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Abstract

We extend the Liouville-type theorems of Gilbarg and Weinberger and of Koch, Nadirashvili, Seregin and Sverák valid for the stationary variant of the classical Navier-Stokes equations in $2D$ to the degenerate power law fluid model.

1 Introduction

To begin with we look at a velocity field u: $\mathbb{R}^2 \to \mathbb{R}^2$ and a pressure function π: $\mathbb{R}^2 \to \mathbb{R}$ satisfying the stationary equations of Navier-Stokes

$$\left.\begin{array}{rcl} -\Delta u + u^k \partial_k u + \nabla \pi & = & 0 \,, \\ \operatorname{div} u & = & 0 \quad \text{on } \mathbb{R}^2 \,, \end{array}\right\} \tag{1.1}$$

which correspond to the flow of an incompressible Newtonian fluid with constant viscosity (w.l.o.g. equal to 1). Here we study entire solutions, and a natural question is the search for suitable conditions which force u (and thereby π) to be constant. We recall two prominent examples of such Liouville-type results for the Navier-Stokes equation (1.1): if u is a finite energy solution, i.e. if we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x < \infty \,, \tag{1.2}$$

then Gilbarg and Weinberger [GW] proved $u = \mathit{const}$ making extensive use of the fact that the vorticity function $\omega := \partial_2 u^1 - \partial_1 u^2$ satisfies a nice elliptic equation. Recently, Koch, Nadirashvili, Seregin and Sverák [KNSS] discussed the instationary variant of (1.1) and, as a byproduct of their investigations, they showed that in the stationary case (1.2) can be replaced by

$$\sup_{x \in \mathbb{R}^2} |u(x)| < \infty \tag{1.3}$$

implying the constancy of the vector field u. In connection with the Navier-Stokes equation we like to remark that according to [Zh] the hypothesis

$$\int_{\mathbb{R}^2} |u|^t \, \mathrm{d}x < \infty \quad \text{for some } t > 1$$

(replacing (1.1) or (1.3)) implies the vanishing of u, whereas in [FZho] it is observed that $u = \mathit{const}$ is still true if the growth of $|u(x)|$ as $|x| \to \infty$ is not too strong.

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In [Fu], [FZha], [Zh] the situation for generalized Newtonian fluids being either of shear thickening or shear thinning type is studied. For this case equation (1.1) has to be replaced by

$$\left.\begin{array}{rcl} -\operatorname{div}\left[DH(\varepsilon(u))\right] + u^k\partial_k u + \nabla\pi & = & 0\ , \\ \operatorname{div} u & = & 0 \quad \text{on } \mathbb{R}^2 \end{array}\right\} \tag{1.4}$$

with a strictly convex potential H of class C^2 acting on symmetric (2×2)-matrices ($\varepsilon(u)$ denoting the symmetric gradient of the velocity field u) and being of the form

$$H(\varepsilon) = h(|\varepsilon|) \tag{1.5}$$

for a function h: $[0,\infty) \to [0,\infty)$ for which

$$\mu(t) := \frac{h'(t)}{t}$$

either decreases or increases. Note that according to (1.5) we have $DH(\varepsilon) = \mu(|\varepsilon|)\varepsilon$, thus μ plays the role of a shear dependent viscosity. For further physical and mathematical explanations we refer to the monographs [La], [Ga1], [Ga2], [MNRR] or [FS].

The most severe restriction concerns the existence and the behaviour of $D^2H(0)$, which in particular means that we require

$$D^2H(0)(\varepsilon,\varepsilon) \geq \lambda|\varepsilon|^2 \tag{1.6}$$

for some positive constant λ. Assuming (1.6) it is shown: suppose that $u \in C^1(\mathbb{R}^2,\mathbb{R}^2)$ is an entire weak solution of (1.4), i.e. it holds $\operatorname{div} u = 0$ together with

$$0 = \int_{\mathbb{R}^2} DH(\varepsilon(u)) : \varepsilon(\varphi)\,\mathrm{d}x + \int_{\mathbb{R}^2} u^k\partial_k u^i\varphi^i\,\mathrm{d}x \tag{1.7}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2,\mathbb{R}^2)$ such that $\operatorname{div}\varphi = 0$. Then we have $u \equiv \mathit{const}$, if either (1.3) holds or if we replace (1.2) through the appropriate hypothesis

$$\int_{\mathbb{R}^2} h(|\nabla u|)\,\mathrm{d}x < \infty\ . \tag{1.8}$$

Clearly these results apply to non-degenerate p-fluids for which $h(t) = (1+t^2)^{p/2}$ (modulo physical constants) with exponent $p \in (1,\infty)$ but not to the degenerate power law model, i.e. to the potential H with function $h(t) = t^p$.

In the present paper we are going to investigate the degenerate p-case, i.e. from now on we assume that H is given by

$$H(\varepsilon) = |\varepsilon|^p$$

for some $1 < p < \infty$ and that $u \in C^1(\mathbb{R}^2,\mathbb{R}^2)$ with $\operatorname{div} u = 0$ solves equation (1.7). Then our results are as follows:

Theorem 1.1 *Suppose that* $1 < p \leq 2$.

i) If u belongs to the space $L^\infty(\mathbb{R}^2, \mathbb{R}^2)$, i.e. if condition (1.3) holds, then u is a constant vector.

ii) If $p < 2$, if

$$0 < \alpha < \frac{2-p}{6+p} \tag{1.9}$$

and if we have

$$\limsup_{|x|\to\infty} |u(x)||x|^{-\alpha} < \infty \ , \tag{1.10}$$

then the conclusion of i) holds.

Remark 1.1 *For the choice $p = 2$ we reproduce the contribution of Koch, Nadirashvili, Seregin and Sverák [KNSS], for $1 < p < 2$ condition (1.10) allows even a certain growth of $|u(x)|$ as $|x| \to \infty$. In Theorem 1.5 we will discuss in more detail the admissible a priori growth rates of u in the case $p = 2$.*

The next two theorems extend the Liouville result of Gilbarg and Weinberger [GW] to exponents p not necessarily equal to 2.

Theorem 1.2 *Let $6/5 < p \leq 2$ and assume that*

$$\int_{\mathbb{R}^2} |\nabla u|^p \, \mathrm{d}x < \infty \ ,$$

which means that (1.8) is satisfied. Then u has to be constant.

Theorem 1.3 *Theorem 1.2 remains valid for exponents $p \in [2, 3]$.*

Theorem 1.4 is the counterpart to Theorem 1.1, *ii*) for $p > 2$ involving formally the same exponent $(p-2)/(p+6)$.

Theorem 1.4 *Let $p > 2$ and let $u_\infty \in \mathbb{R}^2$ denote a vector such that*

i) in case $2 < p < 6$

$$\sup_{|x|\geq R} |u(x) - u_\infty||x|^{\frac{p-2}{p+6}} \to 0 \quad \text{as} \ R \to \infty \ ; \tag{1.11}$$

ii) in case $p = 6$:

$$\limsup_{|x|\to\infty} |u(x) - u_\infty||x|^{\frac{1}{3}} < \infty \ ; \tag{1.12}$$

iii) in case $p > 6$:

$$\sup_{|x|\geq R} |u(x) - u_\infty||x|^{\frac{1}{3}} \to 0 \quad \text{as} \ R \to \infty \ . \tag{1.13}$$

Then $u \equiv u_\infty$ follows.

Remark 1.2 *It remains an open question, if in case $p > 2$ bounded solutions are constant without imposing a decay condition.*

An inspection of the proofs of Theorem 1.1 - 1.4 will show:

Corollary 1.1 *Let $p \in (1, \infty)$ and suppose that u: $\mathbb{R}^2 \to \mathbb{R}^2$ is a solution of the p-Stokes system in the plane, i.e. a solution of (1.7) with $H(\varepsilon) = |\varepsilon|^p$, where now the convective term is neglected. Then u is a constant vector if either $u \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ or if u is of finite energy, i.e. $\int_{\mathbb{R}^2} |\nabla u|^p \, \mathrm{d}x < \infty$.*

Remark 1.3 *Clearly Corollary 1.1 can be generalized in the sense that for $1 < p < 2$ a certain growth of u can be included which might be even stronger in comparison to the formulation given in (1.9) and (1.10). We leave the details to the reader.*

We finish this introduction with an extension of the Liouville results obtained in [KNSS] and [FZho] for the case of the classical Navier-Stokes equation.

Theorem 1.5 *Suppose that u: $\mathbb{R}^2 \to \mathbb{R}^2$ is a solution of (1.1) such that*

$$\limsup_{|x| \to \infty} |u(x)||x|^{-\alpha} < \infty \tag{1.14}$$

for some $\alpha < 1/3$. Then the constancy of u follows.

Remark 1.4 *It would be interesting to know the optimal bound for the number α occurring in (1.14).*

Our paper is organized as follows: in Section 2 we give estimates for the energy $\int_{B_r(x_0)} |\nabla u|^p \, \mathrm{d}x$, $1 < p < \infty$, on disks in terms of the radius under various hypotheses imposed on u. Section 3 is devoted to the case $1 < p < 2$, i.e. we will present the proofs of Theorem 1.1 and of Theorem 1.2 by combining the results of Section 2 with estimates for the "second derivatives" due to Wolf [Wo].

Since these estimates are not available for $p > 2$, we have to find alternatives leading to Theorem 1.3 and to Theorem 1.4. This is done in Section 4.

In Section 5 we give a proof of Theorem 1.5. Moreover, we collect some technical tools in an appendix.

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2 Estimates for the p-energy on disks

In this section we describe the growth of the energy $\int_{B_r(x_0)} |\nabla u|^p \, \mathrm{d}x$ of weak solutions u to (1.7) in terms of the radius of the disk under various conditions concerning the growth of u.

Lemma 2.1 *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, $\operatorname{div} u = 0$, denote a solution of (1.7) for the choice $H(\varepsilon) = |\varepsilon|^p$ with exponent $p \in (1, \infty)$.*

i) Then, for any real number $\beta < 1$, it holds

$$\begin{aligned}\int_{B_r(x_0)} |\nabla u|^p \,\mathrm{d}x \;\leq\; & c\Bigg[r^{-p}\int_{B_{2r}(x_0)} |u|^p \,\mathrm{d}x + r^{-1+\beta}\int_{B_{2r}(x_0)} |u|^2 \,\mathrm{d}x \\ & + r^{-1}\int_{B_{2r}(x_0)} |u|^3 \,\mathrm{d}x + r^{-1-\beta}\int_{B_{2r}(x_0)} |u|^4 \,\mathrm{d}x\Bigg] \qquad (2.1)\end{aligned}$$

for all disks $B_{2r}(x_0)$. Here, the positive constant c is independent of x_0, r and u.

ii) If u is bounded, then it follows by choosing $\beta = 0$

$$\int_{B_r(x_0)} |\nabla u|^p \,\mathrm{d}x \leq c\Big(\|u\|_{L^\infty(\mathbb{R}^2)}\Big)\left[r^{-p}\int_{B_{2r}(x_0)} |u|^p \,\mathrm{d}x + r^{-1}\int_{B_{2r}(x_0)} |u|^2 \,\mathrm{d}x\right] \qquad (2.2)$$

again for all disks. In particular it holds

$$\int_{B_R(0)} |\nabla u|^p \,\mathrm{d}x \leq c\Big(\|u\|_{L^\infty(\mathbb{R}^2)}\Big) R \qquad (2.3)$$

for radii $R \geq 1$.

If $u_\infty \in \mathbb{R}^2$ is some fixed vector, then (2.2) is also valid for the function $\tilde{u} := u - u_\infty$ in place of u.

iii) Suppose that

$$\limsup_{|x|\to\infty} |u(x)||x|^{-\gamma} < \infty$$

for some number γ such that

$$\gamma \in \begin{cases} [0,1)\,, & \text{if } \; 1 < p \leq 2\,, \\ [-1/2, 0)\,, & \text{if } \; p > 2\,. \end{cases} \qquad (2.4)$$

Then it holds for any $R \geq 1$

$$\int_{B_R(0)} |\nabla u|^p \,\mathrm{d}x \leq c R^{1+3\gamma}\,. \qquad (2.5)$$

Proof of Lemma 2.1.

Ad i) & ii).

Consider $\eta \in C_0^\infty(B_{2r}(x_0))$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$ and $|\nabla\eta| \le c/r$. In equation (1.7) we let $\varphi = \eta^{2l}u - w$, where the field w is defined on $B_{2r}(x_0)$, vanishing on $\partial B_{2r}(x_0)$ with the properties

$$\begin{aligned} \operatorname{div} w &= \operatorname{div}(\eta^{2l}u) = \nabla\eta^{2l}\cdot u \quad \text{on } B_{2r}(x_0)\,, \\ \|\nabla w\|_{L^q(B_{2r}(x_0))} &\le c\|\nabla\eta^{2l}\cdot u\|_{L^q(B_{2r}(x_0))}\,. \end{aligned} \tag{2.6}$$

Note that (2.6) holds with the same field w both for the choice $q = 2$ and for the choice $q = p$ (cf. Lemma A.1). The integer l will be determined later. We have

$$\begin{aligned} \int_{B_{2r}(x_0)} DH(\varepsilon(u)) : \varepsilon(u)\eta^{2l}\,\mathrm{d}x &= -\int_{B_{2r}(x_0)} DH(\varepsilon(u)) : (\nabla\eta^{2l}\otimes u)\,\mathrm{d}x \\ &\quad + \int_{B_{2r}(x_0)} DH(\varepsilon(u)) : \varepsilon(w)\,\mathrm{d}x \\ &\quad - \int_{B_{2r}(x_0)} u^k\partial_k u\cdot u\eta^{2l}\,\mathrm{d}x + \int_{B_{2r}(x_0)} u^k\partial_k u\cdot w\,\mathrm{d}x \\ &=: T_1 + T_2 + T_3 + T_4\,. \end{aligned} \tag{2.7}$$

Young's inequality yields for any $\delta > 0$

$$\begin{aligned} |T_1| &\le c\int_{B_{2r}(x_0)} |\varepsilon(u)|^{p-1}\eta^{2l-1}|\nabla\eta||u|\,\mathrm{d}x \\ &\le \delta\int_{B_{2r}(x_0)} |\varepsilon(u)|^p\eta^{(2l-1)\frac{p}{p-1}}\,\mathrm{d}x + c(\delta)\int_{B_{2r}(x_0)} |\nabla\eta|^p|u|^p\,\mathrm{d}x \\ &\le \delta\int_{B_{2r}(x_0)} \eta^{2l}|\varepsilon(u)|^p\,\mathrm{d}x + c(\delta)r^{-p}\int_{B_{2r}(x_0)} |u|^p\,\mathrm{d}x\,, \end{aligned}$$

provided that we choose l so large that $(2l-1)p/(p-1) \ge 2l$. For small enough δ the bound for $|T_1|$ in combination with (2.7) yields

$$\int_{B_{2r}(x_0)} |\varepsilon(u)|^p\eta^{2l}\,\mathrm{d}x \le c\left[r^{-p}\int_{B_{2r}(x_0)} |u|^p\,\mathrm{d}x + |T_2| + |T_3| + |T_4|\right]. \tag{2.8}$$

Next we use (2.6) for $q = p$ and obtain by Young's inequality

$$\begin{aligned} |T_2| &\le \delta\int_{B_{2r}(x_0)} |\varepsilon(u)|^p\,\mathrm{d}x + c(\delta)\int_{B_{2r}(x_0)} |\varepsilon(w)|^p\,\mathrm{d}x \\ &\le \delta\int_{B_{2r}(x_0)} |\varepsilon(u)|^p\,\mathrm{d}x + c(\delta)r^{-p}\int_{B_{2r}(x_0)} |u|^p\,\mathrm{d}x\,, \end{aligned}$$

thus by (2.8)

$$\int_{B_r(x_0)} |\varepsilon(u)|^p\,\mathrm{d}x \le \delta\int_{B_{2r}(x_0)} |\varepsilon(u)|^p\,\mathrm{d}x + c(\delta)r^{-p}\int_{B_{2r}(x_0)} |u|^p\,\mathrm{d}x + \big[|T_3| + |T_4|\big]\,. \tag{2.9}$$

Finally we observe using an integration by parts

$$|T_3| = \frac{1}{2}\left| \int_{B_{2r}(x_0)} u^k |u|^2 \partial_k \eta^{2l} \,\mathrm{d}x \right| \leq c r^{-1} \int_{B_{2r}(x_0)} |u|^3 \,\mathrm{d}x \tag{2.10}$$

and

$$T_4 = -\int_{B_{2r}(x_0)} u^i u^k \partial_k w^i \,\mathrm{d}x \ ,$$

thus

$$|T_4| \leq \left[\int_{B_{2r}(x_0)} |u|^4 \,\mathrm{d}x \right]^{\frac{1}{2}} \left[\int_{B_{2r}(x_0)} |\nabla w|^2 \,\mathrm{d}x \right]^{\frac{1}{2}} ,$$

and the use of (2.6) now with the choice $q = 2$ shows

$$\begin{aligned} |T_4| &\leq \left[\int_{B_{2r}(x_0)} |u|^4 \,\mathrm{d}x \right]^{\frac{1}{2}} \left[r^{-2} \int_{B_{2r}(x_0)} |u|^2 \,\mathrm{d}x \right]^{\frac{1}{2}} \\ &= \left[r^{-1-\beta} \int_{B_{2r}(x_0)} |u|^4 \,\mathrm{d}x \right]^{\frac{1}{2}} \left[r^{-1+\beta} \int_{B_{2r}(x_0)} |u|^2 \,\mathrm{d}x \right]^{\frac{1}{2}} \\ &\leq c r^{-1+\beta} \int_{B_{2r}(x_0)} |u|^2 \,\mathrm{d}x + c r^{-1-\beta} \int_{B_{2r}(x_0)} |u|^4 \,\mathrm{d}x \ . \end{aligned} \tag{2.11}$$

Combining (2.9) with (2.10) and (2.11) and using Lemma A.4 it follows

$$\begin{aligned} \int_{B_r(x_0)} |\varepsilon(u)|^p \,\mathrm{d}x \ \leq \ c \Bigg[& r^{-p} \int_{B_{2r}(x_0)} |u|^p \,\mathrm{d}x + r^{-1+\beta} \int_{B_{2r}(x_0)} |u|^2 \,\mathrm{d}x \\ & + r^{-1} \int_{B_{2r}(x_0)} |u|^3 \,\mathrm{d}x + r^{-1-\beta} \int_{B_{2r}(x_0)} |u|^4 \,\mathrm{d}x \Bigg] . \end{aligned}$$

Applying Korn's inequality in $W^1_p(B_{2r}(x_0), \mathbb{R}^2)$ (cf. Lemma A.2) we arrive at (2.1). From (2.1) the claims (2.2) and (2.3) immediately follow.

For the second statement of *ii*) we observe that $\tilde{u} = u - u_\infty$ solves equation (1.7) with the additional term $\int u^k_\infty \partial_k \tilde{u} \cdot \varphi \,\mathrm{d}x$ and the choice $\varphi = \eta^{2l}\tilde{u} - \tilde{w}$ (with an obvious meaning of $\tilde{w}$) leads to (2.2) for $\tilde{u}$ with the help of elementary identities like

$$u^k_\infty \int_{B_{2r}(x_0)} \partial_k \tilde{u}^i \eta^{2l} \tilde{u}^i \,\mathrm{d}x = -\frac{1}{2} u^k_\infty \int_{B_{2r}(x_0)} |\tilde{u}|^2 \partial_k \eta^{2l} \,\mathrm{d}x \ .$$

Ad iii).

Suppose that we have

$$\limsup_{|x| \to \infty} |u(x)||x|^{-\gamma} < \infty \tag{2.12}$$

with γ satisfying (2.4).

Case 1: $\gamma \in [0,1)$ *and* $1 < p \le 2$*.* In this case (2.12) implies the growth condition

$$\sup_{B_R(0)} |u| \le cR^\gamma \quad \text{for all } \ R \ge 1 \ . \tag{2.13}$$

Quoting inequality (2.1) choosing $x_0 = 0$, $r = R \ge 1$ and $\beta = \gamma$, (2.13) gives

$$\int_{B_R(0)} |\nabla u|^p \, \mathrm{d}x \le c\big[R^{2-p+p\gamma} + R^{1+3\gamma}\big] \ ,$$

and since $2 - p + p\gamma \le 1 + 3\gamma$, we get (2.5).

Case 2: $\gamma \in [-1/2, 0)$ *and* $p > 2$*.* From (2.12) we deduce the boundedness of u together with

$$\sup_{R \le |x| \le 2R} |u| \le R^\gamma \tag{2.14}$$

for R sufficiently large. We return to the beginning of the proof and replace φ through the modified test-function (with η as before and with $w^* \in \overset{\circ}{W}{}^1_q(T_R(0), \mathbb{R}^2)$ given according to Lemma A.1 – again we will make use both of the choice $q = 2$ and of the choice $q = p$ in this Lemma)

$$\varphi^* = \left\{ \begin{array}{ccl} u & \text{on} & B_R(0) \ , \\ \eta^{2l} u - w^* & \text{on} & T_R(0) \ , \end{array} \right.$$

where we always set

$$T_R(x_0) := B_{2R}(x_0) - \overline{B_R(x_0)} \ .$$

We have

$$\begin{aligned} \operatorname{div} w^* &= \operatorname{div}(\eta^{2l} u) = \nabla \eta^{2l} \cdot u \quad \text{on } \ T_R(0) \ , \\ \|\nabla w^*\|_{L^q(T_R(0))} &\le c \|\nabla \eta^{2l} \cdot u\|_{L^q(T_R(0))} \ . \end{aligned}$$

Note that $\int_{T_R(0)} \operatorname{div}(\eta^{2l} \cdot u) \, \mathrm{d}x = 0$. We then obtain a version of (2.7) with $x_0 = 0$, w being replaced by w^* and where in T_2 and T_4 the integration is performed over the annulus $T_R(0)$. In place of (2.9) we get after specifying $c(\delta)$

$$\int_{B_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x \le \delta \int_{T_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x + c\delta^{1-p} R^{-p} \int_{T_R(0)} |u|^p \, \mathrm{d}x + \big[|T_3| + |T_4|\big] \ . \tag{2.15}$$

For T_3 it holds (compare (2.10))

$$|T_3| \le cR^{-1} \int_{T_R(0)} |u|^3 \, \mathrm{d}x$$

and for T_4 we just observe

$$|T_4| \le cR^{-1} \left[\int_{T_R(0)} |u|^4 \, \mathrm{d}x \right]^{\frac{1}{2}} \left[\int_{T_R(0)} |u|^2 \, \mathrm{d}x \right]^{\frac{1}{2}} \ .$$

Thus (2.15) implies (recalling (2.14))

$$\int_{B_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x \le \delta \int_{T_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x + c\big[\delta^{1-p} R^{2-p+p\gamma} + R^{1+3\gamma}\big] \, . \tag{2.16}$$

Since u is bounded, we can apply (2.3) to the first term on the r.h.s. of (2.16), hence

$$\int_{B_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x \le c\big[\delta R + \delta^{1-p} R^{2-p+p\gamma} + cR^{1+3\gamma}\big] \, . \tag{2.17}$$

Suppose now that we have for some $n = 0$, 1, 2

$$\int_{B_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x \le cR^{1+n\gamma} \, , \tag{2.18}$$

which by (2.3) in fact is true in the case $n = 0$. Then, instead of (2.17), we have using assumption (2.18)

$$\int_{B_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x \le c\big[\delta R^{1+n\gamma} + \delta^{1-p} R^{2-p+p\gamma} + cR^{1+3\gamma}\big] \, . \tag{2.19}$$

We choose $\delta = R^\gamma$ in (2.19):

$$\begin{aligned} \int_{B_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x &\le c\big[R^{1+(n+1)\gamma} + R^{\gamma-\gamma p} R^{2-p+p\gamma} + R^{1+3\gamma} \\ &\le cR^{1+(n+1)\gamma} \, , \end{aligned} \tag{2.20}$$

provided that we have $(n+1) \le 3$ (which clearly is true since we suppose $n \le 2$ – recall $\gamma \le 0$ in the case under consideration) and if we have in addition

$$\gamma + 2 - p \le 1 + (n+1)\gamma \quad \Leftrightarrow \quad 1 - p \le \gamma n \, . \tag{2.21}$$

Note that for $\gamma \in [-1/2, 0]$ and $p \ge 2$ (2.21) holds true up to the choice $n = 2$ and as the final result we obtain

$$\int_{B_R(0)} |\varepsilon(u)|^p \, \mathrm{d}x \le cR^{1+3\gamma} \, . \tag{2.22}$$

Applying the version of Korn' s in equality stated in Lemma A.2, *iii*), to (2.22) we obtain

$$\int_{B_R(0)} |\nabla u|^p \, \mathrm{d}x \le c\big[R^{1+3\gamma} + R^{-p+2+p\gamma}\big]$$

and thereby (2.5) which completes the proof of Lemma 2.1. □

From Lemma 2.1 we immediately obtain

Corollary 2.1 *Suppose that $p > 2$ and that*

$$\limsup_{|x|\to\infty} |u(x)||x|^{-\gamma} < \infty$$

holds for some number $\gamma < -1/3$. Then u must be identically zero.

Proof of Corollary 2.1. W.l.o.g. we may assume $\gamma \in [-1/2, -1/3)$ since otherwise we replace the (negative) exponent γ through $-1/2$. But then (2.5) yields the claim by passing to the limit $R \to \infty$. $\square$

3 The case $1 < p < 2$

During this section we always assume that $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ is a solenoidal field satisfying (1.7) for the choice $H(\varepsilon) = |\varepsilon|^p$ with exponent $p \in (1, 2)$. Note that on account of Corollary I in the paper [Wo] of Wolf weak solutions of (1.7) from the space $W^1_{p,\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ are of class C^1 if we require $p > 3/2$.

The proofs of Theorem 1.1 and Theorem 1.2 make extensive use of the following preliminary result, where we let

$$V(\varepsilon) := \begin{cases} |\varepsilon|^{\frac{p-2}{2}} & \text{if } \varepsilon \neq 0 \,, \\ 0 & \text{if } \varepsilon = 0 \,. \end{cases}$$

Lemma 3.1 *The velocity field u is an element of the space $W^2_{p,\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and for any disk $B_r(x_0)$ it holds (recall $T_r(x_0) = B_{2r}(x_0) - \overline{B_r(x_0)}$)*

$$\int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla\varepsilon(u)|^2 \,\mathrm{d}x \leq c\left[r^{-2} \int_{T_r(x_0)} |\nabla u|^p \,\mathrm{d}x + r^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x \right] , \quad (3.1)$$

where c denotes a finite constant independent of u, r and x_0.

Proof of Lemma 3.1. The existence of the second order weak derivatives in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ has been established by Naumann [Na] in Theorem 2 of his paper. Actually Naumann considers slow flows, i.e. the convective term is neglected, but his arguments cover the case of volume forces $f \in L^{p'}_{\mathrm{loc}}$, and since u is a C^1-function, we just put $f := -u^k\partial_k u$.

For proving estimate (3.1) we benefit from the basic inequality (3.24) in Wolf's paper [Wo]: let $\eta \in C^\infty_0(B_{2r}(x_0))$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(x_0)$ and $|\nabla^l\eta| \leq cr^{-l}$, $l = 1$, 2. Choosing

$$S_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}} \,, \quad \lambda = 0 \,, \quad \xi = \eta \,, \quad \tilde{f} := -u^k\partial_k u$$

and using the symbol π for the pressure we obtain from (3.24) in [Wo] (replacing r by $2r$)

$$c(p)\int_{B_{2r}(x_0)} V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\eta^2\,\mathrm{d}x \le \sum_{i=1}^{6} I_i \tag{3.2}$$

with I_i defined exactly as in the above reference and for a constant $c(p)>0$. We have (c denoting positive constants with values varying from line to line but being independent of x_0 and r)

$$\begin{aligned} |I_1| &\le c\int_{T_r(x_0)} |\varepsilon(u)|^{p-1}|\nabla u|\big[|\nabla\eta|^2+|\nabla^2\eta|\big]\,\mathrm{d}x \\ &\le cr^{-2}\int_{T_r(x_0)} |\nabla u|^p\,\mathrm{d}x \end{aligned} \tag{3.3}$$

and by Young's inequality (using also the estimate $|\nabla^2 u| \le c|\nabla\varepsilon(u)|$ and recalling the definition of V)

$$\begin{aligned} |I_2| &\le c\int_{B_{2r}(x_0)} |\varepsilon(u)|^{p-1}|\nabla^2 u|\eta|\nabla\eta|\,\mathrm{d}x \\ &\le c\int_{B_{2r}(x_0)} V(\varepsilon(u))|\nabla\varepsilon(u)|\eta|\varepsilon(u)|^{\frac{p}{2}}|\nabla\eta|\,\mathrm{d}x \\ &\le \delta\int_{B_{2r}(x_0)} V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\eta^2\,\mathrm{d}x + c(\delta)\int_{T_r(x_0)} |\varepsilon(u)|^p|\nabla\eta|^2\,\mathrm{d}x\ . \end{aligned}$$

Choosing δ small enough and quoting (3.3) we deduce from (3.2)

$$\begin{aligned} &\int_{B_{2r}(x_0)} V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\eta^2\,\mathrm{d}x \\ &\quad\le c\left[r^{-2}\int_{T_r(x_0)} |\nabla u|^p\,\mathrm{d}x + |I_3+I_4| + |I_5| + |I_6|\right]. \end{aligned} \tag{3.4}$$

Next we rewrite the quantity $|I_3+I_4|$ in the following form:

$$|I_3+I_4| = \left|\int_{B_{2r}(x_0)} \pi\partial_k(\partial_i\eta^2\partial_k u^i)\,\mathrm{d}x\right| = \left|\int_{B_{2r}(x_0)} \pi\,\mathrm{div}\,\varphi\,\mathrm{d}x\right|,$$

where $\varphi^k := \partial_i\eta^2\partial_k u^i$. From (1.4) it follows that

$$\int_{B_{2r}(x_0)} \pi\,\mathrm{div}\,\varphi\,\mathrm{d}x = \int_{B_{2r}(x_0)} DH(\varepsilon(u)) : \varepsilon(\varphi)\,\mathrm{d}x + \int_{B_{2r}(x_0)} u^k\partial_k u\cdot\varphi\,\mathrm{d}x\ ,$$

hence

$$\begin{aligned} |I_3+I_4| &\le c\Bigg[\int_{B_{2r}(x_0)} |\varepsilon(u)|^{p-1}|\nabla\eta^2||\nabla^2 u|\,\mathrm{d}x + \int_{B_{2r}(x_0)} |\varepsilon(u)|^{p-1}|\nabla^2\eta^2||\nabla u|\,\mathrm{d}x \\ &\qquad + \left|\int_{B_{2r}(x_0)} u^k\partial_k u^i\partial_l\eta^2\partial_i u^l\,\mathrm{d}x\right|\Bigg] \\ &=: c[J_1+J_2+J_3]\ . \end{aligned}$$

J_1 is handled in the same way as I_2, J_2 corresponds to I_1, thus we get from (3.4)

$$\int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla\varepsilon(u)|^2 \,\mathrm{d}x \le c\left[r^{-2}\int_{T_r(x_0)} |\nabla u|^p \,\mathrm{d}x + |I_5| + |I_6| + J_3\right] . \tag{3.5}$$

We estimate I_5:

$$|I_5| = \left|\int_{B_{2r}(x_0)} u^k \partial_k u^i \partial_l u^i \partial_l \eta^2 \,\mathrm{d}x\right| \le r^{-1}\int_{T_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x \,.$$

For I_6 it holds:

$$\begin{aligned}
|I_6| &= \left|\int_{B_{2r}(x_0)} u^k \partial_k u^i \partial_l \partial_l u^i \eta^2 \,\mathrm{d}x\right| = \left|\int_{B_{2r}(x_0)} \partial_l(u^k \partial_k u^i \eta^2) \partial_l u^i \,\mathrm{d}x\right| \\
&= \left|\int_{B_{2r}(x_0)} \partial_l u^k \partial_k u^i \partial_l u^i \eta^2 \,\mathrm{d}x + \int_{B_{2r}(x_0)} u^k \partial_l \partial_k u^i \eta^2 \partial_l u^i \,\mathrm{d}x\right. \\
&\qquad \left. + \int_{B_{2r}(x_0)} u^k \partial_k u^i \partial_l u^i \partial_l \eta^2 \,\mathrm{d}x\right| \\
&=: |K_1 + K_2 + K_3| \,.
\end{aligned}$$

Since we are in the 2 D-case, we have $K_1 = 0$. For K_2 we observe

$$\begin{aligned}
|K_2| &= \left|\int_{B_{2r}(x_0)} \frac{1}{2} u^k \partial_k |\nabla u|^2 \eta^2 \,\mathrm{d}x\right| = \left|\int_{T_r(x_0)} \frac{1}{2} u^k |\nabla u|^2 \partial_k \eta^2 \,\mathrm{d}x\right| \\
&\le cr^{-1}\int_{T_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x \,,
\end{aligned}$$

and clearly the same bound holds for K_3. With (3.5) we therefore arrive at

$$\begin{aligned}
&\int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla\varepsilon(u)|^2 \,\mathrm{d}x \\
&\quad \le c\left[r^{-2}\int_{T_r(x_0)} |\nabla u|^p \,\mathrm{d}x + R^{-1}\int_{T_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x + J_3\right] .
\end{aligned} \tag{3.6}$$

By the definition of J_3 we finally have

$$J_3 \le cr^{-1}\int_{T_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x \,,$$

and our claim (3.1) follows from (3.6). □

With the help of Lemma 3.1 we now give the

Proof of Theorem 1.1. Suppose that $1 < p < 2$ and that we have (1.9) together with (1.10) (the case $p = 2$ together with bounded field u follows by the same arguments setting $\alpha = 0$).

From Lemma 2.1, *iii*), it follows with the choice $x_0 = 0$ on account of $\alpha < 1/3$

$$\lim_{R\to\infty} R^{-2} \int_{B_R(0)} |\nabla u|^p \,\mathrm{d}x = 0 \,. \tag{3.7}$$

Thus (3.1) will imply

$$V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 = 0 \qquad \text{a.e. on } \mathbb{R}^2 \tag{3.8}$$

as soon as we can show that the remaining integral on the r.h.s. of (3.1) can be estimated in a suitable way.

Obviously it is also sufficient to discuss the integral of $|u||\nabla u|^2$ with $T_r(x_0)$ replaced by $\Delta_r(x_0) := B_{3r/2}(x_0) - \overline{B_r(x_0)}$. In fact, inequality (3.1) remains true with $\Delta_r(x_0)$ as domain of integration on the r.h.s., which follows by appropriate choice of η.

In order to estimate the integral $\int_{\Delta_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x$ we choose a new cut-off function $\eta \in C_0^\infty(B_{2r}(x_0))$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $\Delta_r(x_0)$ and $|\nabla \eta| \le c/r$. Moreover, we note that (1.10) implies with a positive constant

$$|u(x)| \le c(1 + |x|^2)^{\frac{\alpha}{2}} =: h(x) \,.$$

Using this bound we obtain after an integration by parts

$$\begin{aligned}
r^{-1} \int_{\Delta_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x \;&\le\; c r^{-1} \int_{B_{2r}(x_0)} h\eta^2 \partial_k u^i \partial_k u^i \,\mathrm{d}x \\
&=\; -c r^{-1} \int_{B_{2r}(x_0)} h u^i \partial_k \partial_k u^i \eta^2 \,\mathrm{d}x \\
&\quad\; -c r^{-1} \int_{B_{2r}(x_0)} h u^i \partial_k u^i \partial_k \eta^2 \,\mathrm{d}x \\
&\quad\; -c r^{-1} \int_{B_{2r}(x_0)} \partial_k h u^i \partial_k u^i \eta^2 \,\mathrm{d}x \\
&\le\; c r^{-1} \int_{B_{2r}(x_0)} (1 + |x|)^{2\alpha} |\nabla \varepsilon(u)| \,\mathrm{d}x \\
&\quad\; + c r^{-2} \int_{B_{2r}(x_0)} (1 + |x|)^{2\alpha} |\nabla u| \,\mathrm{d}x + c r^{-1} |T| \,,
\end{aligned}$$

where

$$T := \int_{B_{2r}(x_0)} \partial_k h u^i \partial_k u^i \eta^2 \,\mathrm{d}x \,.$$

On the set $[\varepsilon(u)=0]$ we clearly have $\nabla\varepsilon(u)=0$, if $\varepsilon(u)\neq 0$, then we use the definition of $V(\varepsilon)$ and obtain from Young's inequality

$$\begin{aligned} r^{-1}\int_{\Delta_r(x_0)}|u||\nabla u|^2\,\mathrm{d}x &\leq cr^{-1}\int_{B_{2r}(x_0)}(1+|x|)^{2\alpha}V(\varepsilon(u))|\nabla\varepsilon(u)||\varepsilon(u)|^{1-\frac{p}{2}}\,\mathrm{d}x \\ &\quad +cr^{-2}\int_{B_{2r}(x_0)}(1+|x|)^{2\alpha}|\nabla u|\,\mathrm{d}x+cr^{-1}|T| \\ &\leq \delta\int_{B_{2r}(x_0)}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x \\ &\quad +c(\delta)r^{-2}\int_{B_{2r}(x_0)}(1+|x|)^{4\alpha}|\varepsilon(u)|^{2-p}\,\mathrm{d}x \\ &\quad +cr^{-2}\int_{B_{2r}(x_0)}(1+|x|)^{2\alpha}|\nabla u|\,\mathrm{d}x+cr^{-1}|T|\,. \end{aligned} \tag{3.9}$$

Let us look at the quantity T: it holds

$$\begin{aligned} T &= \int_{B_{2r}(x_0)}\partial_k h\frac{1}{2}\partial_k|u|^2\eta^2\,\mathrm{d}x \\ &= -\int_{B_{2r}(x_0)}\partial_k\partial_k h\frac{1}{2}|u|^2\eta^2\,\mathrm{d}x-\int_{B_{2r}(x_0)}\partial_k h\frac{1}{2}|u|^2\partial_k\eta^2\,\mathrm{d}x\,, \end{aligned}$$

hence (recalling the bound for $|u|$ and the definition of h)

$$|T|\leq c\left[\int_{B_{2r}(x_0)}(1+|x|)^{3\alpha-2}\,\mathrm{d}x+r^{-1}\int_{B_{2r}(x_0)}(1+|x|)^{3\alpha-1}\,\mathrm{d}x\right].$$

It is worth remarking that the quantity $\int_{B_{2r}(x_0)}hu^i\partial_k u^i\partial_k\eta^2\,\mathrm{d}x$ could have been estimated in a similar way. We insert (3.9) combined with the estimate for $|T|$ into the r.h.s. of (3.1) (in the version for the annulus $\Delta_r(x_0)$ in place of $T_r(x_0)$) with the result

$$\begin{aligned} &\int_{B_r(x_0)}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x \\ &\leq \delta\int_{B_{2r}(x_0)}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x+c(\delta)\Bigg[r^{-2}\int_{B_{2r}(x_0)}|\nabla u|^p\,\mathrm{d}x \\ &\quad +r^{-2}\int_{B_{2r}(x_0)}(1+|x|)^{4\alpha}|\nabla u|^{2-p}\,\mathrm{d}x+r^{-2}\int_{B_{2r}(x_0)}(1+|x|)^{2\alpha}|\nabla u|\,\mathrm{d}x \\ &\quad +r^{-1}\int_{B_{2r}(x_0)}(1+|x|)^{3\alpha-2}\,\mathrm{d}x+r^{-2}\int_{B_{2r}(x_0)}(1+|x|)^{3\alpha-1}\,\mathrm{d}x\Bigg]\,. \end{aligned} \tag{3.10}$$

Note that (3.10) holds for all $\delta>0$ and any disk $B_{2r}(x_0)$. Then Lemma A.4 applied to

(3.10) yields for all disks

$$
\begin{aligned}
&\int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla\varepsilon(u)|^2 \,\mathrm{d}x \\
&\leq c\Bigg[r^{-2}\int_{B_{2r}(x_0)} |\nabla u|^p \,\mathrm{d}x \\
&\quad + r^{-2}\int_{B_{2r}(x_0)} (1+|x|)^{4\alpha}|\nabla u|^{2-p}\,\mathrm{d}x + r^{-2}\int_{B_{2r}(x_0)} (1+|x|)^{2\alpha}|\nabla u|\,\mathrm{d}x \\
&\quad + r^{-1}\int_{B_{2r}(x_0)} (1+|x|)^{3\alpha-2}\,\mathrm{d}x + r^{-2}\int_{B_{2r}(x_0)} (1+|x|)^{3\alpha-1}\,\mathrm{d}x \Bigg] . \qquad (3.11)
\end{aligned}
$$

At this point we make the particular choice $x_0 = 0$. We obtain for $r = R$ sufficiently large

$$
\begin{aligned}
&\int_{B_R(0)} V(\varepsilon(u))^2 |\nabla\varepsilon(u)|^2 \,\mathrm{d}x \\
&\leq c\Bigg[R^{-2}\int_{B_{2R}(0)} |\nabla u|^p \,\mathrm{d}x \\
&\quad + R^{-2+4\alpha}\int_{B_{2R}(0)} |\nabla u|^{2-p}\,\mathrm{d}x + R^{-2+2\alpha}\int_{B_{2R}(0)} |\nabla u|\,\mathrm{d}x \\
&\quad + R^{-1}\int_{B_{2R}(0)} (1+|x|)^{3\alpha-2}\,\mathrm{d}x + R^{-2}\int_{B_{2R}(0)} (1+|x|)^{3\alpha-1}\,\mathrm{d}x \Bigg] . \qquad (3.12)
\end{aligned}
$$

The first integral on the r.h.s. of (3.12) is already discussed in (3.7). For the second one we observe with the help of (2.5):

$$
\begin{aligned}
R^{-2+4\alpha}\int_{B_{2R}(0)} |\nabla u|^{2-p}\,\mathrm{d}x &\leq cR^{-2+4\alpha}\left[\int_{B_{2R}(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{2-p}{p}} R^{2\frac{2p-2}{p}} \\
&= cR^{-2+4\alpha} R^{(1+3\alpha)\frac{2-p}{p}} R^{2\frac{2p-2}{p}} \\
&= cR^{\frac{p-2}{p}} R^{\alpha\frac{p+6}{p}} \to 0 \quad \text{as } R\to\infty ,
\end{aligned}
$$

where we used the fact that (1.9) is equivalent to

$$
\frac{p-2}{p} + \alpha\frac{p+6}{p} < 0 .
$$

Next we note that (1.9) gives by elementary calculations

$$
\alpha < \frac{1}{2p+3} , \qquad (3.13)
$$

which shows

$$\begin{aligned}R^{-2+2\alpha}\int_{B_{2R}(0)}|\nabla u|\,\mathrm{d}x &\leq cR^{-2+2\alpha}\left[\int_{B_{2R}(0)}|\nabla u|^p\,\mathrm{d}x\right]^{\frac{1}{p}}R^{2(1-\frac{1}{p})}\\ &\leq cR^{-2+2\alpha+\frac{1+3\alpha}{p}+2-\frac{2}{p}}\\ &= cR^{-\frac{1}{p}+\alpha\frac{2p+3}{p}}\to 0 \quad \text{as} \ \ R\to\infty\ .\end{aligned}$$

Finally we discuss the last two integrals on the r.h.s. of (3.12): we have

$$\begin{aligned}R^{-1}\int_{B_{2R}(0)}(1+|x|)^{3\alpha-2}\,\mathrm{d}x &= 2\pi R^{-1}\int_0^{2R}(1+t)^{3\alpha-2}t\,\mathrm{d}t\\ &\leq 2\pi R^{-1}\int_0^{2R}(1+t)^{3\alpha-1}\,\mathrm{d}t\\ &= \frac{2\pi}{3\alpha}R^{-1}\big[(1+2R)^{3\alpha}-1\big]\to 0\end{aligned}$$

as $R\to\infty$ on account of $\alpha<1/3$. Moreover,

$$R^{-2}\int_{B_{2R}(0)}(1+|x|)^{3\alpha-1}\,\mathrm{d}x\leq cR^{-2}R^{3\alpha-1}\to 0$$

as $R\to\infty$, and with (3.12) we have shown

$$\int_{\mathbb{R}^2}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x=0\ ,$$

which implies (3.8).

On the set $[\varepsilon(u)=0]$ we once more observe $\nabla\varepsilon(u)=0$, hence $\nabla^2 u=0$ by recalling the inequality $|\nabla^2 u|\leq c|\nabla\varepsilon(u)|$ a.e. On the set $[\varepsilon(u)\neq 0]$ we deduce $\nabla\varepsilon(u)=0$ from (3.8). Thus $\nabla^2 u=0$ on $\mathbb{R}^2$, which means that u is affine. However, since we assume the growth condition (1.10), the constancy of u is established, which completes the proof of Theorem 1.1. $\square$

The proof of Theorem 1.2 additionally needs the following auxiliary results:

Lemma 3.2 *If u is as in Lemma 3.1, then $v:=|\varepsilon(u)|^{\frac{p}{2}}$ belongs to the space $W^1_{2,\mathrm{loc}}(\mathbb{R}^2)$ and*

$$\int_\Omega|\nabla v|^2\,\mathrm{d}x\leq c\int_\Omega V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x$$

for any domain $\Omega\Subset\mathbb{R}^2$.

Proof of Lemma 3.2. Let $v_\delta := (\delta + |\varepsilon(u)|)^{p/2}$, $\delta > 0$. From $u \in W^2_{p,\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ it easily follows that $v_\delta \in W^1_{2,\mathrm{loc}}(\mathbb{R}^2)$ together with

$$|\nabla v_\delta|^2 \begin{cases} \leq & cV(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \quad \text{on the set} \quad [\varepsilon(u) \neq 0] \ , \\ = & 0 \qquad\qquad\qquad\qquad\quad \text{on the set} \quad [\varepsilon(u) = 0] \ , \end{cases} \tag{3.14}$$

so that the sequence $\{v_\delta\}$ is locally uniformly bounded in $W^1_{2,\mathrm{loc}}(\mathbb{R}^2)$, thus

$$v_\delta \rightharpoondown: \tilde{v} \quad \text{in} \ W^1_{2,\mathrm{loc}}(\mathbb{R}^2) \ .$$

Clearly $\tilde{v} = v$, and the desired estimate for $\int_\Omega |\nabla v|^2 \,\mathrm{d}x$ follows from (3.14) and lower semicontinuity. □

Lemma 3.3 *Suppose that $v \in C^1(\mathbb{R}^2)$ satisfies $\int_{\mathbb{R}^2} |\nabla v|^p \,\mathrm{d}x < \infty$ for some $p \in (1,2)$. Then it holds*

$$\limsup_{R\to\infty} R^{-2} \int_{B_R(0)} |v| \,\mathrm{d}x < \infty \ ,$$

in particular we deduce for any $\beta > 2$

$$\lim_{R\to\infty} R^{-\beta} \int_{B_R(0)} |v| \,\mathrm{d}x = 0 \ .$$

Proof of Lemma 3.3. W.l.o.g. let $x_0 = 0$ and fix some real number $\gamma > 0$. Introducing polar coordinates r, θ we define

$$f(r,\theta) = |v(r\cos(\theta), r\sin(\theta))| + \gamma \ .$$

The following calculations are essentially due to Gilbarg and Weinberger (see [GW], proof of Lemma 2.1). We have by Hölder's inequality

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r}\left[\int_0^{2\pi} f(r,\theta)^p \,\mathrm{d}\theta\right]^{\frac{1}{p}} \\ &\leq \left[\int_0^{2\pi} f(r,\theta)^p \,\mathrm{d}\theta\right]^{\frac{1}{p}-1} \int_0^{2\pi} f(r,\theta)^{p-1} |f_r(r,\theta)| \,\mathrm{d}\theta \\ &\leq \left[\int_0^{2\pi} f(r,\theta)^p \,\mathrm{d}\theta\right]^{\frac{1}{p}-1} \left[\int_0^{2\pi} f(r,\theta)^p \,\mathrm{d}\theta\right]^{\frac{p-1}{p}} \left[\int_0^{2\pi} |f_r(r,\theta)|^p \,\mathrm{d}\theta\right]^{\frac{1}{p}} , \end{aligned}$$

where we use the symbol f_r for the partial derivative of f with respect to the variable r. Thus, for any $\gamma > 0$ we have shown (recall that f is depending on the parameter γ)

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\int_0^{2\pi} f(r,\theta)^p \,\mathrm{d}\theta\right]^{\frac{1}{p}} \leq \left[\int_0^{2\pi} |f_r(r,\theta)|^p \,\mathrm{d}\theta\right]^{\frac{1}{p}} . \tag{3.15}$$

Now let

$$\varphi(t) := \left[\int_0^{2\pi} f(t,\theta)^p \,\mathrm{d}\theta\right]^{\frac{1}{p}} .$$

From (3.15) we get for any $R > 1$:

$$\begin{aligned}
\varphi(R) - \varphi(1) &\leq \int_1^R \left[\int_0^{2\pi} |f_r(r,\theta)|^p \,\mathrm{d}\theta\right]^{\frac{1}{p}} \mathrm{d}r \\
&= \int_1^R \left[\int_0^{2\pi} |f_r(r,\theta)|^p \,\mathrm{d}\theta\right]^{\frac{1}{p}} r^{\frac{1}{p}} r^{-\frac{1}{p}} \,\mathrm{d}r \\
&\leq \left[\int_1^R \left[\int_0^{2\pi} |f_r(r,\theta)|^p \,\mathrm{d}\theta\right] r \,\mathrm{d}r\right]^{\frac{1}{p}} \left[\int_1^R r^{-\frac{1}{p}\frac{p}{p-1}} \,\mathrm{d}r\right]^{1-\frac{1}{p}} ,
\end{aligned}$$

where we have used Hölder's inequality once more. This shows (recall $p < 2$)

$$\varphi(R) \leq \varphi(1) + c(p)\left[\int_1^R \int_0^{2\pi} |f_r(r,\theta)|^p r \,\mathrm{d}\theta \,\mathrm{d}r\right]^{\frac{1}{p}}$$

and since

$$|f_r(r,\theta)| \leq |\nabla v|(re^{i\theta}) ,$$

we deduce

$$\varphi(R) \leq \varphi(1) + c(p)\left[\int_{B_R(0)-\overline{B_1(0)}} |\nabla v|^p \,\mathrm{d}x\right]^{\frac{1}{p}} . \tag{3.16}$$

In (3.16) we pass to the limit $\gamma \to 0$ and the finiteness of the energy then yields the inequality

$$\sup_{R\geq 1} \int_0^{2\pi} |v(R\cos(\theta), R\sin(\theta))|^p \,\mathrm{d}\theta < \infty . \tag{3.17}$$

Hence, for any $R > 1$ we obtain from (3.17)

$$\begin{aligned}
\int_{B_R(0)} |v|^p \,\mathrm{d}x &= \int_0^R \int_0^{2\pi} |v(r\cos(\theta), r\sin(\theta))|^p r \,\mathrm{d}\theta \,\mathrm{d}r \\
&\leq c + \int_1^R \int_0^{2\pi} |v(r\cos(\theta), r\sin(\theta))|^p r \,\mathrm{d}\theta \,\mathrm{d}r \\
&\leq c(1+R^2) ,
\end{aligned}$$

which proves Lemma 3.3. □

Proof of Theorem 1.2. Now our assumption on u is

$$\int_{\mathbb{R}^2} |\nabla u|^p \,\mathrm{d}x < \infty , \tag{3.18}$$

and in view of this hypothesis and by quoting Lemma 3.1 we have to discuss the quantity

$$r^{-1}\int_{T_r(x_0)}|u||\nabla u|^2\,\mathrm{d}x$$

in order to verify (3.8) for the situation at hand. Let

$$A := \fint_{T_r(x_0)} u\,\mathrm{d}x\,.$$

Clearly it holds

$$r^{-1}\int_{T_r(x_0)}|u||\nabla u|^2\,\mathrm{d}x \le cr^{-1}\int_{T_r(x_0)}|u-A||\nabla u|^2\,\mathrm{d}x + cr^{-1}|A|\int_{T_r(x_0)}|\nabla u|^2\,\mathrm{d}x\,. \qquad (3.19)$$

In (3.19) we apply Hölder's and Young's inequality and get for any $\delta > 0$

$$\begin{aligned} r^{-1}\int_{T_r(x_0)}|u||\nabla u|^2\,\mathrm{d}x \;&\le\; c\Bigg[\int_{T_r(x_0)}\Big[\frac{|u-A|}{r}\Big]^{\frac{p}{p-1}}\mathrm{d}x\Bigg]^{\frac{p-1}{p}}\Bigg[\int_{T_r(x_0)}|\nabla u|^{2p}\,\mathrm{d}x\Bigg]^{\frac{1}{p}} \\ &\quad +\delta\int_{T_r(x_0)}|\nabla u|^{2p}\,\mathrm{d}x \\ &\quad +c(\delta)r^2\Bigg[r^{-3}\int_{T_r(x_0)}|u|\,\mathrm{d}x\Bigg]^{\frac{p}{p-1}}\,. \end{aligned} \qquad (3.20)$$

To the first integral on the r.h.s. of (3.20) we apply the Sobolev-Poincaré inequality: let $p^* := 2p'/(2+p')$, $p' := p/(p-1)$, so that p' is the Sobolev exponent of p^*.

Let us first consider the case $p \ge 4/3$ for which $p^* \le p$. Then we have

$$\Bigg[\int_{T_r(x_0)}|u-A|^{p'}\,\mathrm{d}x\Bigg]^{\frac{1}{p'}} \le c\Bigg[\int_{T_r(x_0)}|\nabla u|^{p^*}\,\mathrm{d}x\Bigg]^{\frac{1}{p^*}}\,,$$

and by Hölder's inequality

$$\begin{aligned} \Bigg[\int_{T_r(x_0)}|u-A|^{p'}\,\mathrm{d}x\Bigg]^{\frac{1}{p'}} \;&\le\; c\Bigg[\int_{T_r(x_0)}|\nabla u|^{p}\,\mathrm{d}x\Bigg]^{\frac{1}{p}} r^{2(1-\frac{p^*}{p})\frac{1}{p^*}} \\ &=\; cr^{3-\frac{4}{p}}\Bigg[\int_{T_r(x_0)}|\nabla u|^{p}\,\mathrm{d}x\Bigg]^{\frac{1}{p}}\,. \end{aligned}$$

We therefore obtain

$$\begin{aligned} r^{-1}\int_{T_r(x_0)}|u||\nabla u|^2\,\mathrm{d}x &\le cr^{2-\frac{4}{p}}\Big[\int_{T_r(x_0)}|\nabla u|^p\,\mathrm{d}x\Big]^{\frac{1}{p}}\Big[\int_{T_r(x_0)}|\nabla u|^{2p}\,\mathrm{d}x\Big]^{\frac{1}{p}} \\ &\quad +\delta\int_{T_r(x_0)}|\nabla u|^{2p}\,\mathrm{d}x \\ &\quad +c(\delta)r^2\Big[r^{-3}\int_{T_r(x_0)}|u|\,\mathrm{d}x\Big]^{\frac{p}{p-1}}\,. \end{aligned} \tag{3.21}$$

Let $\gamma := 2-4/p$ and assume w.l.o.g. that $p<2$, hence $\gamma<0$. Using our assumption (3.18) in (3.21), we find

$$\begin{aligned} r^{-1}\int_{T_r(x_0)}|u||\nabla u|^2\,\mathrm{d}x &\le \delta\int_{T_r(x_0)}|\nabla u|^{2p}\,\mathrm{d}x + cr^{\gamma}\Big[\int_{T_r(x_0)}|\nabla u|^{2p}\,\mathrm{d}x\Big]^{\frac{1}{p}} \\ &\quad +c(\delta)r^2\Big[r^{-3}\int_{T_r(x_0)}|u|\,\mathrm{d}x\Big]^{\frac{p}{p-1}}\,, \end{aligned}$$

and another application of Young's inequality shows

$$\begin{aligned} r^{-1}\int_{T_r(x_0)}|u||\nabla u|^2\,\mathrm{d}x &\le 2\delta\int_{T_r(x_0)}|\nabla u|^{2p}\,\mathrm{d}x \\ &\quad +c(\delta)\Big[r^{\gamma\frac{p}{p-1}}+r^2\Big[r^{-3}\int_{T_r(x_0)}|u|\,\mathrm{d}x\Big]^{\frac{p}{p-1}}\Big]\,. \end{aligned} \tag{3.22}$$

Next we discuss the quantity $\int_{B_{2r}(x_0)}|\nabla u|^{2p}\,\mathrm{d}x$: by Korn's inequality Lemma A.2, *ii*), we have

$$\int_{B_{2r}(x_0)}|\nabla u|^{2p}\,\mathrm{d}x \le c\Big[\int_{B_{2r}(x_0)}|\varepsilon(u)|^{2p}\,\mathrm{d}x + r^{-2p}\int_{B_{2r}(x_0)}|u|^{2p}\,\mathrm{d}x\Big]\,. \tag{3.23}$$

Since u is a function of class $C^1(\mathbb{R}^2,\mathbb{R}^2)$ and thereby an element of the space $W^1_{2p,\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ we can apply the L^{2p}-variant of Korn's inequality to get (3.23). Let $B := \fint_{B_{2r}(x_0)}u\,\mathrm{d}x$ and $q := 4p/(2+2p)$, i.e. $2p$ is the Sobolev exponent of q. We therefore get from the Sobolev-Poincaré inequality

$$\begin{aligned} \|u\|_{L^{2p}(B_{2r}(x_0))} &\le c\Big[\|u-B\|_{L^{2p}(B_{2r}(x_0))}+|B|r^{\frac{1}{p}}\Big] \\ &\le c\Big[\|\nabla u\|_{L^q(B_{2r}(x_0))}+|B|r^{\frac{1}{p}}\Big] \\ &\le c\Big[\Big[\int_{B_{2r}(x_0)}|\nabla u|^p\,\mathrm{d}x\Big]^{\frac{1}{p}}r^{2(\frac{1}{q}-\frac{1}{p})}+|B|r^{\frac{1}{p}}\Big]\,, \end{aligned}$$

hence (quoting (3.18))

$$r^{-2p}\int_{B_{2r}(x_0)}|u|^{2p}\,\mathrm{d}x\le c\big[r^{-2}+|B|^{2p}r^{2-2p}\big]\,. \tag{3.24}$$

By Lemma 3.2 the function $v:=|\varepsilon(u)|^{p/2}$ is in the local space $W^1_{2,\mathrm{loc}}(\mathbb{R}^2)$, and from Lemma A.3 we obtain

$$\begin{aligned}\int_{B_{2r}(x_0)}|\varepsilon(u)|^{2p}\,\mathrm{d}x &\le c\Bigg[\int_{B_{2r}(x_0)}|\varepsilon(u)|^{p}\,\mathrm{d}x\int_{B_{2r}(x_0)}|\nabla v|^2\,\mathrm{d}x\\ &\quad+r^{-2}\Bigg[\int_{B_{2r}(x_0)}|\varepsilon(u)|^{p}\,\mathrm{d}x\Bigg]^2\Bigg]\,,\end{aligned}$$

thus by (3.18) and the estimate for $\int_{B_{2r}(x_0)}|\nabla v|^2\,\mathrm{d}x$ stated in Lemma 3.2 we find

$$\int_{B_{2r}(x_0)}|\varepsilon(u)|^{2p}\,\mathrm{d}x\le c\Bigg[\int_{B_{2r}(x_0)}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x+r^{-2}\Bigg]\,. \tag{3.25}$$

Inserting (3.23)-(3.25) into (3.22) we get

$$\begin{aligned}r^{-1}\int_{T_r(x_0)}|u||\nabla u|^2\,\mathrm{d}x &\le 2\delta\int_{B_{2r}(x_0)}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x\\ &\quad+c(\delta)\Bigg[r^{-2}+|B|^{2p}r^{2-2p}+r^{\gamma\frac{p}{p-1}}\\ &\quad+r^2\Bigg[r^{-3}\int_{T_r(x_0)}|u|\,\mathrm{d}x\Bigg]^{\frac{p}{p-1}}\Bigg]\,.\end{aligned} \tag{3.26}$$

Next we return to (3.1) estimating the second term on the r.h.s. through (3.26) with the result (replacing δ by $\delta/2$)

$$\begin{aligned}\int_{B_r(x_0)}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x &\le \delta\int_{B_{2r}(x_0)}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x\\ &\quad+c(\delta)\Bigg[r^{-2}+r^{\gamma\frac{p}{p-1}}+r^{2-2p}\Bigg[\fint_{B_{2r}(x_0)}|u|\,\mathrm{d}x\Bigg]^{2p}\\ &\quad+r^2\Bigg[R^{-3}\int_{B_{2r}(x_0)}|u|\,\mathrm{d}x\Bigg]^{\frac{p}{p-1}}\Bigg]\,.\end{aligned}$$

Applying the δ-Lemma A.4 we arrive at (after choosing $r=R\ge 1$ and $x_0=0$)

$$\begin{aligned}\int_{B_R(0)}V(\varepsilon(u))^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x &\le c\Bigg[R^{-2}+R^{\gamma\frac{p}{p-1}}+\Bigg[R^{\frac{1}{p}-3}\int_{B_{2R}(0)}|u|\,\mathrm{d}x\Bigg]^{2p}\\ &\quad+\Bigg[R^{2\frac{p-1}{p}-3}\int_{B_{2R}(0)}|u|\,\mathrm{d}x\Bigg]^{\frac{p}{p-1}}\Bigg]\,.\end{aligned} \tag{3.27}$$

By Lemma 3.3 it follows that the r.h.s. of (3.27) vanishes as $R \to \infty$, thus we obtain (3.8) and, as outlined at the end of the proof of Theorem 1.1, u has to be an affine function. But then (3.18) yields the constancy of u, which proves Theorem 1.2 in the case $p \geq 4/3$.

If $6/5 < p < 4/3$ we return to (3.21) and estimate the r.h.s. of the inequality stated in (3.20) in a different way: observing that by the choice of p

$$p < p^* = \frac{2p}{3p-2} < 2p \, ,$$

we can apply the interpolation inequality

$$\|\nabla u\|_{p^*} \leq \|\nabla u\|_p^{\alpha} \|\nabla u\|_{2p}^{1-\alpha} \, ,$$

where all norms are calculated over $T_r(x_0)$ and where

$$\frac{1}{p^*} = \frac{\alpha}{p} + \frac{1-\alpha}{2p} \, , \qquad \text{hence} \qquad \alpha = \frac{2p}{p^*} - 1 \, .$$

This gives using (3.18)

$$\begin{aligned} \left[\int_{T_r(x_0)} \left| \frac{u-A}{r} \right|^{\frac{p}{p-1}} \mathrm{d}x \right]^{\frac{p-1}{p}} \left[\int_{T_r(x_0)} |\nabla u|^{2p} \, \mathrm{d}x \right]^{\frac{1}{p}} &\leq cr^{-1} \|\nabla u\|_{p^*} \|\nabla u\|_{2p}^2 \\ &\leq cr^{-1} \|\nabla u\|_p^{\alpha} \|\nabla u\|_{2p}^{2+1-\alpha} \\ &\leq cr^{-1} \left[\int_{T_r(x_0)} |\nabla u|^{2p} \, \mathrm{d}x \right]^{\frac{3-\alpha}{2p}} \, . \end{aligned}$$

With elementary calculations one obtains

$$\frac{3-\alpha}{2p} = \frac{6-3p}{2p}$$

and we find that

$$\frac{3-\alpha}{2p} < 1$$

is true under our hypothesis $p > 6/5$. This gives us the flexibility to apply Young's inequality with the result

$$r^{-1} \left[\int_{T_r(x_0)} |\nabla u|^{2p} \, \mathrm{d}x \right]^{\frac{3-\alpha}{2p}} \leq c \left[r^{-\kappa} + \int_{T_r(x_0)} |\nabla u|^{2p} \, \mathrm{d}x \right]$$

with a suitable positive exponent κ. Using this estimate in (3.20) the proof can be finished as before. $\square$

4 The case $p > 2$

We start with an appropriate variant of Lemma 3.1 which is more difficult to establish since now we can no longer benefit from the higher weak differentiability results of Naumann [Na] and Wolf [Wo].

Lemma 4.1 *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ denote a solenoidal field satisfying (1.7) with $H(\varepsilon) = |\varepsilon|^p$ for some exponent $p > 2$. Moreover, let*

$$W := W(\varepsilon(u)) := |\varepsilon(u)|^{\frac{p-2}{2}} \varepsilon(u) \, .$$

Then it holds:

i) W is in the space $W^1_{2,\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^{2\times 2})$.

ii) There exists a finite constant c independent of u such that for any $\delta > 0$ and for each $q > 2$

$$\begin{aligned} \int_{B_r(x_0)} |\nabla W|^2 \,\mathrm{d}x \;\; &\leq \;\; \delta \int_{B_{2r}(x_0)} |\nabla W|^2 \,\mathrm{d}x + c \Bigg[\delta^{-1} r^{-2} \int_{T_r(x_0)} |\nabla u|^p \,\mathrm{d}x \\ &\quad + r^{-1} \Bigg[\int_{T_r(x_0)} |u|^{\frac{q}{q-2}} \,\mathrm{d}x \Bigg]^{1-\frac{2}{q}} \Bigg[\int_{T_r(x_0)} |\nabla u|^q \,\mathrm{d}x \Bigg]^{\frac{2}{q}} \Bigg] \end{aligned} \tag{4.1}$$

for any disk $B_r(x_0)$.

Proof. We use the difference quotient technique and let

$$\Delta_h^\alpha v(x) := \frac{1}{h}\big(v(x + h e_\alpha) - v(x) \big)$$

for functions v, parameters $h \neq 0$ and a coordinate direction e_α, $\alpha = 1, 2$. If $\varphi \in C^1_0(\mathbb{R}^2, \mathbb{R}^2)$ satisfies $\operatorname{div} \varphi = 0$, then we have the equation(1.7) together with the identity

$$0 = \int_{\mathbb{R}^2} DH(\varepsilon(u))(x + h e_\alpha) : \varepsilon(\varphi)(x) \,\mathrm{d}x + \int_{\mathbb{R}^2} (u^k \partial_k u^i)(x + h e_\alpha) \varphi^i(x) \,\mathrm{d}x \, ,$$

hence after subtracting the equations and after dividing by h

$$\int_{\mathbb{R}^2} \Delta_h^\alpha \big(DH(\varepsilon(u)) \big) : \varepsilon(\varphi) \,\mathrm{d}x + \int_{\mathbb{R}^2} \Delta_h^\alpha (u^k \partial_k u) \cdot \varphi \,\mathrm{d}x = 0 \, , \tag{4.2}$$

and (4.2) clearly extends to solenoidal fields from $W^1_{p,\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with compact support. Alternatively – taking into account the pressure function π in the weak form of (1.4) – we can replace (4.2) by

$$\begin{aligned} 0 \;\; &= \;\; \int_{\mathbb{R}^2} \Delta_h^\alpha \big(DH(\varepsilon(u)) \big) : \varepsilon(\varphi) \,\mathrm{d}x + \int_{\mathbb{R}^2} \Delta_h^\alpha (u^k \partial_k u) \cdot \varphi \,\mathrm{d}x - \int_{\mathbb{R}^2} \Delta_h^\alpha \pi \operatorname{div} \varphi \,\mathrm{d}x \\ &=: \;\; T_1 + T_2 + T_3 \end{aligned} \tag{4.3}$$

valid for all $\varphi \in W^1_{p,\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ with compact support. In (4.3) we choose $\varphi := \varphi_\alpha := \eta^2\Delta_h^\alpha u$ with $\alpha = 1, 2$ being fixed (no summation convention w.r.t. α) and with $\eta \in C_0^2(B_{2r}(x_0))$, $0 \le \eta \le 1$, $\eta = 1$ on $B_r(x_0)$, $|\nabla\eta| \le cr^{-1}$. We discuss the quantities T_i from (4.3) related to our choice of φ: it holds

$$\begin{aligned} T_1 &= \int_{B_{2r}(x_0)} \Delta_h^\alpha\big(DH(\varepsilon(u))\big) : \varepsilon(\Delta_h^\alpha u)\eta^2 \,\mathrm{d}x \\ &\quad + \int_{B_{2r}(x_0)} \Delta_h^\alpha\big(DH(\varepsilon(u))\big) : (\nabla\eta^2 \otimes \Delta_h^\alpha u) \,\mathrm{d}x \\ &=: U_1 + U_2\ , \end{aligned}$$

and for U_1 we observe

$$\begin{aligned} &\Delta_h^\alpha\big(|\varepsilon(u)|^{p-2}\varepsilon(u)\big)(x) : \varepsilon(\Delta_h^\alpha u)(x) \\ &= \frac{1}{h}\Big[|\varepsilon(u)|^{p-2}(x+he_\alpha)\varepsilon(u)(x+he_\alpha) - |\varepsilon(u)|^{p-2}(x)\varepsilon(u)(x)\Big] : \\ &\qquad \frac{1}{h}\Big[\varepsilon(u)(x+he_\alpha) - \varepsilon(u)(x)\Big] \\ &\ge c\Big[|\varepsilon(u)|^{p-2}(x+he_\alpha) + |\varepsilon(u)|^{p-2}(x)\Big]\Delta_h^\alpha\varepsilon(u)(x) : \Delta_h^\alpha\varepsilon(u)(x)\ , \end{aligned}$$

where the last inequality can be easily deduced from Lemma A.5, *ii*). At the same time, Lemma A.5, *i*), implies

$$\begin{aligned} &\frac{1}{|h|}\Big||\varepsilon(u)|^{p-2}(x+he_\alpha)\varepsilon(u)(x+he_\alpha) - |\varepsilon(u)|^{p-2}(x)\varepsilon(u)(x)\Big| \\ &\le c\Big[|\varepsilon(u)|^2(x+he_\alpha) + |\varepsilon(u)|^2(x)\Big]^{\frac{p-2}{2}} \frac{1}{|h|}\big|\varepsilon(u)(x+he_\alpha) - \varepsilon(u)(x)\big|\ , \end{aligned}$$

thus using Young's inequality

$$\begin{aligned} |U_2| &\le c\int_{B_{2r}(x_0)} \big[|\varepsilon(u)|(x+he_\alpha) + |\varepsilon(u)|(x)\big]^{p-2}|\Delta_h^\alpha\varepsilon(u)||\Delta_h^\alpha u|\nabla\eta|^2 \,\mathrm{d}x \\ &\le \delta\int_{B_{2r}(x_0)} \big[|\varepsilon(u)|(x+he_\alpha) + |\varepsilon(u)|(x)\big]^{p-2}\Delta_h^\alpha\varepsilon(u) : \Delta_h^\alpha\varepsilon(u)\eta^2 \,\mathrm{d}x \\ &\quad + c\delta^{-1}\int_{B_{2r}(x_0)} \big[|\varepsilon(u)|(x+he_\alpha) + |\varepsilon(u)|(x)\big]^{p-2}|\nabla\eta|^2\Delta_h^\alpha u \cdot \Delta_h^\alpha u \,\mathrm{d}x \end{aligned}$$

for any $\delta > 0$. Combining these estimates, returning to (4.3) and choosing δ small enough we find

$$\begin{aligned} &\int_{B_{2r}(x_0)} \Big[|\varepsilon(u)|^{p-2}(x+he_\alpha) + |\varepsilon(u)|^{p-2}(x)\Big]\eta^2\Delta_h^\alpha\varepsilon(u) : \Delta_h^\alpha\varepsilon(u) \,\mathrm{d}x \\ &\le c\Bigg[\int_{T_r(x_0)} \Big[|\varepsilon(u)|^{p-2}(x+he_\alpha) + |\varepsilon(u)|^{p-2}(x)\Big]|\nabla\eta|^2\Delta_h^\alpha u \cdot \Delta_h^\alpha u \,\mathrm{d}x \\ &\qquad + |T_2| + |T_3|\Bigg]\ . \end{aligned} \tag{4.4}$$

Next we look at the pressure term T_3: we have

$$\operatorname{div}(\eta^2\Delta_h^\alpha u) = \nabla\eta^2 \cdot \Delta_h^\alpha u =: f_h^\alpha$$

where the function f_h^α is compactly supported in $T_r(x_0)$. Moreover, we have by the definition of f_h^α and the properties of η

$$\begin{aligned}
\int_{T_r(x_0)} f_h^\alpha \,\mathrm{d}x &= \int_{T_r(x_0)} \operatorname{div}(\eta^2 \cdot \Delta_h^\alpha u)\,\mathrm{d}x \\
&= -\int_{\partial B_r(x_0)} \Delta_h^\alpha u(x) \cdot \frac{x-x_0}{r}\,\mathrm{d}\mathcal{H}^1(x) \\
&= -\int_{B_r(x_0)} \operatorname{div}(\Delta_h^\alpha u)\,\mathrm{d}x = 0\,,
\end{aligned}$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff-measure. According to Lemma A.1 we find $\psi_h^\alpha \in \overset{\circ}{W}{}^1_p(T_r(x_0),\mathbb{R}^2)$ satisfying $\operatorname{div}\psi_h^\alpha = f_h^\alpha$ on $T_r(x_0)$ and sharing the usual estimates on the annulus $T_r(x_0)$. We get

$$\begin{aligned}
|T_3| &= \left|\int_{T_r(x_0)} \Delta_h^\alpha \pi \operatorname{div}(\eta^2\Delta_h^\alpha u)\,\mathrm{d}x\right| = \left|\int_{T_r(x_0)} \Delta_h^\alpha \pi f_h^\alpha \,\mathrm{d}x\right| \\
&= \left|\int_{T_r(x_0)} \Delta_h^\alpha \pi \operatorname{div}\psi_h^\alpha\,\mathrm{d}x\right|
\end{aligned}$$

and if we use (4.3) with ψ_h^α as test function it follows

$$\begin{aligned}
|T_3| &= \left|\int_{T_r(x_0)} \Delta_h^\alpha\big(DH(\varepsilon(u))\big) : \varepsilon(\psi_h^\alpha)\,\mathrm{d}x + \int_{T_r(x_0)} \Delta_h^\alpha(u^k\partial_k u)\cdot\psi_h^\alpha\,\mathrm{d}x\right| \\
&=: |S_1 + S_2|\,. \qquad (4.5)
\end{aligned}$$

For S_1 we first observe (compare the discussion of U_2)

$$\begin{aligned}
|S_1| &\le c\int_{T_r(x_0)} \big|\Delta_h^\alpha(|\varepsilon(u)|^{p-2}\varepsilon(u))\big||\varepsilon(\psi_h^\alpha)|\,\mathrm{d}x \\
&\le c\int_{T_r(x_0)} \big(|\varepsilon(u)|(x+he_\alpha) + |\varepsilon(u)|(x)\big)^{p-2}|\Delta_h^\alpha\varepsilon(u)||\varepsilon(\psi_h^\alpha)|\,\mathrm{d}x
\end{aligned}$$

and then use Young's inequality to get for any $\delta > 0$

$$\begin{aligned}
|S_1| &\le \delta\int_{T_r(x_0)} \big(|\varepsilon(u)|(x+he_\alpha) + |\varepsilon(u)|(x)\big)^{p-2}|\Delta_h^\alpha\varepsilon(u)|^2\,\mathrm{d}x \\
&\quad + c\delta^{-1}\int_{T_r(x_0)} \big(|\varepsilon(u)|(x+he_\alpha) + |\varepsilon(u)|(x)\big)^{p-2}|\varepsilon(\psi_h^\alpha)|^2\,\mathrm{d}x\,. \qquad (4.6)
\end{aligned}$$

According to [Ga1], Theorem 3.2, p. 130, the support of ψ_h^α is compact in $T_r(x_0)$ and by quoting Lemma 7.23 of [GT] we can estimate using Hölder's inequality

$$\begin{aligned}
& c\delta^{-1}\int_{T_r(x_0)} \big(|\varepsilon(u)|(x+he_\alpha)+|\varepsilon(u)|(x)\big)^{p-2}|\varepsilon(\psi_h^\alpha)|^2\,\mathrm{d}x \\
\leq\ & c\delta^{-1}\left[\int_{T_r(x_0)} |\varepsilon(\psi_h^\alpha)|^p\,\mathrm{d}x\right]^{\frac{2}{p}}\left[\int_{T_r(x_0)} |\nabla u|^p\,\mathrm{d}x\right]^{1-\frac{2}{p}} \\
\leq\ & c\delta^{-1}r^{-2}\int_{T_r(x_0)} |\nabla u|^p\,\mathrm{d}x\ . \qquad (4.7)
\end{aligned}$$

We apply a similar reasoning to the first term on the r.h.s. of (4.4) and get from (4.4)–(4.7)

$$\begin{aligned}
& \int_{B_r(x_0)} \eta^2\big(|\varepsilon(u)|^{p-2}(x+he_\alpha)+|\varepsilon(u)|^{p-2}(x)\big)|\Delta_h^\alpha\varepsilon(u)|^2\,\mathrm{d}x \\
\leq\ & \delta\int_{T_r(x_0)} \big(|\varepsilon(u)|^{p-2}(x+he_\alpha)+|\varepsilon(u)|^{p-2}(x)\big)|\Delta_h^\alpha\varepsilon(u)|^2|\,\mathrm{d}x \\
& +c\delta^{-1}r^{-2}\int_{T_r(x_0)} |\nabla u|^p\,\mathrm{d}x + c\big[|T_2|+|S_2|\big] \qquad (4.8)
\end{aligned}$$

with T_2 defined in (4.3) for the choice $\varphi=\eta^2\Delta_h^\alpha u$ and S_2 from (4.5). Let us look at T_2: we have

$$\begin{aligned}
T_2 &= \int_{B_{2r}(x_0)} \Delta_h^\alpha(u^k\partial_k u^i)\eta^2\Delta_h^\alpha u^i\,\mathrm{d}x \\
&= \int_{B_{2r}(x_0)} \Delta_h^\alpha u^k\partial_k u^i\Delta_h^\alpha u^i\eta^2\,\mathrm{d}x + \int_{B_{2r}(x_0)} u^k\partial_k(\Delta_h^\alpha u^i)\Delta_h^\alpha u^i\eta^2\,\mathrm{d}x \\
&= \int_{B_{2r}(x_0)} \Delta_h^\alpha u^k\partial_k u^i\Delta_h^\alpha u^i\eta^2\,\mathrm{d}x - \frac{1}{2}\int_{B_{2r}(x_0)} u^k\big(\Delta_h^\alpha u\cdot\Delta_h^\alpha u\big)\partial_k\eta^2\,\mathrm{d}x\ ,
\end{aligned}$$

hence

$$|T_2|\leq c\left[\int_{B_{2r}(x_0)}\big(\Delta_h^\alpha u\cdot\Delta_h^\alpha u\big)|\nabla u|\,\mathrm{d}x+\frac{1}{r}\int_{T_r(x_0)}\big(\Delta_h^\alpha u\cdot\Delta_h^\alpha u\big)|u|\,\mathrm{d}x\right]. \qquad (4.9)$$

For estimating S_2 we again use the properties of ψ_h^α as already done after (4.6):

$$\begin{aligned}
S_2 &= -\int_{T_r(x_0)} u^k\partial_k u\cdot\Delta_{-h}^\alpha\psi_h^\alpha\,\mathrm{d}x \\
&\leq \left[\int_{T_r(x_0)} |\nabla\psi_h^\alpha|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{T_r(x_0)} |u|^2|\nabla u|^2\,\mathrm{d}x\right]^{\frac{1}{2}} \\
&\leq cr^{-1}\left[\int_{T_r(x_0)} |\nabla u|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{T_r(x_0)} |u|^2|\nabla u|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\ ,
\end{aligned}$$

thus

$$|S_2| \le c\left[r^{-1}\int_{T_r(x_0)} |\nabla u|^2 \,\mathrm{d}x + r^{-1}\int_{T_r(x_0)} |u|^2|\nabla u|^2 \,\mathrm{d}x\right] . \tag{4.10}$$

Inserting (4.9) and (4.10) into (4.8) and using the δ-Lemma A.4 with suitable functions f, f_j and g (replacing the domain of integration $T_r(x_0)$ through $B_{2r}(x_0)$ on the r.h.s. of the inequalities under consideration), we deduce

$$\int_{B_r(x_0)} \big(|\varepsilon(u)|^{p-2}(x+he_\alpha) + |\varepsilon(u)|^{p-2}(x)\big)|\Delta_h^\alpha \varepsilon(u)|^2 \,\mathrm{d}x \le c(r,u) < \infty \tag{4.11}$$

for a constant $c(r,u)$ being independent of h. Now it is easy to see (cf. Lemma A.5, *i*)) that

$$\Delta_h^\alpha W(\varepsilon(u)) : \Delta_h^\alpha W(\varepsilon(u))$$

can be bounded from above by the quantity

$$\big(|\varepsilon(u)|^{p-2}(x+he_\alpha) + |\varepsilon(u)|^{p-2}(x)\big)|\Delta_h^\alpha \varepsilon(u)|^2 \,,$$

so that (4.11) implies

$$W(\varepsilon(u)) \in W^1_{2,\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^{2\times 2}) \,. \tag{4.12}$$

At the same time we can deduce from (4.8) and the subsequent estimates by taking from now on the sum w.r.t. α (letting $W = W(\varepsilon(u))$ and using the formulas for T_2, S_2)

$$\begin{aligned}
&\int_{B_r(x_0)} \Delta_h^\alpha W : \Delta_h^\alpha W \,\mathrm{d}x \\
&\le\ \delta \int_{B_{2r}(x_0)} \Delta_h^\alpha W : \Delta_h^\alpha W \,\mathrm{d}x + c\Bigg[\delta^{-1} r^{-2}\int_{T_r(x_0)} |\nabla u|^p \,\mathrm{d}x \\
&\quad + \int_{B_{2r}(x_0)} \big|\Delta_h^\alpha u^k \partial_k u^i \Delta_h^\alpha u^i\big| \,\mathrm{d}x + r^{-1}\int_{T_r(x_0)} |u|\big(\Delta_h^\alpha u \cdot \Delta_h^\alpha u\big) \,\mathrm{d}x \\
&\quad + r^{-1}\int_{T_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x + r\int_{T_r(x_0)} |u||\Delta_{-h}^\alpha \psi_h^\alpha|^2 \,\mathrm{d}x\Bigg] \,.
\end{aligned} \tag{4.13}$$

Here the third and the fourth integral on the r.h.s. correspond to T_2, whereas the last two ones are produced by breaking up S_2 with the help of Young's inequality. Using the properties of ψ_h^α we can estimate the last integral on the r.h.s. of (4.13) by Hölder's inequality in order to get for any $q > 2$

$$\begin{aligned}
\int_{T_r(x_0)} |u||\Delta_{-h}^\alpha \psi_h^\alpha|^2 \,\mathrm{d}x \ &\le\ \left[\int_{T_r(x_0)} |u|^{\frac{q}{q-2}}\right]^{1-\frac{2}{q}} \left[\int_{T_r(x_0)} |\Delta_{-h}^\alpha \psi_h^\alpha|^q \,\mathrm{d}x\right]^{\frac{2}{q}} \\
&\le\ cr^{-2}\left[\int_{T_r(x_0)} |u|^{\frac{q}{q-2}}\right]^{1-\frac{2}{q}} \left[\int_{T_r(x_0)} |\nabla u|^q \,\mathrm{d}x\right]^{\frac{2}{q}} \,,
\end{aligned}$$

If we insert this estimate into (4.13), we obtain after passing to the limit $h \to 0$ (using $\partial_\alpha u^k \partial_k u^i \partial_\alpha u^i \equiv 0$)

$$
\begin{aligned}
\int_{B_r(x_0)} |\nabla W(\varepsilon(u))|^2 \,\mathrm{d}x \;\; &\le \;\; \delta \int_{B_{2r}(x_0)} |\nabla W(\varepsilon(u))|^2 \,\mathrm{d}x \\
&\quad + c\Bigg[\delta^{-1} r^{-2} \int_{T_r(x_0)} |\nabla u|^p \,\mathrm{d}x + r^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \,\mathrm{d}x \\
&\quad + r^{-1} \Bigg[\int_{T_r(x_0)} |u|^{\frac{q}{q-2}} \,\mathrm{d}x\Bigg]^{1-\frac{2}{q}} \Bigg[\int_{T_r(x_0)} |\nabla u|^q \,\mathrm{d}x\Bigg]^{\frac{2}{q}}\Bigg] , \qquad (4.14)
\end{aligned}
$$

and (4.14) holds for all $\delta > 0$, all disks $B_r(x_0)$ and for any $q > 2$. Hence, with (4.14) our claim (4.1) is established. □

We also need a substitute for Lemma 3.3.

Lemma 4.2 *Suppose that $v \in C^1(\mathbb{R}^2)$ satisfies $\int_{\mathbb{R}^2} |\nabla v|^p \,\mathrm{d}x < \infty$ for some $p \in (2, \infty)$. Then we have*

$$
\limsup_{R \to \infty} \frac{1}{R^{3-\frac{2}{p}}} \int_{B_R(0)} |v| \,\mathrm{d}x < \infty \, .
$$

Proof of Lemma 4.2. From the proof of Lemma 3.3 we recall the inequality

$$
\varphi(R) - \varphi(1) \le \Bigg[\int_1^R \int_0^{2\pi} |f_r(r,\theta)|^p \,\mathrm{d}\theta r \,\mathrm{d}r\Bigg]^{\frac{1}{p}} \Bigg[\int_1^R r^{-\frac{1}{p}\frac{p}{p-1}} \,\mathrm{d}r\Bigg]^{1-\frac{1}{p}}
$$

being valid also for $p \ge 2$. In place of (3.16) we obtain (recalling $|f_r(r,\theta)| \le |\nabla v(re^{i\theta})|$)

$$
\varphi(R) \le \varphi(1) + c(p) R^{\frac{p-2}{p}} \Bigg[\int_{B_R(0) - B_1(0)} |\nabla u|^p \,\mathrm{d}x\Bigg]^{\frac{1}{p}} ,
$$

provided we choose $R \ge 1$. Using the finiteness of the energy we get after passing to the limit $\gamma \to 0$

$$
\sup_{R \ge 1} R^{2-p} \int_0^{2\pi} |v(R\cos(\theta), R\sin(\theta))|^p \,\mathrm{d}\theta < \infty \, .
$$

This estimate implies for $R \ge 1$

$$
\begin{aligned}
\int_{B_R(0)} |v|^p \,\mathrm{d}x \;\; &= \;\; \int_0^R \int_0^{2\pi} |v(r\cos(\theta), r\sin(\theta))|^p r \,\mathrm{d}\theta \,\mathrm{d}r \\
&\le \;\; c + \int_1^R \int_0^{2\pi} |v(r\cos(\theta), r\sin(\theta))|^p r \,\mathrm{d}\theta \,\mathrm{d}r \\
&\le \;\; c(1 + R^p) \le cR^p \, .
\end{aligned}
$$

Finally we make use of Hölder's inequality

$$\int_{B_R(0)} |v| \, \mathrm{d}x \leq c \left[\int_{B_R(0)} |v|^p \, \mathrm{d}x \right]^{\frac{1}{p}} R^{2(1-\frac{1}{p})} ,$$

hence our claim follows by inserting the previous estimate. □

Next we give the

Proof of Theorem 1.4. W.l.o.g. let $u_\infty = 0$. Let us further assume that

$$\sup_{|x|\geq R} |u(x)||x|^{-\gamma} \to 0 \quad \text{as } R \to \infty \tag{4.15}$$

for some $\gamma \in [-1/3, 0)$, hence we have for all $R \geq 1$:

$$|u(x)| \leq \Theta(R) R^\gamma \quad \text{for all } R \leq |x| \leq 2R \tag{4.16}$$

with some function Θ such that $\Theta(R) \to 0$ as $R \to \infty$. From (4.1) we deduce choosing $q = p$ and applying Young's inequality ($W := W(\varepsilon(u))$)

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla W|^2 \, \mathrm{d}x &\leq \delta \int_{B_{2r}(x_0)} |\nabla W|^2 \, \mathrm{d}x + c \Bigg[\delta^{-1} r^{-2} \int_{B_{2r}(x_0)} |\nabla u|^p \, \mathrm{d}x \\
&\quad + r^{-1} \left[\int_{B_{2r}(x_0)} |u|^{\frac{p}{p-2}} \, \mathrm{d}x \right]^{1-\frac{2}{p}} \left[\int_{B_{2r}(x_0)} |\nabla u|^p \, \mathrm{d}x \right]^{\frac{2}{p}} \Bigg] \\
&\leq \delta \int_{B_{2r}(x_0)} |\nabla W|^2 \, \mathrm{d}x + c \Bigg[\delta^{-1} r^{-2} \int_{B_{2r}(x_0)} |\nabla u|^p \, \mathrm{d}x \\
&\quad + r^{-1} \left[\tau \int_{B_{2r}(x_0)} |\nabla u|^p \, \mathrm{d}x + \tau^{-\frac{2}{p-2}} \int_{B_{2r}(x_0)} |u|^{\frac{p}{p-2}} \, \mathrm{d}x \right] \Bigg]
\end{aligned}$$

for any disk $B_r(x_0)$. Let $\tau := r^\kappa$ for some $\kappa \in (0, 1)$. The δ-Lemma A.4 yields for any disk $B_r(x_0)$

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla W|^2 \, \mathrm{d}x &\leq c \Bigg[r^{-2} \int_{B_{2r}(x_0)} |\nabla u|^p \, \mathrm{d}x + r^{-1+\kappa} \int_{B_{2r}(x_0)} |\nabla u|^p \, \mathrm{d}x \\
&\quad + r^{-\frac{2\kappa}{p-2}-1} \int_{B_{2r}(x_0)} |u|^{\frac{p}{p-2}} \, \mathrm{d}x \Bigg] .
\end{aligned} \tag{4.17}$$

We choose $x_0 = 0$, $r = R > 1$ and insert (2.5) in (4.17), where the last integral on the r.h.s. of (4.17) is handled with the condition $|u| \leq c$. We arrive at

$$\begin{aligned}
\int_{B_R(0)} |\nabla W|^2 \, \mathrm{d}x &\leq c \big[R^{-2+1+3\gamma} + R^{-1+\kappa+1+3\gamma} + R^{-\frac{2\kappa}{p-2}-1} R^2 \big] \\
&\leq c \big[R^{\kappa+3\gamma} + R^{1-\frac{2\kappa}{p-2}} \big] ,
\end{aligned}$$

i.e. we have with some $\nu < 1$ (w.l.o.g. $\nu > 0$)

$$\int_{B_R(0)} |\nabla W|^2 \,\mathrm{d}x \leq c R^{\nu} \quad \text{for all } \; R \geq 1 \, . \tag{4.18}$$

Next we choose $\mu \in (\nu, 1)$ and apply (4.1) with $q = p$ and $\delta = R^{-\mu}$ to obtain

$$\begin{aligned} \int_{B_R(0)} |\nabla W|^2 \,\mathrm{d}x \;\; &\leq \;\; c\Big[R^{-\mu+\nu} + R^{\mu-2+1+3\gamma} \\ &\qquad + R^{-1} R^{2-\frac{4}{p}} \sup_{R \leq |x| \leq 2R} |u| R^{(1+3\gamma)\frac{2}{p}} \Big] \, . \end{aligned} \tag{4.19}$$

By the choice of the above parameters, the first two terms on the r.h.s. of (4.19) converge to zero as $R \to \infty$ and it remains to discuss the quantity (recall (4.16))

$$\zeta_R := R^{1-\frac{4}{p}} \Theta(R) R^{\gamma} R^{(1+3\gamma)\frac{2}{p}} = \Theta(R) R^{1-\frac{2}{p}+\gamma(1+\frac{6}{p})} \, ,$$

where we have to distinguish the three different cases of Theorem 1.4.

Case 1. For $2 < p < 6$ we may choose $\gamma = (2-p)/(p+6)$ in (4.15), where we note that

$$\gamma > -\frac{1}{3} \qquad \Leftrightarrow \qquad p < 6 \, .$$

This particular choice of γ gives

$$1 - \frac{2}{p} + \gamma\big(1 + \frac{6}{p}\big) = 0$$

which implies $\zeta_R \to 0$ as $R \to \infty$, hence the first part of the theorem is established.

Case 2. For $p = 6$ we have by assumption

$$|u(x)| \leq c R^{-\frac{1}{3}} \quad \text{for all } \; |x| \geq R$$

and for all $R \geq 1$. Since the condition $\Theta(R) \to 0$ as $R \to \infty$ is not needed for deriving (4.18), we obtain (4.18) as before. Moreover, (2.5) gives

$$\int_{\mathbb{R}^2} |\nabla u|^p \,\mathrm{d}x < \infty \, . \tag{4.20}$$

As above we let $q = p$ and $\delta = R^{-\mu}$ in (4.1) to obtain (recall (4.18))

$$\int_{B_R(0)} |\nabla W|^2 \,\mathrm{d}x \leq c\left[R^{\nu-\mu} + R^{\mu-2} + R^{-1} \left[\int_{T_R(0)} |u|^{\frac{3}{2}} \,\mathrm{d}x \right]^{\frac{2}{3}} \left[\int_{T_R(0)} |\nabla u|^6 \,\mathrm{d}x \right]^{\frac{1}{3}} \right] . \tag{4.21}$$

Here we observe

$$R^{-1}\left[\int_{T_R(0)}|u|^{\frac{3}{2}}\,\mathrm{d}x\right]^{\frac{2}{3}} \leq cR^{-1}R^{-\frac{1}{3}}R^{2\frac{2}{3}} \leq c$$

and by (4.20) the last integral of (4.21) converges to 0 as $R\to\infty$ which completes the proof in the second case of Theorem 1.4.

Case 3. In the case $p>6$ we again have by assumption the global energy estimate (4.20). We recall (2.15) of Section 2, choose $\delta=1/2$ in this inequality and observe that by the boundedness of u

$$R^{-p}\int_{T_R(0)}|u|^p\,\mathrm{d}x \to 0 \quad \text{as } R\to\infty\,.$$

Moreover we have

$$|T_3|+|T_4| \leq cR\Big[\sup_{R\leq|x|\leq 2R}|u|\Big]^3 \to 0 \quad \text{as } R\to\infty\,.$$

As a consequence we see

$$\int_{\mathbb{R}^2}|\varepsilon(u)|^p\,\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{R}^2}|\varepsilon(u)|^p\,\mathrm{d}x$$

which means $\varepsilon(u)\equiv 0$, hence u is a rigid motion and $u=\mathit{const}$ by the decay assumption. This completes the proof of Theorem 1.4. $\square$

We finish this section with the

Proof of Theorem 1.3. Let $2<p\leq 3$. As above we have (4.17), where we know in the situation at hand that

$$\int_{\mathbb{R}^2}|\nabla u|^p\,\mathrm{d}x < \infty\,,$$

hence for any $R\geq 1$ ($W:=W(\varepsilon(u))$)

$$\int_{B_R(0)}|\nabla W|^2\,\mathrm{d}x \leq c\left[R^{-1+\kappa}+R^{-\frac{2\kappa}{p-2}-1}\int_{B_{2R}(0)}|u|^{\frac{p}{p-2}}\,\mathrm{d}x\right]. \tag{4.22}$$

We insert (4.22) in the r.h.s. of (4.1) choosing $q=p$ there and get for any $\delta>0$

$$\begin{aligned}\int_{B_R(0)}|\nabla W|^2\,\mathrm{d}x \;\leq\;& \delta\left[R^{-1+\kappa}+R^{-\frac{2\kappa}{p-2}-1}\int_{B_{2R}(0)}|u|^{\frac{p}{p-2}}\,\mathrm{d}x\right]\\ &+c\left[\delta^{-1}R^{-2}\int_{T_R(0)}|\nabla u|^p\,\mathrm{d}x\right.\\ &\left.+R^{-1}\left[\int_{T_R(0)}|u|^{\frac{p}{p-2}}\,\mathrm{d}x\right]^{\frac{p-2}{p}}\left[\int_{T_R(0)}|\nabla u|^p\,\mathrm{d}x\right]^{\frac{2}{p}}\right]. \end{aligned}\tag{4.23}$$

Let

$$A := \fint_{B_{2R}(0)} u \,\mathrm{d}x$$

and observe

$$\begin{aligned}\int_{B_{2R}(0)} |u|^{\frac{p}{p-2}} \,\mathrm{d}x &\leq c\Bigg[\int_{B_{2R}(0)} |u-A|^{\frac{p}{p-2}} \,\mathrm{d}x + R^2|A|^{\frac{p}{p-2}}\Bigg] \\ &\leq c\Bigg[\int_{B_{2R}(0)} |u-A|^{\frac{p}{p-2}} \,\mathrm{d}x + \Bigg|R^{-2+2\frac{p-2}{p}}\int_{B_{2R}(0)} u \,\mathrm{d}x\Bigg|^{\frac{p}{p-2}}\Bigg]. \quad (4.24)\end{aligned}$$

To the first integral on the r.h.s. of (4.24) we apply the Sobolev-Poincaré inequality, which is possible on account of $p/(p-2) > 2$: letting

$$1 < q := \frac{2p}{3p-4}$$

and observing $q < p$ on account of $p > 2$, we find

$$\begin{aligned}\Bigg[\int_{B_{2R}(0)} |u-A|^{\frac{p}{p-2}} \,\mathrm{d}x\Bigg]^{\frac{p-2}{p}} &\leq c\Bigg[\int_{B_{2R}(0)} |\nabla u|^q \,\mathrm{d}x\Bigg]^{\frac{1}{q}} \\ &\leq c\Bigg[\Bigg[\int_{B_{2R}(0)} |\nabla u|^p \,\mathrm{d}x\Bigg]^{\frac{q}{p}} R^{2(1-\frac{q}{p})}\Bigg]^{\frac{1}{q}} \\ &= cR^{\frac{2}{q}-\frac{2}{p}}\Bigg[\int_{B_{2R}(0)} |\nabla u|^p \,\mathrm{d}x\Bigg]^{\frac{1}{p}}, \quad (4.25)\end{aligned}$$

where we also made use of Hölder's inequality. With (4.24) and (4.25) we find

$$\begin{aligned}
\xi_1 \;&:=\; R^{-1}\left[\int_{T_R(0)} |u|^{\frac{p}{p-2}}\,\mathrm{d}x\right]^{\frac{p-2}{p}} \left[\int_{T_R(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{2}{p}}\\
&\le\; R^{-1}\left[\int_{B_{2R}(0)} |u|^{\frac{p}{p-2}}\,\mathrm{d}x\right]^{\frac{p-2}{p}} \left[\int_{T_R(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{2}{p}}\\
&\le\; c\Bigg[R^{-1}R^{\frac{2}{q}-\frac{2}{p}}\left[\int_{T_R(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{2}{p}} \left[\int_{B_{2R}(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{1}{p}}\\
&\qquad +R^{-1}R^{-2+2\frac{p-2}{p}}\left|\int_{B_{2R}(0)} u\,\mathrm{d}x\right| \left[\int_{T_R(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{2}{p}}\Bigg]\\
&=\; c\Bigg[R^{2-\frac{6}{p}}\left[\int_{T_R(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{2}{p}} \left[\int_{B_{2R}(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{1}{p}}\\
&\qquad +\left|R^{-1-\frac{4}{p}}\int_{B_{2R}(0)} u\,\mathrm{d}x\right| \left[\int_{T_R(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{2}{p}}\Bigg]\,,
\end{aligned}\tag{4.26}$$

and since

$$\lim_{R\to\infty}\int_{T_R(0)} |\nabla u|^p\,\mathrm{d}x = 0$$

it follows

$$\lim_{R\to\infty}\xi_1 = 0 \tag{4.27}$$

on account of $p \le 3$ and by quoting Lemma 4.2. Using (4.24) and (4.25) one more time we obtain

$$\begin{aligned}
\xi_2 \;&:=\; \delta R^{-\frac{2\kappa}{p-2}-1}\int_{B_{2R}(0)} |u|^{\frac{p}{p-2}}\,\mathrm{d}x\\
&\le\; c\delta R^{-\frac{2\kappa}{p-2}-1}\left[R^{(\frac{2}{q}-\frac{2}{p})\frac{p}{p-2}}\left[\int_{B_{2R}(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{1}{p-2}} + \left|R^{-2+2\frac{p-2}{p}}\int_{B_{2R}(0)} u\,\mathrm{d}x\right|^{\frac{p}{p-2}}\right]\\
&=\; c\delta R^{-\frac{2\kappa}{p-2}-1}\left[R^{3}\left[\int_{B_{2R}(0)} |\nabla u|^p\,\mathrm{d}x\right]^{\frac{1}{p-2}} + \left|R^{-\frac{4}{p}}\int_{B_{2R}(0)} u\,\mathrm{d}x\right|^{\frac{p}{p-2}}\right]\,.
\end{aligned}\tag{4.28}$$

Since $p \le 3$, it holds

$$-\frac{2\kappa}{p-2} - 1 + 3 = 2 - \frac{2\kappa}{p-2} \le 2 - 2\kappa\,.$$

Recalling that $\kappa \in (0,1)$ is arbitrary, we may fix, e.g., $\kappa = 3/4$, hence $2 - 2\kappa = 1/2$. Finally we choose $\delta = 1/R$ in (4.23). This implies

$$\delta R^{-\frac{2\kappa}{p-2}-1} R^3 \left[\int_{B_{2R}(0)} |\nabla u|^p \, \mathrm{d}x \right]^{\frac{1}{p-2}} \to 0$$

as $R \to \infty$ and at the same time by Lemma 4.2

$$\delta R^{-\frac{2\kappa}{p-2}-1} \left| R^{-\frac{4}{p}} \int_{B_{2R}(0)} u \, \mathrm{d}x \right|^{\frac{p}{p-2}} = \left| R^{-2-\frac{2\kappa}{p}} \int_{B_{2R}(0)} u \, \mathrm{d}x \right|^{\frac{p}{p-2}} \to 0$$

as $R \to \infty$, hence

$$\lim_{R\to\infty} \xi_2 = 0 \, . \tag{4.29}$$

Inserting (4.26)–(4.29) into (4.23) and passing to the limit $R \to \infty$, we have shown that $\nabla W = 0$ on $\mathbb{R}^2$, hence u is affine and the finiteness of the p-energy implies the constancy of u. □

5 Proof of Theorem 1.5

Let u denote an entire solution of (1.1) satisfying (1.14). Introducing the vorticity

$$\omega := \partial_2 u^1 - \partial_1 u^2$$

we have for q, $l \in \mathbb{N}$ sufficiently large with $\eta \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned} \int_{\mathbb{R}^2} \omega^{2q} \eta^{2l} \, \mathrm{d}x &= \int_{\mathbb{R}^2} (\partial_2 u^1 - \partial_1 u^2) \omega^{2q-1} \eta^{2l} \, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \mathrm{div}(-u^2, u^1) \omega^{2q-1} \eta^{2l} \, \mathrm{d}x \\ &= - \int_{\mathbb{R}^2} (-u^2, u^1) \cdot \nabla \big[\omega^{2q-1} \eta^{2l} \big] \, \mathrm{d}x \\ &= (2q-1) \int_{\mathbb{R}^2} \nabla \omega \cdot (u^2, -u^1) \omega^{2q-2} \eta^{2l} \, \mathrm{d}x \\ &\quad + 2l \int_{\mathbb{R}^2} (u^2, -u^1) \cdot \nabla \eta \omega^{2q-1} \eta^{2l-1} \, \mathrm{d}x \, , \end{aligned} \tag{5.1}$$

and from $\mathrm{div}\, u = 0$ we infer

$$\begin{aligned} \int_{\mathbb{R}^2} u \cdot \nabla \omega \omega^{2q-3} \eta^{2l} \, \mathrm{d}x &= \frac{1}{2q-2} \int_{\mathbb{R}^2} u \cdot \nabla \omega^{2q-2} \eta^{2l} \, \mathrm{d}x \\ &= -\frac{1}{2q-2} \int_{\mathbb{R}^2} u \cdot \nabla \eta^{2l} \omega^{2q-2} \, \mathrm{d}x \, . \end{aligned} \tag{5.2}$$

Recall that

$$\Delta\omega - u \cdot \nabla\omega = 0 \quad \text{on } \mathbb{R}^2 \, ,$$

hence

$$\int_{\mathbb{R}^2} \nabla\omega \cdot \nabla\varphi \, \mathrm{d}x + \int_{\mathbb{R}^2} u \cdot \nabla\omega\varphi \, \mathrm{d}x = 0$$

for $\varphi \in C_0^1(\mathbb{R}^2)$. We specify $\varphi = \eta^{2l}\omega^{2q-3}$ and get

$$\begin{aligned} & \int_{\mathbb{R}^2} \eta^{2l}(2q-3)|\nabla\omega|^2\omega^{2q-4} \, \mathrm{d}x \\ & \quad = \; -\int_{\mathbb{R}^2} \nabla\omega \cdot \nabla\eta^{2l}\omega^{2q-3} \, \mathrm{d}x - \int_{\mathbb{R}^2} u \cdot \nabla\omega\omega^{2q-3}\eta^{2l} \, \mathrm{d}x \, . \end{aligned} \tag{5.3}$$

By Young's inequality, the first term on the r.h.s. of (5.3) is estimated through

$$\delta \int_{\mathbb{R}^2} |\nabla\omega|^2\omega^{2q-4}\eta^{2l} \, \mathrm{d}x + c(\delta, l) \int_{\mathbb{R}^2} |\nabla\eta|^2\eta^{2l-2}\omega^{2q-2} \, \mathrm{d}x \, ,$$

to the second term on the r.h.s. of (5.3) we apply (5.2). This yields after appropriate choice of δ

$$\begin{aligned} & \int_{\mathbb{R}^2} |\nabla\omega|^2\omega^{2q-4}\eta^{2l} \, \mathrm{d}x \\ & \quad \leq \; c(l,q)\left[\int_{\mathbb{R}^2} \omega^{2q-2}\eta^{2l-2}|\nabla\eta|^2 \, \mathrm{d}x + \int_{\mathbb{R}^2} |u||\nabla\eta^{2l}|\omega^{2q-2} \, \mathrm{d}x\right] . \end{aligned} \tag{5.4}$$

Now we return to (5.1) and estimate

$$\begin{aligned} \int_{\mathbb{R}^2} \omega^{2q}\eta^{2l} \, \mathrm{d}x \; & \leq \; (2q-1) \int_{\mathbb{R}^2} |\nabla\omega||u|\omega^{2q-2}\eta^{2l} \, \mathrm{d}x + 2l \int_{\mathbb{R}^2} |u||\nabla\eta|\omega^{2q-1}\eta^{2l-1} \, \mathrm{d}x \\ & \leq \; \delta \int_{\mathbb{R}^2} \omega^{2q}\eta^{2l} \, \mathrm{d}x + c(\delta, q) \int_{\mathbb{R}^2} |\nabla\omega|^2|u|^2\omega^{2q-4}\eta^{2l} \, \mathrm{d}x \\ & \quad\; +2l \int_{\mathbb{R}^2} |u||\nabla\eta|\omega^{2q-1}\eta^{2l-1} \, \mathrm{d}x \, , \end{aligned}$$

hence for δ sufficiently small

$$\begin{aligned} & \int_{\mathbb{R}^2} \eta^{2l}\omega^{2q} \, \mathrm{d}x \\ & \quad \leq \; c(l,q)\left[\int_{\mathbb{R}^2} |\nabla\omega|^2|u|^2\omega^{2q-4}\eta^{2l} \, \mathrm{d}x + \int_{\mathbb{R}^2} |u||\nabla\eta|\omega^{2q-1}\eta^{2l-1} \, \mathrm{d}x\right] . \end{aligned} \tag{5.5}$$

Next we specify η: let $R \geq 1$ and choose $\eta = 1$ on $B_R(0)$, $0 \leq \eta \leq 1$, $\operatorname{spt}\eta \subset B_{2R}(0)$, $|\nabla\eta| \leq c/R$. From (1.14) we get (w.l.o.g. we assume $\alpha > 0$)

$$|u(x)| \leq cR^\alpha \quad \text{for all } x \in B_R(0) \, . \tag{5.6}$$

We use (5.6) on the r.h.s. of (5.5) and get

$$\begin{aligned}
&\int_{B_{2R}(0)} \eta^{2l}\omega^{2q}\,\mathrm{d}x \\
&\leq\ c(l,q)\left[R^{2\alpha}\int_{B_{2R}(0)} |\nabla\omega|^2\omega^{2q-4}\eta^{2l}\,\mathrm{d}x + R^{\alpha}\int_{B_{2R}(0)} |\nabla\eta|\omega^{2q-1}\eta^{2l-1}\,\mathrm{d}x\right],
\end{aligned}$$

and if we apply (5.4) on the r.h.s. quoting (5.6) one more time it follows

$$\begin{aligned}
&\int_{B_{2R}(0)} \eta^{2l}\omega^{2q}\,\mathrm{d}x \\
&\leq\ c(l,q)\left[R^{2\alpha}\int_{B_{2R}(0)} \omega^{2q-2}\eta^{2l-2}|\nabla\eta|^2\,\mathrm{d}x + R^{3\alpha}\int_{B_{2R}(0)} |\nabla\eta^{2l}|\omega^{2q-2}\,\mathrm{d}x\right. \\
&\qquad \left. + R^{\alpha}\int_{B_{2R}(0)} \omega^{2q-1}|\nabla\eta|\eta^{2l-1}\,\mathrm{d}x\right] \\
&=:\ c(l,q)\big[T_1+T_2+T_3\big]\,. \qquad (5.7)
\end{aligned}$$

Young's inequality yields

$$\begin{aligned}
T_1 &\leq \int_{B_{2R}(0)} \omega^{2q-2}\eta^{2l-2}R^{2\alpha-2}\,\mathrm{d}x \\
&\leq \delta\int_{B_{2R}(0)} \omega^{2q}\eta^{(2l-2)2q/(2q-2)}\,\mathrm{d}x + c(\delta)R^{2+q(2\alpha-2)}
\end{aligned}$$

and

$$\begin{aligned}
T_2 &\leq \int_{B_{2R}(0)} \omega^{2q-2}\eta^{2l-1}R^{3\alpha-1}\,\mathrm{d}x \\
&\leq \delta\int_{B_{2R}(0)} \omega^{2q}\eta^{(2l-1)2q/(2q-2)}\,\mathrm{d}x + c(\delta)R^{2+q(3\alpha-1)}
\end{aligned}$$

as well as

$$\begin{aligned}
T_3 &\leq c\int_{B_{2R}(0)} \omega^{2q-1}\eta^{2l-1}R^{\alpha-1}\,\mathrm{d}x \\
&\leq \delta\int_{B_{2R}(0)} \omega^{2q}\eta^{(2l-1)2q/(2q-1)}\,\mathrm{d}x + c(\delta)R^{2+2q(\alpha-1)}\,.
\end{aligned}$$

Moreover, for $l \gg 1$ we have

$$2l \leq \frac{(2l-2)2q}{2q-2} \quad \text{and} \quad 2l \leq \frac{(2l-1)2q}{2q-1}\,,$$

hence, for δ small enough, we obtain from (5.7)

$$\int_{B_{2R}(0)} \eta^{2l}\omega^{2q}\,\mathrm{d}x \le c(l,q)\Big[R^{2+q(2\alpha-2)} + R^{2+q(3\alpha-1)} + R^{2+2q(\alpha-1)}\Big] . \tag{5.8}$$

Recall that $\alpha < 1/3$. Therefore we can fix a sufficiently large exponent q with the property that

$$2 + q(3\alpha - 1) < 0 ,$$

and (5.8) shows

$$\int_{B_R(0)} \omega^{2q}\,\mathrm{d}x \le c(l,q)R^{2+q(3\alpha-1)} \to 0 \quad \text{as } R \to 0 ,$$

hence $\omega = 0$ on $\mathbb{R}^2$. This together with $\operatorname{div} u = 0$ shows that u is harmonic and the constancy of u then follows from (1.14) and results concerning entire harmonic functions.

□

Appendix. Helpful tools

The following lemma is a well known result. A proof together with further comments can be found in [Ga1], Chapter III, Section 3. Our formulation is taken from [AM], Lemma 2.5.

Lemma A.1 *Suppose that we are given numbers $1 < p_1 \le p \le p_2 < \infty$.*

Then there exists a constant $c = c(p_1, p_2)$ as follows: if $f \in L^p(B_r(x_0))$ satisfies $⨍_{B_r(x_0))} f\,\mathrm{d}x = 0$, then there exists a field v in the space $\overset{\circ}{W}{}^1_p(B_r(x_0), \mathbb{R}^2)$ satisfying $\operatorname{div} v = f$ on the disk $B_r(x_0)$ together with the estimate

$$\int_{B_r(x_0)} |\nabla v|^s\,\mathrm{d}x \le c\int_{B_r(x_0)} |f|^s\,\mathrm{d}x$$

for any exponent $s \in [p_1, p]$. The same is true if the disk is replaced by the annulus $T_r(x_0) = B_{2r}(x_0) - \overline{B_r(x_0)}$.

Our next tool is a collection of Korn-type inequalities. We refer the reader to Lemma 3.0.1 in [FS], where a list of references is given. We note that the last statement follows from the first one by applying *i)* to ηv, where η is a suitable cut-off function.

Lemma A.2 *Let $1 < p < \infty$. Then there exists a constant $c(p)$ such that the following inequalities hold.*

i) For all $v \in \overset{\circ}{W}{}^1_p(B_r(x_0), \mathbb{R}^2)$ we have

$$\|\nabla v\|_{L^p(B_r(x_0))} \le c(p)\|\varepsilon(v)\|_{L^p(B_r(x_0))} .$$

ii) For all $v \in W^1_p(B_r(x_0),\mathbb{R}^2)$ we have

$$\|\nabla v\|_{L^p(B_r(x_0))} \le c(p)\Big[\|\varepsilon(v)\|_{L^p(B_r(x_0))} + r^{-1}\|v\|_{L^p(B_r(x_0))}\Big]\, .$$

iii) For all $v \in W^1_p(B_{2r}(x_0),\mathbb{R}^2)$ we have letting $T_r(x_0) = B_{2r}(x_0) - \overline{B_r(x_0)}$

$$\|\nabla v\|_{L^p(B_r(x_0))} \le c(p)\Big[\|\varepsilon(v)\|_{L^p(B_{2r}(x_0))} + r^{-1}\|v\|_{L^p(T_r(x_0))}\Big]\, .$$

The following lemma originates from the work of Ladyzhenskaya (see [La], Lemma 1, p. 8). Actually it is a local variant of Ladyzhenskaya's lemma established as Lemma 2.6 in part B of [Zh].

Lemma A.3 *Suppose that $u \in W^1_2(B_r(x_0))$, $B_r(x_0) \subset \mathbb{R}^2$. Then there is a constant c independent of u, x_0 and r such that*

$$\int_{B_r(x_0)} |u|^4 \,\mathrm{d}x \le c\Bigg[\int_{B_r(x_0)} |u|^2 \,\mathrm{d}x \int_{B_r(x_0)} |\nabla u|^2 \,\mathrm{d}x + r^{-2}\Bigg[\int_{B_r(x_0)} |u|^2 \,\mathrm{d}x\Bigg]^2\Bigg]\, .$$

The next lemma goes back to Giaquinta and Modica (see [GM1], Lemma 0.5). We state a small extension presented in [FZha] as Lemma 3.1.

Lemma A.4 *Let f, f_1, $\dots$, f_l denote non-negative functions from the space $L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Suppose further that we are given exponents α_1, $\dots$, $\alpha_l > 0$.*

Then we can find a number $\delta_0 > 0$ (depending on α_1, $\dots$, α_l) as follows: if for $\delta \in (0,\delta_0)$ it is possible to calculate a constant $c(\delta) > 0$ such that the inequality

$$\int_{B_r(x_0)} f \,\mathrm{d}x \le \delta \int_{B_{2r}(x_0)} f \,\mathrm{d}x + c(\delta)\sum_{j=1}^{l} r^{-\alpha_j}\int_{B_{2r}(x_0)} f_j \,\mathrm{d}x \tag{A.1}$$

holds for any choice of $B_r(x_0) \subset \mathbb{R}^2$, then there is a constant c with the property

$$\int_{B_r(x_0)} f \,\mathrm{d}x \le c\sum_{j=1}^{l} r^{-\alpha_j}\int_{B_{2r}(x_0)} f_j \,\mathrm{d}x \tag{A.2}$$

for all disks $B_r(x_0) \subset \mathbb{R}^2$.

Finally we recall some well known inequalities.

Lemma A.5 *Let $p > 2$.*

i) With suitable positive constants $c_1 < c_2$ it holds

$$c_1\big[|\xi|^{p-2} + |\eta|^{p-2}\big]|\xi-\eta|^2 \le \Big||\xi|^{\frac{p-2}{2}}\xi - |\eta|^{\frac{p-2}{2}}\eta\Big|^2 \le c_2\Big[|\xi|^{p-2} + |\eta|^{p-2}\Big]|\xi-\eta|^2$$

for any ξ, $\eta \in \mathbb{R}^M$, $M \ge 1$.

ii) There exists a constant $c > 0$ such that

$$\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big) : (\xi - \eta) \geq c\big[|\xi|^{p-2} + |\eta|^{p-2}\big]|\xi - \eta|^2$$

for all ξ, $\eta \in \mathbb{R}^M$, $M \geq 1$.

Proof. $i)$ follows from inequality (2.4) in [GM2] by letting $\mu = 0$, $\delta = p - 2$ in this reference.

For proving $ii)$ we let $F(\xi) = |\xi|^{p-2}\xi$ and observe that

$$\begin{aligned}\big(F(\xi) - F(\eta)\big) : (\xi - \eta) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} F(\eta + t(\xi - \eta))\,\mathrm{d}t : (\xi - \eta)\\ &=: \int_0^1 |\eta + t(\xi - \eta)|^{p-2}\,\mathrm{d}t|\xi - \eta|^2 + A\,,\end{aligned}$$

where A is easily seen to be non-negative. From Lemma 2.2 in [FH] we therefore deduce

$$\big(F(\xi) - F(\eta)\big) : (\xi - \eta) \geq c|\xi - \eta|^2\big[|\xi - \eta|^{p-2} + |\eta|^p\big]\,,$$

and our claim immediately follows from this estimate by considering the cases $|\xi| \geq 2|\eta|$ and $|\xi| < 2|\eta|$, respectively. □

References

[AM] Acerbi, E., Mingione, G., Regularity results for stationary electrorheological fluids. ARMA 164 (2002), 213-259.

[Ad] Adams, R. A., Sobolev spaces. Academic Press, New York-San Francisco-London 1975.

[Fu] Fuchs, M.; Liouville Theorems for stationary flows of shear thickening fluids in the plane. J. Math. Fluid Mech. 14 (2012), 421-444.

[FS] Fuchs, M.; Seregin, G.; Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lecture Notes in Mathematics, 1749. Springer-Verlag, Berlin, 2000.

[FZha] Fuchs, M., Zhang, G., Liouville theorems for entire local minimizers of energies defined on the class $LlogL$ and for entire solutions of the stationary Prandtl-Eyring fluid model. Calc. Var. Partial Differential Equations 44 (2012), no. 1-2, 271-295.

[FZho] Fuchs, M., Zhong, X., A note on a Liouville-type result of Gilbarg and Weinberger for the stationary Navier-Stokes equations in 2D. J. Math. Sci. 178 (6) (2011), 695-703.

[FH] Fusco, N., Hutchinson, J., Partial regularity for minimisers of certain functionals having nonquadratic growth. Ann. Mat. Pura Appl. (4) 155 (1989), 1-24.

[Ga1] Galdi, G.P.; An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems. Springer Tracts in Natural Philosophy, 38. Springer-Verlag, New York, 1994.

[Ga2] Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II. Nonlinear steady problems. Springer Tracts in Natural Philosophy, 39. Springer-Verlag, New York, 1994.

[GM1] Giaquinta, M., Modica, G.; Nonlinear systems of the type of stationary Navier-Stokes system. J. Reine Angew. Math. 330 (1982), 173-214.

[GM2] Giaquinta, M., Modica, G., Remarks on the regularity of the minimizers of certain degenerate functionals. Manus. Math. 57 (1986), no. 1, 55-99.

[GT] Gilbarg, D., Trudinger, N.; Elliptic partial differential equations of second order. Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977.

[GW] Gilbarg, D., Weinberger, H.F.; Asymptotic properties of steady plane solutions of the Navier-Stokes equations with bounded Dirichlet integral. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 (1978), no. 2, 381-404.

[KNSS] Koch, G., Nadirashvili, N., Seregin, G., Sverák, V.; Liouville theorems for the Navier-Stokes equations and applications. Acta Math. 203 (2009), no. 1, 83-105.

[La] Ladyzhenskaya, O.A.; The mathematical theory of viscous incompressible flow. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2 Gordon and Breach, Science Publishers, New York-London-Paris 1969

[MNRR] Málek, J., Necăs, J., Rokyta, M., Růžička, M.; Weak and measure-valued solutions to evolutionary PDEs. Applied Mathematics and Mathematical Computation, 13. Chapman & Hall, London, 1996.

[Na] Naumann, Joachim; On the differentiability of weak solutions of a degenerate system of PDEs in fluid mechanics. Ann. Mat. Pura Appl. (4) 151 (1988), 225-238.

[Wo] Wolf, J.; Interior $C^{1,\alpha}$-regularity of weak solutions to the equations of stationary motions of certain non-Newtonian fluids in two dimensions. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 10 (2007), no. 2, 317-340.

[Zh] Zhang, G.; Liouville theorems for stationary flows of generalized Newtonian fluids. PhD-thesis, Report 135, University of Jyväskylä, Department of Mathematics and Statistics, Jyväskylä 2012.