

Proof. We set $v = \partial_t u^\varepsilon$. It is easy to see that $v_\pm$ are subcaloric in $\mathcal{C}_9$.
Now we may apply the well-known parabolic estimate (see, for example, Lemma 3.1 and Remark 6 from [NU11]) and get

$$\sup_{\mathcal{C}_{8-\delta}} v_\pm \leqslant N \sqrt{\fint_{\mathcal{C}_8} v_\pm^2(z)dz} + \sup_{B_8\times\{0\}} v_\pm,$$

which implies the desired inequality (16). $\square$

Remark 2.3. *By virtue of Lemma 2.1, solutions u^ε of the regularized problem (9)-(11) converge to u as $\varepsilon \to 0$ uniformly in $\mathcal{C}_9$.*

Remark 2.4. *It follows from Lemma 2.2 and Remark 2.3 that the estimate*

$$\sup_{\mathcal{C}_7} |\partial_t u| \leqslant N_1(n, M, \lambda^\pm, \varphi, 1) \tag{17}$$

holds true for a function u satisfying (1)-(2). Here N_1 is the same constant as in Lemma 2.2.

3 Estimates of the second derivatives

Lemma 3.1. *Let the assumptions of Theorem 1 hold, let $z^0 = (x^0, t^0)$ be a point in $\mathcal{C}_1$, and let $R = \sqrt{t^0}$.*
Then there exists a positive constant N_2 completely defined by the values of n, M, $\lambda^\pm$, and the norms of φ such that

$$\sup_{B_{2R}(x^0)} |Du(\cdot, t) - D\varphi(\cdot)| \leqslant N_2 R \quad \textit{for} \quad t \in]0, t^0]. \tag{18}$$

Remark 3.2. *It is evident that $Q_{6R}(z^0) \subset \mathcal{C}_7$.*

Proof. First, we observe that the assumption (3) implies $\Delta\varphi \in L^\infty(B_{10})$. Therefore, for almost all $t \in]0, 1]$ the difference $u(\cdot, t) - \varphi(\cdot)$ can be regarded in B_7 as a solution of an elliptic equation

$$\Delta\left(u(x,t) - \varphi(x)\right) = F(x) \equiv f(u(x,t)) + \partial_t u(x,t) - \Delta\varphi(x). \tag{19}$$

Due to estimate (17), the function $|F|$ is bounded by the known constant up to the bottom of the cylinder $\mathcal{C}_7$.
Thus, for a test function $\eta \in W_0^{1,2}(B_{6R}(x^0))$ we have the integral identity

$$\int\limits_{B_{6R}(x^0)} D(u-\varphi)D\eta dx = \int\limits_{B_{6R}(x^0)} F\eta dx. \tag{20}$$

We set in (20) $\eta = (u-\varphi)\,\zeta_{3R}^2$, where $\zeta_{3R}(x) := \xi_{3R,x^0}(|x|)$ is a standard time-independent cut-off function (see Notation). Then, after integrating by parts and subsequent application of Young's inequality, identity (20) takes the form

$$\begin{aligned}\int\limits_{B_{6R}(x^0)} |D(u-\varphi)|^2\zeta_{3R}^2 dx \leqslant c_1 \int\limits_{B_{6R}(x^0)} (u-\varphi)\zeta_{3R}^2 dx \\ + c_2 \int\limits_{B_{6R}(x^0)} (u-\varphi)^2|D\zeta_{3R}|^2 dx.\end{aligned} \tag{21}$$

It is easy to see that inequality (17) implies the estimate

$$\sup_{B_{6R}(x^0)\times]0,t^0]} |u-\varphi| \leqslant cR^2.$$

Putting together (21) and the above inequality we arrive at

$$\int\limits_{B_{3R}(x^0)} |D(u(x,t)-\varphi(x))|^2 dx \leqslant N_2 R^{n+2}, \qquad t \in [0,t^0]. \tag{22}$$

Now we observe that for almost all $t \in]0,1]$ and for any direction $e \in \mathbb{R}^n$ the difference $D_e u - D_e\varphi$ may be considered as a weak solution of the equation

$$\Delta\left(D_e u(x,t) - D_e\varphi(x)\right) = -D_e F(x)$$

in B_7. The well known results (see [LU68], [GT01]) applied to the difference $D_e u - D_e\varphi$ yield the inequality

$$\sup_{B_{2R}(x^0)} |D_e u - D_e\varphi| \leqslant c\sqrt{\,\fint\limits_{B_{3R}(x^0)} |D_e u - D_e\varphi|^2 dx} + cR\sup_{B_{3R}(x^0)} |F|.$$

Combining the last inequality with the estimate (22), we get (18) and finish the proof. $\square$

Lemma 3.3. *Let the assumptions of Lemma 3.1 hold. Then*

$$\int_0^{R^2}\int_{\mathbb{R}^n} |D^2u(x,t)|^2\zeta_R^2(x)G(x-x^0,t^0-t)dxdt \leqslant N_3(n,M,\lambda^\pm,\varphi)R^2, \quad (23)$$

where $\zeta_R(x) := \xi_{R,x^0}(|x|)$ *is a standard time-independent cut-off function (see Notation) and the heat kernel* G *is defined by the formula (5).*

Proof. Suppose that e is an arbitrary direction in $\mathbb{R}^n$ if $Du(z^0) = 0$ and $e \perp \nu$, where $\nu = Du(z^0)/|Du(z^0)|$, otherwise. From (3), (18) and our choice of e it follows that

$$\sup_{B_{2R}(x^0)} |D_e\varphi| \leqslant cR. \quad (24)$$

According to Fact 1.4, the functions $v = (D_eu)_\pm$ are sub-caloric in $\mathcal{C}_2$, i.e., $H[v] \geqslant 0$ in the sense of distributions. Since $|Dv|^2 + vH[v] = \frac{1}{2}H[v^2]$ we have

$$\begin{aligned}\int_0^{R^2}\int_{B_{2R}(x^0)} &|Dv(x,t)|^2\zeta_R^2(x)G(x-x^0,t^0-t)dxdt \\ &\leqslant \frac{1}{2}\int_0^{R^2}\int_{B_{2R}(x^0)} H[v^2(x,t)]\zeta_R^2(x)G(x-x^0,t^0-t)dxdt.\end{aligned} \quad (25)$$

After successive integration the right-hand side of (25) by parts we get

$$\begin{aligned}\int_0^{R^2}\int_{B_{2R}(x^0)} |Dv|^2\zeta_R^2 G dxdt \leqslant &-\int_{B_{2R}(x^0)} \left(\frac{v^2}{2}\zeta_R^2 G\right)\Big|_0^{R^2} dx \\ &+\int_0^{R^2}\int_{B_{2R}(x^0)} \frac{v^2}{2}\zeta_R^2\left[\partial_t G + \Delta G\right] dxdt \\ &+\int_0^{R^2}\int_{B_{2R}(x^0)} v^2\left[2\zeta_R D\zeta_R DG + G|D\zeta_R|^2 + G\zeta_R\Delta\zeta_R\right] dxdt \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

It is evident that due to (2) we have

$$
\begin{aligned}
I_1 &\leqslant \int\limits_{B_{2R}(x^0)} \frac{|D_e u(x,0)|^2}{2}\zeta_R^2(x)G(x-x^0,t^0)dx \\
&= \int\limits_{B_{2R}(x^0)} \frac{|D_e \varphi(x)|^2}{2}\zeta_R^2(x)G(x-x^0,t^0)dx \\
&\leqslant cR^2,
\end{aligned}
$$

where the last inequality provided by (24).
Taking into account the relation

$$
\partial_t G + \Delta G = \partial_t G(x-x^0,t^0-t) + \Delta G(x-x^0,t^0-t) = 0 \quad \text{for} \quad t < t^0,
$$

we conclude that $I_2 = 0$.
Finally, we observe that the integral in I_3 is really taken over the set $E =]0,R^2]\times\{B_{2R}(x^0)\setminus B_R(x^0)\}$. Therefore, in E we have the following estimates for functions involved into I_3

$$
\begin{aligned}
|G(x-x^0,t^0-t)| &\leqslant c\frac{e^{-\frac{R^2}{4(R^2-t)}}}{(R^2-t)^{n/2}} \leqslant cR^{-n}; \\
|DG(x-x^0,t^0-t)D\zeta(x)| &\leqslant c|G(x-x^0,t^0-t)|\frac{|x-x^0|}{R(R^2-t)} \\
&\leqslant c\frac{e^{-\frac{R^2}{4(R^2-t)}}}{(R^2-t)^{1+n/2}} \leqslant cR^{-n-2}.
\end{aligned}
$$

Consequently,

$$
I_3 \leqslant CR^{-n-2}\iint\limits_E v^2 dxdt \leqslant C\sup_E v^2 \leqslant CR^2,
$$

where the last inequality follows from (18) and (24).
Thus, collecting all inequalities we get

$$
\int\limits_0^{R^2}\int\limits_{B_{2R}(x^0)} |Dv|^2\zeta_R^2 G dz \leqslant CR^2 \tag{26}
$$

Inequalities (26) mean that we obtained the desired integral estimate for all the derivatives $D\,(D_e u)$ with $e \perp \nu$. Similar estimate for the derivative $D_\nu\,(D_\nu u)$ follows from (26) and Eq. (1). □

Corollary 3.4. *Let the assumptions of Lemma 3.1 hold, and let e be an arbitrary direction in $\mathbb{R}^n$. Then*

$$\Phi_e(R) = \Phi(R, (D_e u)_+, (D_e u)_-, \zeta_R, z^0) \leqslant N_4(n, M, \lambda^\pm, \varphi),$$

where the functional Φ is defined by the formula (8), while $\zeta_R(x) := \xi_{R,x^0}(|x|)$ is a standard time-independent cut-off function (see Notation).

Proof. The desired inequality follows immediately from the definition (8) combined with (23). □

Lemma 3.5. *Let the assumptions of Lemma 3.1 hold, let $\nu = Du(z^0)/|Du(z^0)|$, and let e be an arbitrary direction in $\mathbb{R}^n$ if $|Du(z^0)| = 0$ and $e \perp \nu$ otherwise. Then*

$$\|(D_e u)_\pm\|^2_{2,Q_{2R}(z^0)} \leqslant N_5(n, M, \lambda^\pm, \varphi) R^{n+4}. \tag{27}$$

Proof. It is evident that inequalities (18) and (24) imply the estimate (27). □

Proof of Theorem 1. Let $z^0 = (x^0, t^0)$ be an arbitrary point from $\mathcal{C}_1$ such that $|u(z^0)| > 0$, and let $e \in \mathbb{R}^n$ be the same direction as in Lemma 3.5.

Since $D_e u(z^0) = 0$, it follows that

$$C(n)|D(D_e u)(z^0)|^4 \leqslant \lim_{r \to 0} \Phi_e(r),$$

where $\Phi_e(r) = \Phi(r, (D_e u)_+, (D_e u)_-, \zeta_R, z^0)$ with the functional Φ defined by the formula (8), and $\zeta_R(x) = \xi_{R,x^0}(|x|)$. On the other hand, according to Fact 1.5 and Remark 1.6 after that, we have for $0 < r \leqslant R := \sqrt{t^0}$ the inequality

$$\Phi_e(r) \leqslant \Phi_e(R) + \frac{N(n)}{R^{2n+8}} \|(D_e u)_+\|^2_{2,Q_{2R}(z^0)} \|(D_e u)_-\|^2_{2,Q_{2R}(z^0)}.$$

Application of Corollary 3.4 and Lemma 3.5 enable us to estimate the right-hand side of the last inequality by the known constant. This means that we proved in the cylinder $\mathcal{C}_1$ the L^∞-estimate for all the derivatives $D\left(D_e u\right)(z^0)$ with $e \perp \nu$. It is clear that the corresponding estimate of $D_\nu\left(D_\nu u\right)(z^0)$ can be now deduced from Eq. (1).
So, we establish the L^∞-estimates for $D^2 u(z^0)$ for all points $z^0 \in \Omega^\pm \cap \mathcal{C}_1$, and these estimates do not depend on the distance of z^0 from the free boundary $\Gamma(u)$. Since $|D^2 u| = 0$ almost everywhere on $\Lambda(u)$ and the $(n+1)$-dimensional Lebesgue measure of the set $\Gamma^*(u)$ equals zero, we get the uniform estimate of the Lipschitz constant for $Du(\cdot, t)$ for each $t \in [0, 1]$. It remains only to observe that the uniform L^∞-estimate of $\partial_t u$ were established in (17). This finishes the proof. □

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