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# An approximation theorem for vector fields in $\mathbf{BD}_{\mathbf{div}}$

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# Contents




**Abstract.** We consider the equations of slow stationary motion of a perfectly plastic fluid in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$). The proof of the existence of a weak solution to these equations leads to the minimization of a functional of linear growth on the space

$$\mathbf{BD}_{\mathrm{div}}(\Omega) = \left\{ \mathbf{u} \in \mathbf{BD}(\Omega) : \int_\Omega \mathbf{u} \cdot \nabla\varphi \, \mathrm{d}x = 0 \ \ \forall \, \varphi \in W^{1,n}(\Omega) \right\}.$$

The elements of this space are divergence free BD-vector fields with vanishing normal component of the trace in suitable sense. The main result of our paper is an approximation theorem for these vector fields by smooth, compactly supported and divergence free vector fields. This approximation theorem implies the equality between the infimum of the above mentioned functional on its "natural energy space" and the infimum of the extension of this functional on $\mathbf{BD}_{\mathrm{div}}(\Omega)$.


# 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$) be an open bounded set with Lipschitz boundary $\partial\Omega$. The slow motion of a homogeneous incompressible fluid in $\Omega$ is modeled by the following


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system of PDEs

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega\,, \tag{1.1}$$

$$-\nabla \cdot \mathbf{S} + \nabla P = \mathbf{f} \qquad \text{in } \Omega\,, \tag{1.2}$$

where

$$\begin{aligned} \mathbf{u} &= (u_1, \ldots, u_n) \ \text{ velocity}\,, \\ \mathbf{S} &= \{S_{ij}\}_{i,j=1,\ldots,n} \ \text{ deviatoric stress}\,, \\ P &= \text{pressure}\,, \\ \mathbf{f} &= \text{external force}\,. \end{aligned}$$

The full stress tensor of the fluid is then given by $\boldsymbol{\sigma} = \mathbf{S} + P\mathbf{I}$. We complete (1.1), (1.2) by the assumption that the fluid adheres to the boundary of $\Omega$, i.e.

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega\,. \tag{1.3}$$

∎

To proceed, for vector fields $\mathbf{u} = (u_1, \ldots, u_n)$ we introduce the notation

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right) \qquad \text{rate of strain}\,.$$

We then consider constitutive laws of the form $\mathbf{S} = \rho(|\mathbf{D}(\mathbf{u})|)\mathbf{D}(\mathbf{u})$ [1], where $\rho$ is a nonnegative real function defined on $[0, +\infty)$. For notational simplicity, in this section we write $\mathbf{D}$ in place of $\mathbf{D}(\mathbf{u})$.

(I) Power law model.

Let $\nu = const > 0$ be fixed. For $1 < p < +\infty$ we define $\mathbf{S} = \mathbf{S}_p$ by

$(i)$ $\quad 2 \le p < +\infty: \qquad \mathbf{S}_p := \nu|\mathbf{D}|^{p-2}\mathbf{D}\,,$

$(ii)$ $\quad 1 < p < 2: \qquad \mathbf{S}_p := \begin{cases} \mathbf{0} & \text{if } \ \mathbf{D} = \mathbf{0}\,, \\ \nu|\mathbf{D}|^{p-2}\mathbf{D} & \text{if } \ \mathbf{D} \neq \mathbf{0}\,, \end{cases}$

and the fluid is called

$$\begin{array}{ll} \text{dilatant} & \text{if } \ 2 < p < +\infty\,, \\ \text{Newtonian} & \text{if } \ p = 2\,, \\ \text{pseudo-plastic} & \text{if } \ 1 < p < 2\,. \end{array}$$

The above power law model is widely used in chemical engineering to express an elementary non-Newtonian behavior of an incompressible fluid (see, e.g., [10] for a

[1] For $\mathbf{A} = \{A_{ij}\}$, $\mathbf{B} = \{B_{ij}\}$ $(i, j = 1, \ldots, n)$ we define $\mathbf{A} : \mathbf{B} = A_{ij}B_{ij}$, $|A| = (A_{ij}A_{ij})^{1/2}$ (repeated indices imply summation over $1, \ldots, n$). Notice that $\mathbf{D} = \mathbf{D}(\mathbf{u}) = \{D_{ij}(\mathbf{u})\}$, $D_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ $(i, j = 1, \ldots, n)$.

discussion of these fluids from the chemical engineering point of view).

(II) Bingham plastics. (Bingham, Green [9]; Bingham [8])

Let $g = const > 0$ be fixed. For $\nu > 0$, we consider the constitutive law

$$(1.4) \qquad \begin{cases} |\mathbf{S}_{\nu,g}| < g & \text{if} \quad \mathbf{D} = \mathbf{0}\,, \\ \mathbf{S}_{\nu,g} = \left(\nu + \dfrac{g}{|\mathbf{D}|}\right)\mathbf{D} & \text{if} \quad \mathbf{D} \neq \mathbf{0}\,. \end{cases}$$

This constitutive law characterizes an incompressible, visco-plastic fluid, where $g$ is its yield value and $\nu$ its viscosity. The properties of such a fluid can be possibly easier understood by considering the following equivalent formulation of (1.4):

$$(1.5) \qquad \begin{cases} \text{if} \quad |\mathbf{S}_{\nu,g}| < g\,, \quad \text{then} \quad \mathbf{D} = \mathbf{0}\,, \\ \text{if} \quad |\mathbf{S}_{\nu,g}| \geq g\,, \quad \text{then} \quad \mathbf{D} = \dfrac{1}{\nu}\left(1 - \dfrac{g}{|\mathbf{S}_{\nu,g}|}\right)\mathbf{S}_{\nu,g} \end{cases}$$

(cf. also [18], [11], Chap. VI, §1). Thus, inside of the region where $|\mathbf{S}_{\nu,g}| < g$ only rigid motions of the continuum are possible (i.e. the continuum moves as a "*plug of solid*"). On the other hand, if $|\mathbf{S}_{\nu,g}| > g$, then the continuum moves as a Newtonian fluid with $|\mathbf{S}_{\nu,g}| = \nu|\mathbf{D}| + g$ (i.e. $g$ is the value of activation of the viscous flow).

The formal passage to the limit $p \to 1$ in $(ii)$ gives

$$\mathbf{S}_1 = \mathbf{0} \text{ if } \mathbf{D} = \mathbf{0}\,, \qquad \mathbf{S}_1 = \frac{g}{|\mathbf{D}|}\mathbf{D} \text{ if } \mathbf{D} \neq \mathbf{0}\,,$$

while $\nu \to 0$ in (1.4) leads to

$$|\mathbf{S}_{0,g}| < g \text{ if } \mathbf{D} = \mathbf{0}\,, \qquad \mathbf{S}_{0,g} = \frac{g}{|\mathbf{D}|}\mathbf{D} \text{ if } \mathbf{D} \neq \mathbf{0}\,.$$

These limit cases can be viewed as motivation for the following constitutive law.

(III) Perfectly plastic fluids.

Let $g = const > 0$ be fixed. We define $\mathbf{S}$ in terms of $\mathbf{D}$ by

$$(1.6) \qquad |\mathbf{S}| < g \text{ if } \mathbf{D} = \mathbf{0}\,, \qquad \mathbf{S} = \frac{g}{|\mathbf{D}|}\mathbf{D} \text{ if } \mathbf{D} \neq \mathbf{0}\,.$$

With a slightly different notation, this constitutive law has been introduced by von Mises [24] for the first time (cf. also [15], [18]). Incompressible continua which obey (1.6) are also called “von Mises solids”. These continua cannot withstand deviatoric stresses $\mathbf{S}$ such that $|\mathbf{S}| > g$. We notice that (1.6) is equivalent to

$$\begin{cases} \text{if} \quad |\mathbf{S}| < g\ , \quad \text{then} \quad \mathbf{D} = \mathbf{0}\ , \\ \text{if} \quad |\mathbf{S}| = g\ , \quad \text{then} \quad \exists\, \lambda \geq 0 : \quad \mathbf{D} = \lambda \mathbf{S} \end{cases} \tag{1.7}$$

(cf., e.g., [11], Chap VI, §1, Remark 1.2).

# 2. Minimization problems

We start from the above power law model. Given any vector field $\mathbf{u} = (u_1, \ldots, u_n)$ in $\Omega$, we define $\mathbf{S}_p = \mathbf{S}_p(\mathbf{D}(\mathbf{u}))$ according to $(i)$ resp. $(ii)$ above. Inserting $\mathbf{S}_p$ in place of $\mathbf{S}$ into (1.2) we obtain

$$-\nu \nabla \cdot \Big( |\mathbf{D}(\mathbf{u})|^{p-2} \mathbf{D}(\mathbf{u}) \Big) + \nabla P = \mathbf{f}\ . \tag{1.2'}$$

If $1 < p < 2$, this system of PDEs has to be considered within the set

$$\{x \in \Omega :\ \mathbf{D}(\mathbf{u}(x)) \neq \mathbf{0}\}\ .$$

From now on we continue our discussion for any space dimension $n \geq 2$. We then consider the weak formulation of (1.1), (1.2′), (1.3) and introduce the equivalent minimization problem. For notational simplicity, we restrict our discussion to the case $1 < p < n$. The passage to the limit $p \to 1$ leads in a natural way to the minimization problem with (1.2′) for the case $p = 1$ (cf. also (1.6) above). ■

Let $W^{1,r}(\Omega)$ denote the usual Sobolev space. We set

$$W_0^{1,r}(\Omega) := \Big\{ u \in W^{1,r}(\Omega) :\ u = 0 \text{ a.e. on } \partial\Omega \Big\}$$

and [2]

$$\mathbf{W}_{0,\mathrm{div}}^{1,r}(\Omega) := \Big\{ \mathbf{u} \in \mathbf{W}_0^{1,r}(\Omega) :\ \operatorname{div} \mathbf{u} = 0 \text{ a.e. in } \Omega \Big\}\ .$$

We introduce the following

**Definition 2.1** *Let $t = np/(n-p)$ and $\mathbf{f} \in \mathbf{L}^{t'}(\Omega)$ [3] $(1 < p < n)$. Then $\mathbf{u} \in \mathbf{W}_{0,\mathrm{div}}^{1,p}(\Omega)$ is called a weak solution to (1.1), (1.2) (with $\mathbf{S} = \mathbf{S}_p$), (1.3) if* [4]

$$\nu \int_\Omega |\mathbf{D}(\mathbf{u})|^{p-2} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v})\, \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}x \qquad \forall\, \mathbf{v} \in \mathbf{W}_{0,\mathrm{div}}^{1,p}(\Omega)\ . \tag{2.1}$$

[2] By bold capitals we denote spaces of functions with values in $\mathbb{R}^n$, i.e. $\mathbf{L}^p(\Omega) = L^p(\Omega; \mathbb{R}^n)$ etc.
[3] By $t' = t/(t-1)$ we denote the conjugate of $1 < t < \infty$.
[4] Cf. footnote 1 for the notation $\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v})$.

The theory of monotone operators from a reflexive Banach space into its dual implies that for every $\mathbf{f} \in \mathbf{L}^{np/(n-p)}(\Omega)$ there exists exactly one solution $\mathbf{u} \in \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)$ to (2.1).

**Remark 2.1** *Let* $\mathbf{u} \in \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)$ *satisfy (2.1). Then there exists* $P \in L^{p/(p-1)}(\Omega)$ *such that*

$$\begin{aligned} \int_\Omega P \,\mathrm{d}x &= 0 \,, \\ \nu \int_\Omega |\mathbf{D}(\mathbf{u})|^{p-2}\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}) \,\mathrm{d}x &= \int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x + \int_\Omega P \operatorname{div}\mathbf{w}\,\mathrm{d}x \qquad \forall\, \mathbf{w} \in \mathbf{W}^{1,p}_0(\Omega) \,. \end{aligned}$$

*This follows from [13], Chap. III, or [20], Chap. II.*

As above, let $t = np/(n-p)$ and $\mathbf{f} \in \mathbf{L}^{t'}(\Omega)$ $(1 < p < n)$. Given $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)$, we define

$$\mathcal{F}_p(\mathbf{v}) := \frac{\nu}{p}\int_\Omega |\mathbf{D}(\mathbf{v})|^p \,\mathrm{d}x - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x \,.$$

This functional is continuous and strictly convex. Korn's inequality implies that $\mathcal{F}_p$ is coercive.

Let us consider the problem

$$(\mathcal{P}_p) \qquad\qquad \textit{minimize } \mathcal{F}_p \textit{ over } \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega) \,.$$

By standard arguments, one readily proves the existence and uniqueness of a minimizer $\mathbf{u}_p \in \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)$ for $\mathcal{F}_p$. Moreover, the following equivalence is valid:

$$\mathbf{u}_p \in \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega) \textit{ fulfills } (2.1) \quad \Leftrightarrow \quad \mathcal{F}_p(\mathbf{u}_p) = \min_{\mathbf{v}\in\mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)} \mathcal{F}_p(\mathbf{v}) \,.$$

■

Now let us consider the minimum problem $(\mathcal{P}_p)$ for the case $p = 1$. Since Korn's inequality fails in $\mathbf{W}^{1,1}_0(\Omega)$ we introduce the space [5]

$$\mathbf{LD}(\Omega) := \left\{ \mathbf{u} \in \mathbf{L}^1(\Omega) :\; D_{ij}(\mathbf{u}) \in L^1(\Omega),\; i,j = 1,\ldots,n \right\} .$$

It is easily seen that $\mathbf{LD}(\Omega)$ is a Banach space with respect to the norm

$$\|\mathbf{u}\|_{\mathbf{LD}(\Omega)} := \|\mathbf{u}\|_{\mathbf{L}^1(\Omega)} + \sum_{i,j=1}^n \|D_{ij}(\mathbf{u})\|_{L^1(\Omega)} \,.$$

[5] Cf. footnote 1.

Clearly, $\mathbf{W}^{1,1}(\Omega) \subset \mathbf{LD}(\Omega)$ (proper inclusion). We notice that the embedding theorems and the trace theorem for $\mathbf{W}^{1,1}(\Omega)$ continue to hold for $\mathbf{LD}(\Omega)$ (see [23], pp. 20-21, 114-137, for details).
Next we define

$$\mathbf{LD}_{0,\mathrm{div}}(\Omega) := \big\{ \mathbf{u} \in \mathbf{LD}(\Omega) : \ \mathrm{div}\, \mathbf{u} = 0 \text{ a.e. in } \Omega, \ \mathbf{u} = \mathbf{0} \text{ a.e. on } \partial\Omega \big\} .$$

This space is the "natural energy space" for the functional $\mathcal{F}_1$.
Let $\nu = \mathit{const} > 0$, and let $\mathbf{f} \in \mathbf{L}^n(\Omega)$. For $\mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega)$ we define [6]

$$\mathcal{F}(\mathbf{v}) \equiv \mathcal{F}_1(\mathbf{v}) = \nu \int_\Omega |\mathbf{D}(\mathbf{v})| \, \mathrm{d}x - \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x .$$

We consider the minimization problem

$$(\mathcal{P}) \qquad \textit{minimize } \mathcal{F} \textit{ over } \mathbf{LD}_{0,\mathrm{div}}(\Omega) .$$

Since $\mathcal{F}(\mathbf{0}) = 0$, the following two alternatives hold for $(\mathcal{P})$:

$$(1^o) \qquad \inf_{\mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega)} \mathcal{F}(\mathbf{v}) = 0 ;$$

$$(2^o) \qquad \inf_{\mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega)} \mathcal{F}(\mathbf{v}) = -\infty .$$

Clearly, $(1^o)$ is equivalent to

$$\left| \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x \right| \le \nu \int_\Omega |\mathbf{D}(\mathbf{v})| \, \mathrm{d}x \qquad \forall\, \mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega) . \tag{2.2}$$

This inequality represents a "safe load condition" on the pair $(\nu, \mathbf{f})$ (cf. [23] for details within the field of perfectly plastic solids). If (2.2) is satisfied, then a concept of weak solution to (1.1), (1.2) (with $\mathbf{S}$ as in (1.6), cf. Section 1.), (1.3) can be introduced. A detailed discussion will appear in [7] and [17].

On the other hand, $(2^o)$ is equivalent to

$$\exists\, \tilde{\mathbf{v}} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega) : \ \ \mathcal{F}(\tilde{\mathbf{v}}) < 0 .$$

**Remark 2.2** *Let $\nu = const > 0$ and let $\mathbf{f} \in \mathbf{L}^n(\Omega)$ satisfy (2.2). Let $\mathbf{u}_p \in \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)$ ($1 < p < n/(n-1)$) verify (2.1). Choosing $\mathbf{v} = \mathbf{u}_p$ in (2.1) and applying Young's inequality we obtain for all* $1 < p < n/(n-1)$

$$\mathcal{F}(\mathbf{u}_p) - \nu\Big(1 - \frac{1}{p}\Big) \operatorname{meas} \Omega \ \le \ \mathcal{F}_p(\mathbf{u}_p) \ = \ \Big(\frac{1}{p} - 1\Big) \int_\Omega \mathbf{f} \cdot \mathbf{u}_p \, \mathrm{d}x . \tag{2.3}$$

[6] Here, the constant $\nu$ represents the yield value of the perfectly plastic fluid under consideration (cf. the constitutive laws (1.4) and (1.6) of Section 1., where this value was denoted by $g$).

*Hence*

$$\nu \int_\Omega |\mathbf{D}(\mathbf{u}_p)|\,\mathrm{d}x - \nu\Big(1-\frac{1}{p}\Big)\,\mathrm{meas}\,\Omega \;\le\; \frac{1}{p}\int_\Omega \mathbf{f}\cdot\mathbf{u}_p\,\mathrm{d}x \;\le\; \frac{\nu}{p}\int_\Omega |\mathbf{D}(\mathbf{u}_p)|\,\mathrm{d}x$$

*and in conclusion*

$$\int_\Omega |\mathbf{D}(\mathbf{u}_p)|\,\mathrm{d}x \le \mathrm{meas}\;\Omega\;. \tag{2.4}$$

*Moreover, from (2.3) and (2.4) it follows that*

$$\lim_{p\to 1}\mathcal{F}(\mathbf{u}_p) = \lim_{p\to 1}\mathcal{F}_p(\mathbf{u}_p) = 0\;.$$

# 3. The space $\mathbf{BD}(\Omega)$

## 3.1 Basic notions

Definition of $\mathbf{BD}(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We introduce the following notations:

$$\begin{aligned} \mathcal{B}(\Omega) &:= \sigma\text{-algebra of all sets } A = B\cap\Omega,\ B\subset\mathbb{R}^n \text{ Borel}\;,\\ \mathcal{M}(\Omega) &:= \text{set of all signed measures } m \text{ defined on } \mathcal{B}(\Omega) \text{ such that } |m|(\Omega) < +\infty\;. \end{aligned}$$

Next, given $\mathbf{u} = (u_1,\dots,u_n) \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$, we identify $u_i$ with a distribution on $\Omega$ and denote by $\partial_j u_i$ its partial derivative with respect to $x_j$ in the sense of distributions. Then we consider the distributions

$$D_{ij}(\mathbf{u}) := \frac{1}{2}(\partial_i u_j + \partial_j u_i)\;, \qquad i,j = 1,\dots,n\;,$$

and introduce the following

**Definition 3.1** *The space* $\mathbf{BD}(\Omega)$ *is given by*

$$\mathbf{BD}(\Omega) := \Big\{ \mathbf{u}\in\mathbf{L}^1(\Omega) :\ \forall\, i,j\in\{1,\dots,n\}\ \exists\, \mu_{ij}\in\mathcal{M}(\Omega)\ \textit{s.t.}\ \langle D_{ij}(\mathbf{u}),\varphi\rangle = \int_\Omega \varphi\,\mathrm{d}\mu_{ij}\ \forall\,\varphi\in C_c^\infty(\Omega)\Big\}\;.$$

Clearly, the space $\mathbf{LD}(\Omega)$ can be identified with a subspace of $\mathbf{BD}(\Omega)$, namely, for $\mathbf{u}\in\mathbf{LD}(\Omega)$ there holds

$$\langle D_{ij}(\mathbf{u}),\varphi\rangle = \int_\Omega D_{ij}(\mathbf{u})\varphi\,\mathrm{d}x \quad \forall\,\varphi\in C_c^\infty(\Omega)\;,$$

where

$$\mu_{ij}(A) = \int_A D_{ij}(\mathbf{u})\,\mathrm{d}x \qquad \forall\, A \in \mathcal{B}(\Omega)\ .$$

The elements of $\mathbf{BD}(\Omega)$ are called *vector fields of bounded deformation.* This space has been introduced in [21], [22] for the study of mathematical problems in the theory of plasticity of solids. It has been also introduced in [16]. We refer the reader to [2], [12] and [23] for the discussion of the basics of $\mathbf{BD}(\Omega)$. Fine properties of the elements of this space have been investigated in [1], [14].

Our motivation for the use of the space $\mathbf{BD}(\Omega)$ is the study of the limit case $p = 1$ of a power law fluid (cf. Sections 1. and 2.) as well as the vanishing viscosity limit of a Bingham plastic (see [7] and [17], respectively), where we recall that in our discussion the vector field $\mathbf{u}$ represents the velocity of an incompressible fluid. Therefore, in Section 3.2 we consider the subspace of those $\mathbf{u} \in \mathbf{BD}(\Omega)$ such that $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$ (in a sense to be specified). ∎

Let $\mathbf{u} \in \mathbf{BD}(\Omega)$ and let $\mu_{ij} \in \mathcal{M}(\Omega)$ $(i, j = 1, \dots, n)$ be as in Definition 3.1. We define

$$\mu := \begin{pmatrix} \mu_{11} & \dots & \mu_{1n} \\ \vdots & & \vdots \\ \mu_{n1} & \dots & \mu_{nn} \end{pmatrix}\ . \tag{3.1}$$

Then $\mu$ is an $\mathbb{R}^{n^2}$-valued measure on $\mathcal{B}(\Omega)$ [7]. For $A \in \mathcal{B}(\Omega)$ we set

$$\begin{aligned} \|\mu\|(A) \;&:=\; \left\{ \sum_{k=1}^{\infty} \|\mu(A_k)\|_{\mathbb{R}^{n^2}} :\ A_k \in \mathcal{B}(\Omega) \text{ pairwise disjoint},\ A = \bigcup_{k=1}^{\infty} A_k \right\} \\ &=\; \text{total variation of } \mu\ . \end{aligned} \tag{3.2}$$

It is well known that $\|\mu\|$ is a finite measure on $\mathcal{B}(\Omega)$. ∎

[7] For sequences $(a_{ij,k})_{k\in\mathbb{N}} \subset \mathbb{R}$ we define

$$\lim_{k\to\infty} \begin{pmatrix} a_{11,k} & \dots & a_{1n,k} \\ \vdots & & \vdots \\ a_{n1,k} & \dots & a_{nn,k} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

in the sense of the Euclidian norm in $\mathbb{R}^{n^2}$.

Equivalent characterization of $\mathbf{BD}(\Omega)$.

Let $\mathbf{u} \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$. For open sets $A \subset \Omega$ we define

$$|\mathbf{D}(\mathbf{u})|(A) := \sup \left\{ \frac{1}{2} \int_A (u_i \partial_j \varphi_{ij} + u_j \partial_i \varphi_{ij}) \, \mathrm{d}x : \ \varphi_{ij} \in C^1_c(A), \ |\varphi| \le 1 \text{ in } A \right\} .$$

It is easily seen that for every $\mathbf{u} \in \mathbf{LD}(\Omega)$

$$|\mathbf{D}(\mathbf{u})|(\Omega) = \int_\Omega |\mathbf{D}(\mathbf{u})| \, \mathrm{d}x \, . \tag{3.3}$$

By routine arguments, we obtain

**Proposition 3.1** *1)* $\mathbf{BD}(\Omega) = \{\mathbf{u} \in \mathbf{L}^1(\Omega) : \ |\mathbf{D}(\mathbf{u})|(\Omega) < +\infty\}$*;*

*2) for every* $\mathbf{u} \in \mathbf{BD}(\Omega)$ *there holds*

$$\|\mu\|(A) = |\mathbf{D}(\mathbf{u})|(A) \quad \forall\, A \subset \Omega \text{ open}$$

*($\mu$ and $\|\mu\|$ as in (3.1) and (3.2), respectively).*

Some basic results.

From the definition of $|\mathbf{D}(\mathbf{u})|$ one easily deduces

(3.4) *if* $u_k \rightharpoonup u$ *in* $\mathbf{L}^1(\Omega)$ *as* $k \to \infty$, *then* $|\mathbf{D}(\mathbf{u})|(\Omega) \le \liminf_{k\to\infty} |\mathbf{D}(\mathbf{u}_k)|(\Omega)$ .

Clearly, $\mathbf{BD}(\Omega)$ is a Banach space with respect to the norm

$$\|\mathbf{u}\|_{\mathbf{BD}(\Omega)} := \|\mathbf{u}\|_{\mathbf{L}^1(\Omega)} + |\mathbf{D}(\mathbf{u})|(\Omega) \, .$$

We present some results which will be used below.

**Proposition 3.2** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz boundary* $\partial\Omega$*. Then*

*1)* $\mathbf{BD}(\Omega) \subset \mathbf{L}^p(\Omega)$ *compactly if* $1 \le p < n/(n-1)$*, continuously if* $p = n/(n-1)$*.*

*2) (Existence of a trace) There exits a linear mapping* $\boldsymbol{\gamma}$*:* $\mathbf{BD}(\Omega) \to \mathbf{L}^1(\partial\Omega)$ *such that*

$$\boldsymbol{\gamma}(\mathbf{u}) = \mathbf{u}_{|\partial\Omega} \quad \forall\, \mathbf{u} \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{BD}(\Omega)$$

*and*

$$\int_{\partial\Omega} |\boldsymbol{\gamma}(\mathbf{u})| \, \mathrm{d}\mathcal{H}^{n-1} \le \|\mathbf{u}\|_{\mathbf{BD}(\Omega)} \, . \tag{3.5}$$ [8]

[8] Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. A detailed discussion of measure and integration on $k$-dimensional Lipschitz-manifolds in $\mathbb{R}^n$ $(1 \le k < n)$ can be found in: Naumann, J.; Simader, C.G., *Measure and integration on Lipschitz-manifolds. Preprint 2007-15, Inst. f. Math. Humboldt-Univ. Berlin (2007).* Available at: *http://www.mathematik.hu-berlin.de/publ/pre/2007/P-07-15.pdf.*

*3) Let $B \subset \mathbb{R}^n$ be an open ball such that $\overline{\Omega} \subset B$. For $\mathbf{u} \in \mathbf{BD}(B)$ define*

$$\begin{array}{rclcl} \mathbf{u}^+ & := & \gamma^+(\mathbf{u}_{|\Omega}) & = & \textit{trace of } \mathbf{u}_{|\Omega} \textit{ on } \partial\Omega \,, \\ \mathbf{u}^- & := & \gamma^-(\mathbf{u}_{|B\setminus\overline{\Omega}}) & = & \textit{trace of } \mathbf{u}_{|B\setminus\Omega} \textit{ on } \partial\Omega \,. \end{array}$$

*Then*

$$|\mathbf{D}(\mathbf{u})|(\partial\Omega) = \int_{\partial\Omega} |\boldsymbol{\tau}(\mathbf{u}^+ - \mathbf{u}^-)| \, \mathrm{d}\mathcal{H}^{n-1} \,, \tag{3.6}$$

*where*

$$\boldsymbol{\tau}(\xi) = \{\tau_{ij}(\xi)\}_{1\le i,j\le n} \,, \quad \tau_{ij}(\xi) = \frac{1}{2}(\xi_i n_j + \xi_j n_i) \,, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

*and where $\mathbf{n} = (n_1, \dots, n_n)$ denotes the unit outward normal along $\partial\Omega$.*

The proofs of these results can be found in [2], [12], [23]. We notice that besides (3.5), the trace mapping $\boldsymbol{\gamma}$ obeys the following continuity property:

*for every sequence $(\mathbf{u}_k)_{k\in\mathbb{N}} \subset \mathbf{BD}(\Omega)$ such that*

*$\mathbf{u}_k \to \mathbf{u}$ in $\mathbf{L}^1(\Omega)$, $|\mathbf{D}(\mathbf{u}_k)|(\Omega) \to |\mathbf{D}(\mathbf{u})|(\Omega)$ as $k \to \infty$*

*there holds*

*$\boldsymbol{\gamma}(\mathbf{u}_k) \to \boldsymbol{\gamma}(\mathbf{u})$ in $\mathbf{L}^1(\partial\Omega)$ as $k \to \infty$*

(cf. [23], pp. 160-162).

## 3.2 The space $\mathbf{BD}_{\mathrm{div}}(\Omega)$

Let $\partial\Omega$ be of class $C^1$. Observing the embedding $\mathbf{BD}(\Omega) \subset \mathbf{L}^{n/(n-1)}(\Omega)$ (cf. Proposition 3.2, Proposition 3.1), we define

$$\mathbf{BD}_{\mathrm{div}}(\Omega) := \left\{ \mathbf{u} \in \mathbf{BD}(\Omega) : \int_\Omega \mathbf{u} \cdot \nabla\varphi \, \mathrm{d}x = 0 \;\; \forall \varphi \in W^{1,n}(\Omega) \right\} .$$

Of course, $\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)$ implies

$$\int_\Omega \mathbf{u} \cdot \nabla\zeta \, \mathrm{d}x = 0 \quad \forall\, \zeta \in C_c^\infty(\Omega) \,, \tag{3.7}$$

i.e. $\operatorname{div} \mathbf{u} = 0$ in the sense of distributions in $\Omega$. We notice that (3.7) is, however, not sufficient to prove (4.8) below.

To get an information on the normal component of $\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)$ along the boundary of $\partial\Omega$ we consider the space

$$\tilde{\mathbf{W}}^q(\Omega) := \{\mathbf{u} \in \mathbf{L}^q(\Omega) : \operatorname{div} \mathbf{u} \in L^q(\Omega)\} \,, \qquad 1 < q < +\infty \,.$$

Clearly, $\tilde{\mathbf{W}}^q(\Omega)$ is a Banach space with respect to the norm

$$\|\mathbf{u}\|_{\tilde{\mathbf{W}}^q} := \|\mathbf{u}\|_{\mathbf{L}^q} + \|\operatorname{div}\mathbf{u}\|_{L^q} \ .$$

From [13], pp. 113-115, or [19], Theorem 5.3, it follows that there exists a linear continuous and surjective mapping

$$\gamma_{\mathbf{n}} : \ \tilde{\mathbf{W}}^q(\Omega) \to W^{-1/q\,,q}(\partial\Omega)$$

such that

$$\gamma_{\mathbf{n}}(\mathbf{u}) = (\mathbf{u}_{|\partial\Omega}) \cdot \mathbf{n} \quad \ \forall\, \mathbf{u} \in \mathbf{C}^1(\overline{\Omega}) \ ,$$

where $\mathbf{n} = (n_1, \ldots, n_n)$ again denotes the unit outward normal along $\partial\Omega$ (for the case $\tilde{\mathbf{W}}^2(\Omega)$ see also [20], p. 83 ($\partial\Omega$ Lipschitz), and [23], pp. 13-14 ($\partial\Omega$ of class $C^2$)). Moreover, for all $\mathbf{u} \in \tilde{\mathbf{W}}^q(\Omega)$ and all $\varphi \in W^{1,q'}(\Omega)$ there holds the generalized Gauß-Green formula

$$\int_\Omega \mathbf{u} \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega (\operatorname{div}\mathbf{u})\varphi \,\mathrm{d}x = \langle \gamma_{\mathbf{n}}(\mathbf{u}), \varphi \rangle \ ,$$

where the bracket on the right-hand side denotes the dual pairing between $\gamma_{\mathbf{n}}(\mathbf{u}) \in W^{-1/q\,,q}(\partial\Omega)$ and (with a slight abuse of notation) the trace of $\varphi$ (cf. [19], Theorem 5.3).

Now, the above mentioned embedding of $\mathbf{BD}(\Omega)$ gives $\mathbf{BD}_{\mathrm{div}}(\Omega) \subset \tilde{\mathbf{W}}^{n/(n-1)}(\Omega)$. From (3.7) and the generalized Gauß-Green formula it follows that $\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)$ verifies

$$\langle \gamma_{\mathbf{n}}(\mathbf{u}), \varphi \rangle = 0 \quad \ \forall\, \varphi \in W^{1,n}(\Omega) \ .[9] \tag{3.8}$$

Conversely, if $\mathbf{u} \in \mathbf{BD}(\Omega)$ verifies (3.7) and (3.8) then $\mathbf{u}$ is of class $\mathbf{BD}_{\mathrm{div}}(\Omega)$.

We thus obtain the following equivalent definition

$$\mathbf{BD}_{\mathrm{div}}(\Omega) = \{\mathbf{u} \in \mathbf{BD}(\Omega) : \ \mathbf{u} \text{ satisfies (3.7) and (3.8)}\} \ .$$

■

# 4. An approximation theorem for $\mathbf{BD}_{\mathrm{div}}(\Omega)$

The main result of our paper is the following

[9] The surjectivity of the trace mapping of $W^{1,n}(\Omega)$ implies that (3.8) is equivalent to

$$\langle \gamma_{\mathbf{n}}(\mathbf{u}), \chi \rangle = 0 \quad \forall\, \chi \in W^{(n-1)/n,n}(\partial\Omega) \ .$$

**Theorem 4.1** *Let $\Omega \subset \mathbb{R}^n$ be a bounded star-shaped domain with Lipschitz boundary $\partial\Omega$. Then, for every $\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)$ there exists a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}} \subset \mathbf{C}_c^\infty(\Omega)$ such that*

$$\begin{aligned} &(4.1) \qquad & \operatorname{div} \mathbf{u}_k &= 0 \quad \text{in } \Omega \text{ and } \forall\, k \in \mathbb{N}\,, \\ &(4.2) \qquad & \mathbf{u}_k &\to \mathbf{u} \quad \text{in } \mathbf{L}^{n/(n-1)}(\Omega) \text{ as } k \to \infty\,, \\ &(4.3) \qquad & \int_\Omega |\mathbf{D}(\mathbf{u}_k)|\,\mathrm{d}x &\to |\mathbf{D}(\mathbf{u})|(\Omega) + \int_{\partial\Omega} |\boldsymbol{\tau}(\boldsymbol{\gamma}(\mathbf{u}))|\,\mathrm{d}\mathcal{H}^{n-1} \quad \text{as } k \to \infty\,. \end{aligned}$$

We notice that density theorems for $\mathbf{BD}(\Omega)$ are proved in [3], [12] and [23]. ■

Before turning to the proof we make some

Preliminaries.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which is star-shaped with respect to a point $x_0 \in \Omega$ (without loss of generality we may assume that $x_0 = 0$). It follows

1) if $t > t' > 1$, then $\frac{1}{t}\Omega \Subset \frac{1}{t'}\Omega \Subset \Omega$,

2) if $(t_k)$ is any sequence of reals such that $t_k > t_{k+1} > \cdots > 1$ and $\lim_{k\to\infty} t_k = 1$, then
$$\bigcup_{k=1}^{\infty} \left(\frac{1}{t_k}\Omega\right) = \Omega\,.$$

Next, for $u \in L^p(\Omega)$ $(1 \le p < +\infty)$, define

$$\tilde{u}(x) := \begin{cases} u(x) & \text{for a.e. } x \in \Omega\,, \\ 0 & \text{for a.e. } x \in \mathbb{R}^n\backslash\Omega\,, \end{cases}$$

$$\tilde{u}_t(x) := \tilde{u}(tx) \quad \text{for } t > 1\,, \ \text{ for a.e. } x \in \mathbb{R}^n\,.$$

Then $\tilde{u}_t(x) = 0$ for a.e. $x \in \mathbb{R}^n\backslash\big(\frac{1}{t}\Omega\big)$. If $(t_k)$ is any sequence of reals such that $t_k > t_{k+1} > \cdots > 1$ and $\lim_{k\to\infty} t_k = 1$, then by 2)

$$\int_{\Omega\backslash\left(\frac{1}{t_k}\Omega\right)} |u(x)|^p\,\mathrm{d}x \to 0 \quad \text{as } k \to \infty$$

and therefore

$$(4.4) \qquad \int_\Omega |\tilde{u}_{t_k} - u|^p\,\mathrm{d}x = \int_{\frac{1}{t_k}\Omega} |u(t_k x) - u(x)|^p\,\mathrm{d}x + \int_{\Omega\backslash\left(\frac{1}{t_k}\Omega\right)} |u(x)|^p\,\mathrm{d}x \to 0$$

as $k \to \infty$.

Let $\omega \in C^\infty(\mathbb{R}^n)$ be a (fixed) function such that

$$\begin{aligned} &0 \le \omega(x) \le const\ , \qquad \omega(x) = \omega(-x)\ \ \forall\, x \in \mathbb{R}^n\ , \\ &\mathrm{supp}(\omega) \subset \overline{B_1(0)}\ , \qquad \int_{B_1(0)} \omega \,\mathrm{d}x = 1\ . \end{aligned}$$

For $\rho > 0$ and $x \in \mathbb{R}^n$ define

$$\begin{aligned} \omega_\rho(x) &:= \frac{1}{\rho^n}\omega\Big(\frac{x}{\rho}\Big)\ , \\ (\tilde{u}_t)_\rho(x) &:= \int_{\mathbb{R}^n} \omega_\rho(x-y)\tilde{u}_t(y)\,\mathrm{d}y = \int_{B_\rho(x)\cap\left(\frac{1}{t}\Omega\right)} \omega_\rho(x-y)u(ty)\,\mathrm{d}y\ . \end{aligned}$$

Clearly, $(\tilde{u}_t)_\rho \in C^\infty(\mathbb{R}^n)$. Set

$$d_t := \frac{1}{2}\,\mathrm{dist}\,\Big(\frac{1}{t}\Omega, \partial\Omega\Big)$$

(recall $t > 1$). Then, for every $0 < \rho < d_t$ and $x \in \Omega$ with $\mathrm{dist}(x, \partial\Omega) < d_t$ we have $\left(\frac{1}{t}\Omega\right) \cap B_\rho(x) = \emptyset$. Thus

$$(\tilde{u}_t)_\rho(x) = 0 \quad \forall\, 0 < \rho < d_t\ , \quad \forall x \in \Omega \ \text{ with }\ \mathrm{dist}(x, \partial\Omega) < d_t\ . \tag{4.5}$$

■

Now, let $\mathbf{u} \in \mathbf{L}^p(\Omega)$ $(1 < p < +\infty)$ satisfy

$$\int_\Omega \mathbf{u}\cdot\nabla\varphi\,\mathrm{d}x = 0 \quad \forall\varphi \in C^\infty(\mathbb{R}^n)\ . \tag{4.6}$$

As above, let $t > 1$. The substitution $\xi = tx$ $(x \in \mathbb{R}^n)$ implies $\xi \in \Omega$ iff $x \in \frac{1}{t}\Omega$. Next, for any $\varphi \in C^\infty(\mathbb{R}^n)$, define

$$\psi(\xi) := \varphi\Big(\frac{\xi}{t}\Big)\ , \qquad \xi \in \mathbb{R}^n\ .$$

Observing the definition of $\mathbf{BD}_{\mathrm{div}}(\Omega)$, from (4.6) it follows

$$\int_\Omega \tilde{\mathbf{u}}_t\cdot\nabla\varphi\,\mathrm{d}x = \int_{\frac{1}{t}\Omega} \mathbf{u}(tx)\cdot\nabla\varphi(x)\,\mathrm{d}x = t^{1-n}\int_\Omega \mathbf{u}(\xi)\cdot\nabla\psi(\xi)\,\mathrm{d}\xi = 0\ . \tag{4.7}$$

Moreover, we consider the mollification $(\tilde{\mathbf{u}}_t)_\rho$ (componentwise). It follows

$$\mathrm{div}(\tilde{\mathbf{u}}_t)_\rho(x) = 0 \quad \forall\, 0 < \rho < d_t\ ,\ \forall\, x \in \Omega\ . \tag{4.8}$$

Indeed, the function $\varphi$: $y \mapsto \varphi(y) = \omega_\rho(x-y)$ is admissible in (4.7) (notice that $\mathrm{supp}(\varphi)$ need not be included in $\Omega$). We find

$$\begin{aligned}
\mathrm{div}(\tilde{\mathbf{u}}_t)_\rho(x) &= \int_{\mathbb{R}^n} \mathrm{div}_x \big(\omega_\rho(x-y)\tilde{\mathbf{u}}_t(y)\big)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \big(\nabla_x \omega_\rho(x-y)\big)\cdot \tilde{\mathbf{u}}_t(y)\,\mathrm{d}y \\
&= -\int_{\mathbb{R}^n} \big(\nabla_y \omega_\rho(x-y)\big)\cdot \tilde{\mathbf{u}}_t(y)\,\mathrm{d}y \\
&= 0\,.
\end{aligned}$$

■

Proof of the theorem.

To begin with, we fix an open ball $B \subset \mathbb{R}^n$ such that $\overline{\Omega} \subset B$. Let $\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)$. We define $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}}_t$ $(t > 1)$ as above and consider the mollification $(\tilde{\mathbf{u}}_t)_\rho$ (componentwise). Then we fix a sequence $(t_k)$ $(k \in \mathbb{N})$ of reals such that $t_k > t_{k+1} > \cdots > 1$ and $\lim_{k\to\infty} t_k = 1$. Since $\mathbf{u} \in \mathbf{L}^{n/(n-1)}(\Omega)$ we obtain

$$\int_B |\tilde{\mathbf{u}}_{t_k} - \tilde{\mathbf{u}}|^{n/(n-1)}\,\mathrm{d}x = \int_\Omega |\tilde{\mathbf{u}}_{t_k} - \mathbf{u}|^{n/(n-1)}\,\mathrm{d}x \to 0 \qquad \text{as } k \to \infty$$

(c.f (4.4)).

Set

$$d_{t_k} := \frac{1}{2}\,\mathrm{dist}\left(\frac{1}{t_k}\Omega, \partial\Omega\right), \qquad k \in \mathbb{N}\,.$$

Clearly $\lim_{k\to\infty} d_{t_k} = 0$. Then we take $\rho_k$ such that

$$0 < \rho_k < d_{t_k}\,, \qquad \|\tilde{\mathbf{u}}_{t_k} - (\tilde{\mathbf{u}}_{t_k})_{\rho_k}\|_{\mathbf{L}^{n/(n-1)}(B)} \le \frac{1}{k}\,, \qquad k \in \mathbb{N}\,. \tag{4.9}$$

Define

$$\tilde{\mathbf{u}}_k(x) := (\tilde{\mathbf{u}}_{t_k})_{\rho_k}(x)\,,\ \ x \in \mathbb{R}^n\,, \qquad \mathbf{u}_k := \tilde{\mathbf{u}}_{k|\Omega}\,, \qquad k \in \mathbb{N}\,.$$

We obtain

$$\mathbf{u}_k \in \mathbf{C}_c^\infty(\Omega) \qquad \text{by (4.5)}\,, \tag{4.10}$$

$$\mathrm{div}\,\mathbf{u}_k = 0 \qquad \text{in } \Omega \text{ by (4.8)}\,, \tag{4.11}$$

$$\tilde{\mathbf{u}}_k \to \tilde{\mathbf{u}} \qquad \text{in } \mathbf{L}^{n/(n-1)}(B) \text{ as } k \to \infty\,. \tag{4.12}$$

Thus, as shown in (4.10)–(4.12), the sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ satisfies (4.1) and (4.2). It remains to prove (4.3). To this end, we show two inequalities.

Inequality 1.

Since $\mathbf{u}_k \in \mathbf{C}_c^\infty(\Omega)$ we have

$$\int_\Omega |\mathbf{D}(\mathbf{u}_k)|\,\mathrm{d}x = \int_B |\mathbf{D}(\tilde{\mathbf{u}}_k)|\,\mathrm{d}x = |\mathbf{D}(\tilde{\mathbf{u}}_k)|(B)$$

(cf. (3.3)). Combining (4.12) and (3.4), (3.6) ($B$ in place of $\Omega$) one finds

$$\begin{aligned}\liminf_{k\to\infty} \int_\Omega |\mathbf{D}(\mathbf{u}_k)|\,\mathrm{d}x &= \liminf_{k\to\infty} |\mathbf{D}(\tilde{\mathbf{u}}_k)|(B)\\ &\geq |\mathbf{D}(\tilde{\mathbf{u}})|(B)\\ &= |\mathbf{D}(\mathbf{u})|(\Omega) + |\mathbf{D}(\mathbf{u})|(\partial\Omega)\\ &= |\mathbf{D}(\mathbf{u})|(\Omega) + \int_{\partial\Omega} |\boldsymbol{\tau}(\boldsymbol{\gamma}(\mathbf{u}))|\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{4.13}$$

Inequality 2.

Fix $\delta > 0$ such that

$$\Omega_\delta := \{x \in \mathbb{R}^n :\ \mathrm{dist}(x, \overline{\Omega}) \leq \delta\} \subset B\,.$$

Then

$$B_\rho(x) \cap \Omega = \emptyset$$

for all $0 < \rho < \delta/3$ and all $x \in B\backslash\Omega$ such that $\mathrm{dist}(x, \partial\Omega) > 2\delta/3$. Since $t_k > 0$ and $\lim_{k\to\infty} t_k = 1$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$d_{t_k} < \frac{\delta}{3} \tag{4.14}$$

and such that for all $k \geq k_0$

$$B_\rho\Big(\frac{x}{t_k}\Big) \cap \Omega = \emptyset \quad \forall\, 0 < \rho < \frac{\delta}{3}\,,\ \ \forall\, x \in B\backslash\Omega \ \text{ such that }\ \mathrm{dist}(x, \partial\Omega) > \frac{2\delta}{3}\,. \tag{4.15}$$

Let $\varphi_{ij} \in C_c^1(\Omega)$ $(i,\ j = 1,\ \ldots,\ n)$ satisfy $|\varphi|^2 = \sum_{i,j=1}^n \varphi_{ij}^2 \leq 1$ in $\Omega$. For $\rho > 0$ and $x \in \mathbb{R}^n$ consider

$$(\varphi_{ij})_\rho(x) := \int_{\mathbb{R}^n} \omega_\rho(x-y)\varphi_{ij}(y)\,\mathrm{d}y = \int_{B_\rho(x)\cap\mathrm{supp}(\varphi_{ij})} \omega_\rho(x-y)\varphi_{ij}(y)\,\mathrm{d}y\,.$$

Then

$$\sum_{i,j=1}^n \Big((\varphi_{ij})_\rho(x)\Big)^2 \leq 1 \quad \forall\, x \in \mathbb{R}^n\,, \tag{4.16}$$

and for $i,\, j = 1,\, \ldots,\, n$

$$(\varphi_{ij})_\rho\Big(\frac{x}{t_k}\Big) = 0 \forall\, k \geq k_0\ ,\quad \forall\, 0 < \rho < \frac{\delta}{3}\ , \tag{4.17}$$
$$\forall x \in B \backslash \Omega \ \text{ such that }\ \operatorname{dist}(x, \partial\Omega) > \frac{2\delta}{3}\ .$$

Now, for every $k \geq k_0$ and $i,\, j = 1,\, \ldots,\, n$ we have

$$\begin{aligned}\int_\Omega \big(u_{ki}\partial_j\varphi_{ij} + u_{kj}\partial_i\varphi_{ij}\big)\,\mathrm{d}x &= \int_{\mathbb{R}^n}\Bigg[\Bigg(\int_\Omega \omega_{\rho_k}(x-y)\partial_j\varphi_{ij}(x)\,\mathrm{d}x\Bigg)\tilde{u}_{t_k i}(y) \\ &\quad + \Bigg(\int_\Omega \omega_{\rho_k}(x-y)\partial_i\varphi_{ij}(x)\,\mathrm{d}x\Bigg)\tilde{u}_{t_k j}(y)\Bigg]\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\Big[\big(\partial_j(\varphi_{ij})_{\rho_k}\big)(y)\tilde{u}_i(t_k y) + \big(\partial_i(\varphi_{ij})_{\rho_k}\big)(y)\tilde{u}_j(t_k y)\Big]\,\mathrm{d}y\ ,\end{aligned}$$

where we used $\omega_\rho(x-y) = \omega_\rho(y-x)$ and where an integration by parts as well as a change of integration and partial differentiation was made. Hence, recalling $\tilde{\mathbf{u}}(t_k y) = 0$ for all $y \in \mathbb{R}^n \backslash (\frac{1}{t_k}\Omega)$, we obtain

$$\begin{aligned}&\int_\Omega \big(u_{ki}\partial_j\varphi_{ij} + u_{kj}\partial_i\varphi_{ij}\big)\,\mathrm{d}x \\ &= \int_{\frac{1}{t_k}\Omega}\Big[u_i(t_k y)\big(\partial_j(\varphi_{ij})_{\rho_k}\big)(y) + u_j(t_k y)\big(\partial_i(\varphi_{ij})_{\rho_k}\big)(y)\Big]\,\mathrm{d}y \\ &= \frac{1}{t_k^n}\int_\Omega \Big[u_i(z)\big(\partial_j(\varphi_{ij})_{\rho_k}\big)\Big(\frac{z}{t_k}\Big) + u_j(z)\big(\partial_i(\varphi_{ij})_{\rho_k}\big)\Big(\frac{z}{t_k}\Big)\Big]\,\mathrm{d}z \\ &= \frac{1}{t_k^n}\int_B \Big[\tilde{u}_i(z)\big(\partial_j(\varphi_{ij})_{\rho_k}\big)\Big(\frac{z}{t_k}\Big) + \tilde{u}_j(z)\big(\partial_i(\varphi_{ij})_{\rho_k}\big)\Big(\frac{z}{t_k}\Big)\Big]\,\mathrm{d}z\ . \qquad (4.18)\end{aligned}$$

To proceed, for $z \in B$ we define

$$\zeta_{ij}^{(k)}(z) := (\varphi_{ij})_{\rho_k}\Big(\frac{z}{t_k}\Big) = \int_{B_{\rho_k}(\frac{z}{t_k})\cap \operatorname{supp}(\varphi_{ij})} \omega_{\rho_k}\Big(\frac{z}{t_k} - \xi\Big)\varphi_{ij}(\xi)\,\mathrm{d}\xi$$

$(k \geq k_0,\, i,\, j = 1,\, \ldots,\, n)$. It follows

$$\sum_{r,s=1}^n \big(\zeta_{rs}^{(k)}(z)\big)^2 \leq 1 \quad \forall\, z \in B\ ,$$

and for $i,\, j = 1,\, \ldots,\, n$ (recall that $0 < \rho_k < d_{t_k} < \delta/3$, cf. (4.9), (4.14); observe (4.16) and (4.15), (4.17))

$$\zeta_{ij}^{(k)}(z) = 0 \quad \forall\, z \in B\backslash\Omega \ \text{ such that }\ \operatorname{dist}(z, \partial\Omega) > \frac{2\delta}{3}\ .$$

Thus, $\zeta_{ij}^{(k)} \in C_c^1(B)$. Finally, by the definition of $\zeta_{ij}^{(k)}$,

$$\big(\partial_i(\varphi_{ij})_{\rho_k}\big)\Big(\frac{z}{t_k}\Big) = t_k\big(\partial_i\zeta_{ij}^{(k)}\big)(z) \qquad \forall z \in B\ .$$

Inserting this into the integral on the right-hand side of (4.18) we find for every $i$, $j = 1$, $\dots$,$n$ and for all $k \geq k_0$

$$\begin{aligned}
\frac{1}{2}\int_\Omega \big(u_{ki}\partial_j\varphi_{ij} + u_{kj}\partial_i\varphi_{ij}\big)\,\mathrm{d}x &= \frac{t_k^{1-n}}{2}\int_B \big(\tilde{u}_i\partial_j\zeta_{ij}^{(k)} + \tilde{u}_j\partial_i\zeta_{ij}^{(k)}\big)\,\mathrm{d}x \\
&\leq t_k^{1-n}\sup\left\{\frac{1}{2}\int_B \big(\tilde{u}_i\partial_j\eta_{ij} + \tilde{u}_j\partial_i\eta_{ij}\big)\,\mathrm{d}x : \right. \\
&\qquad\qquad \left. \eta_{ij} \in C_c^1(B),\ |\eta| \leq 1 \text{ in } \Omega\right\} \\
&\leq |\mathbf{D}(\tilde{\mathbf{u}})|(B)\ .
\end{aligned}$$

Thus

$$\begin{aligned}
\limsup_{k\to\infty}\int_\Omega |\mathbf{D}(\mathbf{u}_k)|\,\mathrm{d}x &= \limsup_{k\to\infty}|\mathbf{D}(\mathbf{u}_k)|(\Omega) \\
&\leq |\mathbf{D}(\tilde{\mathbf{u}})|(B) \\
&= |\mathbf{D}(\mathbf{u})|(\Omega) + \int_{\partial\Omega} |\boldsymbol{\tau}(\gamma(\mathbf{u}))|\,\mathrm{d}\mathcal{H}^{n-1}\ . \qquad (4.19)
\end{aligned}$$

Combining (4.13) and (4.19) we obtain (4.3). ∎

# 5. The safe load condition in $\mathbf{BD}_{\mathrm{div}}(\Omega)$

Let $\nu = \mathit{const} > 0$ and let $\mathbf{f} \in \mathbf{L}^n(\Omega)$. We define

$$\mathcal{G}(\mathbf{v}) := \nu\left(|\mathbf{D}(\mathbf{v})|(\Omega) + \int_{\partial\Omega} |\boldsymbol{\tau}(\gamma(\mathbf{v}))|\,\mathrm{d}\mathcal{H}^{n-1}\right) - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x\ , \qquad \mathbf{v} \in \mathbf{BD}_{\mathrm{div}}(\Omega)\ .$$

This functional represents the relaxation of $\mathcal{F}$ with respect to $\mathbf{BD}_{\mathrm{div}}(\Omega)$ (cf. [4], pp. 417-420, 437-449; [5], pp. 174-181 and [6]).

**Remark 5.1** *To get more insight into the role of the boundary integral in the definition of $\mathcal{G}(\mathbf{v})$, let us define*

$$\xi_{\mathbf{t}} := \xi - (\xi\cdot\mathbf{n})\mathbf{n}\ , \qquad \xi \in \mathbb{R}^n$$

*($\mathbf{n} = (n_1, \ldots, n_n)$ denoting the unit outward normal along $\partial\Omega$). Then, for every $\mathbf{v} \in \mathbf{BD}_{\mathrm{div}}(\Omega) \cap \mathbf{C}(\overline{\Omega})$ we have*

$$\mathbf{v}_{|\partial\Omega} = \big(\mathbf{v}_{|\partial\Omega}\big)_{\mathbf{t}} \, .$$

*Thus, by Proposition 3.2, 2), for every $\mathbf{v} \in \mathbf{BD}_{\mathrm{div}}(\Omega) \cap \mathbf{C}(\overline{\Omega})$ there holds*

$$\boldsymbol{\gamma}(\mathbf{v}) = \big(\boldsymbol{\gamma}(\mathbf{v})\big)_{\mathbf{t}} \quad \textit{a.e. on } \partial\Omega \, .$$

Associated to the minimization problem $(\mathcal{P})$, the relaxed minimization problem reads as

$$(\mathcal{P}_{\text{relax}}) \qquad \textit{minimize } \mathcal{G} \textit{ over } \mathbf{BD}_{\mathrm{div}}(\Omega) \, .$$

From the definition of $\mathcal{G}$ it follows

$$\begin{aligned} \mathcal{G}(\mathbf{u}) &= \mathcal{F}(\mathbf{u}) \quad \forall \mathbf{u} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega) \, , \\ \inf_{\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)} \mathcal{G}(\mathbf{u}) &\leq \inf_{\mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega)} \mathcal{F}(\mathbf{v}) \, . \end{aligned}$$

■

Now, fix $\mathbf{f} \in \mathbf{L}^n(\Omega)$. We recall the safe load condition

$$\left| \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x \right| \leq \nu \int_\Omega |\mathbf{D}(\mathbf{v})| \, \mathrm{d}x \quad \forall \mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega) \, , \tag{5.1}$$

and suppose that $\Omega$ is a star-shaped Lipschitz domain. Then Theorem 4.1 implies that (5.1) is equivalent to its counterpart

$$\left| \int_\Omega \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \right| \leq \nu \left( |D(\mathbf{u})|(\Omega) + \int_{\partial\Omega} |\boldsymbol{\tau}(\boldsymbol{\gamma}(\mathbf{u}))| \, \mathrm{d}\mathcal{H}^{n-1} \right) \quad \forall \mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega) \tag{5.2}$$

in $\mathbf{BD}_{\mathrm{div}}$, where (5.2) can be seen as a relaxed version of the safe load condition.

We notice that (5.2) is equivalent to

$$\inf_{\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)} \mathcal{G}(\mathbf{u}) = \mathcal{G}(\mathbf{0}) = 0 \, .$$

and that

$$\inf_{\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)} \mathcal{G}(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega)} \mathcal{F}(\mathbf{v}) = \inf_{\mathbf{w} \in \mathbf{W}^{1,1}_{0,\mathrm{div}}(\Omega)} \mathcal{F}(\mathbf{w}) \, .$$

In [7] and [17] it will be shown that (5.1) implies the existence of an approximating sequence $(\mathbf{u}_m)_{m\in\mathbb{N}} \subset \mathbf{W}^{1,1}_{0,\mathrm{div}}(\Omega)$ of physical relevance such that

$$\mathbf{u}_m \rightharpoonup \mathbf{u}^* \quad \text{in} \quad \mathbf{L}^{n/(n-1)}(\Omega) \text{ as } m \to \infty, \quad \mathbf{u}^* \in \mathbf{BD}_{\mathrm{div}}(\Omega) \, ,$$

$$\lim_{n\to\infty} \mathcal{F}(\mathbf{u}_m) = \min_{\mathbf{u} \in \mathbf{BD}_{\mathrm{div}}(\Omega)} \mathcal{G}(\mathbf{u}) = \mathcal{G}(\mathbf{u}^*) = \mathcal{G}(\mathbf{0}) \, .$$

It remains an open question, whether $\mathbf{u}^* \neq \mathbf{0}$ on a set of positive measure.

# Appendix. Inhomogeneous boundary data

The situation is much more involved in the case of inhomogeneous boundary data. Then, as a matter of fact, the corresponding minimization problem in general fails to have a solution even if the functional admits a finite infimum in the natural energy class.

We consider the system of PDEs (1.1), (1.2) with boundary condition

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \partial\Omega. \tag{1.3$'$}$$

Let $\nu = \mathit{const} > 0$, $\mathbf{f} \in \mathbf{L}^n(\Omega)$ and $\mathbf{u}_0 \in \mathbf{LD}_{\mathrm{div}}(\Omega)$ [10]. For $\mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega)$ we define [11]

$$\mathcal{F}^{\mathbf{u}_0}(\mathbf{v}) := \nu \int_\Omega |\mathbf{D}(\mathbf{u}_0 + \mathbf{v})| \,\mathrm{d}x - \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}x$$

(cf. Section 2). The corresponding minimization problem reads

$$\textit{minimize } \mathcal{F}^{\mathbf{u}_0} \textit{ over } \mathbf{LD}_{0,\mathrm{div}}(\Omega)\,. \tag{$\mathcal{P}^{\mathbf{u}_0}$}$$

With the help of elementary arguments we obtain the equivalence of the following conditions (A.1'), (A.1") and (A.2'), (A.2"), respectively:

$$\left| \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}x \right| \le \nu \int_\Omega |\mathbf{D}(\mathbf{v})| \,\mathrm{d}x \quad \forall \mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega) \;\; (\mathit{cf.}\ (2.2))\,, \tag{A.1'}$$

$$-\nu \int_\Omega |\mathbf{D}(\mathbf{u}_0)| \,\mathrm{d}x \le \inf_{\mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega)} \mathcal{F}^{\mathbf{u}_0}(\mathbf{v})\,, \tag{A.1"}$$

and

$$\exists \tilde{\mathbf{v}} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega): \quad \mathcal{F}^{\mathbf{u}_0}(\tilde{\mathbf{v}}) < -\nu \int_\Omega |\mathbf{D}(\mathbf{u}_0)| \,\mathrm{d}x\,, \tag{A.2'}$$

$$\inf_{\mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega)} \mathcal{F}^{\mathbf{u}_0}(\mathbf{v}) = -\infty\,. \tag{A.2"}$$

∎

[10] $\mathbf{LD}_{\mathrm{div}}(\Omega) := \big\{ \mathbf{u} \in \mathbf{LD}(\Omega) : \ \mathrm{div}\,\mathbf{u} = 0 \text{ a.e. in } \Omega \big\}$.

[11] Note that, given boundary values $\mathbf{u}_0$, the constant $\int_\Omega \mathbf{f} \cdot \mathbf{u}_0 \,\mathrm{d}x$ can be neglected in the minimization problem ($\mathcal{P}^{\mathbf{u}_0}$).

We introduce the relaxation of $\mathcal{F}$

$$\mathcal{G}^{\mathbf{u}_0}(\mathbf{v}) := \nu\left[|\mathbf{D}(\mathbf{u}_0+\mathbf{v})|(\Omega) + \int_{\partial\Omega} |\boldsymbol{\tau}(\gamma(\mathbf{v}))|\,\mathrm{d}\mathcal{H}^{n-1}\right] - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x\,, \qquad \mathbf{v}\in\mathbf{BD}_{\mathrm{div}}(\Omega)$$

(cf. [5], pp. 174-181, and [6]), where we clearly have

$$\mathcal{G}^{\mathbf{u}_0}(\mathbf{v}) = \mathcal{F}^{\mathbf{u}_0}(\mathbf{v}) \qquad \forall \mathbf{v}\in\mathbf{LD}_{0,\mathrm{div}}(\Omega)\,.$$

The relaxed minimization problem w.r.t. the boundary values $\mathbf{u}_0$ as above then reads as

$(\mathcal{P}^{\mathbf{u}_0}_{\mathrm{relax}})$ *minimize* $\mathcal{G}^{\mathbf{u}_0}$ *over* $\mathbf{BD}_{\mathrm{div}}(\Omega)$ .

■

Let us first note that the proof of Theorem 4.1 implies with minor changes

**Corollary A.1** *Suppose that we have the assumptions of Theorem 4.1 and that* $\mathbf{u}_0$ *is fixed as above. Then there exists an* $\mathbf{u}$*-approximating sequence* $(\mathbf{w}_k)_{k\in\mathbb{N}}$ *such that* $(\mathbf{w}_k-\mathbf{u}_0)_{k\in\mathbb{N}}\subset \mathbf{C}^\infty_c(\Omega)$ *and such that*

$$\begin{aligned} \operatorname{div}\mathbf{w}_k &= 0 \quad \textit{in } \Omega \textit{ and } \forall\, k\in\mathbb{N}\,,\\ \mathbf{w}_k &\to \mathbf{u} \quad \textit{in } \mathbf{L}^{n/(n-1)}(\Omega) \textit{ as } k\to\infty\,,\\ \int_\Omega |\mathbf{D}(\mathbf{w}_k)|\,\mathrm{d}x &\to |\mathbf{D}(\mathbf{u})|(\Omega) + \int_{\partial\Omega}|\boldsymbol{\tau}(\gamma(\mathbf{u}-\mathbf{u_0}))|\,\mathrm{d}\mathcal{H}^{n-1} \quad \textit{as } k\to\infty\,. \end{aligned}$$

■

Next we observe for all $\mathbf{v}\in\mathbf{BD}_{\mathrm{div}}(\Omega)$ similar to (A.1")

(A.3) $$\mathcal{G}^{\mathbf{u}_0}(\mathbf{v}) \geq \nu\left(|\mathbf{D}(\mathbf{v})|(\Omega) - |\mathbf{D}(\mathbf{u}_0)|(\Omega) + \int_{\partial\Omega}|\boldsymbol{\tau}(\gamma(\mathbf{v}))|\,\mathrm{d}\mathcal{H}^{n-1}\right) - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x\,,$$

i.e. the appropriate safe load condition in the situation at hand is also given by (5.2). ■

To minimize $\mathcal{G}^{\mathbf{u}_0}$ we impose on $(\nu,\mathbf{f})$ the following "**strict safe load condition**"

(A.4) $$\sup_{\mathbf{v}\in\mathbf{BD}_{\mathrm{div}}(\Omega)} \frac{\left|\displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x\right|}{\nu\left(|\mathbf{D}(\mathbf{v})|(\Omega) + \displaystyle\int_{\partial\Omega}|\boldsymbol{\tau}(\gamma(\mathbf{v}))|\,\mathrm{d}\mathcal{H}^{n-1}\right)} < 1\,,$$

which can be formulated as

(A.5) $$\left\{\begin{array}{l} \exists 0<\alpha_0<1 \ \text{ s.t. } \ \forall \mathbf{v}\in\mathbf{BD}_{\mathrm{div}}(\Omega)\\[2mm] \left|\displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x\right| \leq \alpha_0\nu\left(|\mathbf{D}(\mathbf{v})|(\Omega) + \displaystyle\int_{\partial\Omega}|\boldsymbol{\tau}(\gamma(\mathbf{v}))|\,\mathrm{d}\mathcal{H}^{n-1}\right) \end{array}\right. .$$

From (A.5) and (A.3) it follows that for all $\mathbf{v} \in \mathbf{BD}_{\mathrm{div}}(\Omega)$

$$\mathcal{G}^{\mathbf{u}_0}(\mathbf{v}) \geq (1-\alpha_0)\nu\left(|\mathbf{D}(\mathbf{v})|(\Omega) + \int_{\partial\Omega} |\boldsymbol{\tau}(\gamma(\mathbf{v}))|\,\mathrm{d}\mathcal{H}^{n-1}\right) - \nu|\mathbf{D}(\mathbf{u}_0)|(\Omega)\,.$$

Thus, (A.4) permits to prove the existence of a minimizer of $\mathcal{G}^{\mathbf{u}_0}$ over $\mathbf{BD}_{\mathrm{div}}(\Omega)$ provided there exists $\tilde{\mathbf{u}}_0 \in \mathbf{W}^{1,1}_{\mathrm{div}}(B)$ [12] such that $\tilde{\mathbf{u}}_{0|\partial\Omega} = \mathbf{u}_0$ a.e. in $\Omega$, where $B \subset \mathbb{R}^n$ is a fixed open ball such that $\overline{\Omega} \subset B$ (cf. also Proposition 3.2,3).

On account of Corollary A.1 we again have

$$\inf_{\mathbf{u}\in\mathbf{BD}_{\mathrm{div}}(\Omega)} \mathcal{G}^{\mathbf{u}_0}(\mathbf{u}) = \inf_{\mathbf{v}\in\mathbf{LD}_{0,\mathrm{div}}(\Omega)} \mathcal{F}^{\mathbf{u}_0}(\mathbf{v}) = \inf_{\mathbf{w}\in\mathbf{W}^{1,1}_{0,\mathrm{div}}(\Omega)} \mathcal{F}^{\mathbf{u}_0}(\mathbf{w})\,.$$

Here the main open question concerns the existence of generalized minimizers if we just suppose the safe load condition (5.2) instead of its strict variant (A.4). ■

A different approach to solve the minimization problem for $\mathcal{G}^{\mathbf{u}_0}$ over $\mathbf{BD}_{\mathrm{div}}(\Omega)$ consists in minimizing the functional

$$\mathcal{F}^{\mathbf{u}_0}_p(\mathbf{v}) := \frac{\nu}{p}\int_\Omega |\mathbf{D}(\mathbf{u}_0+\mathbf{v})|^p\,\mathrm{d}x - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x\,, \qquad \mathbf{v} \in \mathbf{LD}_{0,\mathrm{div}}(\Omega) \qquad p>1\,,$$

over $\mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)$, proving a priori estimates on the minimizers and carrying out the passage to the limit $p \to 1$ (cf. Section 2).

To sketch an outline of this approach, we assume that $\mathbf{u}_0 \in \mathbf{W}^{1,p}_{\mathrm{div}}(\Omega)$ (for some $1<p<\infty$) Then there exists exactly one $\mathbf{v}_p \in \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)$ such that

$$\mathcal{F}^{\mathbf{u}_0}_p(\mathbf{v}_p) = \min_{\mathbf{v}\in\mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)} \mathcal{F}^{\mathbf{u}_0}_p(\mathbf{v})\,.$$

To proceed, we now assume that

$$\text{(A.6)} \qquad \left|\int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x\right| \leq \nu\int_\Omega |\mathbf{D}(\mathbf{w})|\,\mathrm{d}x \quad \forall \mathbf{w} \in \mathbf{LD}_{\mathrm{div}}(\Omega)\,,$$

$$\text{(A.7)} \qquad \begin{cases} \mathbf{u}_0 \in \mathbf{W}^{1,p_0}_{\mathrm{div}}(\Omega) \quad (1<p_0<\infty \text{ fixed})\,, \\ \dfrac{\nu}{p}\displaystyle\int_\Omega |\mathbf{D}(\mathbf{u}_0)|^p\,\mathrm{d}x \leq \int_\Omega \mathbf{f}\cdot\mathbf{u}_0\,\mathrm{d}x \quad \forall 1<p<p_0\,. \end{cases}$$

Let $\mathbf{v}_p \in \mathbf{W}^{1,p}_{0,\mathrm{div}}(\Omega)$ $(1<p\leq p_0)$ be as above. We obtain by (A.7)

$$\mathcal{F}^{\mathbf{u}_0}_p(\mathbf{v}_p) \leq \mathcal{F}^{\mathbf{u}_0}_p(\mathbf{0}) \leq 0\,.$$

[12] $\mathbf{W}^{1,p}_{\mathrm{div}}(\Omega) := \left\{\mathbf{u} \in W^{1,p}(\Omega) :\ \operatorname{div}\mathbf{u} = 0 \text{ a.e. in } \Omega\right\}$ .

Set $\mathbf{w}_p := \mathbf{u}_0 + \mathbf{v}_p$. Then we have by (A.6)

$$\frac{\nu}{p}\int_\Omega |\mathbf{D}(\mathbf{w}_p)|^p \,\mathrm{d}x \le \int_\Omega \mathbf{f}\cdot\mathbf{w}_p \,\mathrm{d}x \le \nu\int_\Omega |\mathbf{D}(\mathbf{w}_p)|\,\mathrm{d}x\,.$$

Hence, for every $\delta > 0$ there exists $1 < p_\delta \le p_0$ such that

$$\int_\Omega |\mathbf{D}(\mathbf{w}_p)| \le (\delta + e)\mathrm{meas}\,\Omega \quad \forall 1 < p \le p_\delta\,.$$

This estimate forms the basis for the passage to the limit $p \to 1$ in order to get a minimizer for $\mathcal{G}^{\mathbf{u}_0}$ over $\{\mathbf{u}_0\} + \mathbf{BD}_{\mathrm{div}}(\Omega)$. Moreover, this approach gives $\mathbf{S} \in L^\infty(\Omega)$ which satisfies (1.2) and (1.6) in a certain weak sense.

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