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Rate-independent dynamics and Kramers-type phase transitions in nonlocal Fokker-Planck equations with dynamical control

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Abstract

The hysteretic behavior of many-particle systems with non-convex free energy can be modeled by nonlocal Fokker-Planck equations that involve two small parameters and are driven by a time-dependent constraint. In this paper we consider the fast reaction regime related to Kramers-type phase transitions and prove that the dynamics in the small-parameter limit can be described by a rate-independent evolution equation. To this end we derive mass-dissipation estimates from Muckenhoupt constants, establish dynamical peak-stability estimates, and employ moment estimates that encode large deviations results.

Keywords: nonlocal Fokker-Planck equations, gradients flows with dynamical control, multi-scale dynamics of PDE, mass-dissipations estimates, rate-independent models for hysteresis and phase transitions, Kramers' formula in time-dependent potentials

MSC (2000): 35B40, 35Q84, 82C26, 82C31

Contents

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1 Introduction

It is an ubiquitous and intriguing question in the mathematical analysis under which conditions the dynamics of a given high-dimensional systems with small parameters can be described by low-dimensional, reduced evolution equations. In this paper we answer this question, at least partially, for a particular example, namely the Fokker-Planck equation

$$\tau\partial_t\varrho(t,\,x) = \partial_x\Big(\nu^2\partial_x\varrho(t,\,x) + \big(H'(x) - \sigma(t)\big)\varrho(t,\,x)\Big), \tag{FP$_1$}$$

where τ and ν are the small parameters and $x \in \mathbb{R}$ is a one-dimensional state variable. Moreover, H is supposed to be a double-well potential and σ is a dynamical multiplier chosen such that the solution complies with

$$\int_{\mathbb{R}} x\varrho(t,\,x)\,\mathrm{d}x = \ell(t), \tag{FP$_2$}$$

where ℓ is a prescribed control function. This dynamical constraint is, for admissible initial data, equivalent to the mean-field formula

$$\sigma(t) = \int_{\mathbb{R}} H'(x)\varrho(t,\,x)\,\mathrm{d}x + \tau\dot{\ell}(t), \tag{FP$_2'$}$$

which turns (FP$_1$) into a nonlocal, nonlinear, and non-autonomous PDE.

Nonlocal Fokker-Planck equations like (FP$_1$)+(FP$_2$) have been introduced in [DGH11] in order to model the hysteretic behavior of many-particle storage systems such as modern Lithium-ion batteries (for the physical background, we also refer to [DJG$^+$10]). In this context, $x \in \mathbb{R}$ describes the thermodynamic state of a single particle (nano-particle made of iron-phosphate in the battery case), H is the free energy of each particle, and ν accounts for entropic effects. Moreover, ϱ is the probability density of a many-particle ensemble and the dynamical control ℓ reflects that the whole system is driven by some external process (charging or discharging of the battery).

Since H is non-convex, the dynamics of (FP$_1$)+(FP$_2$) can be rather involved as they are related to three different time scales, namely the small relaxation time τ, the time scale of the control ℓ (which is supposed to be of order 1), and the Kramers scale $\tau\exp\big(h(\sigma)/\nu^2\big)$, which corresponds to probabilistic transitions between the different wells of a time-dependent effective potential with energy barrier $h(\sigma)$. The different dynamical regimes for $0 < \nu, \tau \ll 1$ have been investigated by the authors in [HNV12] using formal asymptotic analysis.

In this paper we restrict our considerations to the fast reaction regime, that means we suppose $0 < \nu \ll 1$ and assume that τ is coupled to ν by a certain exponential scaling law implying $0 < \tau \ll \nu$. In the most simple and prototypical case, this scaling law reads

$$\tau = \exp\left(-\frac{h_\#}{\nu^2}\right),$$

where $h_\#$ is some given parameter that is positive but smaller than a certain threshold h_{thres}. We emphasize that there exists also also a slow reaction regime corresponding to $0 < \nu \ll \tau \ll 1$, but then the dynamics is more complicated and neither related to rate-independent evolution nor Kramers-type phase transitions, see the discussion in [HNV12].

Our main result is the proof that the microscopic PDE (FP$_1$)+(FP$_2$) can be replaced, as $\nu \to 0$, by a low-dimensional dynamical system, which turns out to be rate-independent and exhibits hysteresis. These macroscopic equations govern the evolution of the multiplier σ and the phase fraction μ, which is defined by

$$\mu(t) = \int_{\text{right stable region}} \varrho(t,\,x)\,\mathrm{d}x - \int_{\text{left stable region}} \varrho(t,\,x)\,\mathrm{d}x,$$

where 'stable region' refers to a connected component of $\{x\,:\,H''(x) > 0\}$.

The micro-to-macro transition studied here is similar to those in [PT05, Mie11b, MT12], which likewise derive macroscopic models for hysteric behaviour from microscopic gradient flows with non-convex energy and external driving. Our microscopic system, however, is different as it involves the diffusive term $\nu^2\partial_x^2\varrho$, which causes specific effects and necessitates the use of different methods. More precisely, the dominant effect in the fast reaction regime of nonlocal Fokker-Planck equations are Kramers-type phase transitions, which describe that particles can pass through the spinodal region $\{x : H''(x) < 0\}$ due to stochastic fluctuations.

The key observation in our context is that Kramers-type phase transitions can manifest on the macroscopic scale only if the dynamical multiplier σ attains one of two critical values $\sigma_{\#}$ and $\sigma^{\#}$, which are completely determined by H and $h_{\#}$, because otherwise the corresponding microscopic mass flux is either too small or too large. The limit dynamics for $\nu \to 0$ is therefore completely characterized by the flow rule

$$\dot{\mu}(t) \leq 0 \quad \text{for} \quad \sigma(t) = \sigma_{\#}, \qquad \dot{\mu}(t) \geq 0 \quad \text{for} \quad \sigma(t) = \sigma^{\#}, \qquad \dot{\mu}(t) = 0 \quad \text{otherwise},$$

and pointwise relations $\mathcal{C}(\ell(t), \sigma(t), \mu(t)) = 0$ that encode the dynamical constraint. These findings can be summarized as follows.

Main result. *Under natural assumptions on H, the control ℓ, and the initial data, the triple (ℓ, σ, μ) satisfies in the limit $\nu \to 0$ a closed rate-independent evolution equation with hysteresis. Moreover, the limit solution is unique provided that the initial data are well-prepared.*

The rest of the paper is organized as follows. In §2 we give a more detailed introduction into the problem. In particular, in §2.1 and §2.2 we specify our assumptions and review the existence theory for (FP_1)+(FP_2') with arbitrary $\nu, \tau > 0$ as it is developed in Appendix A. Moreover, in §2.3 we heuristically explain the key dynamical features in the fast reaction regime and proceed with a precise formulation of the limit model in §2.4.

A major part of our analytical work is contained in §3. Specifically, we establish mass-dissipation estimates in §3.1 and derive in §3.2 conditional results for the dynamical stability of localized peaks. Afterwards we study the mass transfer between the two stable regions in §3.3 and §3.4.

In §4 we pass to the limit $\nu \to 0$. We continue our investigations concerning the dynamical stability of peaks in §4.1 and obtain uniform Lipschitz estimates for the multiplier σ in §4.2. These ingredients finally enable us to prove our main result in §4.3, see Theorems 29 and 30.

2 Preliminaries

In this section we introduce our assumption on H, ℓ, and the initial data, and summarize some important properties of solutions to the non-local Fokker-Plank equation. Moreover, we discuss the dynamics in the fast reaction regime on a heuristic level and formulate the rate-independent limit model.

2.1 Assumptions on the potential

Throughout this paper we assume that H is a double-well potential with the following properties, see Figure 1 for an illustration.

Assumption 1 (properties of H).

1. *H is three times continuously differentiable, attains a local maximum at $x = 0$ and the global minimum at precisely two points.*
2. *H'' has only two zeros x_*, x^* with $x_* < 0 < x^*$; we set $\sigma^* = H'(x^*)$ and $\sigma_* = H'(x_*)$ and this implies $\sigma_* < 0 < \sigma^*$.*
3. *H' is asymptotically linear in the sense of $\lim_{x\to\pm\infty} H'''(x) = 0$ and $\lim_{x\to\pm\infty} H''(x) = c_\pm$ for some constants $c_\pm$.*

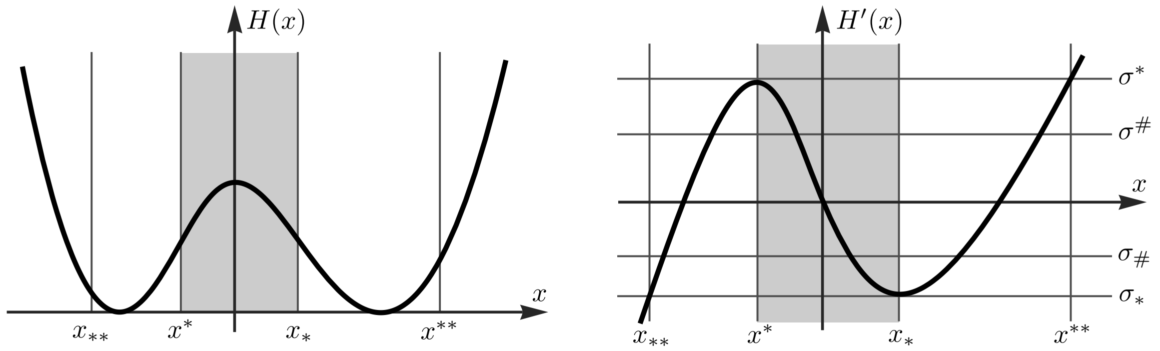

Figure 1: Example of a double-well potential H that satisfies assumption 1 with $\sigma_\#$ and $\sigma^\#$ as in Assumption 4. The shaded regions illustrate the spinodal (or unstable) interval (x^*, x_*).

The assumption that the two wells of H are global minima is not crucial and can always be guaranteed by means of elementary transformations. In fact, (FP$_1$) and (FP$'_2$) are, for any given $c \in \mathbb{R}$, invariant under $H \rightsquigarrow H + cx$, $\sigma \rightsquigarrow \sigma + c$. Moreover, by an appropriate shift in x we can always ensure that the local maximum is attained at $x = 0$. The assumption that H grows quadratically at infinity is of course more restrictive and made in order to keep the presentation as simple as possible. We expect, however, that our convergence result is also true for more general double-well potentials H provided that these grow superquadratically or that the initial data decay sufficiently fast.

As a direct consequence of Assumption 1 we can introduce three functions X_-, X_0, and X_+ such that $H' \circ X_j = \mathrm{id}$.

Remark 2 (functions X_-, X_0, and X_+)**.** *The inverse of H' has three strictly monotone and differentiable branches*

$$X_- : [-\infty,\, \sigma^*) \to (-\infty,\, x^*]\,, \quad X_0 : [\sigma_*,\, \sigma^*] \to [x^*,\, x_*]\,, \quad X_+ : [\sigma_*,\, +\infty) \to [x_*,\, +\infty)\,.$$

In particular, we have

1. $X_+(\sigma) - X_-(\sigma) \geq c$ *for a all* $\sigma \in [\sigma_*,\, \sigma^*]$,
2. $c \leq X'_\pm(\sigma) \leq C_\varepsilon$ *for all* $\sigma \in [\sigma_* + \varepsilon,\, \sigma^* - \varepsilon]$,
3. $|\sigma_2 - \sigma_1| \leq C\big|X_\pm(\sigma_2) - X_\pm(\sigma_1)\big|$ *for all* σ_1, σ_2 *in the domain of* $X_\pm$,

for any ε *with* $0 < \varepsilon < \frac{1}{2}(\sigma^* - \sigma_*)$ *and some constants* c, C *and* C_ε.

In order to describe Kramers-type phase transitions, we further introduce the effective potential

$$H_\sigma(x) := H(x) - \sigma x,$$

and define two functions $h_-, h_+ : (\sigma_*,\, \sigma^*) \to \mathbb{R}$ by

$$h_\pm(\sigma) := H_\sigma(X_0(\sigma)) - H_\sigma(X_\pm(\sigma)).$$

These definitions are motivated by the many-particle interpretation of (FP$_1$). In fact, for frozen σ the particles diffuse in the effective potential and the energy barriers h_- and h_+ appear explicitly in Kramer's formula for the mass fluxes between the two wells of H_σ, see Figure 2 and the discussion in §2.3.

Remark 3 (properties of $h_\pm$)**.** *The functions h_- and h_+ are well-defined and smooth on the interval $[\sigma_*,\, \sigma^*]$ with $h_-(0) = h_+(0) > 0$. Moreover, h_- is strictly decreasing with $h_-(\sigma^*) = 0$ and h_+ is strictly increasing with $h_+(\sigma_*) = 0$.*

We finally describe the coupling between τ and ν and introduce the values $\sigma_\#$ and $\sigma^\#$.

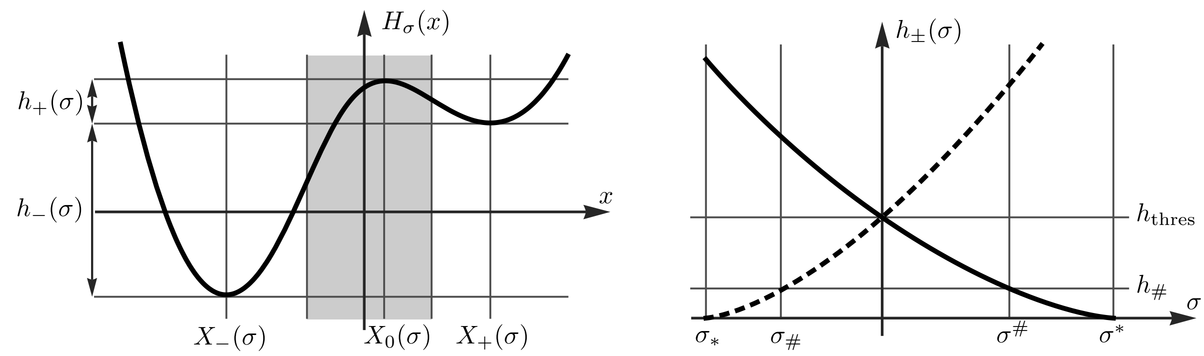

Figure 2: Cartoons of the effective potential H_σ with $\sigma_* < \sigma < 0$ and the functions h_- (solid line) and h_+ (dashed line). The values $\sigma_\#$ and $\sigma^\#$ are defined by $h_-\big(\sigma^\#\big) = h_+(\sigma_\#) = h_\#$ with $h_\# = -\lim_{\nu\to 0}\nu^2 \ln \tau$.

Assumption 4 (coupling between τ and ν)**.** *The parameter τ is positive, depends on ν, and satisfies*

$$\nu^2 \ln \tau \quad \xrightarrow{\nu\to 0} \quad -h_\#$$

for some $h_\#$ with $0 < h_\# < h_{\text{thres}} := h_\pm(0)$. In particular, there exist $\sigma_\#$ and $\sigma^\#$ such that

$$\sigma_* < \sigma_\# < 0 < \sigma^\# < \sigma^*, \qquad h_\# = h_-(\sigma^\#) = h_+(\sigma_\#),$$

and hence $h_\# < h_{\text{thres}} < \min\{h_-(\sigma_\#), h_+(\sigma^\#)\}$.

2.2 Existence and properties of solutions

It is well established, see [JKO97, JKO98], that the linear Fokker-Planck equation without dynamical constraint – that is (FP_1) with $\sigma(t) \equiv 0$ – is the Wasserstein gradient flow to the energy

$$\mathcal{E}(t) := \nu^2 \int \varrho(t,\, x) \ln \varrho(t,\, x)\, \mathrm{d}x + \int_{\mathbb{R}} H(x)\varrho(t,\, x)\, \mathrm{d}x. \tag{1}$$

Similarly, the non-driven variant of the nonlocal Fokker-Planck equations – that is (FP_1)+(FP_2') with $\dot{\ell}(t) \equiv 0$ – can be regarded as the Wasserstein gradient flows for $\mathcal{E}$ restricted to the constraint manifold $\int_{\mathbb{R}} \varrho\, \mathrm{d}x = \ell$, and we easily verify that the corresponding dissipation is given by

$$\mathcal{D}(t) := \int_{\mathbb{R}} \frac{\Big(\nu^2 \partial_x \varrho(t,\, x) + \big(H'(x) - \sigma(t)\big)\varrho(t,\, x)\Big)^2}{\varrho(t,\, x)}\, \mathrm{d}x. \tag{2}$$

In the general case $\dot{\ell} \neq 0$, however, the energy is no longer strictly decreasing but satisfies

$$\tau\dot{\mathcal{E}}(t) = -\mathcal{D}(t) + \tau\sigma(t)\dot{\ell}(t). \tag{3}$$

In particular, we have $\mathrm{d}\mathcal{E} \le \sigma\, \mathrm{d}\ell$ along each trajectory, and this reflects the second law of thermodynamics for the free energy of the many-particle ensemble in the presence of the dynamical control. The energy-dissipation estimate (3) is essential for passing to the limit $\nu \to 0$ as it reveals that the dissipation $\mathcal{D}$ is very small with respect to the L^1-norm and hence, loosely speaking, also small at most of the times. For linear Fokker-Planck equations without constraint, the underlying gradient structure can be used to establish Γ-convergence as $\tau \to 0$. The resulting evolution equation is a one-dimensional reaction ODE for the phase fraction μ and equivalent to Kramers' celebrated formula, see [PSV10, AMP$^+$11, HN11]. However, it is not clear to us whether this variational approach can be adapted to the present case with dynamical constraint; the methods developed here employ the estimate for $\mathcal{D}$ but make no further use of the gradient flow interpretation of (FP_1)+(FP_2).

Since the system (FP_1)+(FP_2') is a nonlinear and nonlocal PDE, it is not clear a priori that the initial value problem is well-posed in an appropriate function space. In the case of a bounded spatial domain and Neumann boundary conditions, the existence and uniqueness of solutions has been established in [Hut12, DHM$^+$11] using an L^q-setting for ϱ with $q > 1$. Since here we are interested in solutions that are defined on the whole real axis, we sketch an alternative existence and uniqueness proof in Appendix A. The key idea there is to obtain solutions as unique fixed points of a rather natural iteration scheme on the state space of all probability measures with bounded variance. Moreover, adapting standard techniques for parabolic PDE we derive several bounds to reveal how these solutions depend on ν.

Our assumptions and key findings concerning the existence and regularity of solutions to the nonlocal Fokker-Planck equation be can be summarized as follows.

Assumption 5 (dynamical control ℓ). *The final time T with $0 < T < \infty$ is independent of ν. The control ℓ is also independent of ν and twice continuously differentiable on $[0, T]$. In particular, we have*

$$\sup_{t\in[0,\,T]} \Big(\big|\ell(t)\big| + \big|\dot{\ell}(t)\big| + \big|\ddot{\ell}(t)\big|\Big) \leq C$$

for some constant C independent of ν.

Assumption 6 (initial data). *The initial data are nonnegative and satisfy*

$$\int_{\mathbb{R}} \varrho(0,\,x)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}} x\varrho(0,\,x)\,\mathrm{d}x = \ell(0), \qquad \int_{\mathbb{R}} x^2\varrho(0,\,x)\,\mathrm{d}x \leq C$$

for some constant C independent of ν.

Lemma 7 (existence and properties of solution). *For any ν with $0 < \nu \leq 1$ and given initial data there exists a unique solution ϱ to the initial value problem* (FP$_1$)+(FP$_2'$) *which is nonnegative and smooth for $t > 0$, and satisfies*

$$\int_{\mathbb{R}} \varrho(t,\,x)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}} x\varrho(t,\,x)\,\mathrm{d}x = \ell(t)$$

for all $t \in [0, T]$. Moreover, each solution satisfies

$$\sup_{t\in[0,\,T]} \left(\big|\sigma(t)\big| + \int_{\mathbb{R}} x^2\varrho(t,\,x)\,\mathrm{d}x\right) + \sup_{t\in[t_*,\,T]} \nu^2 \left\|\varrho(t,\,\cdot)\right\|_\infty + \tau^{-1}\int_{t_*}^{T} \mathcal{D}(t) \leq C$$

with $t_* := \nu^2\tau$ for some constant C which depends only on H, ℓ and $\int_{\mathbb{R}} x^2\varrho(0,\,x)\,\mathrm{d}x$.

Proof. All claims follow from Proposition 31 and Proposition 32 in Appendix A. □

The assertions of Lemma 7 reflect the existence of two small transient time scales. At first we have to wait for the time t_* before we can guarantee that $\|\varrho(t,\,\cdot)\|_\infty \leq C/\nu^2$ and $\int_{t_*}^{T} \mathcal{D}(t)\,\mathrm{d}t \leq C\tau$. The first estimate is needed within §3 in order to show that no mass can penetrate the spinodal region from outside, and that there is no mass flux through the spinodal region for subcritical $\sigma \in \left(\sigma_{\#},\,\sigma^{\#}\right)$. Furthermore, it is in general not before a time of order $\tau^{1-\beta}$ that the dissipation $\mathcal{D}(t)$ is eventually smaller than τ^β (the exponent $0 < \beta < 1$ will be identified below). In §4 we prove that the solutions to the nonlocal Fokker-Planck equations behave nicely after the second time, even though we are not able to exclude that $\mathcal{D}(t)$ becomes large (again) at some later time.

The initial transient regime corresponds to very fast relaxation processes during which the system dissipates a large amount of energy leading to rapid changes of especially the multiplier σ and the phase fraction μ. For generic initial data, we therefore expect to find several limit solutions as $\nu \to 0$ depending on the microscopic details of the initial data. The only possibility to avoid such non-uniqueness is to start with well-prepared initial data.

Definition 8 (well-prepared initial data)**.** *The initial data from Assumption 6 are* well-prepared*, if they additionally satisfy*

$$\nu^2 \|\varrho(0,\,\cdot)\|_\infty + \tau^{-1}\mathcal{D}(0) \le C,$$

for some constant C independent of ν, and if we have

$$\sigma(0) \quad \xrightarrow{\nu\to 0} \quad \sigma_{\rm ini}$$

for some $\sigma_{\rm ini} \in \mathbb{R}$.

Remark 9. *For well prepared initial data we can choose $t_* = 0$ in Lemma 7. Moreover, we have*

$$\varrho(0,\,x) \quad \xrightarrow{\nu\to 0} \quad \varrho_{\rm ini} := \frac{1-\mu_{\rm ini}}{2}\delta_{X_-(\sigma_{\rm ini})}(x) + \frac{1+\mu_{\rm ini}}{2}\delta_{X_+(\sigma_{\rm ini})}(x)$$

in the sense of weak⋆ convergence of measures, where δ_X denotes the Dirac distribution at $X \in \mathbb{R}$ and $\mu_{\rm ini} := \int_{-\infty}^{x^*} \varrho_{\rm ini}(x)\,\mathrm{d}x - \int_{x_*}^{+\infty} \varrho_{\rm ini}(x)\,\mathrm{d}x$.

Proof. The assertions follow from Remark 33 and the mass dissipation estimates formulated in Lemma 17 and Lemma 18. □

2.3 Heuristic description of the fast reaction regime

In order to highlight the key ideas for our convergence proof, we now give an informal overview on the effective dynamics for $\nu \ll 1$. For numerical simulations as well as formal asymptotic analysis we refer to [DGH11, HNV12].

As explained above, the underlying gradient structure ensures that the systems approaches – after a short initial transient regime with large dissipation – at time $0 < t_0 \ll 1$ a state with small dissipation. Assuming both that $\mathcal{D}(t)$ remains small and that σ changes regularly (i.e., on the time scale 1) for all times $t \ge t_0$, we can describe the dynamics for $\nu \ll 1$ as follows.

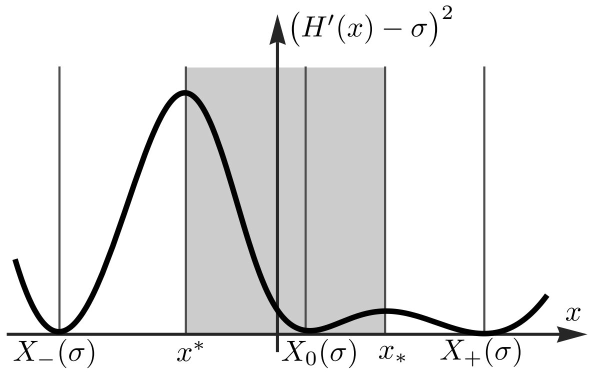

Figure 3: Moment weight for the definition of ξ with $\sigma_* < \sigma < 0$. For $\sigma \in (\sigma_*,\,\sigma^*)$ and $\xi \ll 1$, almost all of the total mass is concentrated in three narrow peaks located at $X_-(\sigma)$, $X_0(\sigma)$, and $X_+(\sigma)$, but only the peaks at $X_\pm(\sigma)$ are dynamically stable. For $\sigma < \sigma_*$ and $\sigma > \sigma_*$, the mass is concentrated for $\xi \ll 1$ in a single stable peak at $X_-(\sigma)$ and $X_+(\sigma)$, respectively.

Formation of peaks The small dissipation assumption implies (see also Remark 19 below) that the moment

$$\xi(t) := \int_{\mathbb{R}} \big(H'(x)-\sigma(t)\big)^2 \varrho(t,\,x)\,\mathrm{d}x \tag{4}$$

is also small, and we conclude that all of the mass of the system must be concentrated in narrow peaks located at the solutions to $H'(x) = \sigma(t)$, see Figure 3. We can therefore (at least in a weak$\star$-sense) approximate

$$\varrho(t,\,x) \approx \delta_{X_-(\sigma(t))} \quad \text{for} \quad \sigma(t) < \sigma_*, \qquad \varrho(t,\,x) \approx \delta_{X_+(\sigma(t))} \quad \text{for} \quad \sigma(t) > \sigma^* \tag{5}$$

as well as

$$\varrho(t,\,x) \approx \sum_{i \in \{-,0,+\}} m_i(t)\delta_{X_-(\sigma(t))}(x) \qquad \text{for} \quad \sigma(t) \in (\sigma_*,\,\sigma^*)\,, \tag{6}$$

where the partial masses are defined by

$$m_-(t) := \int_{-\infty}^{x^*} \varrho(t,\,x)\,\mathrm{d}x, \qquad m_0(t) := \int_{x^*}^{x_*} \varrho(t,\,x)\,\mathrm{d}x, \qquad m_+(t) := \int_{x_*}^{+\infty} \varrho(t,\,x)\,\mathrm{d}x. \tag{7}$$

Notice that $m_-(t)+m_0(t)+m_+(t) = 1$ holds by construction and that the moment ξ can be regarded as the formal limit of the dissipation as $\nu \to 0$.

Thanks to (5), we have $m_0(t) \approx m_+(t) \approx 0$ for $\sigma(t) < \sigma_*$ and the dynamical constraint implies $X_-\big(\sigma(t)\big) \approx \ell(t)$, which determines the evolution of σ. Similarly, with $\sigma(t) > \sigma^*$ we find $m_-(t) \approx m_0(t) \approx 0$ and $X_+\big(\sigma(t)\big) \approx \ell(t)$. These results reflect that H_σ is a single-well potential for both $\sigma < \sigma_*$ and $\sigma > \sigma^*$ attaining the global minimum at $X_-(\sigma)$ and $X_+(\sigma)$, respectively.

In the case of $\sigma(t) \in (\sigma_*,\,\sigma^*)$, the corresponding effective potential has two local minima and a local maximum corresponding to the three possible peak positions in (6). The peaks located at $X_\pm\big(\sigma(t)\big)$ are dynamically stable because adjacent characteristics of the transport operator in (FP_1) approach each other exponentially fast for $H''(x) > 0$. Moreover, asymptotic analysis of the entropic effects reveals that each stable peak is basically a rescaled Gaussian with width of order $\nu/\sqrt{H''}(X_\pm(\sigma))$. A peak at the center position $X_0(\sigma)$, however, is dynamically unstable because the spinodal characteristics separate exponentially fast with local rate proportional to τ, and because the width of each peak is at least of order ν. Each possible peak at $X_0(t)$ therefore disappears rapidly, and by enlarging t_0 if necessary we can assume that $m_0(t) \approx 0$ for all $t \geq t_0$. (This is different to the slow reaction regime, in which unstable peaks can survive for a long time due to $0 < \nu \ll \tau \ll 1$).

In summary, for any time $t > t_0$ with $\sigma(t) \in (\sigma_*,\,\sigma^*)$ we expect that almost all of the mass is concentrated in the two stable peaks at $X_\pm(\sigma(t))$. In the limit $\nu \to 0$, we therefore have $m_0(t) = 0$ and hence

$$m_-(t) + m_+(t) = 0, \qquad \ell(t) = m_-(t)X_-\big(\sigma(t)\big) + m_+(t)X_+\big(\sigma(t)\big),$$

where the last identity stems from the dynamical constraint. Notice that these formulas hold also for $\sigma(t) < \sigma_*$ and $\sigma(t) > \sigma^*$ with $m_+(t) = 0$ and $m_-(t) = 0$, respectively.

Dynamics of peaks It remains to understand the dynamics of the multiplier $\sigma(t)$ and the partial masses $m_-(t)$ and $m_+(t)$ in the case of $\sigma(t) \in (\sigma_*,\,\sigma^*)$. The key observation is that although both peaks are spatially separated they can, at least in principle, exchange mass by a Kramers-type phase transition. In the many-particle picture this means that particles cross the energy barrier between the two wells of H_σ due to stochastic fluctuations. Kramers investigated this large deviations problem in the context of chemical reactions in [Kra40] and derived his seminal formula for the effective mass flux between wells. In our notations, and with respect to our time scaling, this mass flux is, to leading order in ν, given by

$$-\dot{m}_-(t) \approx +\dot{m}_+(t) \approx m_-(t)F_-(t) - m_+(t)F_+(t), \qquad \tau F_\pm \approx C_\pm(\sigma)\exp\left(-\frac{h_\pm(\sigma)}{\nu^2}\right), \tag{8}$$

where the constants $C_\pm(\sigma)$ do not dependent on ν and are provided by Kramers' formula. In our context, however, the particular values of $C_\pm(\sigma)$ are not important because the dominant effects are the dynamical constraint and the time dependence of the energy barriers $h_\pm$. For more details on Kramers' formula and the connection to the theory of large deviations we refer, for instance, to [HTB90, Ber11].

We next discuss the implications of (8) for the different ranges of σ.

<u>Subcritical case:</u> For $\sigma(t) \in \big(\sigma_\#,\, \sigma^\#\big)$, the energy barrier between the two wells is larger than the critical value. This means

$$h_\pm(\sigma(t)) > h_\#, \qquad F_\pm(t) \sim \tau^{-1} \exp\left(-\frac{h_\pm\big(\sigma(t)\big)}{\nu^2}\right) \ll 1,$$

and we conclude that there is virtually no mass exchange between both peaks. In particular, the macroscopic dynamics reduces to

$$\dot m_\pm(t) = 0, \qquad \dot\ell(t) = \dot\sigma(t)\Big(m_-(t)X_-'\big(\sigma(t)\big) + m_+(t)X_+'\big(\sigma(t)\big)\Big)$$

and describes that both peaks are transported by the dynamical constraint, see the right panel in Figure 4.

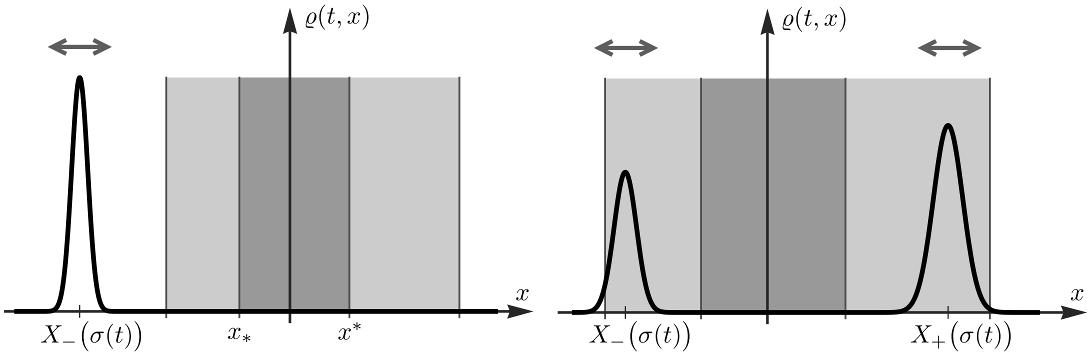

Figure 4: *Left panel:* For supercritical $\sigma < \sigma_\#$, all the mass is contained in a single stable peak, which is located at $X_-(\sigma)$ and transported by the dynamical constraint. (A similar statement holds for supercritical $\sigma > \sigma_\#$.) *Right panel:* For subcritical $\sigma \in \big(\sigma_\#,\, \sigma^\#\big)$, the mass is in general concentrated in two stable peaks, which are located at $X_-(\sigma)$ and $X_+(\sigma)$, and move according to the dynamics of ℓ. *Both panels:* The width of each peak is proportional to $\nu/\sqrt{H''(X)}$, where X denotes the position, and the arrows indicate that the peaks can move to the left (for $\dot\ell < 0$) or to the right (for $\dot\ell > 0$). The shaded regions in light and dark gray represent the intervals $\big[X_-(\sigma_\#),\, X_+\big(\sigma^\#\big)\big]$ and $[x^*,\, x_*]$, respectively.

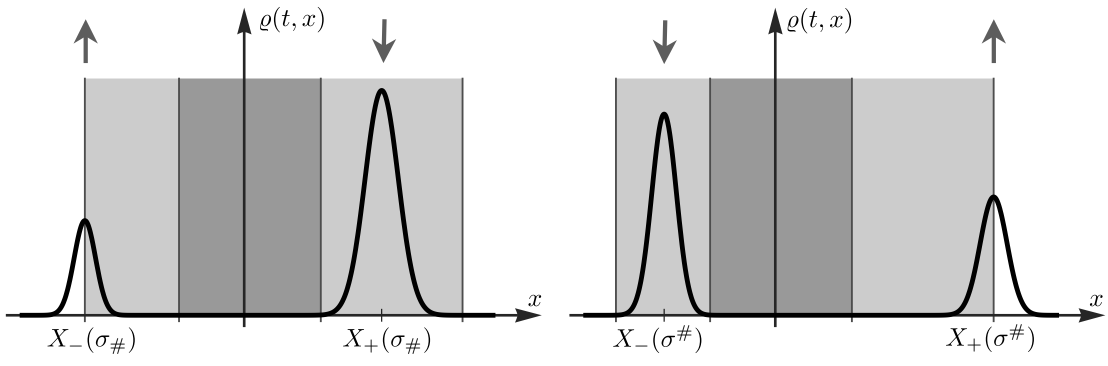

Figure 5: For critical σ, the coexisting stable peaks exchange mass by a Kramers-type phase transition, where $\sigma = \sigma_\#$ (left panel) and $\sigma = \sigma^\#$ (right panel) correspond to negative and positive mass flux, respectively.

<u>Critical cases:</u> For $\sigma(t) = \sigma_\#$, we find

$$h_+(\sigma(t)) = h_\# < h_-(\sigma(t)), \qquad F_-(t) \ll 1 \sim F_+(t),$$

which means particle can move from the well at $X_+\big(\sigma(t)\big)$ towards the well at $X_-\big(\sigma(t)\big)$, but not the other way around. It is therefore possible that in the limit $\nu \to 0$ there exist time intervals of positive length with

$$\sigma(t) = \sigma_\#, \qquad +\dot{m}_+(t) = -\dot{m}_-(t) \leq 0, \qquad \dot{\ell}(t) = \dot{m}_-(t) X_-\big(\sigma_\#\big) + \dot{m}_+(t) X_+\big(\sigma_\#\big).$$

Similarly, it can also happen that the macroscopic dynamics is given by

$$\sigma(t) = \sigma^\#, \qquad +\dot{m}_+(t) = -\dot{m}_-(t) \geq 0, \qquad \dot{\ell}(t) = \dot{m}_-(t) X_-\big(\sigma^\#\big) + \dot{m}_+(t) X_+\big(\sigma^\#\big),$$

reflecting an effective mass flux from the well at $X_-\big(\sigma(t)\big)$ towards the well at $X_+\big(\sigma(t)\big)$. Both critical cases are illustrated in Figure 5.

Supercritical cases: For $\sigma(t) \in (\sigma_*, \sigma_\#)$, we verify

$$h_+(\sigma(t)) < h_\# < h_-(\sigma(t)), \qquad F_-(t) \ll 1 \ll F_+(t),$$

and conclude that particles escape very rapidly from the well at $X_+(\sigma(t))$ but are trapped inside the other well at $X_-(\sigma(t))$. The only consistent choice for the macroscopic dynamics in this case is

$$m_-(t) = 1, \qquad m_+(t) = 0, \qquad \dot{\ell}(t) = X_-'\big(\sigma(t)\big)\dot{\sigma}(t)$$

corresponding to the transport of a single stable peak, see the left panel from Figure 4. To be more precise, for states with $\sigma(t) \in (\sigma_*, \sigma_\#)$ and $m_+(t) > 0$, the mass-dissipation estimates derived below imply that $\mathcal{D}(t)$ is large, and hence we expect that such states cannot be reached dynamically. (If such states are imposed in the initial data, a very rapid mass transfer during the initial transient regime produces $m_-(t_0) \approx 0$.) Similarly, for $\sigma(t) \in \big(\sigma^\#, \sigma^*\big)$ the macroscopic evolution reads

$$m_-(t) = 0, \qquad m_+(t) = 1, \qquad \dot{\ell}(t) = X_+'\big(\sigma(t)\big)\dot{\sigma}(t)$$

and can be justified by analogous arguments. Notice that the limit dynamics in the supercritical cases is the same as in the single-well cases $\sigma(t) < \sigma_*$ and $\sigma(t) > \sigma^*$.

2.4 Rate-independent model for the limit dynamics

The above formulas for the limit dynamics can be translated into closed evolution equations for ℓ, σ, and the phase fraction $\mu := m_+ - m_-$. These equations are illustrated in Figure 6 and are rate-independent because the macroscopic solution corresponding to $\tilde{\ell}(t) = \ell(ct)$ with $c > 0$ is given by $\tilde{\sigma}(t) = \sigma(ct)$ and $\tilde{\mu}(t) = \mu(ct)$. For more details on the general theory of rate-independent systems and the different solution concepts we refer to [Mie11a]. Moreover, the limit dynamics exhibit hysteresis in the sense that the value of the output σ at time t depends not only on the instantaneous value of the input ℓ but also on the history of the evolution (or, equivalently, on the state of the internal variable μ).

In order to give a precise formulation of our limit model, we now define an appropriate notion of solutions. To this end we observe that the parameter constraints

$$\mu \in [-1, 1], \qquad \sigma \in \mathbb{R}, \qquad \ell \in \begin{cases} \{X_-(\sigma)\} & \text{for} \quad \sigma < \sigma_\#, \\ [X_-(\sigma), X_+(\sigma)] & \text{for} \quad \sigma \in \big[\sigma_\#, \sigma^\#\big], \\ \{X_+(\sigma)\} & \text{for} \quad \sigma > \sigma^\# \end{cases} \tag{9}$$

define the macroscopic state space

$$\Omega := \Big\{(\ell, \sigma, \mu) \in \mathbb{R}^3 \quad \text{satisfying} \quad (9)\Big\} \tag{10}$$

and that the dynamical constraint can be written as $\mathcal{C}(\ell, \sigma, \mu) = 0$, where the function

$$\mathcal{C}(\ell, \sigma, \mu) := \frac{1-\mu}{2} X_-(\sigma) + \frac{1+\mu}{2} X_+(\sigma) - \ell. \tag{11}$$

is well-defined and continuously differentiable on Ω. We also recall that any Lipschitz function admits a classical derivative in almost all points (Rademacher's Theorem, see for instance [DiB02, Proposition 23.2]).

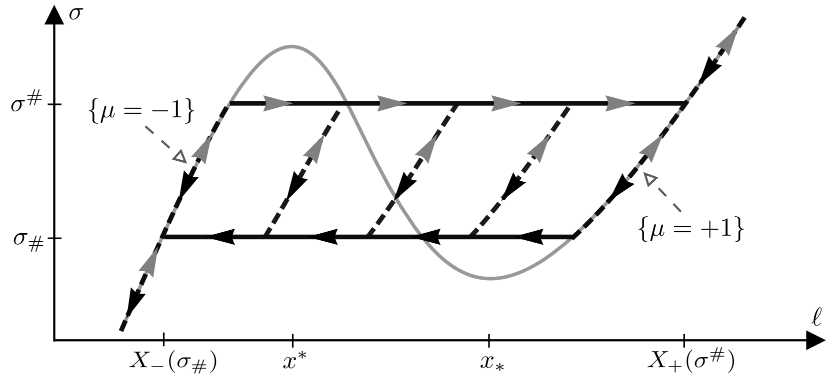

Figure 6: Cartoon of the *macroscopic limit dynamics* in the (ℓ, σ)-plane. The gray solid curve is the graph of H', the dashed black lines represent the level curves of μ, and the solid black lines correspond to the critical values $\sigma_\#$ and $\sigma^\#$, for which mass transfer according to a Kramers-type phase transition is feasible. The black and gray arrows indicate the evolution for decreasing and increasing ℓ, respectively. *Microscopic dynamics for small* ν. The evolution of ϱ along the level sets of μ is illustrated in Figure 4, whereas the panels in Figure 5 correspond to $\sigma(t) = \sigma_\#$ and $\sigma(t) = \sigma^\#$.

Definition 10 (solutions to the limit model)**.** *A pair* $(\sigma, \mu) \in \mathsf{C}^{0,1}\big([0, T]\,;\mathbb{R}^2\big)$ *is called* a solution to the limit problem *for given* $\ell \in \mathsf{C}^{0,1}([0, T])$*, if the pointwise relations*

$$\big(\ell(t), \sigma(t), \mu(t)\big) \in \Omega \quad \textit{with} \quad \mathcal{C}\big(\ell(t), \sigma(t), \mu(t)\big) = 0 \tag{12}$$

are satisfied for all $t \in [0, T]$*, and if the dynamical relations*

$$\dot{\mu}(t) = 0 \quad \textit{if} \quad \sigma(t) \notin \{\sigma_\#, \sigma^\#\}, \qquad \dot{\mu}(t) \le 0 \quad \textit{if} \quad \sigma(t) = \sigma_\#, \qquad \dot{\mu}(t) \ge 0 \quad \textit{if} \quad \sigma(t) = \sigma^\# \tag{13}$$

hold for almost all $t \in [0, T]$.

In Appendix B, Proposition 34 we prove that for each ℓ as in Assumption 5 and any admissible choice of the initial data $(\sigma(0), \mu(0))$ there exists a unique solution to the limit model, which is moreover piecewise continuously differentiable. We also emphasize that the limit model is equivalent to a constrained variational inequality. More precisely, introducing the convex functional

$$\mathcal{R}(\dot{\mu}) := \dot{\mu} \begin{cases} \sigma_\# & \text{if} \quad \dot{\mu} \le 0, \\ \sigma^\# & \text{if} \quad \dot{\mu} \ge 0, \end{cases} \qquad \mathcal{I}(\mu) := \begin{cases} 0 & \text{if} \qquad -1 \le \mu \le +1, \\ +\infty & \text{else}, \end{cases}$$

the dynamical relations (13) can be formulated as

$$\sigma(t) \in \partial_{\dot{\mu}} \mathcal{R}\big(\dot{\mu}(t)\big) + \partial_\mu \mathcal{I}\big(\mu(t)\big).$$

Here, ∂ means the set-valued derivative in the sense of subgradients, and the dynamical constraint enters via the pointwise relations (12).

The heuristic derivation of the limit dynamics in §2.3 relies on two crucial assumptions for $t \ge t_0$, namely (a) that $\mathcal{D}(t)$ is pointwise small, and (b) that $\dot{\sigma}(t)$ is pointwise of order 1. In numerical simulations one observes such a nice behavior but our convergence proof is based on weaker statements, which are, however, sufficient for passing to the limit $\nu \to 0$. Specifically, below we only show (a) that $\xi(t)$ remains small, and (b) that σ is Lipschitz continuous up to some small error terms.

3 Auxiliary results

The quantities c, C, and α always denote positive but generic constants (so their value may change from line to line) which are independent of ν but can depend on H, ℓ, T, the constant from Assumption 6, and other parameters to be introduced below. Notice that the scaling law between τ and ν, see Assumption 4, implies that a given positive quantity is exponentially small with respect to ν if and only if it is bounded by $C\tau^\alpha$ for some constants α and C independent of ν.

3.1 Mass-dissipation estimates

In this section we derive mass-dissipation estimates, that means we show that small dissipation requires the total mass of the system to be concentrated near either both or one of the stable peak positions $X_-(\sigma)$ and $X_+(\sigma)$. These estimates become important in §4 because they guarantee (in combination with the L^1-bound for $\mathcal{D}$) that for each time t_1 there exists another time $t_2 \in \big[t_1,\, t_1 + \tau^\beta\big]$ with $0 < \beta < 1$ at which the data are well-prepared. In the present section, however, all arguments and results hold pointwise in t and thus we omit the time dependence in all quantities.

For the following considerations we introduce, for each $\sigma \in \mathbb{R}$, the relative equilibrium density

$$\gamma_\sigma(x) := \exp\left(-\frac{H_\sigma(x)}{\nu^2}\right), \qquad z_\sigma := \int_{\mathbb{R}} \gamma_\sigma(x)\,\mathrm{d}x$$

see Figure 7 for an illustration, and denote by $\gamma_{\sigma,-}$ and $\gamma_{\sigma,+}$ the restriction of γ_σ to the intervals

$$I_{\sigma,-} := (-\infty,\, X_0(\sigma)) \qquad \text{and} \qquad I_{\sigma,+} := (X_0(\sigma),\, +\infty)\,,$$

respectively. The functions γ_σ are naturally related to states with small dissipations. In fact, γ_σ/z_σ is the global equilibrium of the linear Fokker-Planck equation (FP$_1$) with $\sigma(t) = \sigma = \text{const}$, and the energy functional

$$\mathcal{E}_\sigma(\varrho) := \nu^2 \int_{\mathbb{R}} \varrho(x) \ln \varrho(x)\,\mathrm{d}x + \int_{\mathbb{R}} H_\sigma(x)\varrho(x)\,\mathrm{d}x$$

just gives the relative entropy of ϱ with respect to γ_σ, that is

$$\mathcal{E}_\sigma(\varrho) = \nu^2 \int_{\mathbb{R}} \varrho(x) \ln\left(\frac{\varrho(x)}{\gamma_\sigma(x)}\right)\mathrm{d}x.$$

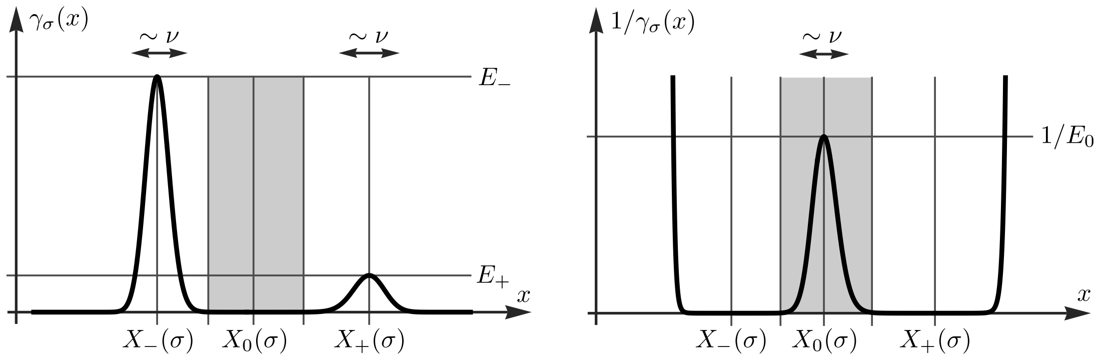

Figure 7: The relative equilibrium density γ_σ for $\sigma_* < \sigma < 0$, where $E_j := \exp\big(-H_\sigma(X_j(\sigma))/\nu^2\big)$. For $\nu \ll 1$, the density γ_σ exhibits two peaks located at $X_-(\sigma)$ and $X_+(\sigma)$. The width of these peaks is of order $O(\nu)$ and the mass ratio between the peaks scales with $\exp\big((h_-(\sigma) - h_+(\sigma))/\nu^2\big)$. The inverse density $1/\gamma_\sigma$ forms a peak at $X_0(\sigma)$ and grows very rapidly for $x \to \pm\infty$.

3.1.1 On Poincaré and Muckenhoupt constants

We now summarize some well-known facts about L^1-measures, which allow us to establish mass-dissipation estimates in §3.1.3. Within this section, let $I = (x_-,\, x_+)$ be some (bounded or unbounded) interval, γ be a positive L^1-function on the interval I, and $C_P(\gamma)$ the Poincaré constant of γ, that means

$$\frac{1}{C_P(\gamma)} := \inf_{w \in \mathsf{L}^2(\gamma\,\mathrm{d}x)} \frac{\displaystyle\int_I \big(w'(x)\big)^2 \gamma(x)\,\mathrm{d}x}{\displaystyle\int_I \big(w(x) - w_{\mathrm{av}}\big)^2 \gamma(x)\,\mathrm{d}x}, \qquad w_{\mathrm{av}} := \frac{\displaystyle\int_I w(x)\gamma(x)\,\mathrm{d}x}{\displaystyle\int_I \gamma(x)\,\mathrm{d}x},$$

where w' abbreviates the derivative of w with respect to x and $\mathsf{L}^2(\gamma\,\mathrm{d}x) := \{w\ :\ \int_I w(x)^2\gamma(x)\,\mathrm{d}x < \infty\}$. For each $x_0 \in \overline{I}$, we also introduce the one-sided Muckenhoupt constants $C_M^{\pm}(\gamma,\, x_0)$ with respect to x_0 by

$$C_M^-(\gamma,\, x_0) := \sup_{x\in(x_-,\,x_0]} \left(\int_x^{x_0} \frac{1}{\gamma(y)}\,\mathrm{d}y\right)\left(\int_{x_-}^{x} \gamma(y)\,\mathrm{d}y\right),$$
$$C_M^+(\gamma,\, x_0) := \sup_{x\in[x_0,\,x_+)} \left(\int_{x_0}^{x} \frac{1}{\gamma(y)}\,\mathrm{d}y\right)\left(\int_{x}^{x_+} \gamma(y)\,\mathrm{d}y\right).$$

It is known, see the discussion in [Fou05, Sch12], that γ admits a finite Poincaré constant if and only if the Muckenhoupt constants are bounded.

Lemma 11 (Muckenhoupt constants bound Poincaré constant)**.** *We have*

$$C_P(\gamma) \le 4\max\left\{C_M^-(\gamma,\, x_0),\, C_M^+(\gamma,\, x_0)\right\}$$

for all γ and any $x_0 \in I$.

Proof. The proof can be found in [Sch12, Proposition 5.21]. □

We also mention the lower bound

$$C_P(\gamma) \ge \tfrac{1}{2}\max\left\{C_M^-(\gamma,\, x_{\mathrm{med}}),\, C_M^+(\gamma,\, x_{\mathrm{med}})\right\},$$

where x_{med} is the median of γ, which is defined by $\int_{x_-}^{x_{\mathrm{med}}} \gamma(y)\,\mathrm{d}y = \int_{x_{\mathrm{med}}}^{x^+} \gamma(y)\,\mathrm{d}y$, and that $C_M^{\pm}$ can easily be estimated for logarithmically concave functions γ.

Remark 12 ($C_M^{\pm}$ for logarithmically concave γ)**.** *Let $\gamma(x) = \exp\big(-V(x)\big)$. For convex and strictly increasing potential $V : [x_0,\, +\infty) \to \mathbb{R}$ we have*

$$C_M^+(\gamma,\, x_0) \le \sup_{x\ge x_0} \frac{x - x_0}{V'(x)}.$$

Similarly, the estimate

$$C_M^-(\gamma,\, x_0) \le \sup_{x\le x_0} \frac{x - x_0}{V'(x)}$$

holds provided that $V : (-\infty,\, x_0] \to \mathbb{R}$ is convex and strictly decreasing.

Proof. For each $x \ge x_0$ we estimate

$$\int_{x_0}^{x} \frac{1}{\gamma(y)}\,\mathrm{d}y = \int_{x_0}^{x} \exp\big(V(y)\big)\,\mathrm{d}y \le \exp\big(V(x)\big)(x - x_0).$$

Moreover, using Taylor-Expansion of V at x as well as the monotonicity of V' we find

$$\begin{aligned}\int_x^{\infty} \gamma(y)\,\mathrm{d}y &= \int_x^{\infty} \exp\big(-V(y)\big)\,\mathrm{d}y \\ &\le \exp\big(-V(x)\big)\int_x^{\infty} \exp\big(-V'(x)(y-x)\big)\,\mathrm{d}y = \frac{\exp\big(-V(x)\big)}{V'(x)}.\end{aligned}$$

The first claim now follows immediately, and the arguments for the second one are similar. □

The mass-dissipation estimates derived below rely on asymptotic expressions for the Muckenhoupt constants of γ_σ and the following observation.

Lemma 13 (variant of Poincaré inequality)**.** *For any γ, the estimate*

$$\int_J w(x)^2\gamma(x)\,\mathrm{d}x \le 2C_P(\gamma)\int_I \big(w'(x)\big)^2\gamma(x)\,\mathrm{d}x + 2C_J(\gamma)\int_I w(x)^2\gamma(x)\,\mathrm{d}x, \qquad C_J(\gamma) := \frac{\displaystyle\int_J \gamma(x)\,\mathrm{d}x}{\displaystyle\int_I \gamma(x)\,\mathrm{d}x}$$

holds for all $w \in \mathsf{L}^2(\gamma\,\mathrm{d}x)$ and any subinterval $J \subseteq I$.

Proof. Thanks to $2ab \le \eta a^2 + \eta^{-1}b^2$ and Hölder's inequality we have

$$2w_{\mathrm{av}}\int_J w(x)\gamma(x)\,\mathrm{d}x \le \eta w_{\mathrm{av}}^2 + \eta^{-1}\Big(\int_J w(x)^2\gamma(x)\,\mathrm{d}x\Big)\Big(\int_J \gamma(x)\,\mathrm{d}x\Big),$$

and with $\eta := 2\int_J \gamma(x)\,\mathrm{d}x$ we find

$$\begin{aligned}\int_I \big(w(x)-w_{\mathrm{av}}\big)^2\gamma(x)\,\mathrm{d}x &\ge \int_J \big(w(x)-w_{\mathrm{av}}\big)^2\gamma(x)\,\mathrm{d}x\\ &= \int_J w(x)^2\gamma(x)\,\mathrm{d}x + w_{\mathrm{av}}^2\int_J \gamma(x)\,\mathrm{d}x - 2w_{\mathrm{av}}\int_J w(x)\gamma(x)\,\mathrm{d}x\\ &\ge \tfrac{1}{2}\int_J w(x)^2\gamma(x)\,\mathrm{d}x - w_{\mathrm{av}}^2\int_J \gamma(x)\,\mathrm{d}x.\end{aligned}$$

Hölders inequality also implies

$$w_{\mathrm{av}}^2 \le \frac{\displaystyle\int_I w(x)^2\gamma(x)\,\mathrm{d}x}{\displaystyle\int_I \gamma(x)\,\mathrm{d}x},$$

and combining the latter two estimates with the Poincaré estimate for w and γ yields the desired result. ∎

3.1.2 Asymptotics of Poincaré constants for γ_σ

In this section we derive upper bounds for the Poincaré constants for γ_σ and $\gamma_{\sigma,\pm}$. For fixed σ, the key observations concerning the ν-dependence can be summarized as follows.

1. For $\sigma > \sigma^*$ or $\sigma < \sigma_*$ we find

$$C_P(\gamma_\sigma) = C_\sigma \nu^2$$

   because H_σ is a regular single-well potential that grows quadratically at infinity.

2. For $\sigma \in (\sigma_*, \sigma^*)$, the Poincaré constant for small ν is given by

$$C_P(\gamma_\sigma) = C_\sigma \exp\left(\frac{\min\big\{h_-(\sigma),\, h_+(\sigma)\big\}}{\nu^2}\right)$$

   because H_σ is a genuine double-well potential. This implies

   (a) $C_P(\gamma_\sigma) \ll 1/\tau$ for supercritical σ as the energy barrier is smaller than the critical barrier $h_\# = h_-\big(\sigma^\#\big) = h_+(\sigma_\#)$, but

   (b) $C_P(\gamma_\sigma) \gg 1/\tau$ for subcritical σ since the energy barrier exceeds $h_\#$.

   Moreover, the Poincaré constants of $\gamma_{\sigma,-}$ and $\gamma_{\sigma,+}$ are bounded by some constant C_σ independent of ν.

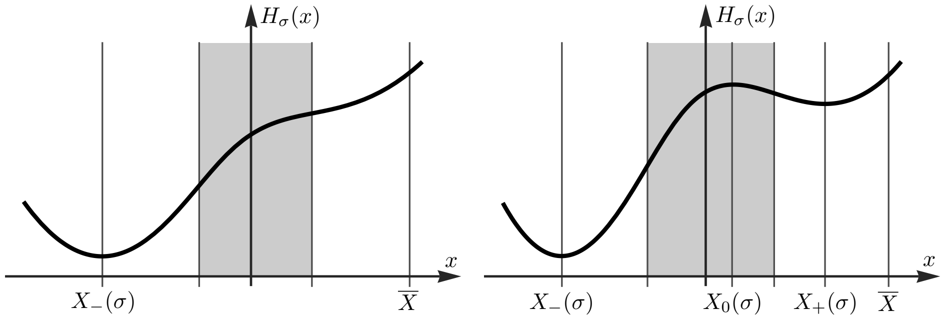

Figure 8: Examples of the effective potential H_σ with $\sigma < \sigma_*$ (left panel) and $\sigma_* < \sigma < 0$ (right panel). The Muckenhoupt constants $C_M^\pm\big(\gamma_\sigma,\, X_-(\sigma)\big)$ are estimated in the proofs of Lemma 15 and Lemma 16

For our purposes, however, we need estimates that hold uniformly in certain ranges of σ and are derived in the subsequent series of lemmata.

Lemma 14 (Poincaré constants of $\gamma_{\sigma,\pm}$ if H_σ is a double-well potential)**.** *For each ε with $0 < \varepsilon < \frac{1}{2}(\sigma^* - \sigma_*)$ there exists a constant C, which depends only on ε and H, such that*

$$C_P(\gamma_{\sigma,\pm}) \le C$$

holds for all $0 < \nu \le 1$ and $\sigma \in [\sigma_* + \varepsilon,\, \sigma^* - \varepsilon]$.

Proof. Let $\sigma \in (\sigma_* + \varepsilon,\, \sigma^* - \varepsilon)$ be given. Since the potential H_σ is strongly convex and strictly decreasing on the interval $(-\infty,\, X_-(\sigma)]$, Remark 12 provides

$$\begin{aligned} C_M^-(\gamma_{\sigma,-}, X_-(\sigma)) &= \sup_{x \le X_-(\sigma)} \left(\int_x^{X_-(\sigma)} \exp\left(+\frac{H_\sigma(y)}{\nu^2}\right) \mathrm{d}y \right) \left(\int_{-\infty}^x \exp\left(-\frac{H_\sigma(y)}{\nu^2}\right) \mathrm{d}y \right) \\ &\le \nu^2 \sup_{x \le X_-(\sigma)} \frac{x - X_-(\sigma)}{H_\sigma'(x)} = \nu^2 \sup_{x \le X_-(\sigma)} \frac{x - X_-(\sigma)}{H'(x) - H'(X_-(\sigma))} \\ &\le \frac{\nu^2}{\inf_{x \le X_-(\sigma)} H''(x)} \le \frac{\nu^2}{\inf_{x \le X_-(\sigma^* - \varepsilon)} H''(x)} = C\nu^2. \end{aligned}$$

Moreover, H_σ is strictly increasing on the interval $[X_-(\sigma),\, X_0(\sigma)]$, and thus we estimate

$$\begin{aligned} C_M^+(\gamma_{\sigma,-}, X_-(\sigma)) &= \sup_{x \in [X_-(\sigma),\, X_0(\sigma)]} \left(\int_{X_-(\sigma)}^x \exp\left(+\frac{H_\sigma(y)}{\nu^2}\right) \mathrm{d}y \right) \left(\int_x^{X_0(\sigma)} \exp\left(-\frac{H_\sigma(y)}{\nu^2}\right) \mathrm{d}y \right) \\ &\le \sup_{x \in [X_-(\sigma),\, X_0(\sigma)]} \left(\exp\left(+\frac{H_\sigma(x)}{\nu^2}\right) \big(x - X_-(\sigma)\big) \right) \left(\exp\left(-\frac{H_\sigma(x)}{\nu^2}\right) \big(X_0(\sigma) - x\big) \right) \\ &= \sup_{x \in [X_-(\sigma),\, X_0(\sigma)]} \big(x - X_-(\sigma)\big)\big(X_0(\sigma) - x\big) \le C. \end{aligned}$$

From Lemma 11 we now conclude that

$$C_P(\gamma_{\sigma,-}) \le \max\big\{C\nu^2,\, C\big\},$$

and the corresponding estimate for $\gamma_{\sigma,+}$ follows by symmetry.

□

Lemma 15 (Poincaré constant of γ_σ if H_σ is a single-well potential)**.** *For each $\varepsilon > 0$ there exists a constant C, which depends only on ε and H, such that*

$$C_P(\gamma_\sigma) \le C$$

holds for all $0 < \nu \le 1$ and $\sigma \in [\sigma_* - 1/\varepsilon,\, \sigma_*] \cup [\sigma^*,\, \sigma^* + 1/\varepsilon]$.

Proof. Suppose that $\sigma \in [\sigma_* - 1/\varepsilon,\, \sigma_*]$. The potential H_σ is strongly convex and strictly decreasing on the interval $(-\infty,\, X_-(\sigma)]$, and hence we show

$$C_M^-(\gamma_\sigma, X_-(\sigma)) \leq \frac{\nu^2}{\inf_{x \leq X_-(\sigma_*)} H''(x)} = C\nu^2$$

as in the proof of Lemma 14. In order to estimate

$$C_M^+(\gamma_\sigma, X_-(\sigma)) = \sup_{x \geq X_-(\sigma)} \left(\int_{X_-(\sigma)}^{x} \exp\left(+\frac{H_\sigma(y)}{\nu^2}\right) \mathrm{d}y \right) \left(\int_{x}^{+\infty} \exp\left(-\frac{H_\sigma(y)}{\nu^2}\right) \mathrm{d}y \right),$$

we choose $\overline{X} > x_*$ and notice that H_σ is strongly convex and strictly increasing on $\big[\overline{X},\, +\infty\big)$, see Figure 8. In particular, we have

$$\inf_{x \geq \overline{X}} \frac{H'_\sigma(x)}{x - \overline{X}} \geq \inf_{x \geq \overline{X}} \frac{H'_\sigma(\bar{X}) + c\big(x - \overline{X}\big)}{x - \overline{X}} \geq c,$$

with $c := \inf_{x \geq \overline{X}} H''(x) > 0$, and Remark 12 yields

$$\sup_{x \geq \overline{X}} \left(\int_{\overline{X}}^{x} \frac{1}{\gamma_\sigma(y)} \, \mathrm{d}y \right) \left(\int_{x}^{+\infty} \gamma_\sigma(y) \, \mathrm{d}y \right) \leq C\nu^2.$$

For $x \geq \overline{X}$ we therefore obtain

$$\left(\int_{X_-(\sigma)}^{x} \frac{1}{\gamma_\sigma(y)} \, \mathrm{d}y \right) \left(\int_{x}^{+\infty} \gamma_\sigma(y) \, \mathrm{d}y \right) \leq C_\sigma + C\nu^2,$$

where

$$C_\sigma := \left(\int_{X_-(\sigma)}^{\overline{X}} \frac{1}{\gamma_\sigma(y)} \, \mathrm{d}y \right) \left(\int_{\overline{X}}^{+\infty} \gamma_\sigma(y) \, \mathrm{d}y \right).$$

Moreover, for $x \in \big[X_-(\sigma),\, \overline{X}\big]$ we estimate

$$\begin{aligned} \left(\int_{X_-(\sigma)}^{x} \frac{1}{\gamma_\sigma(y)} \, \mathrm{d}y \right) \left(\int_{x}^{+\infty} \gamma_\sigma(y) \, \mathrm{d}y \right) &\leq \left(\int_{X_-(\sigma)}^{x} \frac{1}{\gamma_\sigma(y)} \, \mathrm{d}y \right) \left(\int_{x}^{\overline{X}} \gamma_\sigma(y) \, \mathrm{d}y \right) + C_\sigma \\ &\leq \big(x - X_-(\sigma)\big)\big(\overline{X} - x\big) + C_\sigma, \end{aligned}$$

where we used that γ_σ is strictly decreasing on $\big[X_-(\sigma),\, \overline{X}\big]$. Combining all estimates derived so far with Lemma 11 gives

$$C_P(\gamma_\sigma) \leq C\big(1 + \nu^2\big) + C_\sigma,$$

and thus it remains to bound C_σ. To this end we employ the monotonicity properties of H_σ and H'_σ to find

$$\begin{aligned} C_\sigma &= \left(\int_{X_-(\sigma)}^{\overline{X}} \exp\left(\frac{H_\sigma(y) - H_\sigma\big(\overline{X}\big)}{\nu^2} \right) \mathrm{d}y \right) \left(\int_{\overline{X}}^{+\infty} \exp\left(\frac{H_\sigma\big(\overline{X}\big) - H_\sigma(y)}{\nu^2} \right) \mathrm{d}y \right) \\ &\leq \big(\overline{X} - X_-(\sigma)\big) \int_{\overline{X}}^{+\infty} \exp\left(\frac{-H'_\sigma\big(\overline{X}\big)\big(y - \overline{X}\big)}{\nu^2} \right) \mathrm{d}y \leq C \frac{\nu^2}{H'_\sigma\big(\overline{X}\big)} \leq C. \end{aligned}$$

The discussion in the case of $\sigma \in [\sigma^*,\, \sigma^* + 1/\varepsilon]$ is analogous. $\square$

Lemma 16 (Poincaré constant of γ_σ if H_σ is a supercritical double-well potential)**.** *For each ε with $0<\varepsilon<\min\{\sigma_\#-\sigma_*,\,\sigma^*-\sigma^\#\}$ there exist constants α and C which depend only on ε and H such that*

$$C_P(\gamma_\sigma)\le C\tau^{\alpha-1}$$

holds for all $\sigma\in[\sigma_*,\,\sigma_\#-\varepsilon]\cup\big[\sigma^\#+\varepsilon,\,\sigma^*\big]$ and all sufficiently small $\nu>0$.

Proof. By symmetry and continuity, it is sufficient to consider the case $\sigma\in\big(\sigma_*,\,\sigma^\#-\varepsilon\big]$. This implies

$$X_-(\sigma)<x^*<X_0(\sigma)<x_*<X_+(\sigma)<X_+(\sigma_\#),$$

and as in the proofs of Lemma 14 and Lemma 15 we derive the estimates

$$C_M^-(\gamma_\sigma,X_-(\sigma))\le\frac{\nu^2}{\inf_{x\le X_-(\sigma_\#)}H''(x)}=C_0\nu^2$$

and

$$\sup_{x\ge\overline{X}}\left(\int_{\overline{X}}^{x}\frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{+\infty}\gamma_\sigma(y)\,\mathrm{d}y\right)\le C_1\nu^2,$$

where $\overline{X}:=X_+(\sigma_\#)$. Due to the monotonicity properties of H_σ and H'_σ, see Figure 8, we further obtain

$$\left(\int_{X_-(\sigma)}^{\overline{X}}\frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_{\overline{X}}^{+\infty}\gamma_\sigma(y)\,\mathrm{d}y\right)\le\big(\overline{X}-X_-(\sigma)\big)\int_{\overline{X}}^{+\infty}\exp\left(\frac{-H'_\sigma\big(\overline{X}\big)\big(y-\overline{X}\big)}{\nu^2}\right)\mathrm{d}y\le C_2\nu^2$$

as well as

$$\begin{aligned}\left(\int_{X_-(\sigma)}^{X_+(\sigma)}\frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_{X_0(\sigma)}^{\overline{X}}\gamma_\sigma(y)\,\mathrm{d}y\right)&\le\left(C\exp\left(\frac{H_\sigma\big(X_0(\sigma)\big)}{\nu^2}\right)\right)\left(C\exp\left(-\frac{H_\sigma\big(X_+(\sigma)\big)}{\nu^2}\right)\right)\\&\le C_3\exp\left(\frac{h_+(\sigma)}{\nu^2}\right).\end{aligned}$$

We now abbreviate

$$f_\sigma(x):=\left(\int_{X_-(\sigma)}^{x}\frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{+\infty}\gamma_\sigma(y)\,\mathrm{d}y\right)$$

and discuss four different cases: With $x\ge\overline{X}$ we estimate

$$f_\sigma(x)=\left(\int_{X_-(\sigma)}^{\overline{X}}\frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{+\infty}\gamma_\sigma(y)\,\mathrm{d}y\right)+\left(\int_{\overline{X}}^{x}\frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{+\infty}\gamma_\sigma(y)\,\mathrm{d}y\right)\le(C_2+C_1)\nu^2.$$

For $x\in\big[X_+(\sigma),\,\overline{X}\big]$ we find

$$\begin{aligned}f_\sigma(x)&\le\left(\int_{X_-(\sigma)}^{x}\frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{\overline{X}}\gamma_\sigma(y)\,\mathrm{d}y\right)+C_2\nu^2\\&\le\left(\int_{X_+(\sigma)}^{x}\frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{\overline{X}}\gamma_\sigma(y)\,\mathrm{d}y\right)+C_3\exp\left(\frac{h_+(\sigma)}{\nu^2}\right)+C_2\nu^2,\end{aligned}$$

and since H_σ is strictly increasing on the interval $\big[X_+(\sigma),\,\overline{X}\big]$, there exists a constant C_4 such that

$$f_\sigma(x)\le C_4\left(1+\nu^2+\exp\left(\frac{h_+(\sigma)}{\nu^2}\right)\right).$$

In the case of $x \in [X_0(\sigma),\, X_+(\sigma)]$ we verify

$$\begin{aligned} f_\sigma(x) &\le \left(\int_{X_-(\sigma)}^{x} \frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{\overline{X}} \gamma_\sigma(y)\,\mathrm{d}y\right) + C_2\nu^2 \\ &\le C_3 \exp\left(\frac{h_+(\sigma)}{\nu^2}\right) + C_2\nu^2, \end{aligned}$$

and for $x \in [X_-(\sigma),\, X_0(\sigma)]$ we finally get

$$\begin{aligned} f_\sigma(x) &\le \left(\int_{X_-(\sigma)}^{x} \frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{\overline{X}} \gamma_\sigma(y)\,\mathrm{d}y\right) + C_2\nu^2 \\ &\le \left(\int_{X_-(\sigma)}^{x} \frac{1}{\gamma_\sigma(y)}\,\mathrm{d}y\right)\left(\int_x^{X_0(\sigma)} \gamma_\sigma(y)\,\mathrm{d}y\right) + C_3 \exp\left(\frac{h_+(\sigma)}{\nu^2}\right) + C_2\nu^2 \\ &\le C_5\left(1 + \nu^2 + \exp\left(\frac{h_+(\sigma)}{\nu^2}\right)\right), \end{aligned}$$

where the last inequality holds since H_σ is strictly increasing on the interval $[X_-(\sigma),\, X_0(\sigma)]$. Taking the supremum over all $x \ge X_-(\sigma)$ we now obtain, thanks to Lemma 11, the bound

$$C_P(\gamma) \le \max\left\{C_M^-(\gamma_\sigma, X_-(\sigma)),\, C_M^-(\gamma_\sigma, X_-(\sigma))\right\} \le C\left(1 + \nu^2 + \exp\left(\frac{h_+(\sigma)}{\nu^2}\right)\right).$$

The claim now follows with $h_+(\sigma) < h_\#$ and since we have

$$\tau = \exp\left(-\frac{h_\#\big(1 + o(1)\big)}{\nu^2}\right),$$

where $o(1)$ means arbitrary small for small ν. □

3.1.3 Estimates for the mass near the stable peak positions

In order to establish the mass-dissipation estimates, we introduce the dissipation functional

$$\mathcal{D}_\sigma(\varrho) := \int_{\mathbb{R}} \frac{\Big(\nu^2\partial_x\varrho(x) + \big(H'(x) - \sigma\big)\varrho(x)\Big)^2}{\varrho(x)}\,\mathrm{d}x,$$

and observe that

$$\mathcal{D}_\sigma(\varrho) = 4\nu^4 \int_{\mathbb{R}} \big(\partial_x w\big)^2 \gamma_\sigma\,\mathrm{d}x, \qquad \int_J \varrho\,\mathrm{d}x = \int_J w^2\gamma_\sigma\,\mathrm{d}x \qquad \text{for}\quad \varrho = w^2\gamma_\sigma. \tag{14}$$

Our first mass-dissipation estimate implies for each $\sigma \in (\sigma_*,\, \sigma^*)$ that the mass is concentrated near the stable peak positions $X_-(\sigma)$ and $X_+(\sigma)$ provided that the dissipation is sufficiently small.

Lemma 17 (upper bound for mass outside the stable peaks)**.** *For each ε and any η with*

$$0 < \varepsilon < \tfrac{1}{2}(\sigma^* - \sigma_*), \qquad 0 < \eta < \min\Big\{x^* - X_-(\sigma^* - \varepsilon),\, X_+(\sigma_* + \varepsilon) - x_*\Big\}$$

there exist constants α and C, which depend only on ε and η, such that

$$\int_{\mathbb{R}\setminus B_\eta(X_-(\sigma))\cup B_\eta(X_+(\sigma))} \varrho(x)\,\mathrm{d}x \le C\tau^\alpha\left(\frac{\mathcal{D}_\sigma(\varrho)}{\tau} + 1\right)$$

for all $\sigma \in (\sigma_* + \varepsilon,\, \sigma^* - \varepsilon)$, any smooth probability measure ϱ, and all sufficiently small $\nu > 0$.

Proof. Due to the bounds for η, there exist constants C and $\alpha < 1$ such that

$$\frac{\displaystyle\int_{I_{\sigma,-}\setminus B_\eta(X_-(\sigma))} \gamma_\sigma(x)\,\mathrm{d}x}{\displaystyle\int_{I_{\sigma,-}} \gamma_\sigma(x)\,\mathrm{d}x} + \frac{\displaystyle\int_{I_{\sigma,+}\setminus B_\eta(X_+(\sigma))} \gamma_\sigma(x)\,\mathrm{d}x}{\displaystyle\int_{I_{\sigma,+}} \gamma_\sigma(x)\,\mathrm{d}x} \leq C\tau^\alpha$$

for all sufficiently small ν. Using Lemma 13 twice with

$$\gamma = \gamma_{\sigma,\pm}, \qquad I = I_{\sigma,\pm}, \qquad w^2 = \varrho/\gamma_\sigma, \qquad J = I_{\sigma,\pm} \setminus B_\eta\big(X_\pm(\sigma)\big)$$

we therefore arrive, see also (14), at the estimate

$$\int_{\mathbb{R}\setminus\big(B_\eta(X_-(\sigma))\cup B_\eta(X_+(\sigma))\big)} \varrho(x)\,\mathrm{d}x \leq 2\Big(C_P(\gamma_{\sigma,-}) + C_P(\gamma_{\sigma,+})\Big)\frac{\mathcal{D}_\sigma(\varrho)}{4\nu^4} + C\tau^\alpha.$$

Moreover, the combination of Lemma 11 and Lemma 14 yields

$$C_P(\gamma_{\sigma,\pm}) \leq C,$$

and this implies the desired result due to $\int_{I_{\sigma,\pm}} w^2\gamma\,\mathrm{d}x \leq \int_{\mathbb{R}} \varrho\,\mathrm{d}x = 1$ and since we have $\nu^{-4}\tau \leq \tau^\alpha$ for all sufficiently small $\nu > 0$. $\square$

The second mass-dissipation estimate applies to strictly supercritical σ and reveals that the dissipation controls the mass near the global minimizer of H_σ, which is given by $X_-(\sigma)$ and $X_+(\sigma)$ for $\sigma < \sigma_\#$ and $\sigma > \sigma^\#$, respectively.

Lemma 18 (upper bound for mass outside the most stable peak)**.** *For each $\varepsilon > 0$ and any η with*

$$0 < \eta < \Big\{x^* - X_-\big(\sigma^\# - \varepsilon\big),\, X_+\big(\sigma_\# + \varepsilon\big) - x_*\Big\}$$

there exist constants α and C, which depend only on ε and η, such that the implications

$$\sigma \geq \sigma^\# + \varepsilon \quad\Longrightarrow\quad \int_{B_\eta(X_+(\sigma))} \varrho\,\mathrm{d}x \geq 1 - C\tau^\alpha\left(\frac{\mathcal{D}_\sigma(\varrho)}{\tau} + 1\right)$$

and

$$\sigma \leq \sigma_\# - \varepsilon \quad\Longrightarrow\quad \int_{B_\eta(X_-(\sigma))} \varrho\,\mathrm{d}x \geq 1 - C\tau^\alpha\left(\frac{\mathcal{D}_\sigma(\varrho)}{\tau} + 1\right)$$

hold for any smooth probability measure ϱ and all sufficiently small ν.

Proof. We only prove the first implication; the second one follows by analogous arguments. By Lemma 15 and Lemma 16, there exist positive constants C and α such that

$$C_P(\gamma_\sigma) \leq C\frac{\tau^\alpha}{\nu^4\tau}.$$

Making α smaller and C larger (if necessary) we can also assume that

$$\frac{\displaystyle\int_{\mathbb{R}\setminus B_\eta(X_+(\sigma))} \gamma_\sigma(x)\,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}} \gamma_\sigma(x)\,\mathrm{d}x} \leq C\tau^\alpha$$

for all sufficiently small ν. The assertion now follows by applying Lemma 13 with $\gamma = \gamma_\sigma$, $I = \mathbb{R}$, and $J = \mathbb{R} \setminus B_\eta(X_+(\sigma))$. $\square$

3.2 Dynamical stability of peaks

The most fundamental part of our convergence proof is to show that for sufficiently small ν any solution to the nonlocal Fokker-Planck equation (FP_1)+(FP_2') can – at each sufficiently large time t and depending on the value of $\sigma(t)$ – be approximated by either two or one stable peaks located at $X_-\big(\sigma(t)\big)$ and/or $X_+\big(\sigma(t)\big)$. In view of the mass-dissipation estimates from §3.1 it is clear that such an approximation is possible if the dissipation is small, but our approach lacks pointwise estimates for $\mathcal{D}(t)$ (we only have an L^1-bound showing that $\mathcal{D}(t)$ becomes small after a small waiting time).

We therefore control the approximation error by certain combinations of the moment ξ and the partial masses m_-, m_0, and m_+, which all are defined in (4) and (7), because these quantities can be bounded pointwise in time. In order to identify the relevant combinations, we recall that $m_- + m_0 + m_+ = 1$ holds by construction and that any solution to the nonlocal Fokker-Plank equation evolves according to the limit model if and only if

1. $\xi(t) + m_0(t)$ is small for all t,
2. $m_+(t) \approx 0$ for $\sigma(t) < \sigma_\#$,
3. $m_+(t) \approx 1$ for $\sigma(t) > \sigma^\#$,
4. $m_+(t)$ is almost constant for $\sigma_\# < \sigma(t) < \sigma^\#$,
5. $m_+(t)$ is essentially decreasing for $\sigma \approx \sigma_\#$ and essentially increasing for $\sigma \approx \sigma^\#$.

In this section we derive upper bounds for $\xi(t) + m_0(t)$ and discuss the evolution of m_- and m_+ afterwards in §3.3 and §3.4. We start with some auxiliary results which hold pointwise in time and does not rely on dynamical arguments.

Remark 19 (dissipation bounds ξ)**.** *There exists a constant C such that $\xi(t) \le \mathcal{D}(t) + C\nu^2$ for all $t \ge 0$ and $\nu > 0$.*

Proof. The definition of $\mathcal{D}$, see (2), implies

$$\begin{aligned}\mathcal{D}(t) &= \int_{\mathbb{R}} \left(\big(H'(x) - \sigma\big)^2 \varrho(t,\,x) + \nu^4 \frac{\big(\partial_x \varrho(t,\,x)\big)^2}{\varrho(t,\,x)} + 2\nu^2 \big(H'(x) - \sigma\big) \partial_x \varrho(t,\,x) \right) \mathrm{d}x \\ &\ge \int_{\mathbb{R}} \big(H'(x) - \sigma\big)^2 \varrho(t,\,x)\,\mathrm{d}x - 2\nu^2 \int_{\mathbb{R}} H''(x) \varrho(t,\,x)\,\mathrm{d}x,\end{aligned}$$

and this gives the desired result since we have $|H''(x)| \le C$ for all $x \in \mathbb{R}$. $\square$

Lemma 20 (error terms in algebraic relations between ℓ, σ, and $m_\pm$)**.** *For each ε with $0 < \varepsilon < \frac{1}{2}(\sigma^* - \sigma_*)$ there exists a constant C, which depends on ε but not on ν, such that the implications*

$$\sigma(t) \in (-\infty,\, \sigma^* - \varepsilon] \quad \Longrightarrow \quad \big|\ell(t) - X_-\big(\sigma(t)\big)\big| \le C\sqrt{\xi(t) + m_0(t) + m_+(t)}$$

as well as

$$\sigma(t) \in [\sigma_* + \varepsilon,\, \sigma^* - \varepsilon] \quad \Longrightarrow \quad \big|\ell(t) - m_-(t) X_-\big(\sigma(t)\big) - m_+(t) X_+\big(\sigma(t)\big)\big| \le C\sqrt{\xi(t) + m_0(t)}$$

and

$$\sigma(t) \in [\sigma_* + \varepsilon,\, +\infty) \quad \Longrightarrow \quad \big|\ell(t) - X_+\big(\sigma(t)\big)\big| \le C\sqrt{\xi(t) + m_-(t) + m_0(t)}$$

holds for all $0 \le t \le T$ and all $\nu > 0$.

Proof. We only prove the first implication; the derivations of the second and the third one are similar. By definition of the partial masses m_i, $i \in \{-, 0, +\}$, we find

$$\ell(t) - X_-\big(\sigma(t)\big) = -\big(m_0(t) + m_+(t)\big)X_-(\sigma(t)) + \int_{-\infty}^{x^*} \big(x - X_-(\sigma(t))\big)\varrho(t,\,x)\,\mathrm{d}x + \int_{x^*}^{+\infty} x\varrho(t,\,x)\,\mathrm{d}x$$

and hence

$$\big|\ell(t) - X_-\big(\sigma(t)\big)\big| \leq C\left(m_0(t) + m_+(t) + \int_{-\infty}^{x^*} \big|H'(x) - \sigma(t)\big|\varrho(t,\,x)\,\mathrm{d}x + \int_{x^*}^{+\infty} x\varrho(t,\,x)\,\mathrm{d}x\right),$$

where we used $|\sigma(t)| \leq C$, the asymptotic grow of H', and that $|x - X_-(\sigma(t))|$ can be bounded by $|H'(x) - \sigma(t)|$ because of

$$0 < c \leq H''\big(X_-(\sigma)\big) \leq C < \infty \qquad \text{for all} \quad \sigma \in (-\infty,\,\sigma^* - \varepsilon]\,.$$

Moreover, Hölder's inequality yields

$$\int_{-\infty}^{x^*} \big|H'(x) - \sigma(t)\big|\varrho(t,\,x)\,\mathrm{d}x \leq \left(m_-(t)\int_{-\infty}^{x^*} \big|H'(x) - \sigma(t)\big|^2\varrho(t,\,x)\,\mathrm{d}x\right)^{1/2} \leq \sqrt{\xi(t)},$$

thanks to $m_-(t) \leq 1$, as well as

$$\int_{x^*}^{+\infty} x\varrho(t,\,x)\,\mathrm{d}x \leq \big(m_0(t) + m_+(t)\big)^{1/2}\left(\int_{x^*}^{+\infty} x^2\varrho(t,\,x)\,\mathrm{d}x\right)^{1/2} \leq C\sqrt{m_0(t) + m_+(t)},$$

thanks to $\int_{\mathbb{R}} x^2\varrho(t,\,x)\,\mathrm{d}x \leq C$. The first implication now follows from combining all result (notice that $m_i(t) \leq \sqrt{m_i(t)}$). ☐

The assertions and the proof of Lemma 20 can easily be generalized to other moments.

Remark 21. *For any continuous moment weight ψ that grows at most linearly and each ε as in Lemma 20 there exists a constant C, which depends on ε and ψ but not on ν, such that*

$$\left|\int_{\mathbb{R}} \psi(x)\varrho(t,\,x)\,\mathrm{d}x - \sum_{j\in\{-,+\}} m_j(t)\psi\Big(X_j\big(\sigma(t)\big)\Big)\right| \leq C\sqrt{\xi(t) + m_0(t)}$$

holds for all sufficiently small ν as long as $\sigma(t) \in [\sigma_* + \varepsilon,\,\sigma^* - \varepsilon]$. Moreover, similar results hold in the cases $\sigma(t) \in (-\infty,\,\sigma^* - \varepsilon]$ and $\sigma(t) \in [\sigma_* + \varepsilon,\,+\infty)$.

3.2.1 Evolution of the moment ξ

We next study the dynamics of ξ and derive an upper bound for $\xi(t)$ by combining the moment balance for ξ with a simple ODE argument. For the formulation of this result we define

$$m_\eta(t) := \int_{x^*-\eta}^{x_*+\eta} \varrho(t,\,x)\,\mathrm{d}x$$

for all $\eta > 0$.

Lemma 22 (pointwise estimate for ξ)**.** *For each $\eta > 0$ there exists a constant C, which depends on η but not on ν, such that*

$$\sup_{t\in[t_1,\,t_2]} \xi(t) \leq \xi(t_1) + C\left(\nu^2 + \sup_{t\in[t_1,\,t_2]} m_\eta(t)\right)$$

holds for all $0 \leq t_1 < t_2 < T$ and all sufficiently small $\nu > 0$.

Proof. Using the abbreviation $\psi(t, x) := \big(H'(x) - \sigma(t)\big)^2$ as well as (FP$_2'$) and integration by parts, we easily compute

$$\begin{aligned}
\tau\dot{\xi}(t) &= -2\tau\dot{\sigma}(t)\int_{\mathbb{R}} \big(H'(x) - \sigma(t)\big)\varrho(t, x)\,\mathrm{d}x + \int_{\mathbb{R}} \psi(t, x)\tau\partial_t\varrho(t, x)\,\mathrm{d}x \\
&= +2\tau^2\dot{\sigma}(t)\dot{\ell}(t) + \int_{\mathbb{R}} \psi(t, x)\partial_x\Big(\nu^2\partial_x\varrho(t, x) + \big(H'(x) - \sigma(t)\big)\varrho(t, x)\Big)\,\mathrm{d}x \\
&= +2\tau^2\dot{\sigma}(t)\dot{\ell}(t) + \nu^2\int_{\mathbb{R}} \psi''(t, x)\varrho(t, x)\,\mathrm{d}x - 2\int_{\mathbb{R}} H''(x)\psi(t, x)\varrho(t, x)\,\mathrm{d}x,
\end{aligned}$$

as well as

$$\tau\dot{\sigma}(t) = \tau\ddot{\ell}(t) + \nu^2\int_{\mathbb{R}} H'''(x)\varrho(t, x)\,\mathrm{d}x - \int_{\mathbb{R}} H''(x)\big(H'(x) - \sigma(t)\big)\varrho(t, x)\,\mathrm{d}x.$$

In view of

$$\big|\psi''(t, x)\big| + \big|H'(x)\big| + \big|H''(x)\big| + \big|H'''(x)\big| \le C\big(1 + x^2\big)$$

and

$$|\dot{\ell}(t)| + |\ddot{\ell}(t)| + |\sigma(t)| + \int_{\mathbb{R}} \big(1 + x^2\big)\varrho(t, x)\,\mathrm{d}x \le C$$

see Assumption 1, Assumption 5 and Lemma 7, we therefore find

$$\tau\dot{\xi}(t) \le C\big(\nu^2 + \tau\big) - 2\int_{\mathbb{R}} H''(x)\psi(t, x)\varrho(t, x)\,\mathrm{d}x.$$

Moreover, since H is smooth, there exist constants c and C such that

$$H''(x) \ge c \quad \text{for all} \quad x \in \mathbb{R}\setminus[x^* - \eta,\, x_* + \eta], \qquad \big|H''(x)\big| \le C \quad \text{for all} \quad x \in [x^* - \eta,\, x_* + \eta]\,,$$

and this implies

$$\begin{aligned}
\int_{\mathbb{R}} H''(x)\psi(t, x)\varrho(t, x)\,\mathrm{d}x &\ge \int\limits_{\mathbb{R}\setminus[x^*-\eta,\, x_*+\eta]} H''(x)\psi(t, x)\varrho(t, x)\,\mathrm{d}x + \int\limits_{x^*-\eta}^{x_*+\eta} H''(x)\psi(t, x)\varrho(t, x)\,\mathrm{d}x \\
&\ge c\int\limits_{\mathbb{R}\setminus[x^*-\eta,\, x_*+\eta]} \psi(t, x)\varrho(t, x)\,\mathrm{d}x - Cm_\eta(t) = c\,\xi(t) - CM_\eta,
\end{aligned}$$

where M_η is shorthand for $\sup_{t\in[t_1,\, t_2]} m_\eta(t)$. Consequently, find

$$\tau\dot{\xi}(t) \le -c\xi(t) + C\big(\nu^2 + \tau + M_\eta\big)$$

for all $t \in [t_1,\, t_2]$, and Gronwall's Lemma finishes the proof. ☐

3.2.2 Conditional stability estimates

We are now able to investigate the dynamical stability of peaks. More precisely, assuming that $\sigma(t)$ remains confined to certain intervals we now derive estimates that control the evolution of $\xi(t) + m_0(t)$. In the proof we employ, apart from the estimates for ξ, local comparison principles for linear Fokker-Planck equations in order to show that only a very small amount of mass can flow into the unstable interval $(x^*,\, x_*)$.

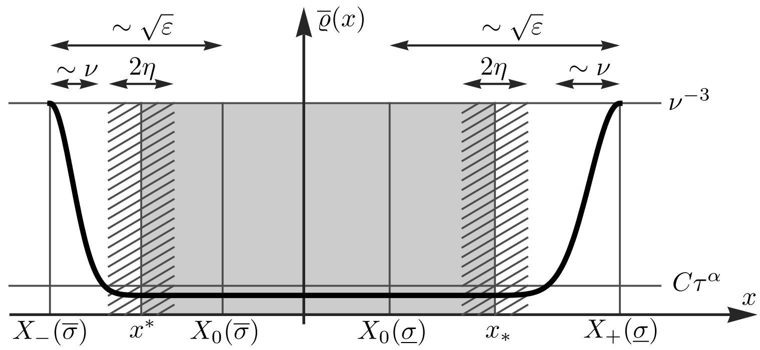

Figure 9: Cartoon of the supersolution $\overline{\varrho}$ and the characteristic lengths from the proof of Lemma 23; the hatched regions indicate intervals of length 2η. The strictly decreasing and increasing branches are given by rescaled equilibrium solutions corresponding to $\overline{\sigma}$ and $\underline{\sigma}$, respectively. For $0 < \nu \ll 1$, $\overline{\varrho}$ is therefore very small in $[x^* - \eta,\, x_* + \eta]$ and exhibits boundary layers with width of order ν near $X_-(\overline{\sigma})$ and $X_+(\underline{\sigma})$.

Lemma 23 (first conditional estimate for $\xi + m_0$)**.** *For each ε with $0 < \varepsilon < \frac{1}{2}(\sigma^* - \sigma_*)$ there exists a positive constant C, which depends only on ε but not on ν, such that the implication*

$$\sigma(t) \in [\sigma_* + \varepsilon,\, \sigma^* - \varepsilon] \quad \textit{for all}\ \ t \in [t_1,\, t_2] \qquad \Longrightarrow \qquad \sup_{t\in[t_1,\, t_2]} \big(\xi(t) + m_0(t)\big) \le C\Big(\xi(t_1) + m_0(t_1) + \nu^2\Big)$$

holds for all $t_* \le t_1 < t_2 \le T$ and all sufficiently small $\nu > 0$.

Proof. Within this proof we regard (FP$_1$) as a non-autonomous but linear PDE for ϱ, that means we ignore (FP$_2'$) and regard σ as a given function of time.

Step 1: We first choose $\underline{\sigma},\, \overline{\sigma} \in (\sigma_*,\, \sigma^*)$ such that

$$h_+(\underline{\sigma}) = h_-(\overline{\sigma}) = \tfrac{1}{2} \min\big\{h_+(\sigma_* + \varepsilon),\ h_-(\sigma^* - \varepsilon)\big\},$$

and the monotonicity properties of h_- and h_+, see Figure 2, ensure that $\underline{\sigma} \in (\sigma_*,\, \sigma_* + \varepsilon)$ and $\overline{\sigma} \in (\sigma^* - \varepsilon,\, \sigma^*)$. Employing the monotonicity of X_-, X_0, and X_+, we easily verify, see also Figure 9, the order relation

$$X_-\big(\sigma(t_1)\big) < X_-(\overline{\sigma}) < x^* < X_0(\overline{\sigma}) < X_0\big(\sigma(t_1)\big) < X_0(\underline{\sigma}) < x_* < X_+(\underline{\sigma}) < X_+\big(\sigma(t_1)\big)$$

and thus we can choose $\eta > 0$ such that the distance between any two adjacent points in this chain is larger than 2η. In particular, by definition of ξ we find

$$\int_{X_-(\overline{\sigma})}^{X_+(\underline{\sigma})} \varrho(t_1,\, x)\,\mathrm{d}x \le C\big(\xi(t_1) + m_0(t_1)\big) \tag{15}$$

for some constant C depending on η.

Step 2: We define a local supersolution $\overline{\varrho}$ on the interval $[X_-(\overline{\sigma}),\, X_+(\underline{\sigma})]$ by combining rescaled versions of the monotone branches of $\gamma_{\overline{\sigma}}$ and $\gamma_{\underline{\sigma}}$. More precisely, we set

$$\overline{\varrho}(x) := \nu^{-3} \begin{cases} \exp\left(\dfrac{H_{\overline{\sigma}}(X_-(\overline{\sigma})) - H_{\overline{\sigma}}(x)}{\nu^2}\right) & \text{for} \quad X_-(\underline{\sigma}) \le x \le X_0(\overline{\sigma}), \\ \exp\left(\dfrac{-h_-(\overline{\sigma})}{\nu^2}\right) & \text{for} \quad X_0(\overline{\sigma}) \le x \le X_0(\underline{\sigma}), \\ \exp\left(\dfrac{H_{\underline{\sigma}}(X_+(\underline{\sigma})) - H_{\underline{\sigma}}(x)}{\nu^2}\right) & \text{for} \quad X_0(\underline{\sigma}) \le x \le X_+(\underline{\sigma}). \end{cases}$$

Our choice of $\overline{\sigma}$ and $\underline{\sigma}$ implies that $\overline{\varrho}$ is continuous with

$$\overline{\varrho}\big(X_-(\overline{\sigma})\big) = \overline{\varrho}\big(X_+(\underline{\sigma})\big) = \nu^{-3},$$

and thanks to our choice of η we readily show that there exist constants α and C such that

$$\overline{\varrho}(x) \le C\tau^{\alpha} \qquad \text{for all} \qquad x \in [x^* - \eta,\, x_* + \eta]$$

and all sufficiently small ν. Moreover, $\overline{\varrho}$ is by construction continuously differentiable and piecewise twice continuously differentiable, where

$$\partial_x\Big(\nu^2\partial_x\overline{\varrho}(x) + \big(H'(x) - \sigma(t)\big)\overline{\varrho}(x)\Big) = \begin{cases} \big(\overline{\sigma} - \sigma(t)\big)\partial_x\overline{\varrho}(x) & \text{for} \quad X_-(\overline{\sigma}) < x < X_0(\overline{\sigma}), \\ H''(x)\overline{\varrho}(x) & \text{for} \quad X_0(\overline{\sigma}) < x < X_0(\underline{\sigma}), \\ \big(\underline{\sigma} - \sigma(t)\big)\partial_x\overline{\varrho}(x) & \text{for} \quad X_0(\underline{\sigma}) < x < X_+(\underline{\sigma}). \end{cases}$$

Combining this with

$$\underline{\sigma} \le \sigma(t) \le \overline{\sigma} \quad \text{for} \quad t \in [t_1,\, t_2]\,, \qquad H''(x) \le 0 \quad \text{for} \quad x \in [X_0(\overline{\sigma}),\, X_0(\underline{\sigma})]$$

and

$$\partial_x\overline{\varrho}(x) \le 0 \quad \text{for} \quad x \in [X_-(\overline{\sigma}),\, X_0(\overline{\sigma})]\,, \qquad \partial_x\overline{\varrho}(x) \ge 0 \quad \text{for} \quad x \in [X_0(\underline{\sigma}),\, X_+(\underline{\sigma})]$$

gives

$$\partial_x\Big(\nu^2\partial_x\overline{\varrho}(x) + \big(H'(x) - \sigma(t)\big)\overline{\varrho}(x)\Big) \le 0 = \tau\partial_t\overline{\varrho}(x),$$

and we conclude that $\overline{\varrho}$ is in fact a weak supersolution to (FP$_1$) on the space-time domain $[t_1,\, t_2] \times [X_-(\overline{\sigma}),\, X_+(\underline{\sigma})]$.

Step 3: We consider three solutions ϱ_-, ϱ_0, and ϱ_+ to (FP$_1$) on the time interval $[t_1,\, t_2]$ defined by the initial conditions

$$\begin{aligned} \varrho_-(t_1,\, x) &= \varrho(t_1,\, x)\chi_{(-\infty,\, X_-(\overline{\sigma})]}(x), \\ \varrho_0(t_1,\, x) &= \varrho(t_1,\, x)\chi_{(X_-(\overline{\sigma}),\, X_+(\underline{\sigma})]}(x), \\ \varrho_+(t_1,\, x) &= \varrho(t_1,\, x)\chi_{[X_+(\underline{\sigma}),\, +\infty)}(x), \end{aligned}$$

where χ_I is the usual indicator function of the interval I. All three functions are nonnegative, and thus we find

$$\varrho_\pm(t,\, x) \le \varrho(t,\, x) \le \frac{C}{\nu^2}$$

for all $x \in \mathbb{R}$ and $t \ge t_1 \ge t_*$ thanks to the L^∞-estimates from Lemma 7. We now conclude that

$$\varrho_\pm(t,\, X_-(\overline{\sigma})) \le \overline{\varrho}(X_-(\overline{\sigma})), \qquad \varrho_\pm(t,\, X_+(\underline{\sigma})) \le \overline{\varrho}(X_+(\underline{\sigma}))$$

for all $t \in [t_1,\, t_2]$, and the comparison principle yields $\varrho_\pm(t,\, x) \le \overline{\varrho}(t,\, x)$ for all $t \in [t_1,\, t_2]$ and almost all $x \in [X_-(\overline{\sigma}),\, X_+(\underline{\sigma})]$. We therefore get

$$\int_{x^*-\eta}^{x_*+\eta} \big(\varrho_-(t,\, x) + \varrho_+(t,\, x)\big)\,\mathrm{d}x \le 2\int_{x^*-\eta}^{x_*+\eta} \overline{\varrho}(x)\,\mathrm{d}x \le C\tau^{\alpha}$$

for all $t \in [t_1,\, t_2]$. On the other hand, using the mass conservation property of (FP$_1$) we estimate

$$\int_{x^*-\eta}^{x_*+\eta} \varrho_0(t,\, x)\,\mathrm{d}x \le \int_{-\infty}^{+\infty} \varrho_0(t_1,\, x)\,\mathrm{d}x = \int_{X_-(\overline{\sigma})}^{X_+(\underline{\sigma})} \varrho(t_1,\, x)\,\mathrm{d}x.$$

With $\varrho = \varrho_- + \varrho_0 + \varrho_+$, the estimate (15), and by taking the supremum over t we therefore get

$$\sup_{t \in [t_1,\, t_2]} m_\eta(t) \le C\left(\int_{X_-(\overline{\sigma})}^{X_+(\underline{\sigma})} \varrho(t_1,\, x)\,\mathrm{d}x + \tau^{\alpha}\right) \le C\Big(\xi(t_1) + m_0(t_1) + \nu^2\Big),$$

where we used $\tau^\alpha \le \nu^2$. Finally, the desired result follows from Lemma 22. $\square$

Lemma 24 (second and third conditional estimate for $\xi + m_0$). *For each ε with $\varepsilon > 0$ there exists a positive constant C, which depends only on ε but not on ν, such that the implications*

$$\sigma(t) \geq \sigma_* + \varepsilon \text{ for all } t \in [t_1, t_2] \quad \Longrightarrow \quad \sup_{t\in[t_1, t_2]} \big(\xi(t) + m_0(t)\big) \leq C\big(\xi(t_1) + m_0(t_1) + m_-(t_1) + \nu^2\big)$$

and

$$\sigma(t) \leq \sigma^* - \varepsilon \text{ for all } t \in [t_1, t_2] \quad \Longrightarrow \quad \sup_{t\in[t_1, t_2]} \big(\xi(t) + m_0(t)\big) \leq C\Big(\xi(t_1) + m_0(t_1) + m_+(t_1) + \nu^2\Big)$$

hold for all $t_* \leq t_1 < t_2 \leq T$ and all sufficiently small $\nu > 0$.

Proof. The proof is very similar to that of Lemma 23, and thus we only sketch the main ideas. For the first implication, we set

$$\underline{\sigma} := \sigma_* - \tfrac{1}{2}\varepsilon, \qquad \eta := \tfrac{1}{2}\min\Big\{x_* - X_0(\underline{\sigma}),\ X_+(\underline{\sigma}) - x_*,\ X_+(\sigma_* + \varepsilon) - X_+\big(\underline{\sigma}\big)\Big\},$$

which in turn implies

$$\int_{-\infty}^{X_+(\underline{\sigma})} \varrho(t_1, x)\,\mathrm{d}x = m_-(t_1) + m_0(t_1) + \int_{x_*}^{X_+(\underline{\sigma})} \varrho(t_1, x)\,\mathrm{d}x \leq m_-(t_1) + m_0(t_1) + C\xi(t_1)$$

for some constant C, which depends on η and hence on ε. We then define a local supersolution $\overline{\varrho}$ in the interval $[x^*, X_+(\underline{\sigma})]$ by

$$\overline{\varrho}(x) := \frac{1}{\nu^3}\begin{cases} \exp\left(-\dfrac{h_+(\underline{\sigma})}{\nu^2}\right) & \text{for} \quad x^* \leq x \leq X_0(\underline{\sigma}), \\ \exp\left(\dfrac{H_{\underline{\sigma}}(X_+(\underline{\sigma})) - H_{\underline{\sigma}}(x)}{\nu^2}\right) & \text{for} \quad X_0(\underline{\sigma}) \leq x \leq X_+(\underline{\sigma}), \end{cases}$$

and consider two solutions ϱ_{-0} and ϱ_+ to (FP$_1$) with

$$\varrho_{-0}(t_1, x) = \varrho(t_1, x)\chi_{(-\infty,\, X_+(\underline{\sigma})]}(x), \qquad \varrho_+(t_1, x) = \varrho(t_1, x)\chi_{[X_+(\underline{\sigma}),\, +\infty)}(x).$$

Employing the comparison principle with respect to the space-time domain $[t_1, t_2] \times [x^*, X_+(\underline{\sigma})]$, we then show that

$$\begin{aligned} m_\eta(t) &= \int_{x^*-\eta}^{x_*+\eta} \varrho_{-0}(t, x)\,\mathrm{d}x + \int_{x^*-\eta}^{x_*+\eta} \varrho_+(t, x)\,\mathrm{d}x \\ &\leq \int_{-\infty}^{+\infty} \varrho_{-0}(t_1, x)\,\mathrm{d}x + \int_{x^*-\eta}^{x_*+\eta} \overline{\varrho}(x)\,\mathrm{d}x \\ &\leq \int_{-\infty}^{X_+(\underline{\sigma})} \varrho(t_1, x)\,\mathrm{d}x + C\tau^\alpha \\ &\leq C\Big(\xi(t_1) + m_-(t_1) + m_0(t_1) + \tau^\alpha\Big), \end{aligned}$$

and the assertion follows from Lemma 22. The second implication can be proven analogously. □

3.3 Mass transfer between the stable regions

We next investigate the evolution of m_- and m_+ by means of appropriate moment balances. The resulting estimates imply for $\nu \ll 1$ that the mass flux from the left stable interval $(-\infty, x^*]$ towards the right one $[x_*, +\infty)$ is – up to small correction terms – positive for $\sigma(t) > \sigma_\#$ but negative for $\sigma(t) < \sigma^\#$, and hence that there is essentially no mass transfer in the subcritical regime $\sigma(t) \in \big(\sigma_\#, \sigma^\#\big)$. These findings perfectly agree with the large deviations results that we obtained in §2.3 by analyzing the orders of magnitude for the different terms in Kramers formula (8).

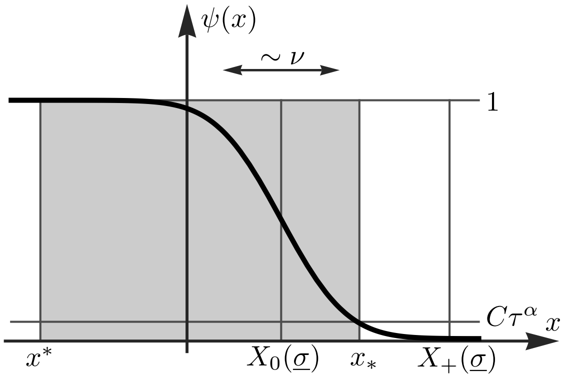

Figure 10: Cartoon of the moment weight ψ that is used in the proof of Lemma 25. The strictly decreasing branch of ψ on $[x^*,\, X_+(\underline{\sigma})]$ is given by the rescaled and shifted primitive of $-1/\gamma_{\underline{\sigma}}$ and has effective width of order ν. For $\nu \ll 1$, the function ψ is therefore close to the indicator function of $(-\infty,\, X_0(\underline{\sigma}))$.

Lemma 25 (almost-monotonicity estimates for $m_\pm$)**.** *For each ε with*

$$0 < \varepsilon < \min\{\sigma^* - \sigma_\#,\ \sigma^\# - \sigma_*\}$$

there exist constants α and C, which only depend on ε, such that the implications

$$\sigma(t) \geq \sigma_\# + \varepsilon \quad \textit{for all} \quad t \in [t_1,\, t_2] \qquad \Longrightarrow \qquad \sup_{t\in[t_1,\,t_2]} m_-(t) \leq m_-(t_1) + m_0(t_1) + C\tau^\alpha$$

and

$$\sigma(t) \leq \sigma^\# - \varepsilon \quad \textit{for all} \quad t \in [t_1,\, t_2] \qquad \Longrightarrow \qquad \sup_{t\in[t_1,\,t_2]} m_+(t) \leq m_+(t_1) + m_0(t_1) + C\tau^\alpha$$

hold for $t_* \leq t_1 < t_2 \leq T$ and all sufficiently small $\nu > 0$.

Proof. We demonstrate the first implication only; the second one follows analogously. In what follows we control the evolution of an upper bound for m_-, namely the moment

$$\overline{m}_-(t) := \int_{\mathbb{R}} \psi(x)\varrho(t,\, x)\,\mathrm{d}x.$$

Here, the weight ψ is defined as piecewise constant continuation of an appropriately rescaled and shifted primitive of $-1/\gamma_{\underline{\sigma}}$, where $\underline{\sigma}$ is shorthand for $\sigma_\# + \varepsilon$. More precisely, we set

$$\psi(x) := \begin{cases} 1 & \text{for} \quad x \leq x^*, \\ \dfrac{\displaystyle\int_x^{X_+(\underline{\sigma})} \exp\left(\frac{H_{\underline{\sigma}}(y)}{\nu^2}\right)\mathrm{d}y}{\displaystyle\int_{x^*}^{X_+(\underline{\sigma})} \exp\left(\frac{H_{\underline{\sigma}}(y)}{\nu^2}\right)\mathrm{d}y} & \text{for} \quad x^* \leq x \leq X_+(\underline{\sigma}), \\ 0 & \text{for} \quad x \geq X_+(\underline{\sigma}), \end{cases}$$

and refer to Figure 10 for an illustration. In particular, ψ is continuous as well as piecewise twice continuously differentiable, and thus we readily verify (using (FP_1) and integration by parts) the moment balance

$$\begin{aligned} \tau\dot{\overline{m}}_-(t) &= -\int_{\mathbb{R}} \psi'(x)\Big(\nu^2\partial_x\varrho(t,\, x) + \big(H'(x) - \sigma(t)\big)\varrho(t,\, x)\Big)\,\mathrm{d}x \\ &= -\int_{x^*}^{X_+(\underline{\sigma})} \psi'(x)\Big(\nu^2\partial_x\varrho(t,\, x) + \big(H'(x) - \sigma(t)\big)\varrho(t,\, x)\Big)\,\mathrm{d}x \\ &= \text{b.t.} + \int_{x^*}^{X_+(\underline{\sigma})} \Big(\nu^2\psi''(x) + \big(\sigma(t) - H'(x)\big)\psi'(x)\Big)\varrho(t,\, x)\,\mathrm{d}x. \end{aligned}$$

Here, the boundary terms are given by

$$\text{b.t.} = \nu^2\psi'\big(x^* + 0\big)\varrho\big(t,\, x^*\big) - \nu^2\psi'\big(X_+(\underline{\sigma}) - 0\big)\varrho\big(t,\, X_+(\underline{\sigma})\big),$$

and the notation ± 0 indicates that the boundary values must be taken with respect to the interval $[x^*,\, X_+(\underline{\sigma})]$.

It remains to estimate $\dot{\overline{m}}_-(t)$ for all $t \in [t_1,\, t_2]$, that means for $\sigma(t) \geq \underline{\sigma}$. We first infer from the definition of ψ that

$$\psi'(x) \leq 0, \quad \nu^2\psi''(x) + \big(\underline{\sigma} - H'(x)\big)\psi'(x) = 0 \quad \text{for all} \quad x \in [x^*,\, X_+(\underline{\sigma})]\,.$$

We next observe that due to $h_+(\underline{\sigma}) < h_+(\sigma_\#)$ the asymptotic properties of γ_σ imply

$$\sup_{x \geq x_*} \psi(x) \leq C\tau^\alpha, \qquad \nu^{-2}\left|\psi'(X_+(\underline{\sigma}) - 0)\right| \leq C\tau^{1+\alpha}$$

for some positive constants α, C and all sufficiently small ν. Finally, we have $\varrho\big(t,\, X_+(\underline{\sigma})\big) \leq C\nu^{-2}$ according to Lemma 7. Combining all these estimates we therefore find

$$\dot{\overline{m}}_-(t) \leq \nu^2\big|\psi'\big(X_+(\underline{\sigma}) - 0\big)\big|\varrho\big(t,\, X_+(\underline{\sigma})\big) \leq C\tau^\alpha$$

for all sufficiently small ν, and hence

$$\sup_{t \in [t_1,\, t_2]} m_-(t) \leq \sup_{t \in [t_1,\, t_2]} \overline{m}_-(t) \leq \overline{m}_-(t_1) + C\tau^\alpha \leq m_-(t_1) + m_0(t_1) + C\tau^\alpha,$$

where we used that $t_2 \leq T$ and $\overline{m}_-(t_1) \leq m_-(t_1) + m_0(t_1) + \sup_{x \geq x_*} \psi(x)$. ☐

3.4 Monotonicity relations between σ and ℓ

As an important consequence of the almost-monotonicity relations for m_- and m_+ we now establish, up to some (small) error terms, monotonicity relations between ℓ and σ. These results have three important implications, which can informally be summarized as follows:

1. If $\sigma(t) \approx \sigma_\#$ or $\sigma(t) \approx \sigma^\#$ holds for all t in some interval $[t_1,\, t_2]$, then ℓ must be essentially decreasing or increasing, respectively, on this interval. The dynamical constraint then implies in the limit $\nu \to 0$ that the phase fraction μ is decreasing and increasing for $\sigma = \sigma_\#$ and $\sigma = \sigma^\#$, respectively.
2. If $[t_1,\, t_2]$ is some time interval such that σ behaves nicely with
   (a) (crossing $\sigma_\#$ from above) $\sigma(t_2) < \sigma_\# < \sigma(t_1) < \sigma^\#$, or
   (b) (crossing $\sigma^\#$ from below) $\sigma_\# < \sigma(t_1) < \sigma^\# < \sigma(t_2)$,

   then $t_2 - t_1$ can be bounded from below by $|\sigma(t_2) - \sigma(t_1)|$. This implies, roughly speaking, that solutions for small ν cannot change too rapidly from subcritical σ to supercritical σ.
3. If $[t_1,\, t_2]$ is some time interval such that σ stays inside the subcritical range $\big(\sigma_\#,\, \sigma^\#\big)$, then $|\sigma(t_2) - \sigma(t_1)|$ can be bounded from above by $|\ell(t_2) - \ell(t_1)|$, and this gives rise to Lipschitz estimates for subcritical σ in the limit $\nu \to 0$.

Lemma 26 (conditional monotonicity relations)**.** *Let ε be fixed with*

$$0 < \varepsilon < \tfrac{1}{2}\min\{\sigma^* - \sigma_\#,\, \sigma^\# - \sigma_*\}.$$

Then the implications

$$\sigma(t) \in \big[\sigma_* + \varepsilon,\, \sigma^\# - \varepsilon\big] \quad \textit{for all} \quad t \in [t_1,\, t_2] \quad \Longrightarrow \quad g\big(\sigma(t_1) - \sigma(t_2)\big) \leq \ell(t_1) - \ell(t_2) + \text{e.t.}$$

and

$$\sigma(t)\in\big[\sigma_{\#}+\varepsilon,\,\sigma^{*}-\varepsilon\big]\quad \textit{for all}\quad t\in[t_1,\,t_2]\qquad\Longrightarrow\qquad g\big(\sigma(t_2)-\sigma(t_1)\big)\le \ell(t_2)-\ell(t_1)+\text{e.t.}$$

hold with error terms

$$\text{e.t.}:=C\Big(\sqrt{\xi(t_1)+m_0(t_1)}+\sqrt{\xi(t_2)+m_0(t_2)}\Big)+C\tau^{\alpha}$$

for all $t_*\le t_1\le t_2\le T$ and all sufficiently small $\nu>0$. Here, g is the increasing and piecewise linear function $g(s)=C_{\mathrm{sgn}(s)}s$, where C_-, $C_+>0$ are independent of both ε and ν, and α, C denote two constants which depend on ε but not on ν.

Proof. We only derive the first implication; the arguments for the second one are similar. For the proof we set $x_\pm(t):=X_\pm\big(\sigma(t)\big)$ as well as

$$\tilde{\ell}(t):=m_-(t)x_-(t)+m_+(t)x_+(t),\qquad \bar{e}:=\sqrt{\xi(t_1)+m_0(t_1)}+\sqrt{\xi(t_2)+m_0(t_2)},$$

and suppose that $\sigma(t)\in\big[\sigma_*+\varepsilon,\,\sigma^{\#}-\varepsilon\big]$ holds for all $t\in[t_1,\,t_2]$. Lemma 20 yields $\big|\ell(t_i)-\tilde{\ell}(t_i)\big|\le\bar{e}$, and we conclude that

$$\begin{aligned}C\bar{e}+\ell(t_1)-\ell(t_2)&\ge\tilde{\ell}(t_1)-\tilde{\ell}(t_2)\\&=\sum_{j\in\{-,+\}}\big(m_j(t_1)-m_j(t_2)\big)x_j(t_1)+\sum_{j\in\{-,+\}}m_j(t_2)\big(x_j(t_1)-x_j(t_2)\big).\end{aligned}\tag{16}$$

Thanks to $m_-(t)+m_0(t)+m_+(t)=1$ we find

$$m_-(t_1)-m_-(t_2)=-\big(m_0(t_1)-m_0(t_2)\big)-\big(m_+(t_1)-m_+(t_2)\big)$$

and hence

$$\sum_{j\in\{-,+\}}\big(m_j(t_1)-m_j(t_2)\big)x_j(t_1)\ge\big(m_+(t_1)-m_+(t_2)\big)\big(x_+(t_1)-x_-(t_1)\big)-C\bar{e},$$

where we used that $|x_\pm(t_1)|\le C$ and $m_0(t_1),m_0(t_2)\le\bar{e}$. Moreover, Lemma 25 provides constants α and C such that

$$m_+(t_1)-m_+(t_2)\ge -C\big(\bar{e}+\tau^{\alpha}\big)$$

holds for all sufficiently small ν, and in view of $x_+(t_1)>x_-(t_1)$, see Remark 2, we arrive at

$$\sum_{j\in\{-,+\}}\big(m_j(t_1)-m_j(t_2)\big)x_j(t_1)\ge -C\big(\bar{e}+\tau^{\alpha}\big).\tag{17}$$

On the other hand, we have $x_j(t_1)-x_j(t_2)=X_j'(\tilde{\sigma}_j)\big(\sigma(t_1)-\sigma(t_2)\big)$ for some intermediate value $\tilde{\sigma}_j$, and the monotonicity properties of X_- and X_+ (again Remark 2) ensure the validity of

$$x_j(t_1)-x_j(t_2)\ge g\big(\sigma(t_1)-\sigma(t_2)\big),$$

where

$$C_-:=\max_{\tilde{\sigma}\in I}\max_{j\in\{-,+\}}X_j'(\tilde{\sigma}),\qquad C_+:=\min_{\tilde{\sigma}\in I}\min_{j\in\{-,+\}}X_j'(\tilde{\sigma}),\qquad I:=\big[\tfrac12(\sigma_*+\sigma_{\#}),\,\tfrac12(\sigma^{\#}+\sigma^{*})\big].$$

We therefore find

$$\begin{aligned}\sum_{j\in\{-,+\}}m_j(t_2)\big(x_j(t_1)-x_j(t_2)\big)&\ge g\big(\sigma(t_1)-\sigma(t_2)\big)\big(m_-(t_1)+m_+(t_1)\big)\\&\ge g\big(\sigma(t_1)-\sigma(t_2)\big)-C\bar{e},\end{aligned}$$

and the desired implication follows by combining this estimate with (16) and (17). ☐

4 Passage to the limit $\nu \to 0$

The arguments used in §3 to characterize the dynamics of the partial masses m_- and m_+ depend crucially on the range of σ. In particular, we have have quite strong results for the subcritical case $\sigma \in \big(\sigma_\#, \sigma^\#\big)$ because here both m_- and m_+ are, to leading order in ν, constant in time. We can also control the evolution in the supercritical cases $\sigma \in (-\infty, \sigma_\#)$ $\sigma \in (\sigma_\#, +\infty)$ because then either m_- or m_+ is always very small. In the critical cases $\sigma \approx \sigma_\#$ and $\sigma \approx \sigma^\#$, however, we can use only relatively weak monotonicity relations, and this complicates the analysis of the limit $\nu \to 0$. Our strategy is therefore as follows. We introduce a small parameter ε with

$$0 < \varepsilon < \varepsilon_* := \tfrac{1}{2} \min\{\sigma^* - \sigma^\#,\ \sigma_\# - \sigma_*,\ \sigma^\# - \sigma_\#\}$$

and accept to have only incomplete control over the dynamics as long as $\sigma(t)$ is inside the ε-neighborhood of either $\sigma_\#$ or $\sigma^\#$. Afterwards we pass to the limit $\nu \to 0$ along sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ and $0 < \nu_n \le \bar{\nu}(\varepsilon_n) \to 0$, where the critical value $\bar{\nu}(\varepsilon_n)$ will be identified below.

4.1 Approximation by stable peaks

Heuristically it is clear that the small parameter dynamics evolves according to the rate-independent limit model if and only if the state of the system can be approximated

1. by two narrow peaks located at $X_-(t)$ and $X_+(t)$ as long as $\sigma(t) \in \big(\sigma_\#, \sigma^\#\big)$,
2. by a single narrow peak located at $X_-(t)$ or $X_+(t)$ for $\sigma(t) \in (-\infty, \sigma_\#)$ or $\sigma(t) \in \big(\sigma^\#, +\infty\big)$, respectively.

In this section we combine all partial results from §3 to show that these assertions are satisfied for all sufficiently small ν and ε, and all sufficiently large times t. Specifically, we consider

$$\zeta(t) := \xi(t) + m_0(t) + \begin{cases} m_+(t) & \text{for} \quad \sigma(t) \in (-\infty, \sigma_\# - \varepsilon]\,, \\ 0 & \text{for} \quad \sigma(t) \in \big(\sigma_\# - \varepsilon, \sigma^\# + \varepsilon\big)\,, \\ m_-(t) & \text{for} \quad \sigma(t) \in \big[\sigma^\# + \varepsilon, +\infty\big)\,, \end{cases}$$

and prove that $\zeta(t)$ is small for all times $t \ge t_1$ provided that the dissipation is small at time t_1. This conclusion is in fact at the very core or our approach as it allows us to convert the L^1-bound for the dissipation into moment estimates that hold pointwise in time.

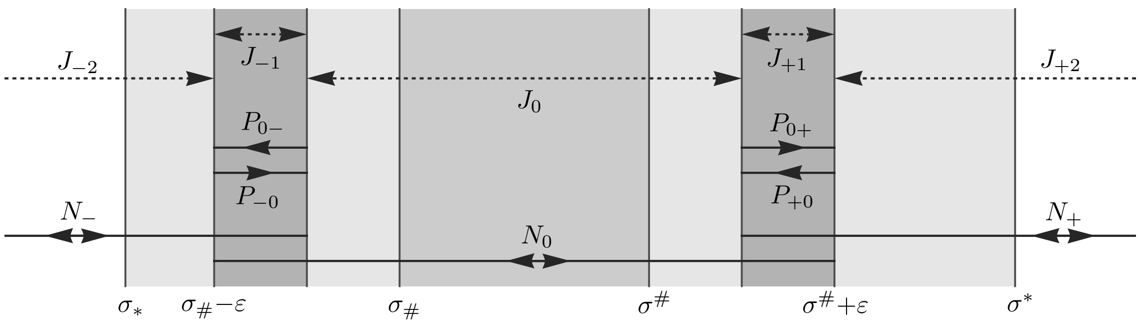

Figure 11: Schematic representation of the intervals J_i as well as the σ-domains for the cases N_j and P_j as used in the proof of Lemma 27. For $\varepsilon \to 0$, we have $J_{-2} \to (-\infty, \sigma_\#]$, $J_0 \to \big[\sigma_\#, \sigma^\#\big]$, $J_{+2} \to \big[\sigma^\#, +\infty\big)$ as well as $J_{-1} \to \{\sigma_\#\}$ and $J_{+1} \to \{\sigma^\#\}$.

Lemma 27 (pointwise upper bound for ζ)**.** *For each $\varepsilon \in [0, \varepsilon_*]$ there exist positive constants $\beta < 1$ and C, which depend on ε but not on ν, such that the implication*

$$\mathcal{D}(t_0) \le \tau^\beta \quad \Longrightarrow \quad \sup_{t\in[t_0, T]} \zeta(t) \le C\nu^2 \tag{18}$$

holds for all $t_* \le t_0 \le T$ and all sufficiently small $\nu > 0$.

Proof. We consider the intervals

$$J_{-2} = (-\infty,\, \sigma_{\#} - \varepsilon]\,, \qquad J_{-1} = \big[\sigma_{\#} - \varepsilon,\, \sigma_{\#} - \tfrac{1}{2}\varepsilon\big]\,,$$

as well as $J_0 := \big[\sigma_{\#} - \tfrac{1}{2}\varepsilon,\, \sigma^{\#} + \tfrac{1}{2}\varepsilon\big]$ and

$$J_{+1} = \Big[\sigma^{\#} + \tfrac{1}{2}\varepsilon,\, \sigma^{\#} + \varepsilon\Big]\,, \qquad J_{+2} = \Big[\sigma^{\#} + \varepsilon,\, +\infty\Big)\,.$$

These intervals and the different cases considered within this proof are illustrated in Figure 11.

<u>*Part 1:*</u> We first prove the assertion under the assumption that σ remains confined to at most two or three neighboring intervals from $\{J_{-2},\, J_{-1},\, J_0,\, J_{+1},\, J_{+2}\}$, and start with the case

$$\sigma(t) \in J_{-2} \cup J_{-1} \quad \text{for all} \quad t \in [t_1,\, t_2]\,, \tag{N_-}$$

where $t_0 \le t_1 < t_2 \le T$. Under this assumption, Lemma 24 combined with Lemma 25 provides constants α_1 and C_1 such that

$$\sup_{t\in[t_1,\, t_2]} \big(\xi(t) + m_0(t)\big) \le C_1\Big(\xi(t_1) + m_+(t_1) + m_0(t_1) + \nu^2\Big)$$

as well as

$$\sup_{t\in[t_1,\, t_2]} m_+(t) \le m_+(t_1) + m_0(t_1) + C_1\tau^{\alpha_1}.$$

Moreover, by Lemma 18, there exist constants α_2 and C_2 such that

$$m_0(t_1) + m_+(t_1) \le C_2\tau^{\alpha_2}\Big(\tau^{-1}\mathcal{D}(t_1) + 1\Big),$$

and Remark 19 yields a constant C_3 such that

$$\xi(t_1) \le \mathcal{D}(t_1) + C_3\nu^2.$$

We now choose $\beta_1 \in (0,\, 1)$ sufficiently large such that $\alpha_2 + \beta_1 - 1 > 0$ and this guarantees (via $\xi(t_1) \le (C_3 + 1)\nu^2$ and $m_+(t_1) + m_0(t_1) \le \nu^2$), that the implication

$$(N_-) \quad \text{and} \quad \mathcal{D}(t_1) \le \tau^{\beta_1} \qquad \Longrightarrow \qquad \sup_{t\in[t_1,\, t_2]} \big(\xi(t) + m_0(t) + m_+(t)\big) \le C_4\nu^2 \tag{19}$$

holds for all sufficiently small $\nu > 0$, where $C_4 := C_1\big(C_3 + 3\big)$.

The arguments for the case

$$\sigma(t) \in J_{+1} \cup J_{+2} \quad \text{for all} \quad t \in [t_1,\, t_2] \tag{N_+}$$

are entirely similar. In particular, possibly changing all constants introduced so far, we readily demonstrate that

$$(N_+) \quad \text{and} \quad \mathcal{D}(t_1) \le \tau^{\beta_1} \qquad \Longrightarrow \qquad \sup_{t\in[t_1,\, t_2]} \big(\xi(t) + m_-(t) + m_0(t)\big) \le C_4\nu^2 \tag{20}$$

holds for all sufficiently small $\nu > 0$.

We next study the case

$$\sigma(t) \in J_{-1} \cup J_0 \cup J_{+1} \quad \text{for all} \quad t \in [t_1,\, t_2]\,, \tag{N_0}$$

and observe that Lemma 23 provides a constant C_5 such that

$$\sup_{t\in(t_1,\, t_2)} \big(\xi(t) + m_0(t)\big) \le C_5\big(\xi(t_1) + m_0(t_1) + \nu^2\big).$$

By Lemma 17 we find further constants α_6 and C_6 such that

$$m_0(t_1) \le C_6 \tau^{\alpha_6}\Big(\tau^{-1}\mathcal{D}(t_1) + 1\Big),$$

and we choose $\beta_2 \in (0,\,1)$ sufficiently close to 1 such that $\alpha_6 + \beta_2 - 1 > 0$. This ensures (via $\xi(t_1) \le (C_3+1)\nu^2$ and $m_0(t_1) \le \nu^2$) that the implication

$$(N_0) \quad \text{and} \quad \mathcal{D}(t_1) \le \tau^{\beta_2} \qquad \Longrightarrow \qquad \sup_{t\in[t_1,\,t_2]} \big(\xi(t) + m_0(t)\big) \le C_7\nu^2. \tag{21}$$

holds for all sufficiently small $\nu > 0$, where $C_7 := C_5(C_3+3)$.

<u>*Part 2:*</u> We set

$$\beta := \max\big\{\beta_1,\, \beta_2\big\}, \qquad C := \max\big\{C_4,\, C_7\big\}.$$

Our next goal is to demonstrate that whenever the systems passes for $t \in [t_3,\, t_4] \subseteq [t_*,\, T]$ through the entire interval $J_{\pm1}$, then there exist at least one time t in between t_3 and t_4, at which the data are well prepared in the sense of $\mathcal{D}(t) \le \tau^\beta$. To this end, we have to discuss the four cases

$$\sigma(t) \in J_{-1} \quad \text{for all} \quad t \in [t_3,\, t_4]\,, \qquad \sigma(t_3) = \sigma_\# - \varepsilon, \qquad \sigma(t_4) = \sigma_\# - \tfrac{1}{2}\varepsilon \tag{P_{-0}}$$

and

$$\sigma(t) \in J_{-1} \quad \text{for all} \quad t \in [t_3,\, t_4]\,, \qquad \sigma(t_3) = \sigma_\# - \tfrac{1}{2}\varepsilon, \qquad \sigma(t_4) = \sigma_\# - \varepsilon \tag{P_{0-}}$$

as well as

$$\sigma(t) \in J_{+1} \quad \text{for all} \quad t \in [t_3,\, t_4]\,, \qquad \sigma(t_3) = \sigma^\# + \varepsilon, \qquad \sigma(t_4) = \sigma^\# + \tfrac{1}{2}\varepsilon \tag{P_{+0}}$$

and

$$\sigma(t) \in J_{+1} \quad \text{for all} \quad t \in [t_3,\, t_4]\,, \qquad \sigma(t_3) = \sigma^\# + \tfrac{1}{2}\varepsilon, \qquad \sigma(t_4) = \sigma^\# + \varepsilon \tag{P_{0+}}$$

but by symmetry it is sufficient to study (P_{-0}) and (P_{0-}). We first discuss the case (P_{0-}) and suppose that

$$\zeta(t_3) = \xi(t_3) + m_0(t_3) \le C\nu^2.$$

Lemma 26 combined with the uniform bounds for $|\dot{\ell}(t)|$ yields constants c_8 and C_9 such that

$$t_4 - t_3 \ge c_8\big(\sigma(t_3) - \sigma(t_4)\big) - C_9\nu = \tfrac{1}{2}c_8\varepsilon - C_9\nu,$$

and hence there exists a positive constant c_{10} such that $t_4 - t_3 \ge c_{10}$ for all sufficiently small ν. Since we have $\int_{t_3}^{t_4} \mathcal{D}(t)\,\mathrm{d}t \le C_{11}\tau$, there exists at least one time $t \in [t_3,\, t_4]$ (which depends on ν) such that

$$\mathcal{D}(t) \le \frac{C_{11}}{c_{10}}\tau \le \tau^\beta$$

for all sufficiently small $\nu > 0$. We have thus proven that the implications

$$(P_{0\mp}) \quad \text{and} \qquad \zeta(t_3) \le C\nu^2 \qquad \Longrightarrow \qquad \mathcal{D}(t) \le \tau^\beta \quad \text{for some} \quad t \in [t_3,\, t_4] \tag{22}$$

hold for all sufficiently small $\nu > 0$.

For the case (P_{0-}) we assume that

$$\zeta(t_3) = \xi(t_3) + m_0(t_3) + m_+(t_3) \le C\nu^2.$$

Similar to the above discussion of the case (N_-), we use of Lemma 24 and Lemma 25 to show that there is a constant C_{12} such that

$$\zeta(t_4) = \xi(t_4) + m_0(t_4) + m_+(t_4) \le C_{12}\nu^2$$

holds for all sufficiently small $\nu > 0$. From Lemma 20 we further infer that there is a constant C_{13} such that

$$\big|X_-\big(\sigma(t_4)\big) - X_-\big(\sigma(t_3)\big)\big| \le |\ell(t_4) - \ell(t_3)| + C_{13}\nu^2,$$

and the properties of X_- and ℓ imply that $t_4 - t_3 \ge c_{14}$ holds for all sufficiently small $\nu > 0$ and some constant c_{14} independent of ε. In particular, using $\int_{t_3}^{t_4} \mathcal{D}(t)\,\mathrm{d}t \le C_{11}\tau$ once more, we show that implications

$$(P_{\mp 0}) \quad \text{and} \qquad \zeta(t_3) \le C\nu^2 \qquad \Longrightarrow \qquad \mathcal{D}(t) \le \tau^\beta \quad \text{for some} \quad t \in [t_3,\, t_4] \tag{23}$$

holds for all sufficiently small $\nu > 0$. Finally, we recall that we have

$$(P_{0\mp}) \quad \text{or} \quad (P_{\mp 0}) \qquad \Longrightarrow \qquad t_4 - t_3 \ge c \tag{24}$$

for some constant $c > 0$ and all sufficiently small $\nu > 0$.

Part 3: We finally return to discussing the time interval $[t_0,\, T]$ and show in a preparatory step that there exists a sufficiently large time $\bar{t}_0 \in (t_0,\, T)$ such that

$$\sup_{t\in[t_0,\,\bar{t}_0]} \zeta(t) \le C\nu^2, \qquad \mathcal{D}(\bar{t}_0) \le \tau^\beta. \tag{25}$$

Suppose at first that $\sigma(t_0) \in J_{-2}$. If $\sigma(t) \in J_{-2} \cup J_{-1}$ holds for all $t \in [t_0,\, T]$, then we are done with $\bar{t}_0 = T$ as (19) implies (18). Otherwise we consider the times

$$t_4 := \inf\Big\{t \in [t_0,\, T] \;:\; \sigma(t) = \sigma_\# - \tfrac{1}{2}\varepsilon\Big\}, \qquad t_3 := \sup\Big\{t \in [t_0,\, t_4] \;:\; \sigma(t) = \sigma_\# - \varepsilon\Big\},$$

which are well-defined as σ is continuous. By construction, the intervals $[t_0,\, t_3]$ and $[t_3,\, t_4]$ corresponds to the cases (N_-) and (P_{-0}), respectively, and the existence of $\bar{t}_0 \in [t_3,\, t_4]$ is a consequence of (19) and (22). Similarly, the case $\sigma(t_0) \in J_{+2}$ can be traced back to the cases (N_+) and (P_{+0}), and $\bar{t}_0$ is provided by (20) and (22).

Now suppose that $\sigma(t_0) \in J_0$. If $\sigma(t_0) \in J_{-1} \cup J_0 \cup J_{+1}$ holds for all $t \in [t_0,\, T]$, we set $\bar{t}_0 = T$ and are done by (21). Otherwise we find times $t_3 < t_4$ such that $[t_0,\, t_3]$ corresponds to (N_0) and $[t_3,\, t_4]$ to either (P_{0-}) or (P_{0+}), and the existence of $\bar{t}_0$ is implied by (21) and (23).

For $\sigma(t_0) \in J_{\pm 1}$, we are either done via $\sigma(t) \in J_{\pm 1}$ for all $[t_0,\, T]$, or we find a time t_1 with $\sigma(t) \in J_{\pm 1}$ for all $t \in [t_0,\, t_1]$ and either $\sigma(t_1) = \sigma_\# \pm \varepsilon$ or $\sigma(t_1) = \sigma_\# \pm \frac{1}{2}\varepsilon$. Depending on the value of $\sigma(t_1)$, we can now argue as for $\sigma(t_0) \in J_{\pm 2}$ or $\sigma(t_0) \in J_0$.

In summary, we have now proven the existence of $\bar{t}_0$ with (25). Our arguments can easily be iterated, and since (24) provides a lower bound for the time covered by two subsequent iterations, we finally arrive at (18). □

4.2 Continuity estimates for σ

As a further key ingredient to the derivation of the limit dynamics we now show that σ is, up to some error terms, globally Lipschitz continuous in time. These estimates become important when establishing the limit $\nu \to 0$ because they imply the existence of convergent subsequences as well as the Lipschitz continuity of any limit function.

Lemma 28 (Lipschitz continuity of σ up to small error terms). *For each $\varepsilon \in [0,\, \varepsilon_*]$ there exist constants α and C, which depend on ε but not on ν, as well as a constant C_0, which is independent of both ε and ν, such that*

$$\big|\sigma(t_2) - \sigma(t_1)\big| \le C_0\big(|t_2 - t_1| + \varepsilon\big) + C\big(\tau^\alpha + \sup_{t\in[t_1,\,t_2]} \sqrt{\zeta(t)}\big)$$

holds for all $t_* \le t_1 \le t_2 \le T$ and all sufficiently small ν.

Proof. <u>*Step 0:*</u> We introduce appropriate cut offs in σ-space. More precisely, we define

$$\sigma_{-2}(t) := \Pi_{\left(-\infty,\,\sigma_{\#}-\varepsilon\right)}\sigma(t), \qquad \sigma_0(t) := \Pi_{\left(\sigma_{\#}+\varepsilon,\,\sigma^{\#}-\varepsilon\right)}\sigma(t), \qquad \sigma_{+2}(t) := \Pi_{\left(\sigma^{\#}+\varepsilon,\,+\infty\right)}\sigma(t),$$

as well as

$$\sigma_{-1}(t) := \Pi_{\left(\sigma_{\#}-\varepsilon,\,\sigma_{\#}+\varepsilon\right)}\sigma(t), \qquad \sigma_{+1}(t) := \Pi_{\left(\sigma^{\#}-\varepsilon,\,\sigma^{\#}+\varepsilon\right)}\sigma(t),$$

where the nonlinear projectors $P_{(\underline{\sigma},\overline{\sigma})}$ are given by $P_{(\underline{\sigma},\overline{\sigma})}(\sigma) := \max\{\min\{\sigma,\overline{\sigma}\},\underline{\sigma}\}$. These definitions imply

$$\sum_{j=-2}^{+2}\sigma_j(t) = \sigma(t) + 2\big(\sigma^{\#} - \sigma_{\#}\big), \tag{26}$$

and since σ is (for any given $\nu > 0$) continuous in time, all projected functions σ_j depend continuously on t as well.

<u>*Step 1:*</u> To show that σ_0 is almost Lipschitz continuous, we assume without loss of generality that $\sigma_0(t_1) < \sigma_0(t_2)$ and consider at first the special case of $\sigma_0(t) = \sigma(t) \in [\sigma_0(t_1),\, \sigma_0(t_2)]$ for all $t \in [t_1,\, t_2]$. Under this assumption, Lemma 26 provides constants α, C and C_0 such that

$$|\sigma_0(t_2) - \sigma_0(t_1)| \le C_0\,|\ell(t_2) - \ell(t_1)| + C\Big(\sqrt{\zeta(t_1)} + \sqrt{\zeta(t_2)} + \tau^{\alpha}\Big) \tag{27}$$

holds for all sufficiently small $\nu > 0$. In the general case, we introduce two times $\hat{t}_1$ and $\hat{t}_2$, which both depend on ν, by

$$\hat{t}_1 := \max\big\{t \in [t_1,\, t_2] \;:\; \sigma_0(t) = \sigma_0(t_1)\big\}, \qquad \hat{t}_2 := \min\big\{t \in \big[\hat{t}_1,\, t_2\big] \;:\; \sigma_0(t) = \sigma_0(t_2)\big\}, \tag{28}$$

and notice that the Intermediate Value Theorem (applied to the continuous function σ_0) ensures that σ_0 is a bijective map between the intervals $\big[\hat{t}_1,\, \hat{t}_2\big]$ and $[\sigma_0(t_1),\, \sigma_0(t_2)]$. In particular, our result for the special case applied to the interval $\big[\hat{t}_1,\, \hat{t}_2\big]$ combined with $\big|\hat{t}_2 - \hat{t}_1\big| \le |t_2 - t_1|$ yields again (27).

<u>*Step 2:*</u> We next derive a Lipschitz estimate for σ_{+2}. As above, we suppose that $\sigma_{+2}(t_1) < \sigma_{+2}(t_2)$ and consider at first the special case of $\sigma_{+2}(t) = \sigma(t) \in [\sigma_{+2}(t_1),\, \sigma_{+2}(t_2)]$ for all $t \in [t_1,\, t_2]$. From Lemma 20 we then infer that

$$\big|\ell(t_i) - X_+\big(\sigma_{+2}(t_i)\big)\big| \le C\sqrt{\zeta(t_i)}, \qquad i = 1, 2,$$

for some constant C and all sufficiently small $\nu > 0$, and hence we get

$$\big|X_+\big(\sigma_{+2}(t_2)\big) - X_+\big(\sigma_{+2}(t_1)\big)\big| \le \big|\ell(t_2) + \ell(t_1)\big| + C\Big(\sqrt{\zeta(t_1)} + \sqrt{\zeta(t_2)}\Big). \tag{29}$$

On the other hand, thanks to $\sigma_{+2}(t_i) \ge \sigma^{\#} + \varepsilon > \sigma_*$ and the properties of X_+, cf. Remark 2, we have

$$\big|\sigma_{+2}(t_2) - \sigma_{+2}(t_1)\big| \le C_0\big|X_+\big(\sigma_+(t_2)\big) - X_+\big(\sigma_{+2}(t_1)\big)\big|,$$

and combining this (29) gives

$$\big|\sigma_{+2}(t_2) - \sigma_{+2}(t_1)\big| \le C_0\big|\ell(t_2) + \ell(t_1)\big| + C\Big(\sqrt{\zeta(t_1)} + \sqrt{\zeta(t_2)}\Big). \tag{30}$$

In the general case we introduce again two times $\hat{t}_1$ and $\hat{t}_2$ by using (28) with σ_{+2} instead of σ_0, and argue as above. Moreover, the estimate

$$\big|\sigma_{-2}(t_2) - \sigma_{-2}(t_1)\big| \le C_0\big|\ell(t_2) + \ell(t_1)\big| + C\Big(\sqrt{\zeta(t_1)} + \sqrt{\zeta(t_2)}\Big). \tag{31}$$

can be proven similarly.

Step 3: By construction, we have

$$|\sigma_{-1}(t_2) - \sigma_{-1}(t_1)| \leq 2\varepsilon, \qquad |\sigma_{+1}(t_2) - \sigma_{+1}(t_1)| \leq 2\varepsilon,$$

and Assumption 5 implies

$$|\ell(t_2) - \ell(t_1)| \leq \left(\sup_{t \in [t_1,\, t_2]} \left|\dot{\ell}(t)\right| \right) |t_2 - t_1| \leq C_0 \, |t_2 - t_1| \, .$$

The desired result now follows from the algebraic relation (26) as well as the estimates (27), (30), and (31). □

4.3 Compactness results and convergence to limit model

In this section we finally pass to the limit $\nu \to 0$ and verify the validity of the limit model. We therefore write

$$\tau_\nu \text{ instead of } \tau, \quad \varrho_\nu \text{ instead of } \varrho, \quad \sigma_\nu \text{ instead of } \sigma, \quad m_{j,\,\nu} \text{ instead of } m_j, \quad \zeta_{\varepsilon,\,\nu} \text{ instead of } \zeta,$$

and define the phase fraction by $\mu_\nu := m_{+,\,\nu} - m_{-,\,\nu}$.

Theorem 29 (convergence to limit model along subsequences)**.** *There exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ with $\nu_n \to 0$ as $n \to \infty$ as well as two Lipschitz functions σ_0, $\mu_0 \in \mathsf{C}^{0,\,1}([0,\, T])$ such that*

$$\|\sigma_{\nu_n} - \sigma_0\|_{\mathsf{C}(I)} \xrightarrow{n\to\infty} 0, \qquad \|\mu_{\nu_n} - \mu_0\|_{\mathsf{C}(I)} \xrightarrow{n\to\infty} 0, \tag{32}$$

for each compact interval $I \subset (0,\, T]$. Moreover, we have

$$\varrho_{\nu_n}(t,\, x) \quad \xrightharpoonup{n\to\infty} \quad \frac{1-\mu_0(t)}{2} \delta_{X_-(\sigma_0(t))}(x) + \frac{1+\mu_0(t)}{2} \delta_{X_+(\sigma_0(t))}(x)$$

weakly⋆ for all $t > 0$, and the triple $(\ell,\, \sigma_0,\, \mu_0)$ is a solution to the limit model in the sense of Definition 10.

Proof. Convergence of σ: We choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n < \varepsilon_*$ for all n and $\varepsilon_n \to 0$ as $n \to \infty$. According to Lemma 27 and Lemma 28, there exist – for any given n – positive constant C_0, C_n, α_n, and $\beta_n < 1$ such that

$$\mathcal{D}_\nu(t_0) \leq \tau^{\beta_n} \qquad \Longrightarrow \qquad \sup_{t\in[t_0,\, T]} \zeta_{\varepsilon_n,\nu}(t) \leq C_n \nu^2$$

and

$$\bigl|\sigma_\nu(t_2) - \sigma_\nu(t_1)\bigr| \leq C_0 \Bigl(|t_2 - t_1| + \varepsilon_n \Bigr) + C_n \Bigl(\tau_\nu^{\alpha_n} + \sup_{t\in[t_1,\, t_2]} \sqrt{\zeta_{\varepsilon_n,\nu}(t)} \Bigr)$$

holds for all n, all times $t_0, t_1, t_2 \in (0,\, T]$, and all sufficiently small $\nu > 0$, where C_0 is in fact independent of n. Moreover, making C_0 larger (if necessary) we can also assume that

$$C_0 \tau_\nu \geq \int_{t_*}^{T} \mathcal{D}_\nu(t) \, \mathrm{d}t \geq \tau_\nu^{\beta_n} \Bigl| \bigl\{ t \in [t_*,\, T] \; : \; \mathcal{D}_\nu(t) > \tau_\nu^{\beta_n} \bigr\} \Bigr|$$

holds for all $\nu > 0$ and $n \in \mathbb{N}$, and hence there exists for any choice of ν and n a time

$$S_{n,\nu} \in \Bigl[t_*,\, t_* + C_0 \tau_\nu^{(1-\beta_n)} \Bigr] \quad \text{with} \quad \mathcal{D}_\nu(S_{n,\nu}) \leq \tau_\nu^{\beta_n}.$$

For each n we next choose $\nu_n > 0$ such that

$$\max \bigl\{ C_n \nu_n^2,\; C_n \tau_{\nu_n}^{\alpha_n} + C_n^2 \nu_n^2,\; \tau_{\nu_n}^{\beta_n} \bigr\} \leq \varepsilon_n.$$

In particular, using the abbreviations $\sigma_n := \sigma_{\nu_n}$, $m_{j,n} := m_{j,\nu_n}$, $\zeta_n := \zeta_{\varepsilon_n,\nu_n}$, and $S_n := S_{n,\nu_n}$ we have

$$\sup_{t\in[S_n,T]} \zeta_n(t) \le \varepsilon_n, \qquad S_n \le \varepsilon_n \tag{33}$$

as well as

$$\big|\sigma_n(t_2) - \sigma_n(t_1)\big| \le C_0\,|t_2 - t_1| + (C_0+1)\varepsilon_n \quad \text{for all} \quad t_1, t_2 \in [S_n,\, T]\,. \tag{34}$$

Let $t_0 > 0$ be fixed and notice that $S_n \le t_0$ for almost all n. The compactness result from Appendix C, Proposition 35, guarantees the existence of a continuous function σ_0 defined on $[t_0,\, T]$ and a not relabeled subsequence such that $\|\sigma_n - \sigma_0\|_{\mathsf{C}([t_0,T])} \to 0$ as $n \to \infty$. Moreover, by the usual diagonal argument we can extract a further subsequence such that $\|\sigma_n - \sigma_0\|_{\mathsf{C}(I)} \to 0$ for any compact $I \subset (0,\, T]$, and the estimate (34) implies that σ_0 is Lipschitz continuous on the whole interval $[0,\, T]$.

Convergence of μ and ϱ: In what follows, we denote by C_0 any generic constant independent of n, and assume (without saying so explicitly) that n is sufficiently large. We also define

$$\underline{\sigma} := \tfrac{1}{2}\big(\sigma_* + \sigma_\#\big), \qquad \overline{\sigma} := \tfrac{1}{2}\big(\sigma^\# + \sigma^*\big),$$

and introduce a function μ_0 as follows: For $\sigma_0(t) \in (-\infty,\, \underline{\sigma}]$ we set $\mu_0(t) = -1$, and since we have $\sum_{j\in\{-,0,+\}} m_{j,n}(t) = 1$ as well as

$$m_{0,n}(t) + m_{+,n}(t) \le \zeta_n(t) \quad \text{for} \quad \varepsilon_n \le \overline{\sigma} - \sigma^\#$$

we find

$$\big|\mu_n(t) - \mu_0(t)\big| \le \zeta_n(t). \tag{35}$$

Similarly, for $\sigma_0(t) \in [\overline{\sigma},\, +\infty)$ we set $\mu_0(t) = +1$ and find again (35). In the case $\sigma_0(t) \in (\underline{\sigma},\, \overline{\sigma})$, we employ Lemma 20 – applied with $\varepsilon = \min\{\sigma^* - \overline{\sigma}, \underline{\sigma} - \sigma_*\}$, which does not depend on n – to find

$$\Big|\ell(t) - \frac{1-\mu_n(t)}{2} X_-\big(\sigma_n(t)\big) - \frac{1+\mu_n(t)}{2} X_+(\sigma_n(t))\Big| \le C_0\zeta_n(t),$$

and hence $\ell(t) \in [X_-(\sigma_*),\, X_+(\sigma^*)]$. In particular, we can define $\mu_0(t) \in [-1,\, +1]$ as the unique solution to

$$\ell(t) = \frac{1-\mu_0(t)}{2} X_-\big(\sigma_0(t)\big) - \frac{1+\mu_0(t)}{2} X_+(\sigma_0(t)), \tag{36}$$

and using the properties of $X_\pm$, see Remark 2, we prove that (35) holds also in this case.

In summary, we have now defined $\mu_0(t)$ for all $t \in [0,\, T]$, and (35) combined with (33) and $S_n \to 0$ yields $\|\mu_n - \mu_0\|_{\mathsf{C}(I)} \to 0$ as $n \to 0$. Moreover, the claimed weak$\star$ convergence of ϱ_n is a direct consequence of $\xi_n(t) + m_{0,n}(t) \le \varepsilon_n \to 0$, see Remark 21.

Verification of limit dynamics: Using Lemma 20 once more we find

$$X_-\big(\sigma(t)\big) = \ell(t) \quad \text{for} \quad \sigma_0(t) < \sigma_\#, \qquad X_+\big(\sigma(t)\big) = \ell(t) \quad \text{for} \quad \sigma_0(t) > \sigma^\#.$$

and this implies

$$\mu_0(t) = -1 \quad \text{for} \quad \sigma_0(t) < \sigma_\#, \qquad \mu_0(t) = +1 \quad \text{for} \quad \sigma_0(t) > \sigma^\#. \tag{37}$$

Combing these results with (36) we readily verify the algebraic relations

$$\big(\ell(t),\, \sigma_0(t),\, \mu_0(t)\big) \in \Omega, \qquad \mathcal{C}\big(\ell(t),\, \sigma_0(t),\, \mu_0(t)\big) = 0, \tag{38}$$

where Ω and $\mathcal{C}$ are defined in (10)+(11). Therefore, and thanks to the properties of ℓ and the functions $X_\pm$, see Definition 5 and Remark 2, the pointwise identities (38) imply that μ_0 belongs in fact to $\mathsf{C}^{0,1}([0,T])$ and satisfies

$$\dot{\ell} = \Big(X_+(\sigma) - X_-(\sigma)\Big)\dot{\mu} + \left(\frac{1-\mu}{2}X_-'(\sigma) + \frac{1+\mu}{2}X_+'(\sigma)\right)\dot{\sigma}$$

for almost all $t \in [0, T]$.

It remains to establish the dynamical relations from Definition 10. We first observe that Lemma 25 yields $\dot{\mu} = 0$ for almost all t with $\sigma(t) \in \big(\sigma_\#, \sigma^\#\big)$, and combining this with (37) we conclude that $\sigma(t) \notin \{\sigma_\#, \sigma^\#\}$ implies $\dot{\mu}(t) = 0$. Now let t be a time such that $\sigma(t) = \sigma_\#$ and $\mu(t) \in (-1, +1)$. The set constraint $(\mu, \sigma) \in \Xi$ then implies $\dot{\sigma}(t) \geq 0$, and in the case of $\dot{\sigma}(t) = 0$ we can employ Lemma 26 to show that $\dot{\ell}(t) \leq 0$ and hence $\dot{\mu}(t) \leq 0$. The derivation of $\dot{\mu}(t) \geq 0$ for $\sigma(t) = \sigma^\#$ is similar. ☐

Notice that Theorem 29 neither implies $\sigma_{\nu_n}(0) \to \sigma_0(0)$ nor $\mu_{\nu_n}(0) \to \mu_0(0)$. This is not surprising because we expect, as explained within §2, that each solution with generic initial data exhibits a small initial transition layer. More precisely, if the mass at time $t = 0$ is not yet concentrated in two narrow peaks, the systems undergoes a fast initial relaxation process during which σ and μ may change rapidly. After this process, that means at some time of order at most $\tau_\nu^{1-\beta}$, $0 < \beta < 1$, the dissipation is of order τ_ν^β and our peak stability estimates imply that afterwards the state ϱ_ν can be described by two narrow peaks, which in turn are either transported by the dynamical constraint or exchange mass by a Kramers-type phase transition.

The above arguments reveal that the limit functions σ_0 and μ_0 can (and in general they do) depend on the subsequence, or equivalently, on the microscopic details of the initial data. For well-prepared initial data, however, we can improve our result as follows.

Theorem 30 (convergence for well-prepared initial data)**.** *For well-prepared initial data in the sense of Definition 8, we can choose $I = [0, T]$ in* (32)*. In particular, the whole family $\big((\ell, \sigma_\nu, \mu_\nu)\big)_{\nu>0}$ converges as $\nu \to 0$ to a solution to the limit model.*

Proof. By assumption, there exist values $\sigma_{\mathrm{ini}} \in \mathbb{R}$ and $\mu_{\mathrm{ini}} \in [-1, 1]$ such that $\sigma_\nu(0) \to \sigma_{\mathrm{ini}}$ as well as $\mu_\nu(0) \to \mu_{\mathrm{ini}}$ as $\nu \to 0$. Now let $((\sigma_n, \mu_n))_{n\in\mathbb{N}}$ be any sequence as provided by Theorem 29. Since the initial data are well-prepared, we can choose $S_n = 0$ in the proof of Theorem 29, see also Remark 9. This implies

$$\|\sigma_n - \sigma_0\|_{\mathsf{C}([0,T])} + \|\mu_n - \mu_0\|_{\mathsf{C}([0,T])} \quad \xrightarrow{\nu\to 0} \quad 0$$

and hence $\sigma_n(0) \to \sigma_{\mathrm{ini}}$ as well as $\mu_n(0) \to \mu_{\mathrm{ini}}$. Since the limit model has only one solution with initial data $(\sigma_{\mathrm{ini}}, \mu_{\mathrm{ini}})$, see Proposition 34, we conclude that each sequence from Theorem 29 has the same limit, and standard arguments (compactness+uniqueness of accumulation points=convergence) provide the claimed convergence. ☐

A Solutions to the nonlocal Fokker-Planck equation

In this appendix we show that the initial value problem to the nonlocal Fokker-Planck equation (FP_1) and (FP_2') is well-posed with state space

$$\mathsf{P}^2(\mathbb{R}) := \Big\{\text{probability measures on } \mathbb{R} \text{ with bounded variance}\Big\}.$$

We emphasize that all results derived in this section apply to arbitrary (i.e., uncoupled) parameters $\nu > 0$ and $\tau > 0$.

Our existence and uniqueness proof is based on a fixed point argument that allows to construct solutions to the nonlocal problem by iterating the solution operator of a linear PDE with a nonlinear integral operator. The key ideas are as follows. Let τ, ν be fixed and ϱ_{ini} be given. For any $\sigma \in \mathsf{C}\big([0,\, T]\big)$, we denote by $\mathcal{R}[\sigma]$ the solution to the linear PDE (FP$_1$). In other words, for each σ the function $\mathcal{R}[\sigma]$ satisfies the initial value problem

$$\tau\partial_t\mathcal{R}[\sigma](t,\,x) = \nu^2\partial_x^2\mathcal{R}[\sigma](t,\,x) + \partial_x\Big(\big(H'(x) - \sigma(t)\big)\mathcal{R}[\sigma](t,\,x)\Big), \qquad \mathcal{R}[\sigma](0,\,x) = \varrho_{\text{ini}}(x) \tag{39}$$

with $x \in \mathbb{R}$ and $t \in [0,\, T]$. Using $\mathcal{R}$, we now observe that the dynamical constraint (FP$_2'$) is equivalent to the fixed point equation $\sigma = \mathcal{S}[\sigma]$, where the operator $\mathcal{S}$ is defined by

$$\mathcal{S}[\sigma](t) := \int_{\mathbb{R}} H'(x)\mathcal{R}[\sigma](t,\,x)\,\mathrm{d}x + \tau\dot{\ell}(t).$$

Notice that (FP$_2'$) implies (FP$_2$) if and only if the initial data are admissible in the sense of $\int_{\mathbb{R}} x\varrho_{\text{ini}}(x)\,\mathrm{d}x = \ell(0)$.

Our first result in this section employs Banach's Fixed Point Theorem in order to show that $\mathcal{S}$ admits a unique fixed point in the space of continuous functions. Afterwards we derive some bounds for these solutions which are uniform with respect to τ and ν.

Proposition 31 (Existence and uniqueness of solution)**.** *For any $\tau > 0$, $\nu > 0$ and all initial data $\varrho_{\text{ini}} \in \mathsf{P}^2(\mathbb{R})$ there exists a unique solution to* (FP$_1$)+(FP$_2'$). *In particular, ϱ is smooth in $(0,\, T]\times\mathbb{R}$ as well as continuous in t with respect to the weak$\star$ topology in $\mathsf{P}^2(\mathbb{R})$, and σ is continuously differentiable on $[0,\, T]$.*

Proof. *Operators and moment balances:* For given $\sigma \in \mathsf{C}\big([0,\, 1]\big)$, the existence, uniqueness and regularity of $\mathcal{R}[\sigma]$ can be established by adapting standard methods. For instance, [Fri75, Section 6, Corollary 4.2 and Theorem 4.5] guarantees the existence and uniqueness of smooth solutions under slightly stronger assumptions (boundedness of H'). For linearly increasing H', we are only aware of results concerning the stochastic Langevin equation $\tau\,\mathrm{d}x = \big(\sigma(t) - H'(x)\big)\,\mathrm{d}t + \nu^2\,\mathrm{d}W$, see e.g. [Fri75, Section 5, Theorem 1.1]. The solution $\mathcal{R}[\sigma]$ to (39) is then provided by the resulting probability distribution function for finding a particle at $(t,\,x)$. We also refer to [JKO98, ASZ09], which study the existence and uniqueness problem for similar equations in the framework of Wasserstein gradient flows.

Using the PDE as well as integration by parts we verify the moment balance

$$\tau\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\psi(x)\varrho(t,\,x)\,\mathrm{d}x = \nu^2\int_{\mathbb{R}}\psi''(x)\varrho(t,\,x)\,\mathrm{d}x + \int_{\mathbb{R}}\psi'(x)\big(\sigma(t) - H'(x)\big)\varrho(t,\,x)\,\mathrm{d}x \tag{40}$$

for any ψ with $|\psi(x)| + |\psi''(x)| \le C\big(1 + x^2\big)$ and $|\psi'(x)| \le C(1 + |x|)$ for all $x \in \mathbb{R}$, and this implies the desired continuity of moments with respect to t. For $\psi(x) = 1$ we obtain $\int \varrho(t,\,x)\,\mathrm{d}x = 1$ and with $\psi(x) = x^2$ we verify that

$$\int_{\mathbb{R}} x^2\varrho(t,\,x)\,\mathrm{d}x \le \left(1 + \int_{\mathbb{R}} x^2\varrho_{\text{ini}}(x)\,\mathrm{d}x\right)\exp\left(C\frac{1 + \nu^2 + \|\sigma\|_\infty}{\tau}t\right).$$

Moreover, the choice $\psi = H'(x)$ reveals that the operator $\mathcal{S}$ is well defined since $H'(x)$ grows linearly as $x \to \pm\infty$, see Assumption 1.

Lipschitz estimates: We next consider two functions $\sigma_1, \sigma_2 \in \mathsf{C}\big([0,\, T]\big)$, abbreviate $\varrho_i := \mathcal{R}[\sigma_i]$, and introduce functions R_1 and R_2 by

$$R_i(t,\,x) := \int_{-\infty}^{x}\varrho_i(t,\,y)\,\mathrm{d}y.$$

The function $R := R_2 - R_1$ then satisfies

$$\tau\partial_t R(t,\,x) = \nu^2\partial_x^2 R(t,\,x) + \big(H'(x) - \sigma_2(t)\big)\partial_x R(t,\,x) - \big(\sigma_2(t) - \sigma_1(t)\big)\varrho_1(t,\,x).$$

In view of $\varrho_i(t,\,\cdot) \in \mathsf{P}^2(\mathbb{R})$ we readily verify that

$$x^2\,|R(t,\,x)| \quad \xrightarrow{x\to\pm\infty} \quad 0,$$

and this implies $R(t,\,\cdot) \in \mathsf{L}^1(\mathbb{R})$ for all t as well as

$$\Big|\mathcal{S}[\sigma_2](t) - \mathcal{S}[\sigma_1](t)\Big| = \left|\int_{\mathbb{R}} H'(x)\partial_x R(t,\,x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} H''(x)R(t,\,x)\,\mathrm{d}x\right| \le C\int_{\mathbb{R}} |R(t,\,x)|\,\mathrm{d}x.$$

In order to derive L^1-bounds for R, we fix some $\varepsilon > 0$ and approximate the modulus function by $h_\varepsilon(r) := \sqrt{\varepsilon + r^2}$. Thanks to $-1 \le h_\varepsilon'(r) \le 1$ and $h_\varepsilon''(r) \ge 0$ for all $r \in \mathbb{R}$, we obtain the moment estimate

$$\begin{aligned}\tau\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} h_\varepsilon\big(R(t,\,x)\big)\,\mathrm{d}x &\le -\int_{\mathbb{R}} H''(x)h_\varepsilon\big(R(t,\,x)\big)\,\mathrm{d}x - \big(\sigma_2(t)-\sigma_1(t)\big)\int_{\mathbb{R}} h_\varepsilon'\big(R(t,\,x)\big)\varrho_1(t,\,x)\\ &\le C\int_{\mathbb{R}} h_\varepsilon\big(R(t,\,x)\big)\,\mathrm{d}x + \Big|\sigma_2(t)-\sigma_1(t)\Big|,\end{aligned}$$

where $C := \|H''\|_\infty$. Using the comparison principle for ODEs and passing to the limit $\varepsilon \to 0$ we therefore get

$$\int_{\mathbb{R}} \big|R(t,\,x)\big|\,\mathrm{d}x \le \tau^{-1}\exp\big(C\tau^{-1}t\big)\int_0^t \Big|\sigma_2(s)-\sigma_1(s)\Big|\,\mathrm{d}s,$$

where we used that $R(0,\,\cdot) = 0$ holds by construction.

Fixed point argument: The estimates derived so far ensure that

$$\Big|\mathcal{S}[\sigma_2](t) - \mathcal{S}[\sigma_1](t)\Big| = C\int_0^t \Big|\sigma_2(s)-\sigma_1(s)\Big|\,\mathrm{d}s,$$

and this implies that $\mathcal{S}$ is a contraction with respect to $\|\sigma\|_\tau := \sup_{t\in[0,\,T]}\exp(-2Ct)\,|\sigma(t)|$, which is equivalent to the standard norm. The existence of a unique fixed point is therefore granted by Banach's Contraction Principle. Now suppose that $\mathcal{S}[\sigma] = \sigma$. From (40) with $\psi(x) = H'(x)$ and $\psi(x) = x$ we then conclude that σ is continuously differentiable and that (FP_2') is satisfied, respectively. □

Proposition 32 (Uniform bounds for solutions)**.** *Let τ and ν be fixed with $0 < \tau < \bar\tau$ and $0 < \nu < \bar\nu$. Then, each solution to the nonlocal Fokker-Planck equation* (FP_1)+(FP_2') *satisfies*

$$\sup_{t\in[0,\,T]}\left(\big|\sigma(t)\big| + \int_{\mathbb{R}} x^2\varrho(t,\,x)\,\mathrm{d}x\right) \le C$$

as well as

$$\sup_{t\in[\nu^2\tau,\,T]}\|\varrho(t,\,\cdot)\|_\infty \le \frac{C}{\nu^2}$$

and

$$\int_{\nu^2\tau}^T \mathcal{D}(t)\,\mathrm{d}t \le C\tau$$

where C is some constant which is independent of τ and ν but depends on H, $\bar\tau$, $\bar\nu$, ℓ, and $\int_{\mathbb{R}} x^2\varrho_{\mathrm{ini}}(x)\,\mathrm{d}x$.

Proof. <u>*Moment estimates:*</u> Due to the constraint (FP_2'), the moment balance (40) with $\psi(x) = x$ implies

$$\tau \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} x^2 \varrho(t,\, x)\,\mathrm{d}x \le 2\nu^2 + 2\|\sigma\|_\infty \|\ell\|_\infty + 2c - c\int_{\mathbb{R}} x^2 \varrho(t,\, x)\,\mathrm{d}x,$$

where c is chosen such that $xH'(x) \ge c\big(x^2 - 1\big)$ holds for all x. Employing the comparison principle for scalar ODEs we therefore find

$$\begin{aligned}\int_{\mathbb{R}} x^2 \varrho(t,\, x)\,\mathrm{d}x &\le 2\nu^2 + 2\|\sigma\|_\infty \|\ell\|_\infty + 2c + \int_{\mathbb{R}} x^2 \varrho_{\mathrm{ini}}(x)\,\mathrm{d}x \\ &\le C\big(1 + \|\sigma\|_\infty\big).\end{aligned}$$

Moreover, by applying Hölder's inequality to (FP_2') we get

$$\begin{aligned}\big|\sigma(t)\big| &\le \tau\big|\dot{\ell}(t)\big| + \Big(\int_{\mathbb{R}} \big|H'(x)\big|^2 \varrho(t,\, x)\,\mathrm{d}x\Big)^{1/2} \Big(\int_{\mathbb{R}} \varrho(t,\, x)\,\mathrm{d}x\Big)^{1/2} \\ &\le C + C\Big(\int_{\mathbb{R}} x^2 \varrho(t,\, x)\,\mathrm{d}x\Big)^{1/2}\end{aligned}$$

where C is some constant independent of τ and ν. The combination of both estimates gives

$$\|\sigma\|_\infty \le C\sqrt{1 + \|\sigma\|_\infty},$$

and the desired moment bounds follow immediately.

<u>L^∞*-estimate after waiting time* $\nu^2\tau$*:*</u> Parabolic regularity theory implies that $\|\varrho(t,\, \cdot)\|_\infty$ is well-defined for all $t > 0$, and thus we only have to understand how this quantity depends on t, τ, ν, and the initial data. To this end we fix t_0 with $0 < t_0 < T$, consider the function

$$M_{t_0}(t) := \sup_{0 \le s \le t} \|\sqrt{s}\varrho(t_0 + s,\, \cdot)\|_\infty,$$

and denote by C any generic constant that is independent of τ, ν and t_0. Using the rescaled heat kernel

$$K(t,\, x) := \sqrt{\frac{\tau}{4\pi\nu^2 t}}\exp\Big(-\frac{\tau x^2}{4\nu^2 t}\Big),$$

as well as Duhamel's Principle, any solution to (FP_1)+(FP_2') can be written as

$$\varrho(t_0 + t,\, x) = I_{1,\, t_0}(t,\, x) + I_{2,\, t_0}(t,\, x),$$

where

$$I_{1, t_0}(t,\, x) := \int_{\mathbb{R}} K(t,\, x - y)\varrho(t_0,\, y)\,\mathrm{d}y$$

and

$$I_{2,\, t_0}(t,\, x) := \frac{1}{\tau}\int_0^t \int_{\mathbb{R}} K_x(t - s,\, x - y) f(t_0 + s,\, y)\,\mathrm{d}y\,\mathrm{d}s, \qquad f(t,\, x) := \big(H'(x) - \sigma(t)\big)\varrho(t,\, x).$$

The first term can be estimated by

$$\big|I_{1,\, t_0}(t,\, x)\big| \le \|K(t,\, \cdot)\|_\infty \int_{\mathbb{R}} \varrho(t_0,\, y)\,\mathrm{d}y \le \frac{C}{\nu}\sqrt{\frac{\tau}{t}},$$

whereas for the second term we employ Hölder's inequality to find

$$\big|I_{2,\, t_0}(t,\, x)\big| \le \frac{1}{\tau}\int_0^t \Big(\int_{\mathbb{R}} K_x(t - s,\, y)^2\,\mathrm{d}y\Big)^{1/2} \Big(\int_{\mathbb{R}} f(t_0 + s,\, y)^2\,\mathrm{d}y\Big)^{1/2}.$$

By direct computations we verify

$$\int_{\mathbb{R}} K_x(t-s,\,y)^2\,\mathrm{d}y = \left(\frac{\tau}{\nu^2(t-s)}\right)^{3/2}\left(\frac{1}{2\pi}\int_{\mathbb{R}} |y|^2 \exp\left(-2y^2\right)\mathrm{d}x\right),$$

and using $|H'(x)| \leq C(1+|x|)$, $\int_{\mathbb{R}} \varrho(t,\,x)\,\mathrm{d}x = 1$ as well as the uniform moment bounds we get

$$\begin{aligned}\int_{\mathbb{R}} f(t_0+s,\,y)^2\,\mathrm{d}y &\leq C\|\varrho(t_0+s,\,\cdot)\|_\infty\left(\left|\sigma(t_0+s)\right|^2 + 1 + \int_{\mathbb{R}} y^2\varrho(t_0+s,\,y)\,\mathrm{d}y\right)\\ &\leq Cs^{-1/2}M_{t_0}(s).\end{aligned}$$

The latter three estimates imply

$$|I_{2,\,t_0}(t,\,x)| \leq \frac{C}{\nu^{3/2}\tau^{1/4}}\int_0^t (t-s)^{-3/4}s^{-1/4}\sqrt{M_{t_0}(s)}\,\mathrm{d}s \leq \frac{C\sqrt{M_{t_0}(t)}}{\nu^{3/2}\tau^{1/4}},$$

where we used $\int_0^t (t-s)^{-3/4}s^{-1/4}\,\mathrm{d}s = \int_0^1 (1-s)^{-3/4}s^{-1/4}\,\mathrm{d}s < \infty$ and that M_{t_0} is an increasing function in t. We therefore get

$$\sqrt{t}\|\varrho(t_0+t,\,\cdot)\|_\infty \leq \frac{C\sqrt{\tau}}{\nu} + \frac{C\sqrt{tM_{t_0}(t)}}{\nu^{3/2}\tau^{1/4}},$$

and since an analogous estimate holds for all $0 \leq s \leq t$, we arrive at the estimate

$$M_{t_0}(t) \leq \frac{C\sqrt{\tau}}{\nu} + \frac{C\sqrt{tM_{t_0}(t)}}{\nu^{3/2}\tau^{1/4}}.$$

This implies

$$\sqrt{t}\|\varrho(t_0+t,\,\cdot)\|_\infty \leq M_{t_0}(t) \leq C\max\left\{\frac{\sqrt{\tau}}{\nu},\,\frac{t}{\nu^3\sqrt{\tau}}\right\}, \tag{41}$$

and for $t = \nu^2\tau$ we get

$$\|\varrho\big(t_0+\nu^2\tau,\,\cdot\big)\|_\infty \leq \frac{C}{\nu^2}.$$

The claimed L^∞-estimate now follows since t_0 was arbitrary and C independent of t_0.

Bound for dissipation: The energy balance (3) implies

$$\begin{aligned}\int_{\nu^2\tau}^T \mathcal{D}(t)\,\mathrm{d}t &= \tau\left(\mathcal{E}\big(\nu^2\tau\big) - \mathcal{E}(T) + \int_{\nu^2\tau}^T \sigma(t)\dot{\ell}(t)\,\mathrm{d}t\right)\\ &\leq \tau\big(\mathcal{E}\big(\nu^2\tau\big) - \mathcal{E}(T) + C\big),\end{aligned}$$

and from the definition of the energy (1), the above L^∞-bounds, and $H(x) \leq C\big(1+x^2\big)$ we infer that

$$\begin{aligned}\mathcal{E}\big(\nu^2\tau\big) &\leq \nu^2\int_{\mathbb{R}} \varrho\big(\nu^2\tau,\,x\big)\ln\varrho\big(\nu^2\tau,\,x\big)\,\mathrm{d}x + C\int_{\mathbb{R}} \big(1+x^2\big)\varrho\big(\nu^2\tau,\,x\big)\,\mathrm{d}x\\ &\leq \left(\nu^2\ln\frac{C}{\nu^2}\right) + C \leq C.\end{aligned}$$

In order to derive a lower for $\mathcal{E}(T)$, we assume (without loss of generality) that the global minimum of H is normalized to 0. The properties of H, see Assumption 1, then guarantee the existence of constants $c > 0$ as well as $\bar{x}_- < 0$ and $\bar{x}_+ > 0$ such that

$$H(x) \geq c\begin{cases} (x-\bar{x}_-)^2 & \text{for} \quad x \leq 0,\\ (x-\bar{x}_+)^2 & \text{for} \quad x \geq 0,\end{cases}$$

and hence we estimate

$$\int_{\mathbb{R}} \gamma_0(x)\,\mathrm{d}x \le \int_{-\infty}^{0} \exp\left(-\frac{c(x-\bar{x}_-)^2}{\nu^2}\right)\mathrm{d}x + \int_{0}^{+\infty} \exp\left(-\frac{c(x-\bar{x}_+)^2}{\nu^2}\right)\mathrm{d}x \le C\nu,$$

where $\gamma_0(x) := \exp\bigl(-H(x)/\nu^2\bigr)$. This implies

$$\begin{aligned}\mathcal{E}(T) &= \nu^2 \int_{\mathbb{R}} \varrho(t,\,x) \ln\left(\frac{\varrho(t,\,x)}{\gamma_0(x)}\right)\mathrm{d}x \\ &\ge \nu^2 \int_{\mathbb{R}} \varrho(t,\,x)\left(\ln\left(\frac{\varrho(t,\,x)}{\gamma_0(x)}\right) + \frac{\gamma_0(x)}{\varrho(t,\,x)} - 1\right)\mathrm{d}x - \nu^2\int_{\mathbb{R}}\gamma_0(x)\,\mathrm{d}x + \nu^2\int_{\mathbb{R}}\varrho(t,\,x)\,\mathrm{d}x \\ &\ge 0 - C\nu^3 + \nu^2,\end{aligned}$$

where we used that $\ln z + 1/z \ge 1$ holds for all $z > 0$, and the desired L^1-estimate for the dissipation follows immediately. ☐

Remark 33. *For initial data $\varrho_{\mathrm{ini}} \in \mathsf{L}^\infty(\mathbb{R})$ we have*

$$\sup_{t\in[0,\,T]} \|\varrho(t,\,\cdot)\|_\infty \le \frac{C}{\nu^2}, \qquad \int_0^T \mathcal{D}(t)\,\mathrm{d}t \le C\tau$$

for some constant C which depends only on H, $\bar{\tau}$, $\bar{\nu}$, ℓ, $\int_{\mathbb{R}} x^2\varrho_{\mathrm{ini}}(x)\,\mathrm{d}x$, and $\nu^2\|\varrho_{\mathrm{ini}}\|_\infty$.

Proof. In this case we can estimate

$$I_{1,\,0}(t,\,x) \le \|\varrho_{\mathrm{ini}}\|_\infty \int_{\mathbb{R}} K(t,\,x)\,\mathrm{d}x = \|\varrho_{\mathrm{ini}}\|_\infty.$$

Moreover, for $0 \le s \le t \le \nu^2\tau$ we infer from (41) that

$$\sqrt{s}\|\varrho(s,\,\cdot)\|_\infty \le M_0(s) \le M_0(t) \le \frac{C\sqrt{\tau}}{\nu}$$

and this implies

$$I_{2,\,0}(t,\,x) \le \frac{C}{\nu^{3/2}\tau^{1/4}} \int_0^t (t-s)^{-3/4}\sqrt{\|\varrho(s,\,\cdot)\|_\infty}\,\mathrm{d}s \le \frac{C}{\nu^2}.$$

The claimed L^∞-estimate now follows from summing both inequalities (for $0 \le t \le \nu^2\tau$) and using Proposition 32 (for $\nu^2\tau \le t \le T$). Moreover, the L^1-bound for the dissipation can be derived as in the proof of Proposition 32. ☐

B Solutions to the limit model

We prove that the initial value problem for the limit model has always a unique solution.

Proposition 34. *For any ℓ as in Assumption 5, and any given initial data $\sigma(0)$ and $\mu(0)$ with $(\ell(0),\,\sigma(0),\,\mu(0)) \in \Omega$, there exist two functions σ and ν on $[0,\,T]$ such that*

1. *both σ and μ are continuous and piecewise continuously differentiable,*
2. *both functions attain the initial data,*
3. *the triple $(\ell,\,\sigma,\,\mu)$ is a solution to the limit model in the sense of Definition 10.*

Moreover, σ and ν are uniquely determined by ℓ, $\sigma(0)$, and $\mu(0)$.

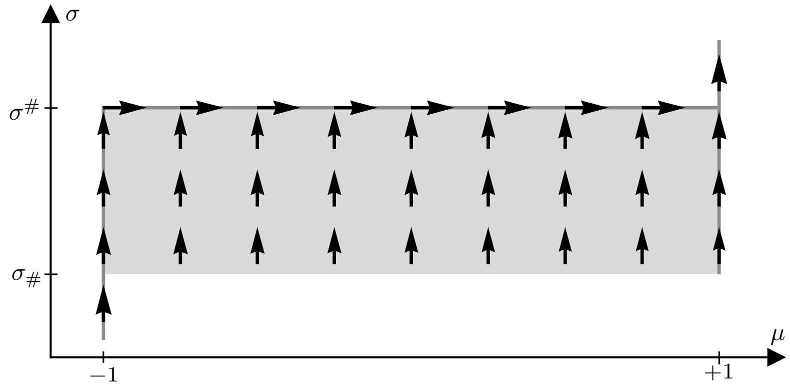

Figure 12: Cartoon of the piecewise smooth vector field $\mathcal{V}_+$ (arrows) on the set Ξ (gray area) as used in the proof of Proposition 34. For given initial data from Ξ, there exists a unique integral curve which is continuous and piecewise continuous differentiable.

Proof. We observe that

$$(\ell,\, \sigma,\, \mu) \in \Omega \qquad \Longrightarrow \qquad (\mu,\, \sigma) \in \Xi$$

where the closed set Ξ is defined by

$$\Xi := \{-1\} \times \Big(-\infty,\, \sigma^{\#}\Big] \;\cup\; (-1,\, +1) \times \Big[\sigma_{\#},\, \sigma^{\#}\Big] \;\cup\; \{+1\} \times [\sigma_{\#},\, +\infty)\,,$$

see Figure 12 for an illustration. Moreover, for each point $(\mu,\, \sigma) \in \Xi$ there exists a unique value for ℓ such that $\mathcal{C}(\ell,\, \sigma,\, \mu) = 0$. We proceed with discussing three special cases: If $\ell(t) = \ell(0)$ holds for all $t \in [0,\, T]$, then the unique solution to the limit model is given by $\sigma(t) = \sigma(0)$ and $\mu(t) = \mu(0)$. In the case of $\dot{\ell}(t) > 0$ for all $t \in (0,\, T)$, we argue as follows. By reparametrization of time, we can assume that $\dot{\ell}(t) = 1$. The pointwise constraint $\mathcal{C}\big(\ell(t),\, \sigma(t),\, \mu(t)\big) = 0$ then implies that any solution to the limit model satisfies

$$\big(\dot{\mu}(t),\, \dot{\sigma}(t)\big) = \mathcal{V}_+\big(\mu(t),\, \sigma(t)\big)$$

for almost all $t \in [0,\, T]$, where the vector field $\mathcal{V}_+ : \Xi \to \mathbb{R}^2$ is defined by

$$\mathcal{V}_+(\mu,\, \sigma) = \begin{cases} \left(\Big(X_+(\sigma) - X_-(\sigma)\Big)^{-1},\, 0\right) & \text{for } -1 \le \mu < +1 \text{ and } \sigma = \sigma^{\#}, \\ \left(0,\, \left(\dfrac{1-\mu}{2} X'_-(\sigma) + \dfrac{1+\mu}{2} X'_+(\sigma)\right)^{-1}\right) & \text{for all other points in } \Xi. \end{cases}$$

Since $\mathcal{V}_+$ is piecewise continuously differentiable with derivative on Ξ, there exists a unique continuous integral curve emanating from the initial data, and this integral curve is obviously piecewise continuously differentiable. The arguments for the third case, that is $\dot{\ell}(t) < 0$ for all $t \in (0,\, T)$, are entirely similar. For arbitrary ℓ, we introduce times $0 = T_0 < T_1 < \ldots < T_N = T$ such that for any $i = 1 \ldots N$ and all $t \in (T_{i-1},\, T_i)$ we have either $\dot{\ell}(t) < 0$, or $\dot{\ell}(t) = 0$, or $\dot{\ell}(t) > 0$. The assertion now follows by iterating the arguments for the special cases. ☐

C Non-standard compactness criterion for continuous functions

In the proof of Theorem 29 we utilize the following, non-standard compactness result in the space of continuous functions.

Proposition 35. *Let I be some compact interval, $g \in \mathsf{C}(I)$ a continuous function on I, and $(c_n)_{n\in\mathbb{N}}$ be a positive sequence with $c_n \to 0$ as $n \to 0$. Moreover, let $(f_n)_{n\in\mathbb{N}} \subset \mathsf{C}(I)$ be a bounded sequence such that*

$$\big|f_n(t_2) - f_n(t_1)\big| \le \big|g(t_2) - g(t_1)\big| + c_n \tag{42}$$

holds for all $t_1, t_2 \in I$ *and all* $n \in \mathbb{N}$. *Then, the sequence* $(f_n)_{n\in\mathbb{N}}$ *is compact in* $\mathsf{C}(I)$ *and hence equicontinuous.*

Proof. Without loss of generality we assume that $I = [0,\,1]$. Since the sequence $(f_n(t))_{n\in\mathbb{N}} \subset \mathbb{R}$ is compact for any $t \in \mathbb{R}$, we can – by the usual diagonal argument – extract a (not relabeled) subsequence, such that $f_n(t)$ converges as $n \to \infty$ for all $t \in I \cap \mathbb{Q}$. Our assumptions imply that the function $\bar{f}_\infty : I \cap \mathbb{Q} \to \mathbb{R}$ defined by

$$\bar{f}_\infty(\bar{t}) := \lim_{n\to\infty} f_n(\bar{t}) \qquad \text{for all} \quad \bar{t} \in I \cap \mathbb{Q},$$

satisfies

$$\big|\bar{f}_\infty(\bar{t}_2) - \bar{f}_\infty(\bar{t}_1)\big| \le \big|g(\bar{t}_2) - g(\bar{t}_1)\big| \qquad \text{for all} \quad \bar{t}_1, \bar{t}_2 \in I \cap \mathbb{Q},$$

and we conclude that $\bar{f}_\infty$ admits a unique continuous extension $f_\infty \in \mathsf{C}(I)$, which obviously satisfies

$$\big|f_\infty(t_2) - f_\infty(t_1)\big| \le \big|g(t_2) - g(t_1)\big| \qquad \text{for all} \quad t_1, t_2 \in I. \tag{43}$$

We next show that f_n converges to f_∞ as $n \to \infty$ strongly in $\mathsf{C}(I)$. To this end let $\delta > 0$ be fixed. Exploiting the continuity of g as well as (42) and (43), we first choose $n_0 \in \mathbb{N}$ and $N \in \mathbb{N}$ such that

$$n \ge n_0, \quad |t_2 - t_1| \le \frac{1}{N} \qquad \Longrightarrow \qquad \big|f_n(t_2) - f_n(t_1)\big| + \big|f_\infty(t_2) - f_\infty(t_1)\big| \le \delta/2, \tag{44}$$

where both n_0 and N can depend on δ. We next divide $I = [0,\,1]$ into N subintervals of length $1/N$, that means we introduce

$$0 = t_0 < t_1 < t_2 < ... < t_N = 1 \qquad \text{with} \qquad t_j := j/N.$$

For each $t \in I$ there exists $j = j(\delta,\,t) \in \{0, 1, ...N\}$ such that $|t - t_j| \le 1/N$, and (44) ensures that

$$n \ge n_0 \qquad \Longrightarrow \qquad \big|f_n(t) - f_n(t_j)\big| + \big|f_\infty(t) - f_\infty(t_j)\big| \le \delta/2.$$

We finally choose n_1 such that

$$n \ge n_1 \qquad \Longrightarrow \qquad \sup_{j\in\{0,1,...N\}} \big|f_n(t_j) - f_\infty(t_j)\big| \le \delta/2,$$

and combining the latter two implications gives

$$n \ge \max\{n_0, n_1\} \qquad \Longrightarrow \qquad \big|f_n(t) - f_\infty(t)\big| \le \delta$$

for all $t \in I$. Since δ was arbitrary, we have thus proven that $\|f_n - f_\infty\|_\infty \to 0$ as $n \to \infty$, and the equicontinuity follows from the Arzelá-Ascoli Theorem (e.g. [DiB02, Proposition 19.1]). □

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